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GENERALIZED SCHUR COMPLEMENTS OF MATRICES AND COMPOUND MATRICES*

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Abstract. In this paper, we obtain some formulas for compound matrices of generalized Schur complements of matrices. Further, we give some Löwner partial orders for compound matrices of Schur complements of positive semidefinite Hermitian matrices, and obtain some estimates for eigenvalues of Schur complements of sums of positive semidefinite Hermitian matrices.

Key words. Löwner partial order, generalized Schur complement, compound matrix, eigenvalues.

AMS subject classifications. 15A45, 15A57.

1. Introduction. Many important results for compound matrices and Schur complements of matrices have been obtained in [2,3,9,10]. Recently, Smith [11], Liu et al. [5,6], Li et al. [4] obtained some estimates of eigenvalues and singular values for Schur complements of matrices. Liu and Wang [7], Wang and Zhang et al. [12] obtained some Löwner partial orders of Schur complements for positive semidefinite Hermitian matrices respectively. Wang and Zhang [13] obtained some Löwner partial orders for Hadamard products of Schur complements of positive definite Hermitian matrices. Liu [8] obtained some Löwner partial orders for Kronecker products of Schur complements of matrices. In this paper, we study Schur complements of matrices and compound matrices.

Let $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of complex, real, and positive real numbers respectively. Let $\mathbb{C}^{m\times n}$ denote the set of $m\times n$ complex matrices. Let N_n denote the set of $n\times n$ normal matrices. Let H_n denote the set of $n\times n$ Hermitian matrices, and let $H_n^{\geq}(H_n^{>})$ denote the subset consisting of positive semidefinite (positive definite)

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Hermitian matrices. For $A \in \mathbb{C}^{m\times n}$, the rank of A is denoted by $r(A)$. Denote by A^* the conjugate transpose matrix of A. For $A, B \in H_n^{\geq}$, write $B \geq A$ if $B - A \in H_n^{\geq}$. The relation "$\geq$" is called the Löwner partial order. For $A \in \mathbb{C}^{n\times n}$, we always arrange the eigenvalues of A as $|\lambda_1(A)| \geq \cdots \geq |\lambda_n(A)|$. For $A \in \mathbb{C}^{m\times n}$, denote the column space of A by $\Re(A)$.

Let k be an integer with $1 \leq k \leq n$. Define

$$Q_{k,n} = \{\omega = \{\omega_1, \ldots, \omega_k\} : \omega_i \in \mathbb{R} \text{ and } 1 \leq \omega_1 < \cdots < \omega_k \leq n\}. \tag{1.1}$$

Given a matrix $A = (a_{ij}) \in \mathbb{C}^{m\times n}$. Let k and r be integers satisfying $1 \leq k \leq m$ and $1 \leq r \leq n$, respectively. If $\alpha \in Q_{k,m}$ and $\beta \in Q_{r,n}$, then $A(\alpha, \beta)$ denotes the $k \times r$ matrix whose (i, j) entry is a_{α_i,β_j}. If α is equal to β, $A(\alpha|\alpha)$ is abbreviated to $A(\alpha)$. Let $A \in \mathbb{C}^{m\times n}$, $l = \min\{m, n\}$, $k \in L = \{1, 2, \ldots, l\}$. We denote by $C_k(A)$ the kth compound matrix. Let all the elements of $Q_{k,m}$ be ordered lexicographically; "$\prec$" denotes the lexicographical order. Let $Q_{r,m} = \left\{\alpha_i | i = 1, \ldots, \begin{pmatrix} m \\ r \end{pmatrix}\right\}$ satisfy $\alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_{\begin{pmatrix} m \\ r \end{pmatrix}}$. Define a mapping $\sigma : \sigma(\alpha_i) = i$; it is a one to one correspondence. We denote $\sigma(\alpha)$ by j_α if $\alpha \in Q_{r,m}$. If $\alpha \in Q_{r,m}$ and $\beta \in Q_{k,n}$, then A_{j_α,j_β} denotes the (j_α, j_β) entry of $C_k(A)$.

Let $A \in \mathbb{C}^{m\times n}$. If $X \in \mathbb{C}^{n\times m}$ satisfies the equations

$$(i)\ AXA = A, \quad (ii)\ XAX = X, \quad (iii)\ (XA)^* = XA, \quad (iv)\ (AX)^* = AX, \tag{1.2}$$

then X is called the Moore-Penrose (MP) inverse of A.

