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FACES OF THE SIGNED BIRKHOFF POLYTOPES[*]

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Abstract. We study faces of the signed Birkhoff polytopes, denoted by $\Omega_n^{\pm}$. We describe its nonempty faces, 1-dimensional faces, 2-dimensional faces, and facets. Moreover, we study the diameter and Hamiltonian connectivity of the graph of $\Omega_n^{\pm}$. In the end, we show that the reduced Gröbner basis of the toric ideal of the signed Birkhoff polytope $\Omega_n^{\pm}$ with respect to the graded reverse lexicographic order induced by rank orders has square-free initial monomials of degree $\leq n$.

Key words. Faces, Facets, Signed Birkhoff polytopes, Signed permutation matrices, Toric ideals.

AMS subject classifications. 15B51, 52B05, 52B20.

1. Introduction.

The hyperoctahedral group B_n is the semi-direct product $\mathbb{Z}_2^n \rtimes S_n$, where $\mathbb{Z}_2 = \{-1, 1\}$ is the multiplicative group of two elements and S_n is the symmetric group. It is the Coxeter group of type-B and can be considered as the group of all permutations and sign changes of the coordinates in $\mathbb{R}^n$. Elements of B_n are called *signed permutations*, each of which can be expressed in one-line notation $\sigma = (\sigma_1, \ldots, \sigma_n)$, where $\sigma_i \in \{1, \ldots, n, -1, \ldots, -n\}$ and $\{|\sigma_1|, \ldots, |\sigma_n|\} = \{1, \ldots, n\}$. Note that $|B_n| = 2^n n!$. It is useful to represent a signed permutation σ by a *signed permutation matrix* $P_\sigma = [p_{i,j}]$, where

$$p_{i,j} = \begin{cases} sgn(\sigma_i), & \text{if } j = |\sigma_i|, \\ 0, & \text{otherwise.} \end{cases} \quad (1 \leq i, j \leq n),$$

If every entry of P_σ is positive, then P_σ is a *permutation matrix* corresponding to σ in S_n. The convex hull of all $n \times n$ signed permutation matrices is called the *signed* (or *type-B*) *Birkhoff polytope* $\Omega_n^{\pm}$, whose extreme points were given in [4, 18, 12, 15].

THEOREM 1.1. [4, 18, 12, 15] *Let n be a positive integer. The extreme points of the polytope $\Omega_n^{\pm}$ are all $n \times n$ signed permutation matrices.*

The hyperplane description of $\Omega_n^{\pm}$ is given in the following theorem.

THEOREM 1.2. [18, 12] *The polytope $\Omega_n^{\pm}$ can be given by the intersection of the following $2^{n+1}n$ half-spaces:*

$$\begin{cases} \sum_{j=1}^{n} a_{i,j} x_{i,j} \leq 1, & \forall 1 \leq i \leq n, \\ \sum_{i=1}^{n} b_{i,j} x_{i,j} \leq 1, & \forall 1 \leq j \leq n, \end{cases}$$

where $a_{i,j}, b_{i,j} \in \{1, -1\}$.

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Thus, $\Omega_n^{\pm}$ is a closed bounded convex polyhedron in Euclidean n^2-space whose dimension is n^2.

Definition 1.3. (i) *An $n \times n$ real matrix $A = [a_{i,j}]$ is called a signed doubly substochastic matrix (or absolutely bistochastic matrix in [4, 18]), if for all $1 \leq i, j \leq n$,*

$$\sum_{j=1}^{n} |a_{i,j}| \leq 1, \ \sum_{i=1}^{n} |a_{i,j}| \leq 1. \tag{1.1}$$

(ii) *An $n \times n$ nonnegative real matrix $B = [b_{i,j}]$ is called a doubly substochastic matrix if it satisfies for all $1 \leq i, j \leq n, \sum_{j=1}^{n} a_{i,j} \leq 1, \sum_{i=1}^{n} a_{i,j} \leq 1$.*

(iii) *An $n \times n$ nonnegative real matrix $C = [c_{i,j}]$ is called a doubly stochastic matrix if it satisfies for all $1 \leq i, j \leq n, \sum_{j=1}^{n} c_{i,j} = 1, \sum_{i=1}^{n} c_{i,j} = 1$.*

By Theorem 1.2, the polytope $\Omega_n^{\pm}$ consists of all $n \times n$ signed doubly substochastic matrices. By Definition 1.3, if we restrict $\Omega_n^{\pm}$ to the positive orthant, we will get the convex polytope consisting of all $n \times n$ doubly substochastic matrices, denoted by ω_n, i.e., $\omega_n = \Omega_n^{\pm} \cap \mathbb{R}_{\geq 0}^{n \times n}$. The dimension of ω_n is n^2, which is the same as the dimension of $\Omega_n^{\pm}$ [11]. All $n \times n$ doubly stochastic matrices form the widely studied *Birkhoff polytope*, usually denoted by Ω_n [2, 6, 7, 8, 9, 20, 10]. It is a closed bounded convex polytope in Euclidean n^2-space whose dimension is $(n-1)^2$. The extreme points of Ω_n and ω_n are well known due to Birkhoff [3] and Mirsky [19], respectively.

Theorem 1.4. [3] *Let n be a positive integer, the extreme points of Ω_n are all $n \times n$ permutation matrices.*

Theorem 1.5. [19] *Let n be a positive integer, the extreme points of ω_n are all $n \times n$ subpermutation matrices, which are $(0, 1)$-matrices satisfying that each row and column contains at most one 1.*

From definition, we have $\Omega_n \subset \omega_n \subset \Omega_n^{\pm}$. In this paper, we study the facial structure of $\Omega_n^{\pm}$. In section 2, we construct facial matrices to characterize faces of $\Omega_n^{\pm}$. Based on this result, we give formulas to calculate the dimension and number of vertices of a face in $\Omega_n^{\pm}$. In section 3, we characterize all 1-dimensional faces, 2-dimensional faces, and facets of $\Omega_n^{\pm}$. Furthermore, we study some properties of the vertex-edge graph $G(\Omega_n^{\pm})$ of $\Omega_n^{\pm}$ in section 4. We show that the diameter of $G(\Omega_n^{\pm})$ is 2 and $G(\Omega_n^{\pm})$ is Hamilton connected. In the last section, we show that the reduced Gröbner bases of the toric ideal of $\Omega_n^{\pm}$ have at most degree n with respect to the graded reverse lexicographic order induced by rank orders.

2. Nonempty faces of $\Omega_n^{\pm}$.

A *face* $\mathfrak{F}$ of $\Omega_n^{\pm}$ is the set of form

$$\mathfrak{F} = \Omega_n^{\pm} \bigcap \left\{ Y = [y_{i,j}] \in \mathbb{R}^{n \times n} \,\middle|\, \sum_{i,j=1}^{n} c_{i,j} y_{i,j} = c_0 \right\},$$

where for all $X = [x_{i,j}] \in \Omega_n^{\pm 1}, \sum_{i,j=1}^{n} c_{i,j} x_{i,j} \leq c_0$ [24]. To better describe the nonempty faces of $\Omega_n^{\pm}$, we consider $n \times n$ $(0, \pm 1)$-matrices whose entries consist of either $0, 1, -1$, or ± 1. Given a $(0, \pm 1)$-matrix $A = [a_{i,j}]$, define the index set $K_A = \{i | a_{i,j} = \pm 1$ for some $1 \leq i, j \leq n\}$ and $L_A = \{j | a_{i,j} = \pm 1$ for some $1 \leq i, j \leq n\}$. Let $[n] := \{1, \ldots, n\}$ and $\bar{K}_A = [n] \setminus K_A, \bar{L}_A = [n] \setminus L_A$. Denote by $\mathfrak{F}(A)$ the set of all $n \times n$ real matrices $X = [x_{i,j}]$ satisfying that

$$\begin{cases} x_{i,j} = 0, & \text{if } a_{i,j} = 0, \\ -1 \leq x_{i,j} \leq 0, & \text{if } a_{i,j} = -1, \\ 0 \leq x_{i,j} \leq 1, & \text{if } a_{i,j} = 1, \\ -1 \leq x_{i,j} \leq 1, & \text{if } a_{i,j} = \pm 1, \end{cases} \quad \text{and} \quad \begin{cases} \forall i \in \bar{K}_A, & \sum_{j=1}^{n} |x_{i,j}| = 1, \\ \forall j \in \bar{L}_A, & \sum_{i=1}^{n} |x_{i,j}| = 1, \\ \forall i \in K_A, & \sum_{j=1}^{n} |x_{i,j}| \leq 1, \\ \forall j \in L_A, & \sum_{i=1}^{n} |x_{i,j}| \leq 1. \end{cases}$$

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Note that $\mathfrak{F}(A)$ is a subpolytope of $\Omega_n^{\pm}$ whose vertices are all $n \times n$ signed permutation matrices $P = [p_{i,j}]$ satisfying that

$$p_{i,j} = \begin{cases} 0, & \text{if } a_{i,j} = 0, \\ 0 \text{ or } a_{i,j}, & \text{if } a_{i,j} = 1 \text{ or } -1, \\ 0,\ 1 \text{ or } -1, & \text{if } a_{i,j} = \pm 1. \end{cases}$$

We call such a signed permutation matrix *dominated* by A. Suppose there exist $r, s \in [n]$ such that $a_{r,s} \neq 0$, but there is no signed permutation matrix dominated by A satisfying that

$$p_{r,s} = \begin{cases} a_{r,s}, & \text{if } a_{r,s} = 1 \text{ or } -1, \\ 1 \text{ or } -1, & \text{if } a_{r,s} = \pm 1. \end{cases}$$

Then $\mathfrak{F}(A) = \mathfrak{F}(A')$, where A' is obtained from A by replacing $a_{r,s}$ by 0. Thus, we need to consider those $(0, \pm 1)$-matrices with no redundant nonzero elements.

