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# NEW PROPERTIES OF A SPECIAL MATRIX RELATED TO POSITIVE-DEFINITE MATRICES*

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**Abstract.** Let $H$ be a $2n \times 2n$ real symmetric positive-definite matrix. Suppose that $H \circ H = (H_{ij})_{2n\times 2n}$ is a partitioned matrix, in which $\circ$ represents the Hadamard product and the block $H_{ij}$ has order $n \times n$, $1 \le i, j \le 2$. Several new properties on the matrix $\widetilde{H}$ are derived including inequalities that involve the symplectic eigenvalues and the usual eigenvalues, where $2\widetilde{H} = H_{11} + H_{22} + H_{12} + H_{21}$.

**Key words.** Positive-definite matrix, Symplectic eigenvalue, Principal submatrix, Schur inequality.

**AMS subject classifications.** 15A18, 15A42.

**1. Introduction.** Let $\mathbb{R}^{n\times n}$, $\mathbb{P}(n)$, $\mathbb{P}_0(n)$ and $I_n$ be the set of $n \times n$ real matrices, and the set of $n \times n$ real symmetric positive-definite matrices, and the set of $n \times n$ real symmetric positive-semidefinite matrices, and the $n \times n$ identity matrix, respectively. Denote by $J$ the $2n \times 2n$ matrix $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$, we define the set of symplectic matrices $\mathrm{Sp}(2n)$ and the set of rectangular symplectic matrices $\mathrm{Sp}(2k, 2n)$ to be $\mathrm{Sp}(2n) = \{M \in \mathbb{R}^{2n\times 2n} : M^T J M = J\}$ and $\mathrm{Sp}(2k, 2n) = \{M \in \mathbb{R}^{2n\times 2k} : M^T J_{2n} M = J_{2k}\}$ for some $k$ with $1 \le k \le n$, respectively. Williamson's theorem (see [1, 11]) says that for every element $A \in \mathbb{P}(2n)$, there exists a symplectic matrix $M$ such that

$$M^T A M = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix},$$

where $D = \mathrm{diag}(d_1(A), \dots, d_n(A))$ with diagonal elements $0 < d_1(A) \le d_2(A) \le \cdots \le d_n(A)$. The diagonal entries of $D$ are known as symplectic eigenvalues of $A$.

Let $H \in \mathbb{R}^{2n\times 2n}$ have a block decomposition

$$H = \begin{pmatrix} L & Y_h \\ C & Z \end{pmatrix},$$

where $C, L, Y_h, Z$ are $n \times n$ matrices. Let $\widetilde{H}$ be the $n \times n$ matrix whose entries are given by

$$\widetilde{h}_{ij} = \frac{1}{2}\left(c_{ij}^2 + l_{ij}^2 + y_{ij}^2 + z_{ij}^2\right), \tag{1}$$

where $C = (c_{ij})_{i,j=1}^n, L = (l_{ij})_{i,j=1}^n, Y_h = (y_{ij})_{i,j=1}^n, Z = (z_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n}$.

In Bhatia and Jain [3], the matrix $\widetilde{H}$ was introduced and several properties were obtained. Our recent paper [5] also presented another proof of [3, Theorem 6]. Meanwhile, we established an analog of Schur-Horn

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theorem via the matrix $\widetilde{H}$ (see [4]). In this paper, new properties on the matrix $\widetilde{H}$ are derived including inequalities that involve the symplectic eigenvalues and the usual eigenvalues.

**2. New properties of $\widetilde{H}$.** Bhatia and Jain [3] showed that $\widetilde{H}$ is doubly superstochastic for the symplectic matrix $H$. Our first result provides an analog of [3, Theorem 6] for the positive-definite matrix $H$.