Let $A \in \mathbb{C}^{m\times n}$, $\alpha \subset M$, $\beta \subset N$, $\alpha' = M - \alpha$, and $\beta' = N - \beta$. Then

$$A/_+(\alpha, \beta) = A(\alpha', \beta') - A(\alpha', \beta)[A(\alpha, \beta)]^+ A(\alpha, \beta') \tag{1.3}$$

is called the generalized Schur complement with respect to $A(\alpha, \beta)$. If $A(\alpha, \beta)$ is a nonsingular matrix, then $A/_+(\alpha, \beta) = A/(\alpha, \beta)$ is called the Schur complement with respect to $A(\alpha, \beta)$. If $\alpha = \beta$, we define $A/_+(\alpha, \beta) = A/_+\alpha$ and $A/(\alpha, \beta) = A/\alpha$ respectively. In [1], Ando shows that if $A, B \in H_n^{\geq}$, then

$$(A + B)/\alpha \geq A/\alpha + B/\alpha,$$

and

$$A^{\frac{1}{2}}/\alpha \geq (A/\alpha)^{\frac{1}{2}}.$$

In this paper, we provide some similar results for compound matrices of the Schur complements of positive semidefinite Hermitian matrices and obtain some estimates for eigenvalues.

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2. Some formulae for compound matrices of generalized Schur complements of matrices. In this section, using properties of compound matrices and MP inverses, we obtain some formulae for compound matrices of generalized Schur complements of matrices.

Lemma 2.1. *Let $A \in \mathbb{C}^{m\times n}$. Then*

$$C_k(A^+) = [(C_k(A)]^+. \tag{2.1}$$

Proof. By properties of compound matrices, we have

i. $C_k(A) = C_k(AA^+A) = C_k(A)C_k(A^+)C_k(A)$,
ii. $C_k(A^+) = C_k(A^+AA^+) = C_k(A^+)C_k(A)C_k(A^+)$,
iii. $C_k(A^+A) = C_k(A^+)C_k(A)$,
iv. $C_k(AA^+) = C_k(A)C_k(A^+)$.

Thus, by equations (i)-(iv) of (1.2), we easily get that $C_k(A^+) = [C_k(A)]^+$. ∎

Lemma 2.2. ([4]) *Let $A \in \mathbb{C}^{m\times n}$ be partitioned as*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{2.2}$$

where

$$r\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = r(A_{11}, A_{12}) = r(A_{11}),$$

and

$$r\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} = r(A_{21}, A_{22}) = r(A_{22} - A_{21}A_{11}^+A_{12}).$$

Then

$$A^+ = \begin{pmatrix} A_{11}^+ + A_{11}^+A_{12}S^+A_{21}A_{11}^+ & -A_{11}^+A_{12}S^+ \\ -S^+A_{21}A_{11}^+ & S^+ \end{pmatrix}, \tag{2.3}$$

where $S = A_{22} - A_{21}A_{11}^+A_{12}$.

Lemma 2.3. *Let $A \in H_n^{\geq}$ be partitioned as (2.2) with*

$$r(A) = r(A_{11}) + r(A_{22}).$$

Then (2.3) holds.

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THEOREM 2.4. *Let $A \in \mathbb{C}^{m\times n}$, $\alpha \subset M$, $\beta \subset N$, $\alpha^{'} = M-\alpha$ and $\beta^{'} = N-\beta$. Set $1 \le k \le \min\{|M-\alpha|, |N-\beta|\}$. Suppose that the following conditions are satisfied:*

$$r\begin{pmatrix} A(\alpha^{'},\beta^{'}) \\ A(\alpha,\beta^{'}) \end{pmatrix} = r(A(\alpha^{'},\beta^{'}), A(\alpha^{'},\beta)) = r(A(\alpha^{'},\beta^{'})), \tag{2.4}$$

$$r\begin{pmatrix} A(\alpha^{'},\beta) \\ A(\alpha,\beta) \end{pmatrix} = r(A(\alpha,\beta^{'}), A(\alpha,\beta)) = r(A/_{+}(\alpha,\beta)); \tag{2.5}$$

and

$$r\begin{pmatrix} C_k[A(\alpha^{'},\beta^{'})] \\ C_k(A)(\gamma,\delta^{'}) \end{pmatrix} = r(C_k[A(\alpha^{'},\beta^{'})], C_k(A)(\gamma^{'},\delta)) = r\{C_k[A(\alpha^{'},\beta^{'})]\}, \tag{2.6}$$