Definition 2.1. *An $n \times n$ nonzero $(0, \pm 1)$-matrix $A = [a_{i,j}]$ is said to have total support, if for any $a_{r,s} \neq 0$, there exists a signed permutation matrix $P = [p_{i,j}]$ dominated by A with $p_{r,s} \neq 0$.*

Remark 2.2. *For $n \times n$ real matrices A and B, we write $A \geq B$ if and only if $A - B$ is a nonnegative matrix. Definition 2.1 coincides with the definition for a nonzero $(0,1)$-matrix $B = [b_{i,j}]$ to have total support [22, 6], i.e., for any $b_{r,s} = 1$, there exists a permutation matrix $P = [p_{i,j}]$ with $p_{r,s} = 1$ and $P \leq B$.*

Remark 2.3. *For a matrix $A = [a_{i,j}]$, let $|A| := [|a_{i,j}|]$. If A is a $(0, \pm 1)$-matrix, then $|A|$ is a $(0,1)$-matrix. According to Definition 2.1, A has total support if and only if $|A|$ has total support.*

Define the operation "$+_b$" on $\{0, 1, -1, \pm 1\}$ satisfying the commutative law and the following rules.

$$\begin{cases} 0 +_b 0 = 0, \\ 0 +_b 1 = 1 +_b 1 = 1, \\ 0 +_b (-1) = (-1) +_b (-1) = -1, \\ 0 +_b (\pm 1) = 1 +_b (\pm 1) = (-1) +_b (\pm 1) = (\pm 1) +_b (\pm 1) = 1 +_b (-1) = \pm 1. \end{cases}$$

Let $A = [a_{i,j}]$ and $B = [b_{i,j}]$ are both $(0, \pm 1)$-matrices of order n. Their *Boolean sum* $A +_b B := [a_{i,j} +_b b_{i,j}]$ is again a $(0, \pm 1)$-matrix. Specifically, if $P_1, \ldots, P_k$ are signed permutation matrices and $A = P_1 +_b \cdots +_b P_k$, then A has total support.

Definition 2.4. *A $(0, \pm 1)$-matrix $A = [a_{i,j}]$ is called a facial matrix of $\Omega_n^{\pm}$ if*

(i) *A has total support;*
(ii) *$a_{i,j} = \pm 1$ for all $i \in K_A, j \in L_A$.*

Lemma 2.5. *If A is a facial matrix, then $\mathfrak{F}(A)$ is a nonempty face of $\Omega_n^{\pm}$ whose vertices are those signed permutation matrices dominated by A.*

Proof. For each $u \in \bar{K}_A$, define the hyperplane

$$H_{r_u} := \left\{ Y = [y_{i,j}] \in \mathbb{R}^{n \times n} \,\middle|\, \sum_{j=1}^{n} sgn(a_{u,j}) y_{u,j} = 1 \right\},$$

and for each $v \in \bar{L}_A$, define the hyperplane

$$H_{c_v} := \left\{ Y = [y_{i,j}] \in \mathbb{R}^{n\times n} \,\middle|\, \sum_{i=1}^{n} sgn(a_{i,v}) y_{i,v} = 1 \right\}.$$

To show $\mathfrak{F}(A)$ is a face of $\Omega_n^{\pm}$, we need to show that

$$\mathfrak{F}(A) = \Omega_n^{\pm} \bigcap \left(\bigcap_{u \in \bar{K}_A} H_{r_u} \right) \bigcap \left(\bigcap_{v \in \bar{L}_A} H_{c_v} \right),$$

and for all $X = [x_{i,j}] \in \Omega_n^{\pm}$,

$$\left\{ \begin{array}{ll} \sum_{j=1}^{n} sgn(a_{u,j}) x_{u,j} \le 1, & \text{for each } u \in \bar{K}_A, \text{ and} \\ \sum_{i=1}^{n} sgn(a_{i,v}) x_{i,v} \le 1, & \text{for each } v \in \bar{L}_A. \end{array} \right. \tag{2.1}$$

Note that

$$\mathfrak{F}(A) \subseteq \Omega_n^{\pm} \bigcap \left(\bigcap_{u \in \bar{K}_A} H_{r_u} \right) \bigcap \left(\bigcap_{v \in \bar{L}_A} H_{c_v} \right).$$

To show the inclusion of the opposite direction, let $Z = [z_{i,j}] \in \Omega_n^{\pm} \bigcap \left(\bigcap_{u \in \bar{K}_A} H_{r_u} \right) \bigcap \left(\bigcap_{v \in \bar{L}_A} H_{c_v} \right)$. On one hand, for each $u \in \bar{K}_A, \sum_{j=1}^{n} sgn(a_{u,j}) z_{u,j} = 1$. On the other hand, $\sum_{j=1}^{n} sgn(a_{u,j}) z_{u,j} \le \sum_{j=1}^{n} |z_{u,j}| \le 1$, which implies $\sum_{j=1}^{n} |z_{u,j}| = 1$. Therefore, either $sgn(z_{u,j}) = sgn(a_{u,j})$ or $z_{u,j} = 0$, where $1 \le j \le n$. Same results hold for $z_{i,v}$, where $1 \le i \le n, v \in \bar{L}_A$. Therefore, $Z \in \mathfrak{F}(A)$.

Inequalities (2.1) follow from Theorem 1.2. Since A has total support, $\mathfrak{F}(A)$ is nonempty. Note that a signed permutation matrix P is in $\mathfrak{F}(A)$ if and only if P is dominated by A. The vertices of $\mathfrak{F}(A)$ are all signed permutation matrices dominated by A. ☐

LEMMA 2.6. *Let $\mathfrak{F}$ be a nonempty face of $\Omega_n^{\pm}$ and $P_1, \ldots, P_k$ be all vertices of $\mathfrak{F}$. Then $A = P_1 +_b \cdots +_b P_k$ is a facial matrix and $\mathfrak{F}(A) = \mathfrak{F}$.*

Proof. Since A is a $(0, \pm 1)$-matrix having total support, we only need to show that A satisfies condition (ii) in Definition 2.4. For any $u \in K_A$ and $v \in L_A$, we use contradiction method to show that $a_{u,v} = \pm 1$. Suppose $a_{u,v} \ne \pm 1$, since $u \in K_A$, there exists some $s \in L_A$ such that $a_{u,s} = \pm 1$. For the same reason, there exists some $t \in K_A$ such that $a_{t,v} = \pm 1$. Moreover, there exist signed permutation matrices $P_1 = [p_{i,j}^{(1)}], P_2 = [p_{i,j}^{(2)}], P_3 = [p_{i,j}^{(3)}], P_4 = [p_{i,j}^{(4)}] \in \mathfrak{F}$ such that $p_{u,s}^{(1)} = 1, p_{u,s}^{(2)} = -1, p_{t,v}^{(3)} = 1, p_{t,v}^{(4)} = -1$. Consider $X = [x_{i,j}] = \frac{1}{4}(P_1 + P_2 + P_3 + P_4) \in \mathfrak{F}$, which satisfies that $x_{u,v} = x_{t,s} = 0$, and $\sum_{j=1}^{n} |x_{u,j}| \le 1/2, \sum_{j=1}^{n} |x_{t,j}| \le 1/2, \sum_{i=1}^{n} |x_{i,s}| \le 1/2, \sum_{i=1}^{n} |x_{i,v}| \le 1/2$. Let

$$0 < \epsilon \le \min \left\{ 1 - \sum_{j=1}^{n} |x_{u,j}|, 1 - \sum_{j=1}^{n} |x_{t,j}|, 1 - \sum_{i=1}^{n} |x_{i,s}|, 1 - \sum_{i=1}^{n} |x_{i,v}| \right\}.$$

Let $X_1 = [x_{i,j}^{(1)}]$ and $X_2 = [x_{i,j}^{(2)}]$ be two $n \times n$ matrices having the same entries as X has except for $x_{u,v}^{(1)} = x_{t,s}^{(1)} = \epsilon$ and $x_{u,v}^{(2)} = x_{t,s}^{(2)} = -\epsilon$. Note that $X_1, X_2 \in \Omega_n^{\pm}$ and $\frac{1}{2}(X_1 + X_2) = X$. Since $X \in \mathfrak{F}$, this implies that X_1 and X_2 are also in $\mathfrak{F}$ (c.f. [5]). There must exist signed permutation matrices $P_5 = [p_{i,j}^{(5)}]$

and $P_6 = [p^{(6)}_{i,j}]$ satisfying that $p^{(5)}_{u,v} = 1$ and $p^{(6)}_{u,v} = -1$, which implies $a_{u,v} = \pm 1$ and gets the contradiction. Thus, A is a facial matrix.

By Lemma 2.5, since A is a facial matrix, $\mathfrak{F}(A)$ is a face of $\Omega^{\pm}_n$ containing vertices $P_1, \ldots, P_k$. Therefore, $\mathfrak{F} \subseteq \mathfrak{F}(A)$. Moreover, let $P = [p_{i,j}]$ be any signed permutation matrix dominated by A. We can choose $P_{l_1}, \ldots, P_{l_w} \in \{P_1, \ldots, P_k\}$, such that for any $p_{i_0,j_0} \neq 0$, there exists some $P_{l_s} = [p^{(l_s)}_{i,j}] \in \{P_{l_1}, \ldots, P_{l_w}\}$ such that $p^{(l_s)}_{i_0,j_0} = p_{i_0,j_0}$, where $1 \leq s \leq w$. Now consider $Y = \frac{1}{w}(P_{l_1} + \cdots + P_{l_w}) \in \mathfrak{F}$. Let $Z = [z_{i,j}] = Y - \frac{1}{w}P$. Note that for all $1 \leq i, j \leq n$, $\sum_{j=1}^{n} |z_{i,j}| \leq (1 - \frac{1}{w})$ and $\sum_{i=1}^{n} |z_{i,j}| \leq (1 - \frac{1}{w})$. Therefore, $Z' = \frac{w}{w-1} Z \in \Omega^{\pm}_n$, and $Y = \frac{1}{w}P + \frac{w-1}{w}Z'$. Since $Y \in \mathfrak{F}$, this implies that $P \in \mathfrak{F}$. Thus, we have $\mathfrak{F} = \mathfrak{F}(A)$. □

Lemma 2.5 and Lemma 2.6 imply the following theorem.