THEOREM 2.1. *Let $H \in \mathbb{P}(2n)$ and $\widetilde{H}$ be the $n \times n$ matrix associated with $H$ according to the rule (1). Then $\widetilde{H}$ is a positive-definite matrix.*

*Proof.* The condition $H \in \mathbb{P}(2n)$ implies $H \circ H$ is positive-definite (see [7, Theorem 5.2.1]). We partition $H \circ H$ in the form

$$H \circ H = \begin{pmatrix} P_1 & P_3 \\ P_3^T & P_2 \end{pmatrix},$$

where $P_i \in \mathbb{R}^{n\times n}, i = 1, 2, 3$. Note that

$$\begin{pmatrix} 2\widetilde{H} & P_2 + P_3 \\ P_2 + P_3^T & P_2 \end{pmatrix} = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} (H \circ H) \begin{pmatrix} I & 0 \\ I & I \end{pmatrix},$$

is a positive-definite matrix, which implies $\widetilde{H}$ is positive-definite. ∎

We note that the proof of Theorem 2.1 also holds for the positive-semidefinite matrices.

COROLLARY 2.2. *Let $H \in \mathbb{P}_0(2n)$ and $\widetilde{H}$ be the $n \times n$ matrix associated with $H$ according to the rule (1). Then $\widetilde{H}$ is a positive-semidefinite matrix.*

Let $H \in \mathrm{Sp}(2n)$ and $X = (x_{ij})_{l\times k}$ be any submatrix of $\widetilde{H}$. [3, Theorem 6] implies $\sum_{i,j} x_{ij} \geq \max\{k + l - n, 0\}$ (see [2, Theorem 1] or [9, p.44. D.4.Theorem]). We present a similar result for the positive-definite matrix $H$. For ease of presentation, let $\tau \subseteq \Omega = \{1, 2, \ldots, n\}$ with $|\tau| = l$ and

$$I_\tau = (e_{ij})_{n\times n}, e_{ij} = \begin{cases} 1, & i = j \in \tau; \\ 0, & \text{else.} \end{cases}$$

In addition, write $|\tau|$ to indicate the cardinality of $\tau$. Our proof of the first inequality in Theorem 2.3 relies on the Schur inequality: Let $A \in \mathbb{C}^{n\times n}$ have eigenvalues $\{\lambda_i(A)\}$. Then,

$$\|A\|_F^2 = \sum_{i=1}^{n} \sigma_i^2(A) \geq \sum_{i=1}^{n} |\lambda_i(A)|^2,$$

where $\sigma_i(A)$ denotes the $i$th singular value of $A$.

THEOREM 2.3. *Let $H \in \mathbb{P}(2n)$ and $\widetilde{H}$ be the $n \times n$ matrix associated with $H$ according to the rule (1). Suppose $Y = (y_{ij})_{l\times l}$ be any $l \times l$ principal submatrix of $\widetilde{H}$, $1 \leq l \leq n$, we have*

$$\sum_{i=1}^{l} d_i^2(H) \leq \sum_{i,j=1}^{l} y_{ij} \leq \frac{1}{2} \sum_{i=2n-2l+1}^{2n} \lambda_i^2(H).$$

*Proof.* Note that

$$2\sum_{i,j=1}^{n} y_{ij} = \mathrm{tr}\left[\begin{pmatrix} I_{\tau_1} & 0 \\ 0 & I_{\tau_1} \end{pmatrix} H \begin{pmatrix} I_{\tau_1} & 0 \\ 0 & I_{\tau_1} \end{pmatrix} H\right],$$

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for some $\tau_1 \subseteq \Omega$ and there exists a permutation matrix $P$ such that

$$I_{\tau_1} = P \begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix} P^T \triangleq P I_{\tau_2} P^T.$$

Let $\mathcal{I}_t = \operatorname{diag}(I_t, I_t)$, $t \in (\tau_1, \tau_2)$ and $\mathcal{P} = \operatorname{diag}(P, P)$. We have $2\sum_{i,j=1}^n y_{ij} = \operatorname{tr}(\mathcal{I}_{\tau_2} B \mathcal{I}_{\tau_2} B)$, where $B = \mathcal{P}^T H \mathcal{P}$ with $d(H) = d(B)$ and $\lambda(H) = \lambda(B)$.