$$r\begin{pmatrix} C_k(A)(\gamma^{'},\delta) \\ C_k(A)(\gamma,\delta) \end{pmatrix} = r(C_k(A)(\gamma,\delta^{'}), C_k(A)(\gamma,\delta)) = r(C_k(A)/_{+}(\gamma,\delta)), \tag{2.7}$$

where $\gamma^{'} = \{j_{\bar{\alpha'}} | \bar{\alpha'} \subset \alpha^{'},\ |\bar{\alpha'}| = k, \bar{\alpha'} \in Q_{k,m}\}$, $\delta^{'} = \{j_{\bar{\beta'}} | \bar{\beta'} \subset \beta^{'},\ |\bar{\beta'}| = k, \bar{\beta'} \in Q_{k,n}\}$*; and* $\gamma = \{1,2,\ldots,\begin{pmatrix} m \\ k \end{pmatrix}\} - \gamma^{'}$, $\delta = \{1,2,\ldots,\begin{pmatrix} n \\ k \end{pmatrix}\} - \delta^{'}$.

Then

$$C_k[A/_{+}(\alpha,\beta)] = C_k(A)/_{+}(\gamma,\delta). \tag{2.8}$$

Proof. For $A \in \mathbb{C}^{m\times n}$, there exist permutation matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ such that

$$PAQ = \begin{pmatrix} A(\alpha^{'},\beta^{'}) & A(\alpha^{'},\beta) \\ A(\alpha,\beta^{'}) & A(\alpha,\beta) \end{pmatrix}.$$

Let

$$\tilde{\alpha'} = \{1,2,\ldots,|\alpha^{'}|\}, \tilde{\beta'} = \{1,2,\ldots,|\beta^{'}|\}, \tilde{\alpha} = M - \tilde{\alpha'},$$

$$\tilde{\beta} = N - \tilde{\beta'}, \tilde{\gamma'} = \{1,2,\ldots,|\gamma^{'}|\}, \tilde{\delta'} = \{1,2,\ldots,|\delta^{'}|\},$$

$$\tilde{\gamma} = \{1,2,\ldots,\begin{pmatrix} m \\ k \end{pmatrix}\} - \tilde{\gamma'}, \tilde{\delta} = \{1,2,\ldots,\begin{pmatrix} n \\ k \end{pmatrix}\} - \tilde{\delta'}.$$

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Thus, by (2.3), (2.4) and Lemma 2.2, we have

$$[A^+(\alpha^{'},\beta^{'})]^+ = A/_+(\alpha,\beta). \tag{2.9}$$

By (2.5), (2.6) and Lemma 2.2, we have

$$\{[C_k(A)]^+(\gamma^{'},\delta^{'})\}^+ = C_k(A)/_+(\gamma,\delta). \tag{2.10}$$

Therefore, from (1.3), (2.8) and (2.9), it follows that

$$\begin{aligned} C_k[A/_+(\alpha,\beta)] &= C_k[(PAQ)/_+(\tilde{\alpha},\tilde{\beta})] \\ &= C_k\{[(PAQ)^+(\tilde{\alpha}^{'},\tilde{\beta}^{'})]^+\} \quad \text{(by (2.8))} \\ &= \{C_k[(PAQ)^+(\tilde{\alpha}^{'},\tilde{\beta}^{'})]\}^+ \quad \text{(by (1.3))} \\ &= \{C_k[(PAQ)^+](\tilde{\gamma}^{'},\tilde{\delta}^{'})\}^+ \\ &= \{[C_k(PAQ)]^+(\tilde{\gamma}^{'},\tilde{\delta}^{'})\}^+ \quad \text{(by (1.3))} \\ &= C_k(PAQ)/_+(\tilde{\gamma},\tilde{\delta}) \quad \text{(by (2.9))} \\ &= C_k(A)/_+(\gamma,\delta). \qquad \square \end{aligned}$$

Corollary 2.5. *Let $A \in \mathbb{C}^{n\times n}$, $\alpha \subset N$ and $\alpha^{'} = N-\alpha$. Set $1 \le k \le n-|\alpha|$. If A, $A(\alpha)$, and $A(\alpha^{'})$ are nonsingular respectively, then*

$$C_k(A/\alpha) = C_k(A)/\gamma \tag{2.11}$$

where $\gamma^{'} = \{j_{\bar{\alpha}^{'}} | \bar{\alpha}^{'} \subset \alpha^{'},\ |\bar{\alpha}^{'}| = k, \bar{\alpha}^{'} \in Q_{k,n}\}$ and $\gamma = \{1,2,\ldots,\binom{n}{k}\} - \gamma^{'}$.

In a manner similar to the proof of Theorem 2.4, we obtain the following result by using Lemma 2.3.