Theorem 2.7. *There is a bijection between faces of $\Omega^{\pm}_n$ and $n \times n$ facial matrices.*

Corollary 2.8. *Let $P_1, \ldots, P_k$ be distinct $n \times n$ signed permutation matrices and $A = [a_{i,j}] = P_1 +_b \cdots +_b P_k$. The smallest face of $\Omega^{\pm}_n$ containing vertices $P_1, \ldots, P_k$ is $\mathfrak{F}(\bar{A})$, where $\bar{A} = [\bar{a}_{i,j}]$ is defined by*

$$\bar{a}_{i,j} = \begin{cases} a_{i,j}, & \text{if } i \notin K_A \text{ or } j \notin L_A, \\ \pm 1, & \text{if } i \in K_A \text{ and } j \in L_A. \end{cases}$$

Proof. Although A may not necessarily be a facial matrix, $\bar{A}$ must be a facial matrix by which $P_1, \ldots, P_k$ are dominated. From Lemma 2.5, $\mathfrak{F}(\bar{A})$ is a face containing $P_1, \ldots, P_k$. Suppose B is a facial matrix such that $\mathfrak{F}(B)$ contains $P_1, \ldots, P_k$, then $P_1, \ldots, P_k$ are dominated by B and $K_A \subseteq K_B, L_A \subseteq L_B$. Hence, $\mathfrak{F}(\bar{A}) \subseteq \mathfrak{F}(B)$. Therefore, $\mathfrak{F}(\bar{A})$ is the smallest face containing $P_1, \ldots, P_k$. □

Let $A = [a_{i,j}]$ be an $n \times n$ real matrix. The *permanent* of A is defined by

$$\mathrm{per}A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

Theorem 2.9. *Given a facial matrix A, define $\|A\| = [\|a_{i,j}\|]$ where*

$$\|a_{i,j}\| = \begin{cases} 0, & \text{if } a_{i,j} = 0, \\ 1, & \text{if } a_{i,j} = 1 \text{ or } -1, \\ 2, & \text{if } a_{i,j} = \pm 1. \end{cases}$$

Then the number of vertices of $\mathfrak{F}(A)$ is equal to $\mathrm{per}(\|A\|)$.

Proof. The permanent of $\|A\|$ counts the number of signed permutation matrices dominated by A. Thus, it gives the number of vertices of $\mathfrak{F}(A)$. □

An $n \times n (n \geq 2)$ matrix A is *fully indecomposable* if there do not exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix},$$

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where A_1 and A_2 are square matrices [6]. For example, the following matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix},$$

is fully indecomposable. A $(0, \pm 1)$-matrix has *total support* if and only if there exist permutation matrices P and Q such that PAQ is a direct sum of fully indecomposable matrices [6, 21], i.e.

$$PAQ = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{bmatrix} = A_1 \oplus \cdots \oplus A_k,$$

where $A_1, \ldots, A_k$ are fully indecomposable.

THEOREM 2.10. *Let A be a facial matrix which is fully indecomposable. Then the dimension of $\mathfrak{F}(A)$, denoted by $\dim \mathfrak{F}(A)$, is given by*

$$\dim \mathfrak{F}(A) = \begin{cases} \sigma(A) - |\bar{K}_A| - |\bar{L}_A|, & \text{if } K_A, L_A \neq \emptyset, \\ \sigma(A) - 2n + 1, & \text{if } K_A = L_A = \emptyset, \end{cases}$$

where $\sigma(A)$ denotes the number of nonzero entries in A.

Proof. Since A is a facial matrix, by Theorem 2.7, $\mathfrak{F}(A)$ is a nonempty face whose inequality expression contains $\sigma(A)$ indeterminates. If $K_A = L_A = \emptyset$, i.e., $\bar{K}_A = \bar{L}_A = [n]$, since A is fully indecomposable, the coefficient matrix of the system of equations defining $\mathfrak{F}(A)$ has rank equal to $2n - 1$. This is because

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |x_{i,j}| = \sum_{j=1}^{n} \sum_{i=1}^{n} |x_{i,j}|.$$

Hence, $\mathfrak{F}(A)$ has dimension $\sigma(A) - 2n + 1$. If $K_A, L_A \neq \emptyset$, the coefficient matrix of the system of equations defining $\mathfrak{F}(A)$ has rank equal to $|\bar{K}_A| + |\bar{L}_A|$. Thus, the dimension of $\mathfrak{F}(A)$ is $\sigma(A) - |\bar{K}_A| - |\bar{L}_A|$. Therefore, we have the above dimension formula. ∎

Given two faces $\mathfrak{F}_1$ and $\mathfrak{F}_2$, if there exist permutation matrices P and Q such that for any $M \in \mathfrak{F}_1, PMQ \in \mathfrak{F}_2$ and vice versa, then $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are said to be *congruent*, denoted by $\mathfrak{F}_1 \cong \mathfrak{F}_2$. For any faces $\mathfrak{F}_1$ and $\mathfrak{F}_2$, define $\mathfrak{F}_1 \oplus \mathfrak{F}_2 := \{M_1 \oplus M_2 | M_1 \in \mathfrak{F}_1, M_2 \in \mathfrak{F}_2\}$.

COROLLARY 2.11. *For a facial matrix A, let P and Q be permutation matrices such that $PAQ = A_1 \oplus \cdots \oplus A_s$, where A_i is a fully indecomposable facial matrix for $1 \leq i \leq s$. Then*

$$\mathfrak{F}(A) \cong \mathfrak{F}(A_1) \oplus \cdots \oplus \mathfrak{F}(A_s),$$

and

$$\dim(\mathfrak{F}(A)) = \dim(\mathfrak{F}(A_1)) + \cdots + \dim(\mathfrak{F}(A_s)).$$

Moreover, the number of vertices of $\mathfrak{F}(A)$ is the product of the number of vertices of $\mathfrak{F}(A_i)$, where $1 \leq i \leq s$.

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Example 2.12. *Let*

$$A=\begin{bmatrix} \pm1 & 1 \\ 1 & -1 \end{bmatrix} \oplus \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & \pm1 & \pm1 \end{bmatrix}.$$

Notice that $K_{A_1}=L_{A_1}=\{1\}, K_{A_2}=\{3\}, L_{A_2}=\{2,3\}$, *we have* $\dim(\mathfrak{F}(A))=\dim(\mathfrak{F}(A_1))+\dim(\mathfrak{F}(A_2))=4-2+(7-3)=6$. *Moreover, the number of vertices of* $\mathfrak{F}(A)$ *is*

$$\operatorname{per}(\|A_1\|)\operatorname{per}(\|A_2\|)=\operatorname{per}\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\right)\operatorname{per}\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\right)=(2+1)(2+2+2)=18.$$

Given a $(0,\pm1)$-matrix $A=[a_{i,j}]$, define the *positive pattern* (resp. *negative pattern*) of A to be the set

$$\mathcal{P}_+(A)=\{(i,j)|a_{i,j}=1 \text{ or } \pm1\} \text{ (resp. } \mathcal{P}_-(A)=\{(i,j)|a_{i,j}=-1 \text{ or } \pm1\}).$$

The *pattern* of A, denoted by $\mathcal{P}(A)$, is the set of positions of A containing nonzero elements. Thus,

$$\mathcal{P}(A)=\mathcal{P}_+(A)\cup\mathcal{P}_-(A).$$

Lemma 2.13. *Let A and B be two facial matrices satisfying that $\mathcal{P}_+(A)\subseteq\mathcal{P}_+(B)$ and $\mathcal{P}_-(A)\subseteq\mathcal{P}_-(B)$. Then $\mathfrak{F}(A)\subseteq\mathfrak{F}(B)$, and vice versa.*

Proof. If $\mathcal{P}_+(A)\subseteq\mathcal{P}_+(B)$ and $\mathcal{P}_-(A)\subseteq\mathcal{P}_-(B)$, then every signed permutation matrix dominated by A must also be dominated by B, which implies $\mathfrak{F}(A)\subseteq\mathfrak{F}(B)$, and vice versa. □

The following theorem follows from Lemma 2.13.

Theorem 2.14. *The face lattice of $\Omega_n^{\pm}$ is isomorphic to the lattice of all $n\times n$ facial matrices ordered by inclusion of positive and negative patterns.*

3. One-dimensional faces, two-dimensional faces, and facets of $\Omega_n^{\pm}$.

To represent the structure of nonzeros of an $n\times n$ $(0,\pm1)$-matrix A by a bipartite graph, let $U=\{\mathbf{r}_1,\ldots,\mathbf{r}_n\}$ and $V=\{\mathbf{s}_1,\ldots,\mathbf{s}_n\}$ be sets of cardinality n, where $U\cap V=\emptyset$. The *bipartite graph* associated with A, denoted by $G(A)$, is a bipartite graph in which $\mathbf{r}_i$ and $\mathbf{s}_j$ are represented by dots, where $1\le i,j\le n$, and there is an edge between $\mathbf{r}_i$ and $\mathbf{s}_j$ if and only if $a_{i,j}\neq 0$.

Theorem 3.1. *Given two $n\times n$ signed permutation matrices P and Q, the dimension of the smallest face containing P and Q is equal to the number of cycles in $G(P+_b Q)$ plus the square of the number of ±1's in $P+_b Q$.*

Proof. Let $A=P+_b Q$. There exist permutation matrices R and S such that

$$RAS=\begin{bmatrix} L_1 & & & & & \\ & \ddots & & & & \\ & & L_s & & & \\ & & & (\pm1)I_t & & \\ & & & & I_u & \\ & & & & & -I_v \end{bmatrix},$$

where for each $1 \leq i \leq s, L_i$ has order $n_i \geq 2$ and is of the form

$$L_i = \begin{bmatrix} \epsilon_{1,1} & & & \epsilon_{1,n_i} \\ \epsilon_{2,1} & \epsilon_{2,2} & & \\ & \ddots & \ddots & \\ & & \epsilon_{n_i-1,n_i} & \epsilon_{n_i,n_i} \end{bmatrix},$$

where each $\epsilon_{i,j} = 1$ or -1.