Let

$$\mathcal{I}_{\tau_2} B \mathcal{I}_{\tau_2} = \begin{pmatrix} B_{11} & 0 & B_{12} & 0 \\ 0 & 0 & 0 & 0 \\ B_{12}^T & 0 & B_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B_{[l]} = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{pmatrix}.$$

Now, it can be easily seen that $B_{[l]} \in \mathbb{P}(2l)$. Therefore, by the Schur inequality and the interlacing theorem for symplectic eigenvalues (see [3, (42)]),

$$\begin{aligned} \operatorname{tr}(\mathcal{I}_{\tau_2} B \mathcal{I}_{\tau_2} B) &= \operatorname{tr}(\mathcal{I}_{\tau_2} B \mathcal{I}_{\tau_2} B \mathcal{I}_{\tau_2}) = \operatorname{tr}\left(B_{[l]} B_{[l]}\right) \\ &= \operatorname{tr}\left(J B_{[l]} (J B_{[l]})^T\right) = \left\| J B_{[l]} \right\|_F^2 \\ &\geq \sum_{i=1}^{2l} \left| \lambda_i (J B_{[l]}) \right|^2 = 2 \sum_{i=1}^{l} d_i^2 (B_{[l]}) \geq 2 \sum_{i=1}^{l} d_i^2 (B) \\ &= 2 \sum_{i=1}^{l} d_i^2 (H). \end{aligned}$$

On the other hand, by [6, Corollary 4.3.37] and $\mathcal{L}\mathcal{L}^T = I_{2l}$,

$$\begin{aligned} \operatorname{tr}(\mathcal{I}_l B \mathcal{I}_l B \mathcal{I}_l) &= \operatorname{tr}\left(\mathcal{L} B \mathcal{L}^T \mathcal{L} B \mathcal{L}^T\right) = \sum_{i=1}^{2l} \lambda_i^2 \left(\mathcal{L} B \mathcal{L}^T\right) \\ &\leq \sum_{i=2n-2l+1}^{2n} \left[\lambda_i^{\uparrow}(B)\right]^2 = \sum_{i=2n-2l+1}^{2n} \left[\lambda_i^{\uparrow}(H)\right]^2, \end{aligned}$$

where $\mathcal{L} = \left(\begin{array}{cc|c} I_l & 0 & 0_{l\times n} \\ \hline \multicolumn{2}{c|}{0_{l\times n}} & I_l \quad 0 \end{array}\right)$. ∎

As a direct consequence of Theorem 2.3, we obtain the following result. Trace minimizations are useful tools in studying matrix inequalities. One may see [3, 8, 10] and references therein. We prove it again by a trace minimization theorem.

Theorem 2.4. *Let $H \in \mathbb{P}(2n)$ and $\widetilde{H}$ be the $n \times n$ matrix associated with $H$ according to the rule (1). Suppose $Y = (y_{ij})_{l\times l}$ be any $l \times l$ principal submatrix of $\widetilde{H}$, $1 \leq l \leq n$, we have*

$$\sum_{i,j=1}^{l} y_{ij} \geq \frac{1}{l} \left(d_1(H) + d_2(H) + \cdots + d_l(H)\right)^2.$$

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*Proof.* We proceed to adopt notations in the above theorem. As in the proof of Theorem 2.3, we obtain

$$2\sum_{i,j=1}^{n} y_{ij} = \mathrm{tr}\left(\mathcal{L}B\mathcal{L}^T\mathcal{L}B\mathcal{L}^T\right) = \mathrm{tr}\left(\mathcal{L}B\mathcal{L}^T\right)^2,$$

for some $\tau_1 \subseteq \Omega$. For any square matrix $N$ with all eigenvalues real, we have

$$\mathrm{tr}(N^2) \geq \frac{1}{\mathrm{rank}(N)}(\mathrm{tr}N)^2.$$

So, by [3, Theorem 5],

$$\begin{aligned} 2\sum_{i,j=1}^{n} y_{ij} &= \mathrm{tr}\left(\mathcal{L}B\mathcal{L}^T\right)^2 \geq \frac{1}{2l}\left[\mathrm{tr}\left(\mathcal{L}B\mathcal{L}^T\right)\right]^2 \geq \frac{1}{2l}\min_{Z\in \mathrm{Sp}(2l,2n)}\left[\mathrm{tr}\left(ZBZ^T\right)\right]^2 \\ &= \frac{1}{2l}\left[\min_{Z\in \mathrm{Sp}(2l,2n)}\mathrm{tr}\left(ZBZ^T\right)\right]^2 = \frac{1}{2l}\left(2\sum_{i=1}^{l} d_i(B)\right)^2. \end{aligned}$$