Theorem 2.6. *Let $A \in H_n^{\geq}$, $\alpha \subset N$ and $\alpha^{'} = N-\alpha$. Set $1 \le k \le n-|\alpha|$. Suppose that the following conditions are satisfied:*

$$\begin{aligned} r(A) &= r(A(\alpha)) + r(A(\alpha^{'})), \\ r[C_k(A)] &= r[C_k(A)(\gamma)] + r[C_k(A)(\gamma^{'})], \end{aligned}$$

where $\gamma^{'} = \{j_{\bar{\alpha}^{'}} | \bar{\alpha}^{'} \subset \alpha^{'}, |\bar{\alpha}^{'}| = k, \bar{\alpha}^{'} \in Q_{k,n}\}$, $\gamma = \{1,2,\ldots,\binom{n}{k}\} - \gamma^{'}$. Then

$$C_k(A/_+\alpha) = C_k(A)/_+\gamma. \tag{2.12}$$

Corollary 2.7. *Let $A \in H_n^{>}$, $\alpha \subset N$ and $\alpha^{'} = N-\alpha$. Set $1 \le k \le n-|\alpha|$. Then*

$$C_k(A/\alpha) = C_k(A)/\gamma, \tag{2.13}$$

where $\gamma^{'} = \{j_{\bar{\alpha}^{'}} | \bar{\alpha}^{'} \subset \alpha^{'}, |\bar{\alpha}^{'}| = k, \bar{\alpha}^{'} \in Q_{k,n}\}$, $\gamma = \{1,2,\ldots,\binom{n}{k}\} - \gamma^{'}$.

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3. Some Löwner partial orders for compound matrices of sums of matrices. In this section, we obtain some Löwner partial orders for compound matrices of Schur complements of positive semidefinite Hermitian matrices. Further, we obtain some estimates for eigenvalues of Schur complements of sums of positive semidefinite Hermitian matrices.

Lemma 3.1. ([2, p. 184]) *Let $A, B \in H_n$. Then*

$$\lambda_t(A+B) \geq \max_{i+j=n+t}\{\lambda_i(A)+\lambda_j(B)\}. \tag{3.1}$$

Lemma 3.2. (i) *Let $A \in H_n^{>}, k \in N$ and $r \in R$. Then*

$$C_k(A^r) = [C_k(A)]^r. \tag{3.2}$$

(ii) *Let $A \in H_n^{\geq}, k \in N$ and $r \in R_+$. Then* (3.2) *holds.*

Proof. Since $A \in H_n^{>}$, there exists an unitary matrix U such that

$$A = U\mathrm{diag}(\lambda_1(A), \ldots, \lambda_n(A))U^*$$

where $\lambda_i(A) > 0$ $(i = 1, 2, \ldots, n)$. Thus

$$\begin{aligned}
C_k(A^r) &= C_k[U\mathrm{diag}(\lambda_1^r(A), \ldots, \lambda_n^r(A))U^*]\\
&= C_k(U)C_k[\mathrm{diag}(\lambda_1^r(A), \ldots, \lambda_n^r(A)][C_k(U)]^*\\
&= C_k(U)\mathrm{diag}([\lambda_1(A)\ldots\lambda_k(A)]^r, \ldots, [\lambda_{n-k+1}(A)\ldots\lambda_n(A)]^r)[C_k(U)]^*\\
&= \{C_k(U)\mathrm{diag}(\lambda_1(A)\ldots\lambda_k(A), \ldots, \lambda_{n-k+1}(A)\ldots\lambda_n(A))[C_k(U)]^*\}^r\\
&= \{C_k(U)C_k[\mathrm{diag}(\lambda_1(A), \ldots, \lambda_n(A))][C_k(U)]^*\}^r\\
&= \{C_k[U\mathrm{diag}(\lambda_1(A), \ldots, \lambda_n(A))U^*]\}^r\\
&= [C_k(A)]^r.
\end{aligned}$$

In a manner similar to the proof of (i), we obtain (ii). $\square$

Lemma 3.3. *Let $A, B \in H_n^{\geq}, k \in N$. Then*

$$C_k(A+B) \geq C_k(A) + C_k(B). \tag{3.3}$$

Proof. Since $A, B \in H_n^{\geq}$, we have

$$\begin{aligned}
C_k(A+B) &= C_k(A^{\frac{1}{2}}A^{\frac{1}{2}} + B^{\frac{1}{2}}B^{\frac{1}{2}}) \qquad (3.4)\\
&= C_k[(A^{\frac{1}{2}}, B^{\frac{1}{2}})(A^{\frac{1}{2}}, B^{\frac{1}{2}})^*]\\
&= C_k[(A^{\frac{1}{2}}, B^{\frac{1}{2}})]\{C_k[(A^{\frac{1}{2}}, B^{\frac{1}{2}})]\}^*.
\end{aligned}$$