Let $\bar{A}$ be the matrix obtained from RAS by replacing the submatrix $(\pm1)I_t$ by $(\pm1)J_t$, where J_t is a $t \times t$ all one matrix, i.e.

$$\bar{A} = \begin{bmatrix} L_1 & & & & & \\ & \ddots & & & & \\ & & L_s & & & \\ & & & (\pm1)J_t & & \\ & & & & I_u & \\ & & & & & -I_v \end{bmatrix}.$$

By Corollary 2.8, $\bar{A}$ is a facial matrix and $\mathfrak{F}(\bar{A})$ is the smallest face containing P and Q. By Theorem 2.9, the number of vertices of $\mathfrak{F}(\bar{A})$ is $2^{s+t}t!$. By Theorem 2.10 and Corollary 2.11,

$$\dim(\mathfrak{F}(\bar{A})) = \dim(\mathfrak{F}(L_1)) + \cdots \dim(\mathfrak{F}(L_s)) + \dim(\mathfrak{F}((\pm1)J_t)) = s + t^2.$$

∎

Corollary 3.2. *The signed permutation matrices P and Q correspond to an edge of $\Omega_n^{\pm}$ if and only if either $G(P +_b Q)$ contains exactly one cycle and no ±1 in $P +_b Q$ or P and Q only differ on the sign of exactly one element, i.e., there exist permutation matrices R and S such that either*

$$RAS = \begin{bmatrix} L_1 & & \\ & I_u & \\ & & (-1)I_v \end{bmatrix},$$

or

$$RAS = \begin{bmatrix} \pm1 & & \\ & I_{u'} & \\ & & (-1)I_{v'} \end{bmatrix}.$$

Proof. According to Theorem 3.1 and the proof, P and Q form an edge of $\Omega_n^{\pm}$ if and only if $s + t^2 = 1$, i.e., either $s = 1, t = 0$ or $s = 0, t = 1$. Therefore, we have the corollary. ∎

Let $E_{i,j}$ denote the $(0,1)$-matrix with exactly one 1 on the (i,j)th entry.

Theorem 3.3. *Let A be an $n \times n$ facial matrix. Then the followings are equivalent.*

(i) $\dim(\mathfrak{F}(A)) = 2$.
(ii) $\mathfrak{F}(A)$ *is either a triangle or a rectangle.*

(iii) *There exist permutation matrices R and S such that RAS is one of the following forms:*

$$\begin{bmatrix} L_{n_1} & & & \\ & L_{n_2} & & \\ & & I_{u_1} & \\ & & & -I_{v_1} \end{bmatrix}, \begin{bmatrix} L_{n_3} & & & \\ & \pm 1 & & \\ & & I_{u_2} & \\ & & & -I_{v_2} \end{bmatrix}, \begin{bmatrix} L_{n_4} +_b (\pm 1)E_{1,1} & & \\ & I_{u_3} & \\ & & -I_{v_3} \end{bmatrix},$$

and

$$\begin{bmatrix} L_{n_5} +_b \epsilon E_{i,j} & & \\ & I_{u_4} & \\ & & -I_{v_4} \end{bmatrix},$$

where $\epsilon = 1$ or -1, and $j \neq i-1, i$.

Proof. "$(ii) \Rightarrow (i)$" and "$(iii) \Rightarrow (i)$" are easy to check due to Theorem 3.1. We now prove "$(i) \Rightarrow (ii)$" and "$(i) \Rightarrow (iii)$". By Theorem 2.10 and Corollary 2.11, there exist permutation matrices R and S such that $RAS = A_1 \oplus \cdots \oplus A_s \oplus I_u \oplus -I_v$, where $A_1, \ldots, A_s$ are fully indecomposable and $\dim(\mathfrak{F}(A_i)) \geq 1$ for all $1 \leq i \leq s$. Since $\dim(\mathfrak{F}(A)) = 2$, either $s = 2$ and $\dim(\mathfrak{F}(A_1)) = \dim(\mathfrak{F}(A_2)) = 1$ or $s = 1$ and $\dim(\mathfrak{F}(A_1)) = 2$.

In the case that $s = 2$, RAS is one of the first two matrices listed in (iii). Let P_1, P_2 be the signed permutation matrices such that $P_1 +_b P_2 = A_1$, and Q_1, Q_2 be the signed permutation matrices such that $Q_1 +_b Q_2 = A_2$. Then, the four vertices of $\mathfrak{F}(RAS)$ are

$$V_1 = P_1 \oplus Q_1 \oplus I_u \oplus -I_v,\ V_2 = P_1 \oplus Q_2 \oplus I_u \oplus -I_v,$$
$$V_3 = P_2 \oplus Q_1 \oplus I_u \oplus -I_v,\ V_4 = P_2 \oplus Q_2 \oplus I_u \oplus -I_v,$$

which implies that $\mathfrak{F}(A)$ is a rectangle.

In the case that $s = 1$, $\mathfrak{F}(A_1)$ contains a one-dimensional face $\mathfrak{F}(A_1')$ of $\Omega_n^{\pm}$. If $K_{A_1} = K_{A_1'} = \emptyset, L_{A_1} = L_{A_1'} = \emptyset$, then A_1 has one more 1 or -1 than A_1', and RAS is the last matrix listed in (iii). By Theorem 2.9, the number of vertices of $\mathfrak{F}(A)$ is three, which implies that $\mathfrak{F}(A)$ is a triangle. If $K_{A_1} \neq \emptyset, L_{A_1} \neq \emptyset$, then RAS is the third matrix listed in (iii). The number of vertices of $\mathfrak{F}(A)$ is also three in this case, and $\mathfrak{F}(A)$ is a triangle. ☐

Theorem 1.2 and the hyperplane description of $\Omega_n^{\pm}$ give the facets of $\Omega_n^{\pm}$ [18, 12]. Here, we use facial matrices to characterize the facets of $\Omega_n^{\pm}$ and count the number of facets, which coincides with the results in Theorem 1.2.

THEOREM 3.4. *Let A be a facial matrix. Then $\mathfrak{F}(A)$ is a facet of $\Omega_n^{\pm}$ if and only if there exist permutation matrices R and S such that*

$$RAS = \begin{bmatrix} \epsilon_{1,1} & \epsilon_{1,2} & \cdots & \epsilon_{1,n} \\ \pm 1 & \pm 1 & \cdots & \pm 1 \\ \vdots & \vdots & & \vdots \\ \pm 1 & \pm 1 & \cdots & \pm 1 \end{bmatrix},$$

where $\epsilon_{1,j} = 1$ or -1 for all $1 \leq j \leq n$. Moreover, there are $n2^{n+1}$ facets, and the number of vertices of a facet is $2^{n-1}n!$.

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Proof. Since A is a facial matrix, $\mathfrak{F}(A)$ is a facet of $\Omega_n^{\pm}$ if and only if $\dim(\mathfrak{F}(A)) = n^2 - 1$. Thus, $|K_A| + |L_A| = 2n - 1$ and A can be permutated to the matrix of the above form. The restricted line can be chosen from any one of rows or columns, and each $\epsilon_{1,j}$ has two possibilities which is either 1 or -1. There are $2n \cdot 2^n = n2^{n+1}$ facets. The number of vertices of $\mathfrak{F}(A)$ can be computed by

∎

$$\operatorname{per}(\|A\|) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 \\ \vdots & \vdots & & \vdots \\ 2 & 2 & \cdots & 2 \end{bmatrix} = 2^{n-1}n!.$$

4. Graph of $\Omega_n^{\pm}$.

The vertex-edge graph of $\Omega_n^{\pm}$, denoted by $G(\Omega_n^{\pm})$, is the finite undirected graph having vertices of $\Omega_n^{\pm}$ as its vertices and the one-dimensional faces of $\Omega_n^{\pm}$ as its edges. Two vertices P and Q of a graph are *similar* if there exists some automorphism ϕ of the graph such that $\phi(P) = Q$. A graph is called *vertex symmetric* if every pair of vertices are similar [1].

THEOREM 4.1. *$G(\Omega_n^{\pm})$ is vertex symmetric and each vertex has degree*

$$\sum_{k=2}^{n} \binom{n}{k} (k-1)!2^k + n.$$

Proof. Let P and Q be two signed permutation matrices. For all $X \in \Omega_n^{\pm}$, define $\phi(X) = XP^{-1}Q$, which permutes the vertices of $G(\Omega_n^{\pm})$ and satisfies $\phi(P) = Q$. Note that ϕ also preserves adjacency since $P^{-1}Q$ is a signed permutation matrix. For any facial matrix A, multiplying $P^{-1}Q$ from right-hand side will only permute columns of A and possibly change the sign of entries 1 and -1 in some columns, but the entries ± 1's will not be changed. So $\mathfrak{F}(A)$ and $\mathfrak{F}(\phi(A))$ are combinatorially equivalent, and

$$\dim(\mathfrak{F}(A)) = \dim(\mathfrak{F}(\phi(A))).$$

All vertices of $G(\Omega_n^{\pm})$ have the same valency. Taking $P = I_n$, we look for all possible vertices adjacent to I_n. By Corollary 3.2, there are two cases. Either we can find permutation matrices R and S such that

$$R(I_n +_b Q)S = \begin{bmatrix} L_{n_1} & \\ & I_{n_2} \end{bmatrix},$$

where L_{n_1} has all 1's on the main diagonal and $2 \le n_1 \le n$. The number of such Q's is $\sum_{k=2}^{n} \binom{n}{k}(k-1)!2^k$. Or we can also find permutation matrices R and S such that

$$R(I_n +_b Q)S = \begin{bmatrix} \pm 1 & \\ & I_{n-1} \end{bmatrix}.$$

The number of such Q's is n. So the number of adjacency vertices is $\sum_{k=2}^{n} \binom{n}{k}(k-1)!2^k + n$. ∎

EXAMPLE 4.2. *Consider $G(\Omega_2^{\pm})$ and take vertex I_2. The vertices that can form an edge with I_2 are*

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The degree of the vertex I_2 is 6, which coincides with the result in Theorem 4.1.

The *distance* between two vertices in a connected graph is the number of edges in the shortest path joining the vertices. The *diameter* of a connected graph is the maximum distance between any two vertices of the graph [1].

THEOREM 4.3. *$G(\Omega_n^{\pm})$ has diameter* 2.