Note that $d(H) = d(B)$. This completes the proof. ∎

**3. Remarks.** Suppose two real vectors $x, y \in \mathbb{R}^n$, we say that $x$ is weakly majorized by $y$, denoted by $x \prec_w y$, if the sum of the $k$ largest entries of $x$ is not larger than that of $y$ for each $k = 1, \ldots, n$. If in addition the sum of the entries of each of the vectors is the same, we say that $x$ is majorized by $y$, and write $x \prec y$. Let $x = (x_1, \ldots, x_n)$ be a vector in $\mathbb{R}^n$. We rearrange the components of $x$ in decreasing order and obtain a vector $x^{\downarrow} = (x_1^{\downarrow}, \ldots, x_n^{\downarrow})$, where $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \cdots \geq x_n^{\downarrow}$. Similarly, let $x_1^{\uparrow} \leq x_2^{\uparrow} \leq \cdots \leq x_n^{\uparrow}$ denote the components of $x$ in increasing order and write $x^{\uparrow} = (x_1^{\uparrow}, x_2^{\uparrow}, \ldots, x_n^{\uparrow})$. If $x, y$ are two $n$-vectors with positive coordinates, then we say that $x$ is log majorized by $y$, in symbols $x \prec_{\log} y$, if

$$\prod_{i=1}^{k} x_j^{\downarrow} \leq \prod_{i=1}^{k} y_j^{\downarrow}, 1 \leq k \leq n \quad \text{and} \quad \prod_{i=1}^{n} x_j^{\downarrow} = \prod_{i=1}^{n} y_j^{\downarrow}.$$

Next we recall an important result from [3].

THEOREM 3.1. [3, Theorem 11] *Let $A \in \mathbb{P}(2n)$. Then*

$$\widehat{d}(A) \prec_{\log} \lambda(A) \quad \text{and} \quad \lambda_j^{\uparrow}(A) \leq d_j(A) \leq \lambda_{n+j}^{\uparrow}(A), 1 \leq j \leq n,$$

*where $\widehat{d}(A) = \{d_1(A), d_1(A), \ldots, d_n(A), d_n(A)\}$ and $\lambda(A) = \{\lambda_1(A), \ldots, \lambda_{2n}(A)\}$.*

We consider two special cases in Theorem 2.3:

Case 1: $l = 1$. We have $\min_{1\leq l\leq n}\{h_{ii}\} \geq d_1^2(H)$, which also is a special case in [4, Theorem 2.1].

Case 2: $l = n$. We have $2\sum_{i=1}^{n} d_i^2(H) \leq \sum_{i=1}^{2n} \lambda_i^2(H)$, which can be followed by [3, Theorem 11(i)].

In view of Theorem 2.3 and [3, Theorem 11], we have

THEOREM 3.2. *Let $H \in \mathbb{P}(2n)$ and $\widetilde{H}$ be the $n \times n$ matrix associated with $H$ according to the rule (1). Suppose $Y = (y_{ij})_{l\times l}$ be any $l \times l$ principal submatrix of $\widetilde{H}$, $1 \leq l \leq n$, we have*

$$\sum_{i=2n-2l+1}^{2n} \lambda_i^2(H) \geq \max\left\{2\sum_{i,j=1}^{l} y_{ij}, 2\sum_{i=n-l+1}^{n} d_i^2(H)\right\}.$$

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*Proof.* Since weak log majorization implies weak majorization, the result follows from [3, Theorem 11(i)] combined with the operator convexity of $f(x) = x^2$ on $(0, +\infty)$ (see [9, p.644, B.3.c] and [9, p.167, A.2.Theorem]). ∎

In the following, we give a numerical example to illustrate the result obtained in the above theorem.

Example 3.3. Let $H = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 6 & 5 \\ 1 & 1 & 5 & 6 \end{pmatrix}$. For $l = 1$, we have

$$\sum_{i=3}^{4} \lambda_i^2(H) \approx 138.38 \begin{cases} \geq 2\widetilde{h}_{22} = 47 \text{ (by Theorem 3.2)}, \\ \geq 2d_2^2(H) \approx 37.3205 \text{ (by [3, Theorem 11(i)])}. \end{cases}$$

It is obvious that the bound of Theorem 3.2 is sharper.

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