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It is not difficult to show that there exist X and a permutation matrix U such that

$$C_k[(A^{\frac{1}{2}}, B^{\frac{1}{2}})] = (C_k(A^{\frac{1}{2}}), C_k(B^{\frac{1}{2}}), X)U, \tag{3.5}$$

where X is $\begin{pmatrix} n \\ k \end{pmatrix} \times \left[\begin{pmatrix} 2n \\ k \end{pmatrix} - 2\begin{pmatrix} n \\ k \end{pmatrix}\right]$ and U is $\begin{pmatrix} 2n \\ k \end{pmatrix} \times \begin{pmatrix} 2n \\ k \end{pmatrix}$.

Therefore, by (3.3) and (3.4), we have

$$\begin{aligned} C_k(A+B) &= [(C_k(A^{\frac{1}{2}}), C_k(B^{\frac{1}{2}}), X)U][C_k(A^{\frac{1}{2}}), C_k(B^{\frac{1}{2}}), X)U]^* \\ &= ([C_k(A)]^{\frac{1}{2}}, [C_k(B)]^{\frac{1}{2}}, X)([C_k(A)]^{\frac{1}{2}}, [C_k(B)]^{\frac{1}{2}}, X)^* \\ &= C_k(A) + C_k(B) + XX^* \\ &\geq C_k(A) + C_k(B). \qquad \square \end{aligned}$$

Lemma 3.4. *Let $A, B \in H_n^{\geq}, A \geq B$ and $k \in N$. Then*

$$C_k(A) \geq C_k(B). \tag{3.6}$$

Proof. Lemma 3.3 ensures that

$$C_k(A) = C_k[B + (A-B)] \geq C_k(B) + C_k(A-B) \geq C_k(B). \qquad \square$$

Theorem 3.5. *Let $A, B \in H_n^{\geq}$, $\alpha \subset N$, and $\alpha^{'} = N - \alpha$. Set $1 \leq k \leq n - |\alpha|$. Suppose that the following conditions are satisfied:*

$$r(A) = r(A(\alpha)] + r[A(\alpha^{'})], \quad r(B) = r[B(\alpha)] + r[B(\alpha^{'})], \tag{3.7}$$

$$r[C_k(A)] = r[C_k(A)(\gamma)] + r[C_k(A)(\gamma^{'})], \quad r[C_k(B)] = r[C_k(B)(\gamma)] + r[C_k(B)(\gamma^{'})], \tag{3.8}$$

where $\gamma^{'} = \{j_{\bar{\alpha}^{'}} | \bar{\alpha}^{'} \subset \alpha^{'}, |\bar{\alpha}^{'}| = k, \bar{\alpha}^{'} \in Q_{k,n}\}$ and $\gamma = \{1, 2, \ldots, \begin{pmatrix} n \\ k \end{pmatrix}\} - \gamma^{'}$. Then

$$C_k[(A+B)/_+\alpha] \geq C_k(A)/_+\gamma + C_k(B)/_+\gamma. \tag{3.9}$$

Proof. By [8, Theorem 3.1], it follows that

$$(A+B)/_+\alpha \geq A/_+\alpha + B/_+\alpha. \tag{3.10}$$

Thus, by (3.5), (3.9), (3.2), (3.6), (3.7) and (2.13), we conclude that

$$\begin{aligned} C_k[(A+B)/_+\alpha] &\geq C_k(A/_+\alpha + B/_+\alpha) \ \ \text{(by (3.5) and (3.9))} \\ &\geq C_k(A/_+\alpha) + C_k(B/_+\alpha) \ \ \text{(by (3.2))} \\ &= C_k(A)/_+\gamma + C_k(B)/_+\gamma. \ \ \text{(by (3.6), (3.7) and (2.13))}. \qquad \square \end{aligned}$$