Proof. Without loss of generality, we can assume that

$$P = \begin{bmatrix} \epsilon_1(P) & & \\ & \ddots & \\ & & \epsilon_{n_1}(P) \end{bmatrix}_{n_1} \oplus \cdots \oplus \begin{bmatrix} \epsilon_{\sum_{i=1}^{s-1} n_i+1}(P) & & \\ & \ddots & \\ & & \epsilon_{\sum_{i=1}^{s} n_i}(P) \end{bmatrix}_{n_s}$$
$$\oplus \begin{bmatrix} \epsilon_{\sum_{i=1}^{s} n_i+1}(P) & & \\ & \ddots & \\ & & \epsilon_{\sum_{i=1}^{s} n_i+t}(P) \end{bmatrix}_{t} \oplus I_u \oplus -I_v,$$

and

$$Q = \begin{bmatrix} 0 & 0 & & \epsilon_1(Q) \\ \epsilon_2(Q) & 0 & & 0 \\ & \ddots & \ddots & \\ & & \epsilon_{n_1}(Q) & 0 \end{bmatrix}_{n_1} \oplus \cdots \oplus \begin{bmatrix} 0 & 0 & & \epsilon_{\sum_{i=1}^{s-1} n_i+1}(Q) \\ \epsilon_{\sum_{i=1}^{s-1} n_i+2}(Q) & 0 & & 0 \\ & \ddots & \ddots & \\ & & \epsilon_{\sum_{i=1}^{s} n_i}(Q) & 0 \end{bmatrix}_{n_s}$$
$$\oplus \begin{bmatrix} -\epsilon_{\sum_{i=1}^{s} n_i+1}(P) & & \\ & \ddots & \\ & & -\epsilon_{\sum_{i=1}^{s} n_i+t}(P) \end{bmatrix}_{t} \oplus I_u \oplus -I_v,$$

where $\epsilon_j(P)$ and $\epsilon_j(Q)$ take either 1 or -1 for all $1 \leq j \leq \sum_{i=1}^{s} n_i + t$, such that

$$P +_b Q = L_{n_1} \oplus \cdots \oplus L_{n_s} \oplus (\pm 1) I_t \oplus I_u \oplus -I_v.$$

Choose the signed permutation matrix

$$R = \begin{bmatrix} 0 & & & & & & & & & & 1 \\ \epsilon_2(Q) & 0 & & & & & & & & & \\ & \ddots & \ddots & & & & & & & & \\ & & \epsilon_{n_1}(Q) & 0 & & & & & & & \\ & & & 1 & 0 & & & & & & \\ & & & & \epsilon_{n_1+2}(Q) & 0 & & & & & \\ & & & & & \ddots & \ddots & & & & \\ & & & & & & \epsilon_{\sum_{i=1}^{s} n_i}(Q) & 0 & & & \\ & & & & & & & 1 & 0 & & \\ & & & & & & & & \ddots & \ddots & \\ & & & & & & & & & 1 & 0 \end{bmatrix}_{\sum_{i=1}^{s} n_i+t} \oplus I_u \oplus -I_v.$$

We can check that $G(P +_b R)$ contains exactly one cycle of length $\sum_{i=1}^{s} n_i + t$, and $G(Q +_b R)$ contains exactly one cycle of length $s+t$, respectively. Moreover, $P +_b R$ and $Q +_b R$ contain no ± 1. By Corollary 3.2, P and R form an edge of $\Omega_n^{\pm}$ and the same for Q and R. Hence, the diameter of $G(\Omega_n^{\pm})$ is 2. ∎

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A graph is said to be *Hamiltonian* if it contains a circuit including all vertices of the graph. For any signed permutation $\sigma = (\sigma_1, \ldots, sgn(i)i, \ldots, \sigma_n)$, define the operator R_i on σ by $R_i\sigma = (\sigma_1, \ldots, -sgn(i)i, \ldots, \sigma_n)$ for $1 \leq i \leq n$. Here for convenience, we identify the signed permutation with its corresponding signed permutation matrix.

Lemma 4.4. *For any signed permutation σ, there exist 2^n operators $R_{i_1}, \ldots, R_{i_{2^n}}$ such that by consecutively applying these operators, we can get the signed permutations whose corresponding signed permutation matrices form a circuit in $G(\Omega_n^{\pm})$. Moreover, any matrix arising from changing arbitrary signs of elements in P_σ will be in this circuit.*

Proof. Without loss of generality, we can assume that $\sigma = (1, 2, \ldots, n)$. For $n = 2$, a circuit is

$$(1,2) \xrightarrow{R_1} (-1,2) \xrightarrow{R_2} (-1,-2) \xrightarrow{R_1} (1,-2) \xrightarrow{R_2} (1,2),$$

since each time when we apply the operator R_i there is only one element changing its sign. For $n = 3$, a circuit is

$$\begin{aligned}(1,2,3) \xrightarrow{R_1} (-1,2,3) \xrightarrow{R_2} (-1.-2,3) \xrightarrow{R_1} (1,-2,3) \xrightarrow{R_3} (1,-2,-3)\\ \xrightarrow{R_1} (-1,-2,-3) \xrightarrow{R_2} (-1,2,-3) \xrightarrow{R_1} (1,2,-3) \xrightarrow{R_3} (1,2,3).\end{aligned}$$

Assume inductively that we have a circuit $\sigma^{(1)}, \ldots, \sigma^{(2^{n-1})}$ containing all possible sign changes for elements in $(1, \ldots, n-1)$ by consecutively applying operators $R_{i'_1}, \ldots, R_{i'_{2^{n-1}}-1}, R_{i'_{2^{n-1}}} = R_{n-1}$. Then we obtain a circuit $\sigma^{(1)}, \ldots, \sigma^{(2^n)}$ containing all possible sign changes for elements in $(1, \ldots, n)$ by consecutively applying operators

$$\begin{aligned}&R_{i_1} = R_{i'_1}, R_{i_2} = R_{i'_2}, \ldots, R_{i_{2^{n-1}}-1} = R_{i'_{2^{n-1}}-1}, R_{i_{2^{n-1}}} = R_n,\\ &R_{i_{2^{n-1}}+1} = R_{i_1}, R_{i_{2^{n-1}}+2} = R_{i_2}, \ldots, R_{i_{2^n}-1} = R_{i_{2^{n-1}}-1}, R_{i_{2^n}} = R_n.\end{aligned}$$ □

Theorem 4.5. *$G(\Omega_n^{\pm})$ is Hamiltonian.*

Proof. According to [1, 7], we can assume $\sigma^{(1)}, \ldots, \sigma^{(n!)}, \sigma^{(1)}$ is a Hamiltonian circuit of $G(\Omega_n)$. Based on this result and Lemma 4.4, we construct a Hamiltonian circuit of $G(\Omega_n^{\pm})$ in the following way.

Step 1. Begin with $\sigma^{(1)}, \ldots, \sigma^{(n!)}$. Then apply the operator R_{i_1} in Lemma 4.4 consecutively on $\sigma^{(n!)}, \ldots, \sigma^{(1)}$ to obtain $R_{i_1}\sigma^{(n!)}, \ldots, R_{i_1}\sigma^{(1)}$. Note that $\sigma^{(n!)}$ and $R_{i_1}\sigma^{(n!)}$ are neighboring vertices of $G(\Omega_n^{\pm})$.

Step 2. Apply the operator R_{i_2} in Lemma 4.4 consecutively on $R_{i_1}\sigma^{(1)}, \ldots, R_{i_1}\sigma^{(n!)}$. We can obtain $R_{i_2}R_{i_1}\sigma^{(1)}, \ldots, R_{i_2}R_{i_1}\sigma^{(n!)}$. Note that $R_{i_1}\sigma^{(1)}$ and $R_{i_2}R_{i_1}\sigma^{(1)}$ are neighboring vertices of $G(\Omega_n^{\pm})$.

...

Step (2^n-1). Apply the operator $R_{i_{2^n-1}}$ in Lemma 4.4 consecutively on $R_{i_{2^n-2}} \cdots R_{i_1}\sigma^{(n!)}, \ldots, R_{i_{2^n-2}} \cdots R_{i_1}\sigma^{(1)}$ and get $R_{i_{2^n-1}} \cdots R_{i_1}\sigma^{(n!)}, \ldots, R_{i_{2^n-1}} \cdots R_{i_1}\sigma^{(1)}$. Note that $R_{i_{2^n-2}} \cdots R_{i_1}\sigma^{(n!)}$ and $R_{i_{2^n-1}}R_{i_{2^n-2}} \cdots R_{i_1}\sigma^{(n!)}$ are neighboring vertices of $G(\Omega_n^{\pm})$. Then applying R_{2^n} on $R_{i_{2^n-1}} \cdots R_{i_1}\sigma^{(1)}$, we get $\sigma^{(1)}$. Based on the above process, by consecutively applying operators $R_{i_1}, \ldots, R_{i_{2^n}}$ we can get a Hamiltonian circuit of $G(\Omega_n^{\pm})$. □

Example 4.6. *We construct a Hamiltonian circuit for $\Omega_3^{\pm}$ in the following way.*

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Step 1. *Begin with*

$$(1,2,3) \to (1,3,2) \to (3,1,2) \to (3,2,1) \to (2,3,1) \to (2,1,3). \tag{4.1}$$

Apply R_1 *on* $(2,1,3)$ *and keep doing this backward on elements in* (4.1)*, we get*

$$(2,-1,3) \to (2,3,-1) \to (3,2,-1) \to (3,-1,2) \to (-1,3,2) \to (-1,2,3). \tag{4.2}$$

Step 2. *Apply* R_2 *on* $(-1,2,3)$ *and keep doing this backward on elements in* (4.2)*, we get*

$$(-1,-2,3) \to (-1,3,-2) \to (3,-1,-2) \to (3,-2,-1) \to (-2,3,-1) \to (-2,-1,3). \tag{4.3}$$

Step 3. *Apply* R_1 *on* $(-2,-1,3)$ *and keep doing this backward on elements in* (4.3)*, we get*

$$(-2,1,3) \to (-2,3,1) \to (3,-2,1) \to (3,1,-2) \to (1,3,-2) \to (1,-2,3). \tag{4.4}$$

Step 4. *Apply* R_3 *on* $(1,-2,3)$ *and keep doing this backward on elements in* (4.4)*, we get*

$$(1,-2,-3) \to (1,-3,-2) \to (-3,1,-2) \to (-3,-2,1) \to (-2,-3,1) \to (-2,1,-3). \tag{4.5}$$

Step 5. *Apply* R_1 *on* $(-2,1,-3)$ *and keep doing this backward on elements in* (4.5)*, we get*

$$(-2,-1,-3) \to (-2,-3,-1) \to (-3,-2,-1) \to (-3,-1,-2) \to (-1,-3,-2) \to (-1,-2,-3). \tag{4.6}$$

Step 6. *Apply* R_2 *on* $(-1,-2,-3)$ *and keep doing this backward on elements in* (4.6)*, we get*

$$(-1,2,-3) \to (-1,-3,2) \to (-3,-1,2) \to (-3,2,-1) \to (2,-3,-1) \to (2,-1,-3). \tag{4.7}$$

Step 7. *Apply* R_1 *on* $(2,-1,-3)$ *and keep doing this backward on elements in* (4.7)*, we get*

$$(2,1,-3) \to (2,-3,1) \to (-3,2,1) \to (-3,1,2) \to (1,-3,2) \to (1,2,-3). \tag{4.8}$$

Then apply R_3 *on* $(1,2,-3)$ *and we get* $(1,2,3)$.