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COROLLARY 3.6. *Let all assumptions of* Theorem 3.5 *be satisfied. If* $A-B \in H_n^{\geq}$, *then*

$$C_k[(A-B)/_+\alpha] \leq C_k(A)/_+\gamma - C_k(B)/_+\gamma. \tag{3.11}$$

Proof. Since $A-B \in H_n^{\geq}$, Theorem 3.5 ensures that

$$C_k(A)/_+\gamma = C_k(A/_+\alpha) = C_k[(B+(A-B))/_+\alpha]$$

is at least

$$C_k(B/_+\alpha) + C_k[(A-B)/_+\alpha] = C_k(B)/_+\gamma + C_k[(A-B)/_+\alpha],$$

which means that (3.11) holds. □

THEOREM 3.7. *Let all the assumptions of* Theorem 3.5 *be satisfied. Then*

$$\prod_{t=1}^{k} \lambda_t[(A+B)/_+\alpha] \tag{3.12}$$

is bounded below by the maximum of

$$\prod_{t=1}^{k} \lambda_t(A/_+\alpha) + \prod_{t=1}^{k} \lambda_{n-|\alpha|-t+1}(B/_+\alpha)$$

and

$$\prod_{t=1}^{k} \lambda_{n-|\alpha|-t+1}(A/_+\alpha) + \prod_{t=1}^{k} \lambda_t(B/_+\alpha).$$

Proof. Theorem 3.5 and (2.13) imply that

$$\prod_{t=1}^{k} \lambda_t[(A+B)/_+\alpha] \lambda_1\{C_k[(A+B)/_+\alpha]\} \geq \lambda_1[C_k(A/_+\alpha) + C_k(B/_+\alpha),$$

which is bounded below by the maximum of

$$\lambda_1[(C_k(A/_+\alpha)] + \lambda_{\binom{n-|\alpha|}{k}}[C_k(B/_+\alpha)]$$

and

$$\lambda_{\binom{n-|\alpha|}{k}}[C_k(A/_+\alpha) + \lambda_1[(C_k(B/_+\alpha)],$$

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and hence by the maximum of

$$\prod_{t=1}^{k} \lambda_t(A/_+\alpha) + \prod_{t=1}^{k} \lambda_{n-|\alpha|-t+1}(B/_+\alpha)$$

and

$$\prod_{t=1}^{k} \lambda_{n-|\alpha|-t+1}(A/_+\alpha) + \prod_{t=1}^{k} \lambda_t(B/_+\alpha). \qquad \square$$

Remark 3.8. In a manner similar to the proof of Theorem 3.7, we get the following result

$$\prod_{t=1}^{k} \lambda_{n-|\alpha|-t+1}[(A+B)/_+\alpha] \geq \prod_{t=1}^{k} \lambda_{n-|\alpha|-t+1}(A/_+\alpha)+\prod_{t=1}^{k} \lambda_{n-|\alpha|-t+1}(B/_+\alpha). \tag{3.13}$$

Theorem 3.9. *Let* $B \in H_n^{\geq}$, $L = \{1, 2, \ldots, \min\{m, n\}\}$, $\alpha \subset L$, *and* $\alpha^{'} = M - \alpha$, $\beta^{'} = N - \alpha$. *Set* $1 \leq k \leq |L - \alpha|$. *If* $A \in \mathbb{C}^{m\times n}$ *satisfies conditions* (2.3)-(2.6) *and*

$$\Re[A(\alpha, \alpha^{'})] \subseteq \Re[A(\alpha)], \tag{3.14}$$

then

$$C_k[(ABA^*)/_+\alpha] \leq [C_k(A)/_+\gamma]C_k(B)(\delta^{'})[C_k(A)/_+\gamma]^*, \tag{3.15}$$

where $\gamma^{'} = \{j_{\bar{\alpha^{'}}} | \bar{\alpha^{'}} \subset \alpha^{'}, |\bar{\alpha^{'}}| = k, \bar{\alpha^{'}} \in Q_{k,m}\}$, $\delta^{'} = \{j_{\bar{\beta^{'}}} | \bar{\beta^{'}} \subset \beta^{'}, |\bar{\beta^{'}}| = k, \bar{\beta^{'}} \in Q_{k,m}\}$, *and* $\gamma = \{1, 2, \ldots, \begin{pmatrix} m \\ k \end{pmatrix}\} - \gamma^{'}$, $\delta = \{1, 2, \ldots, \begin{pmatrix} n \\ k \end{pmatrix}\} - \delta^{'}$.

Proof. Using (3.13), in a manner similar to the proof of [10, Theorem 3] and [9, Theorem 2], it follows that

$$(ABA^*)/_+\alpha \leq (A/_+\alpha)B(\alpha^{'})(A/_+\alpha)^*. \tag{3.16}$$

Thus, from (3.15), (3.5) and (2.7), we obtain

$$\begin{aligned} C_k[(ABA^*)/_+\alpha] &\leq C_k[(A/_+\alpha)B(\beta^{'})(A/_+\alpha)^*] \quad \text{(by (3.15) and (3.5))} \\ &= C_k(A/_+\alpha)C_k[B(\beta^{'})]C_k[(A/_+\alpha)^*] \\ &= [C_k(A)/_+\gamma]C_k(B)(\delta^{'})[C_k(A)/_+\gamma]^* \quad \text{(by (2.7))}. \qquad \square \end{aligned}$$

Theorem 3.10. *Let all assumptions of* Corollary 2.7 *be satisfied, and* $0 \leq l \leq 1$. *Then*

$$C_k(A^l/\alpha) \geq [C_k(A)/\gamma]^l. \tag{3.17}$$

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Proof. By [13], for $1 \leq t \leq +\infty$, we have

$$A(\alpha') \leq [A^t(\alpha')]^{\frac{1}{t}}. \tag{3.18}$$

Replace A with $(A^{-1})^{\frac{1}{t}}$ in Eqs. (3.17), and let $l = \frac{1}{t}$. Then

$$(A^l)^{-1}(\alpha') = (A^{-1})^l(\alpha') \leq [A^{-1}(\alpha')]^l. \tag{3.19}$$

It is known by [3, p. 474] that for $B \in H_n^>$,

$$B^{-1}(\alpha') = (B/\alpha)^{-1}. \tag{3.20}$$

Thus, by (3.18) and (3.19), we get

$$A^l/\alpha \geq (A/\alpha)^l. \tag{3.21}$$

Therefore, from (3.20), (3.5), (3.1), and (2.12), we have

$$C_k(A^l/\alpha) \geq C_k[(A/\alpha)^l] = [C_k(A/\alpha)]^l = [C_k(A)/\gamma]^l. \qquad \square$$

4. Some Löwner partial orders for compound matrices of Schur complements of two types matrices. Let $A \in \mathbb{C}^{n\times n}$. Then

$$H_A = \frac{A+A^*}{2},\ S_A = \frac{A-A^*}{2}. \tag{4.1}$$

In this section, we study compound matrices of Schur complements of complex square matrices that are either normal or have positive definite Hermitian part.

THEOREM 4.1. *Let all assumptions of* Corollary 2.5 *be satisfied. If* $(A+A^*)(\alpha) \in H_{|\alpha|}^>$, *then*

$$C_k(A+A^*)/\gamma \leq C_k[A/\alpha + (A/\alpha)^*] \leq \left[\frac{|\det A|}{\det H_A}\right]^{2k} C_k(A+A^*)/\gamma. \tag{4.2}$$

Proof. By [9, Theorem 7 and Theorem 8], we have

$$\left[\frac{\det H_A}{|\det A|}\right]^2 [A/\alpha + (A/\alpha)^*] \leq (A+A^*)/\alpha \leq A/\alpha + (A/\alpha)^*. \tag{4.3}$$

Thus

$$(A+A^*)/\alpha \leq A/\alpha + (A/\alpha)^* \leq \left[\frac{|\det A|}{\det H_A}\right]^2 (A+A^*)/\alpha. \tag{4.4}$$

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By (4.3), (3.5) and (2.10), we get that

$$\begin{aligned} C_k[A/\alpha+(A/\alpha)^*] &\le C_k\left\{\left[\frac{|\det A|}{\det H_A}\right]^2 (A+A^*)/\alpha\right\} \quad \text{(by(4.3) and (3.5))} \\ &= \left[\frac{|\det A|}{\det H_A}\right]^{2k} C_k[(A+A^*)/\alpha] \\ &= \left[\frac{|\det A|}{\det H_A}\right]^{2k} C_k(A+A^*)/\gamma \quad \text{(by (2.10))}, \end{aligned}$$

and

$$\begin{aligned} C_k[A/\alpha+(A/\alpha)^*] &\ge C_k[(A+A^*)/\alpha] \\ &= C_k(A+A^*)/\gamma. \qquad \square \end{aligned}$$