5. Toric ideals and Gröbner bases of $\Omega_n^{\pm}$. For the lattice polytope $\Omega_n^{\pm} \subset \mathbb{R}^{n\times n}$, the *point configuration* $\mathcal{A} = \Omega_n^{\pm} \cap \mathbb{Z}^{n\times n} = \{P_1,\dots,P_r\}$, where $r = \sum_{k=0}^{n} \binom{n}{k}^2 k! 2^k$. For each $1 \le s \le r$, we associate a variable X_{P_s} to $P_s = [p_{i,j}^{(s)}]$. Define a ring homomorphism

$$\begin{aligned} \hat{\pi} : \mathbb{K}[X_{P_1},\dots,X_{P_r}] &\to \mathbb{K}[t_0^{\pm 1}, t_{1,1}^{\pm 1},\dots,t_{n,n}^{\pm 1}] \\ X_{P_s} &\to t_0 t_{1,1}^{p_{1,1}^{(s)}} \cdots t_{n,n}^{p_{n,n}^{(s)}}. \end{aligned}$$

Its kernel is the homogeneous ideal $I_{\Omega_n^{\pm}}$ defined by

$$I_{\Omega_n^{\pm}} = \langle \mathbf{X}^{\mu} - \mathbf{X}^{\nu} : \mu,\nu \in \mathbb{N}^r, \sum_{s=1}^{r} \mu_s P_s = \sum_{s=1}^{r} \nu_s P_s, \sum_{s=1}^{r} \mu_s = \sum_{s=1}^{r} \nu_s \rangle,$$

which is called the *toric ideal* associated to $\Omega_n^{\pm}$. It is generated by homogeneous binomials and any reduced Gröbner basis of $I_{\Omega_n^{\pm}}$ consists of homogeneous binomials [23]. The Gröbner basis of toric ideals has intimate relation with the triangulation of the convex polytope, which is one of the important topics in the theory of convex polytopes.

Given an integral convex polytope $\mathcal{P}$, a simplex δ is called *belonging to* $\mathcal{P}$ if the set of vertices of δ is contained in $\mathcal{P} \cap \mathbb{Z}^n$. A set Δ of simplices belonging to $\mathcal{P}$ is called a *covering* of $\mathcal{P}$ if $\cup_{\delta \in \Delta} \delta = \mathcal{P}$. A covering Δ of $\mathcal{P}$ is called a *triangulation* of $\mathcal{P}$ if Δ is a simplicial complex. A triangulation Δ of $\mathcal{P}$ is called *unimodular* if the normalized volume of each maximal simplex in Δ is equal to 1 [14]. A triangulation of a polytope $\mathcal{P}$ in $\mathbb{R}^n$ is called *regular* if it can be obtained by projecting the lower envelope of a lifting of $\mathcal{P}$ to $\mathbb{R}^{n+1}$ [16, 23].

Theorem 5.1. [14] *An integral convex polytope $\mathcal{P}$ has a regular unimodular triangulation if and only if there exists a monomial order such that the initial ideal of the toric ideal of $\mathcal{P}$ is generated by square-free monomials.*

In [15], Kohl, Olsen, and Sanyal showed that the polytope $\Omega_n^{\pm}$ has a regular unimodular triangulation based on the fact that $\Omega_n^{\pm} \cap \mathbb{R}_{\geq 0}^{n\times n} = \omega_n$, which is compressed and by extending the corresponding heights on ω_n to the heights on $\Omega_n^{\pm}$ which induce a regular unimodular triangulation on $\Omega_n^{\pm}$. In this paper, we use a different approach to show this result by applying Theorem 5.1, which explicitly give the monomial orders under which the lattice polytope $\Omega_n^{\pm}$ has a regular unimodular triangulation. Later on from Example 5.10, we will see that these monomial orders are not the only ones that can lead to regular unimodular triangulations.

The following Lemma is due to Birkhoff's Theorem 1.4.

Lemma 5.2. [23] *Every nonnegative integer $n \times n$ matrix A with each row and column sum equal to k can be written as a sum of k permutation matrices.*

For an $n \times n$ matrix A, denote by $R_i(A)$ the ith row sum of A, and by $S_j(A)$ the jth column sum of A. Lemma 5.2 can be generalized to the following case.

Lemma 5.3. *Every nonnegative integer $n \times n$ matrix A with* $\max\{R_1(A), \ldots, R_n(A), S_1(A), \ldots, S_n(A)\} = k$ *can be written as a sum of k subpermutation matrices.*

Proof. Every nonnegative integer $n \times n$ matrix $A = [a_{i,j}]$ with $\max\{R_1(A), \ldots, R_n(A), S_1(A), \ldots, S_n(A)\} = k$ can be "completed" into a nonnegative integer matrix $\bar{A}$ with all row and column sums equal to k in the following way.

Step 1. Pick any i, j satisfying that $R_i(A) < k, S_j(A) < k$ and add $\min\{k - R_i(A), k - S_j(A)\}$ to $a_{i,j}$. Denote the resultant matrix by A_1, which has one more line sum equal to k than A has.

Step 2. Apply the process in Step 1 on A_1 and keep doing this until we get $\bar{A}$.

By Lemma 5.2, we can write $\bar{A} = P_1 + \cdots P_k$, where P_i is a permutation matrix for $1 \leq i \leq k$. Let $D = [d_{i,j}] = \bar{A} - A$ and for each $d_{i,j} > 0$, in the summand of $\bar{A}$, we pick $d_{i,j}$ permutation matrices whose (i, j)th entries are all equal to 1. Replace these 1's by 0's and eventually, we get $A = Q_1 + \cdots Q_k$, where Q_i is a subpermutation matrix for $1 \leq i \leq k$. ∎

Definition 5.4. *A signed subpermutation matrix $P = [p_{i,j}]$ is an $n \times n$ matrix whose entries are either* $0, -1$ *or* 1*, such that $|P|$ is a subpermutation matrix.*

By Lemma 5.3, we can get the following corollary.

Corollary 5.5. *Every integer $n\times n$ matrix A with* $\max\{R_1(|A|), \ldots, R_n(|A|), S_1(|A|), \ldots, S_n(|A|)\} = k$ *can be written as a sum of k signed subpermutation matrices.*

There are totally $r!$ orders of the variables $X_{P_1},\dots,X_{P_r}$. Among them we are interested in so called the *rank orders* which satisfy that if the rank of P_i is greater than the rank of P_j, then $X_{P_i}\succ X_{P_j}$. Rank orders have no requirements on the order of variables whose associated sign permutation matrices have the same rank.

Theorem 5.6. *Let $\succ$ be the graded reverse lexicographic term order induced by the rank order on $\mathbb{K}[X_{P_1},\dots,X_{P_r}]$. Then the initial ideal $in_{\prec}(I_{\Omega_n^{\pm}})$ is generated by square-free monomials of degree $\le n$. (Assume $n\ge 2$.)*

Proof. Let $\mathcal{G}$ be the reduced Gröbner bases of $\Omega_n^{\pm}$ with respect to the order "$\succ$". Consider any element $\mathbf{X}^{\mu}-\mathbf{X}^{\nu}$ of $\mathcal{G}$, where $\mathbf{X}^{\mu}$ is the initial term. Thus, $\sum_{s=1}^{r}\mu_sP_s=\sum_{s=1}^{r}\nu_sP_s, \sum_{s=1}^{r}\mu_s=\sum_{s=1}^{r}\nu_s$.

Case 1. Suppose there exist $1\le s,t\le r$ such that $\mu_s\neq 0,\mu_t\neq 0$, with $P_s=[p_{i,j}^{(s)}]$ and $P_t=[p_{i,j}^{(t)}]$ satisfying that there exist some $1\le i_0,j_0\le n$ such that either $p_{i_0,j_0}^{(s)}=1,p_{i_0,j_0}^{(t)}=-1$ or $p_{i_0,j_0}^{(s)}=-1,p_{i_0,j_0}^{(t)}=1$. Let s' and t' be the indices such that $P_{s'}$ and $P_{t'}$ can be obtained from P_s and P_t, respectively, by replacing the (i_0,j_0)th element by 0. With the property of the rank order, $X_{P_s}\succ X_{P_{s'}}$ and $X_{P_t}\succ X_{P_{t'}}$. Note that $P_s+P_t=P_{s'}+P_{t'}$, which implies that $X_{P_s}X_{P_t}\in in_{\prec}(I_{\Omega_n^{\pm}})$. Since $\mathcal{G}$ is the reduced Gröbner bases of $\Omega_n^{\pm}, \mathbf{X}^{\mu}=X_{P_s}X_{P_t}$, which is a square-free monomial of degree 2.

Case 2. Suppose for any $1\le s,t\le r$ such that $\mu_s\neq 0,\mu_t\neq 0$, there exist no entries on which P_s and P_t have the opposite signs.

Case 2a. If for all $\nu_s\neq 0$, $P_s\neq \mathbf{0}_{n\times n}$ and X_{P_k} is the smallest variable which divides $\mathbf{X}^{\nu}$. Since we choose the reverse lexicographic order, it is smaller than any variable appearing in $\mathbf{X}^{\mu}$. Let $A=\sum_{s=1}^{r}\nu_sP_s, A'=A-P_k$ and $P_k=[p_{i,j}^{(k)}]$.