Theorem 4.2. *Let $A \in N_n$, $\alpha \subset N$ and $\alpha^{'} = N-\alpha$. Set $1 \le k \le n-|\alpha|$. Suppose that the following conditions are satisfied:*

$$\begin{aligned} &r(H_A^2) = r[H_A^2(\alpha)] + r[H_A^2(\alpha^{'})], \\ &r(S_A^2) = r[S_A^2(\alpha)] + r[S_A^2(\alpha^{'})], \\ &r[C_k(H_A^2)] = r[C_k(H_A^2)(\gamma)] + r[C_k(H_A^2)(\gamma^{'})], \\ &r[C_k(S_A^2)] = r[C_k(S_A^2)(\gamma)] + r[C_k(S_A^2)(\gamma^{'})], \end{aligned}$$

where $\gamma^{'} = \{j_{\bar{\alpha}'} | \bar{\alpha}' \subset \alpha^{'}, |\bar{\alpha}'| = k, \bar{\alpha}' \in Q_{k,n}\}$, $\gamma = \{1, 2, \ldots, \begin{pmatrix} n \\ k \end{pmatrix}\} - \gamma^{'}$. *Then*

$$C_k[(AA^*)/_+\alpha] \ge [C_k(H_A)]^2/_+\gamma + (-1)^k[C_k(S_A)]^2/_+\gamma. \tag{4.5}$$

Proof. Since $A \in N_n$, we have

$$H_A S_A = S_A H_A. \tag{4.6}$$

Noting that $S_A^* = -S_A$, by (4.6), we obtain

$$AA^* = (H_A+S_A)(H_A+S_A)^* = (H_A+S_A)(H_A+S_A^*)$$

$$= H_A^2 + H_A S_A^* + S_A H_A + S_A S_A^* = H_A^2 + S_A S_A^*. \tag{4.7}$$

By (4.6), (3.9), (3.5), (3.2) and (2.11), we have

$$C_k[(AA^*)/_+\alpha] = C_k[(H_A^2 + S_A S_A^*)/_+\alpha] \quad \text{(by (4.6))}$$

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ELA

$$
\begin{aligned}
&\geq C_k[H_A^2/_+\alpha + (S_AS_A^*)/_+\alpha] \quad \text{(by (3.9) and (3.5))}\\
&\geq C_k(H_A^2/_+\alpha) + C_k[(S_AS_A^*)/_+\alpha] \quad \text{(by (3.2))}\\
&= C_k(H_A^2)/_+\gamma + C_k(S_AS_A^*)/_+\gamma \quad \text{(by (2.11))}\\
&= [C_k(H_A)]^2/_+\gamma + C_k(-S_A^2)/_+\gamma\\
&= [C_k(H_A)]^2/_+\gamma + (-1)^k[C_k(S_A)]^2/_+\gamma. \qquad \square
\end{aligned}
$$

THEOREM 4.3. *Let all the assumptions of* Corollary 2.5 *be satisfied. Suppose* $A \in N_n$ *is nonsingular and each of* H_A *and* S_A *is nonsingular. Then*

$$
\begin{aligned}
C_k[(AA^*)^{-1}/\alpha] \geq\ & [C_k(H_A - S_AH_A^{-1}S_A)]^{-2}/\gamma\\
&+(-1)^k[C_k(S_A - H_AS_A^{-1}H_A)]^{-2}/\gamma.
\end{aligned} \tag{4.8}
$$

Proof. Since $A \in N_n$, we have $A^{-1} \in N_n$. Further

$$
\begin{aligned}
H_{A^{-1}} &= \frac{1}{2}(A^{-1} + (A^{-1})^*)\\
&= \frac{1}{2}A^{-1}(A + A^*)(A^{-1})^*)\\
&= (A^*H_A^{-1}A)^{-1}\\
&= [(H_A + S_A)^*H_A^{-1}(H_A + S_A)]^{-1}\\
&= (H_A - S_AH_A^{-1}S_A)^{-1}.
\end{aligned}
$$

Similarly, we have

$$S_{A^{-1}} = \frac{1}{2}(A^{-1} - A^{-1*}) = (S_A - H_AS_A^{-1}H_A)^{-1}.$$

Thus, in a manner similar to the proof of Theorem 4.2, we obtain (4.8). $\square$

5. Conclusions. We have obtained some formulae for compound matrices of generalized Schur complements of matrices. Using these results, we studied some Löwner partial orders for compound matrices of Schur complements of positive semi-definite Hermitian matrices. If $A, B \in H_n^{\geq}$, we extend some results in [1] and show that

$$C_k[(A + B)/_+\alpha] \geq C_k(A)/_+\gamma + C_k(B)/_+$$

under some restrictive conditions, as shown in Theorem 3.5, as well as

$$C_k(A^l/\alpha) \geq [C_k(A)/\gamma]^l$$

if $A \in H_n^{>}$, as shown in Theorem 3.10. In addition, we provide some results for compound matrices of Schur complements of complex square matrices that are either

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normal or have positive definite Hermitian part. We obtained some estimates for eigenvalues.

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