Suppose we have

$$\begin{aligned}&\max\{R_1(|A'|),\dots,R_n(|A'|),S_1(|A'|),\dots,S_n(|A'|)\}\\&\quad=\max\{R_1(|A|),\dots,R_n(|A|),S_1(|A|),\dots,S_n(|A|)\}-1.\end{aligned}$$

Since $\sum_{s=1}^{r}\mu_sP_s=\sum_{s=1}^{r}\nu_sP_s$, we can find $P_{s_1}=[p_{i,j}^{(s_1)}],\dots,P_{s_l}=[p_{i,j}^{(s_l)}]$ with no repetition, where $2\le l\le n$, such that for each $p_{i,j}^{(k)}\neq 0$, there exists some $1\le d\le l$ such that $p_{i,j}^{(s_d)}=p_{i,j}^{(k)}$. Let $Q=P_{s_1}+\cdots+P_{s_l}$. Therefore,

$$\begin{aligned}&\max\{R_1(|Q-P_k|),\dots,R_n(|Q-P_k|),S_1(|Q-P_k|),\dots,S_n(|Q-P_k|)\}\\&\quad=\max\{R_1(|Q|),\dots,R_n(|Q|),S_1(|Q|),\dots,S_n(|Q|)\}-1.\end{aligned}$$

By Corollary 5.5, we can write

$$P_{s_1}+\cdots+P_{s_l}=P_k+P_{t_1}+\cdots+P_{t_{l-1}},$$

where $P_{t_1},\dots,P_{t_{l-1}}$ are signed permutation matrices. Thus, $X_{P_{s_1}}\cdots X_{P_{s_l}}-X_{P_k}X_{P_{t_1}}\cdots X_{P_{t_{l-1}}}\in I_{\Omega_n^{\pm}}$. Since X_{P_k} is smaller than variables $X_{P_{s_1}},\dots,X_{P_{s_l}}$, $X_{P_{s_1}}\cdots X_{P_{s_l}}\in in_{\prec}(I_{\Omega_n^{\pm}})$. This implies $\mathbf{X}^{\mu}=X_{P_{s_1}}\cdots X_{P_{s_l}}$, which is square-free with degree $l\le n$.

Suppose we have

$$\max\{R_1(|A'|),\dots,R_n(|A'|),S_1(|A'|),\dots,S_n(|A'|)\}=\max\{R_1(|A|),\dots,R_n(|A|),S_1(|A|),\dots,S_n(|A|)\}.$$

Since $A' = A - P_k, \max\{R_1(|A|), \dots, R_n(|A|), S_1(|A|), \dots, S_n(|A|)\} = \sum_{s=1}^{r} \nu_s - 1$. By Corollary 5.5, we have

$$\sum_{s=1}^{r} \mu_s P_s = \sum_{s=1}^{r} \nu_s P_s = \mathbf{0}_{n\times n} + P_{j_1} + \cdots + P_{j_w},$$

where $P_{j_1}, \dots, P_{j_w}$ are signed permutation matrices and $w = \sum_{s=1}^{r} \nu_s - 1$. With the property of the rank order, $X_{\mathbf{0}_{n\times n}}$ is the smallest variable. Thus, $\mathbf{X}^{\nu} - X_{\mathbf{0}_{n\times n}} X_{P_{j_1}} \cdots X_{P_{j_w}} \in I_{\Omega_n^{\pm}}$ and $\mathbf{X}^{\nu} \in in_{\prec}(I_{\Omega_n^{\pm}})$, which contradicts with the fact that $\mathcal{G}$ is a reduced Gröbner bases of $\Omega_n^{\pm}$.

Case 2b. Suppose $\mathbf{0}_{n\times n}$ is in the summand $\sum_{s=1}^{r} \nu_s P_s$. If there exists some $\nu_i \neq 0$ such that $P_i \neq \mathbf{0}_{n\times n}$ and

$$\begin{aligned}&\max\{R_1(|A-P_i|), \dots, R_n(|A-P_i|), S_1(|A-P_i|), \dots, S_n(|A-P_i|)\}\\ &\quad = \max\{R_1(|A|), \dots, R_n(|A|), S_1(|A|), \dots, S_n(|A|)\},\end{aligned}$$

then $\max\{R_1(|A|), \dots, R_n(|A|), S_1(|A|), \dots, S_n(|A|)\} \leq \sum_{s=1}^{r} \nu_s - 2$. By Corollary 5.5, we have

$$\sum_{s=1}^{r} \nu_s P_s = \sum_{s=1}^{r-1} \nu_s P_s + \mathbf{0}_{n\times n} = \mathbf{0}_{n\times n} + \mathbf{0}_{n\times n} + P_{j_1} + \cdots + P_{j_v},$$

where $P_{j_1}, \dots, P_{j_v}$ are signed permutation matrices and $v = \sum_{s=1}^{r} \nu_s - 2$. Thus,

$$\prod_{s=1}^{r-1} X_{P_s}^{\nu_s} - X_{\mathbf{0}_{n\times n}} X_{P_{j_1}} \cdots X_{P_{j_v}} \in I_{\Omega_n^{\pm}},$$

and $\prod_{s=1}^{r-1} X_{P_s}^{\nu_s} \in in_{\prec}(I_{\Omega_n^{\pm}})$, which contradicts with the fact that $\mathcal{G}$ is a reduced Gröbner bases of $\Omega_n^{\pm}$.

Therefore, for all $\nu_s \neq 0$ such that $P_s \neq \mathbf{0}_{n\times n}$, we have

$$\begin{aligned}&\max\{R_1(|A-P_s|), \dots, R_n(|A-P_s|), S_1(|A-P_s|), \dots, S_n(|A-P_s|)\}\\ &\quad = \max\{R_1(|A|), \dots, R_n(|A|), S_1(|A|), \dots, S_n(|A|)\} - 1.\end{aligned}$$

Since

$$\sum_{s=1}^{r} \mu_s P_s = \sum_{s=1}^{r-1} \nu_s P_s + \mathbf{0}_{n\times n},$$

there exists some signed permutation matrix $P_k = [p_{i,j}^{(k)}]$ in the summand of $\sum_{s=1}^{r-1} \nu_s P_s$, which contains maximum nonzero elements. So the rank of P_k is greater than the rank of any signed permutation matrices in the summand of $\sum_{s=1}^{r} \mu_s P_s$. Since $\sum_{s=1}^{r} \mu_s P_s = \sum_{s=1}^{r-1} \nu_s P_s + \mathbf{0}_{n\times n}$, we can find $P_{s_1} = [p_{i,j}^{(s_1)}], \dots, P_{s_l} = [p_{i,j}^{(s_l)}]$ with no repetition, where $2 \leq l \leq n$, such that for each $p_{i,j}^{(k)} \neq 0$, there exists some $1 \leq d \leq l$ such that $p_{i,j}^{(s_d)} = p_{i,j}^{(k)}$. Let $Q = P_{s_1} + \cdots + P_{s_l}$. We have

$$\begin{aligned}&\max\{R_1(|Q-P_k|), \dots, R_n(|Q-P_k|), S_1(|Q-P_k|), \dots, S_n(|Q-P_k|)\}\\ &\quad = \max\{R_1(|Q|), \dots, R_n(|Q|), S_1(|Q|), \dots, S_n(|Q|)\} - 1.\end{aligned}$$

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By Corollary 5.5, we can write

$$P_{s_1}+\cdots+P_{s_l}=P_k+P_{t_1}+\cdots+P_{t_{l-1}},$$

where $P_{t_1},\dots,P_{t_{l-1}}$ are signed permutation matrices. Moreover, since P_k has more nonzero elements than P_{s_i} has, where $1\le i\le l$, there exists some P_{t_j} which has less nonzero elements than P_{s_i} has. Thus, $X_{P_{s_1}}\cdots X_{P_{s_l}}-X_{P_k}X_{P_{t_1}}\cdots X_{P_{t_{l-1}}}\in I_{\Omega_n^\pm}$. Since the rank of P_{t_j} is smaller than the rank of P_{s_i} for all $1\le i\le l$, $X_{P_{t_j}}$ has smaller order than variables $X_{P_{s_1}},\dots,X_{P_{s_l}}$. Thus, we have $X_{P_{s_1}}\cdots X_{P_{s_l}}\in in_\prec(I_{\Omega_n^\pm})$. This implies $\mathbf{X}^\mu=X_{P_{s_1}}\cdots X_{P_{s_l}}$, which is square-free with degree $l\le n$. ∎

Remark 5.7. *The rank order is just the sufficient condition to obtain square-free monomials generating the initial ideal $I_{\Omega_n^\pm}$. However, it is not necessary, which can be seen from the following example. Besides, rank orders may not coincide with the monomial orders, which were yielded by the extended heights in [15], since there are no requirements on the order of variables whose associated sign permutation matrices have the same rank. In the case of the monomial order yielded by the extended heights, the order of variables whose associated sign permutation matrices are in the same orthant should be coincide with the order of variables whose associated matrices are subpermutation matrices, while in the rank order case there is no such requirement. We use Example 5.10 to illustrate this fact.*

Remark 5.8. *It was shown in [15] by Kohl, Olsen and Sanyal that $\omega_n=\Omega_n^\pm\cap\mathbb{R}_{\ge0}^{n\times n}$ is compressed, so all pulling triangulations of ω_n are regular and unimodular. However, $\Omega_n^\pm$ is not compressed, which is also illustrated by Example 5.10.*

Remark 5.9. *By restricting the rank order on the variables whose associated sign permutation matrices are subpermutation matrices, and using the graded reverse lexicographic term order induced by the restriced rank order, we get the initial ideal $in_\prec(I_{\omega_n})$, which is also generated by square-free monomials of degree $\le n$. Thus, the regular unimodular triangulation of $\Omega_n^\pm$ obtained from the rank order restricts to a triangulation of ω_n, which is also regular and unimodular.*

Example 5.10. *For $\Omega_2^\pm$, the point configuration $\mathcal{A}=\Omega_2^\pm\cap\mathbb{Z}^{2\times2}$ is the set containing*

$$P_1=\begin{bmatrix}1&0\\0&1\end{bmatrix},P_2=\begin{bmatrix}-1&0\\0&1\end{bmatrix},P_3=\begin{bmatrix}1&0\\0&-1\end{bmatrix},P_4=\begin{bmatrix}-1&0\\0&-1\end{bmatrix},$$
$$P_5=\begin{bmatrix}0&1\\1&0\end{bmatrix},P_6=\begin{bmatrix}0&-1\\1&0\end{bmatrix},P_7=\begin{bmatrix}0&1\\-1&0\end{bmatrix},P_8=\begin{bmatrix}0&-1\\-1&0\end{bmatrix},$$
$$P_9=\begin{bmatrix}1&0\\0&0\end{bmatrix},P_{10}=\begin{bmatrix}-1&0\\0&0\end{bmatrix},P_{11}=\begin{bmatrix}0&1\\0&0\end{bmatrix},P_{12}=\begin{bmatrix}0&-1\\0&0\end{bmatrix},$$
$$P_{13}=\begin{bmatrix}0&0\\1&0\end{bmatrix},P_{14}=\begin{bmatrix}0&0\\-1&0\end{bmatrix},P_{15}=\begin{bmatrix}0&0\\0&1\end{bmatrix},P_{16}=\begin{bmatrix}0&0\\0&-1\end{bmatrix},$$
$$P_{17}=\begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

With respect to the graded reverse lexicographic term order induced by the rank order $X_{P_1}\succ X_{P_2}\succ\cdots\succ X_{P_{17}}$, we use Maple software [17] to calculate the initial ideal $I_{\Omega_n^\pm}$, which is generated by the following monomials

$$\begin{gathered}\{X_{P_1}X_{P_2}, X_{P_1}X_{P_3}, X_{P_1}X_{P_4}, X_{P_1}X_{P_{10}}, X_{P_1}X_{P_{16}}, X_{P_2}X_{P_3}, X_{P_2}X_{P_4}, X_{P_2}X_{P_9}, X_{P_2}X_{P_{16}}, X_{P_3}X_{P_4}, X_{P_3}X_{P_{10}},\\
X_{P_3}X_{P_{15}}, X_{P_4}X_{P_9}, X_{P_4}X_{P_{15}}, X_{P_5}X_{P_6}, X_{P_5}X_{P_7}, X_{P_5}X_{P_8}, X_{P_5}X_{P_{12}}, X_{P_5}X_{P_{14}}, X_{P_6}X_{P_7}, X_{P_6}X_{P_8}, X_{P_6}X_{P_{11}},\\
X_{P_6}X_{P_{14}}, X_{P_7}X_{P_8}, X_{P_7}X_{P_{12}}, X_{P_7}X_{P_{13}}, X_{P_8}X_{P_{11}}, X_{P_8}X_{P_{13}}, X_{P_9}X_{P_{10}}, X_{P_9}X_{P_{15}}, X_{P_9}X_{P_{16}}, X_{P_{10}}X_{P_{15}},\\
X_{P_{10}}X_{P_{16}}, X_{P_{11}}X_{P_{12}}, X_{P_{11}}X_{P_{13}}, X_{P_{11}}X_{P_{14}}, X_{P_{12}}X_{P_{13}}, X_{P_{12}}X_{P_{14}}, X_{P_{13}}X_{P_{14}}, X_{P_{15}}X_{P_{16}}\},\end{gathered}$$

which are square-free and of degree 2.

Notice that the rank order of X_{P_9} *is less than the rank order of* X_{P_7} *and* X_{P_8}*. If we take the graded reverse lexicographic term order induced by the order* $X_{P_1} \succ X_{P_2} \succ \cdots \succ X_{P_6} \succ X_{P_9} \succ X_{P_7} \succ X_{P_8} \succ X_{P_{10}} \succ \cdots \succ X_{P_{17}}$*, the initial ideal can still be generated by square-free monomials as follows:*

$$\begin{gathered}\{X_{P_1}X_{P_2}, X_{P_1}X_{P_3}, X_{P_1}X_{P_4}, X_{P_1}X_{P_{10}}, X_{P_1}X_{P_{16}}, X_{P_2}X_{P_3}, X_{P_2}X_{P_4}, X_{P_2}X_{P_9}, X_{P_2}X_{P_{16}}, X_{P_3}X_{P_4}, X_{P_3}X_{P_{10}},\\
X_{P_3}X_{P_{15}}, X_{P_4}X_{P_9}, X_{P_4}X_{P_{15}}, X_{P_5}X_{P_6}, X_{P_5}X_{P_7}, X_{P_5}X_{P_8}, X_{P_5}X_{P_{12}}, X_{P_5}X_{P_{14}}, X_{P_6}X_{P_7}, X_{P_6}X_{P_8}, X_{P_6}X_{P_{11}},\\
X_{P_6}X_{P_{14}}, X_{P_9}X_{P_{10}}, X_{P_9}X_{P_{15}}, X_{P_9}X_{P_{16}}, X_{P_7}X_{P_8}, X_{P_7}X_{P_{12}}, X_{P_7}X_{P_{13}}, X_{P_8}X_{P_{11}}, X_{P_8}X_{P_{13}}, X_{P_{10}}X_{P_{15}},\\
X_{P_{10}}X_{P_{16}}, X_{P_{11}}X_{P_{12}}, X_{P_{11}}X_{P_{13}}, X_{P_{11}}X_{P_{14}}, X_{P_{12}}X_{P_{13}}, X_{P_{12}}X_{P_{14}}, X_{P_{13}}X_{P_{14}}, X_{P_{15}}X_{P_{16}}\}.\end{gathered}$$

The above two different monomial orders generate the same initial ideal basis, which correspond to the same triangulation.

If we interchange the order of variables X_{P_4} *and* X_{P_8} *and use the graded reverse lexicographic term order induced by the order* $X_{P_1} \succ X_{P_2} \succ X_{P_3} \succ X_{P_8} \succ X_{P_5} \succ X_{P_6} \succ X_{P_7} \succ X_{P_4} \succ X_{P_9} \succ \cdots \succ X_{P_{17}}$*, which cannot be generated by the extended heights in* [15] *since* $X_{P_1} \succ X_{P_5}$*, we sill get the same initial ideal basis as above.*

If we choose the graded reverse lexicographic term order induced by the order $X_{P_{17}} \succ X_{P_1} \succ X_{P_2} \succ \cdots \succ X_{P_{16}}$*, the initial ideal is generated by*

$$\begin{gathered}\{X_{P_1}^2, X_{P_1}X_{P_2}, X_{P_1}X_{P_3}, X_{P_1}X_{P_4}, X_{P_1}X_{P_5}, X_{P_1}X_{P_6}, X_{P_1}X_{P_7}, X_{P_1}X_{P_8}, X_{P_1}X_{P_9}, X_{P_1}X_{P_{10}}, X_{P_1}X_{P_{11}},\\
X_{P_1}X_{P_{12}}, X_{P_1}X_{P_{13}}, X_{P_2}X_{P_3}, X_{P_2}X_{P_4}, X_{P_2}X_{P_5}, X_{P_2}X_{P_{11}}, X_{P_3}X_{P_4}, X_{P_3}X_{P_5}, X_{P_3}X_{P_{10}}, X_{P_4}X_{P_5}, X_{P_4}X_{P_{11}},\\
X_{P_4}X_{P_{16}}, X_{P_5}X_{P_{10}}, X_{P_5}X_{P_{16}}, X_{P_6}X_{P_7}, X_{P_6}X_{P_8}, X_{P_6}X_{P_9}, X_{P_6}X_{P_{13}}, X_{P_6}X_{P_{15}}^2, X_{P_7}X_{P_8}, X_{P_7}X_{P_9}, X_{P_7}X_{P_{12}},\\
X_{P_7}X_{P_{15}}^2, X_{P_8}X_{P_9}, X_{P_8}X_{P_{13}}, X_{P_8}X_{P_{14}}, X_{P_9}X_{P_{12}}, X_{P_9}X_{P_{14}}, X_{P_{10}}X_{P_{11}}, X_{P_{10}}^2X_{P_{16}}, X_{P_{11}}^2X_{P_{16}}, X_{P_{12}}X_{P_{13}},\\
X_{P_{12}}^2X_{P_{14}}, X_{P_{12}}^2X_{P_{15}}^2, X_{P_{13}}^2X_{P_{14}}, X_{P_{13}}^2X_{P_{15}}^2, X_{P_{14}}X_{P_{15}}, X_{P_2}X_{P_6}X_{P_{15}}, X_{P_2}X_{P_7}X_{P_{15}}, X_{P_2}X_{P_{12}}X_{P_{14}},\\
X_{P_2}X_{P_{12}}X_{P_{15}}, X_{P_2}X_{P_{13}}X_{P_{14}}, X_{P_2}X_{P_{13}}X_{P_{15}}, X_{P_3}X_{P_6}X_{P_{15}}, X_{P_3}X_{P_7}X_{P_{15}}, X_{P_3}X_{P_{12}}X_{P_{14}}, X_{P_3}X_{P_{12}}X_{P_{15}},\\
X_{P_3}X_{P_{13}}X_{P_{14}}, X_{P_3}X_{P_{13}}X_{P_{15}}, X_{P_4}X_{P_6}X_{P_{15}}, X_{P_4}X_{P_7}X_{P_{15}}, X_{P_4}X_{P_{12}}X_{P_{14}}, X_{P_4}X_{P_{12}}X_{P_{15}}, X_{P_4}X_{P_{13}}X_{P_{14}},\\
X_{P_4}X_{P_{13}}X_{P_{15}}, X_{P_5}X_{P_6}X_{P_{15}}, X_{P_5}X_{P_7}X_{P_{15}}, X_{P_5}X_{P_{12}}X_{P_{14}}, X_{P_5}X_{P_{12}}X_{P_{15}}, X_{P_5}X_{P_{13}}X_{P_{14}}, X_{P_5}X_{P_{13}}X_{P_{15}},\\
X_{P_6}X_{P_{10}}X_{P_{15}}, X_{P_6}X_{P_{11}}X_{P_{15}}, X_{P_7}X_{P_{10}}X_{P_{15}}, X_{P_{10}}X_{P_{12}}X_{P_{14}}, X_{P_{10}}X_{P_{12}}X_{P_{15}}, X_{P_{10}}X_{P_{13}}X_{P_{14}}, X_{P_{10}}X_{P_{13}}X_{P_{15}},\\
X_{P_7}X_{P_{11}}X_{P_{15}}, X_{P_{11}}X_{P_{12}}X_{P_{14}}, X_{P_{11}}X_{P_{12}}X_{P_{15}}, X_{P_{11}}X_{P_{13}}X_{P_{14}}, X_{P_{11}}X_{P_{13}}X_{P_{15}}\}.\end{gathered}$$

It contains not only square terms but also terms of degree 3 *and degree* 4*, which shows that* $\Omega_2^{\pm}$ *is not compressed.*

Theorem 5.1 and 5.6 lead to the following corollary, which was shown in [15] in a different method.

Corollary 5.11. [15] *The polytope* $\Omega_n^{\pm}$ *has a regular unimodular triangulation.*

Question 5.12. *In Example 5.10, we only consider the case when* $n = 2$*. The toric ideal of the* 3×3 *Birkhoff polytope* Ω_3 *is not generated by quadratic polynomials* [13]*. It is worth to study whether or not the* *toric ideal $I_{\Omega_n^\pm}$ has an initial ideal which is generated by square-free quadratic monomials when $n \geq 3$, i.e., the existence of a regular, unimodular flag triangulation of $\Omega_n^\pm$.*

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