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# THE SIGNLESS LAPLACIAN SPECTRAL RADIUS OF GRAPHS WITHOUT INTERSECTING ODD CYCLES*

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**Abstract.** Let $F_{a_1,\ldots,a_k}$ be a graph consisting of $k$ cycles of odd length $2a_1+1,\ldots,2a_k+1$, respectively, which intersect in exactly one common vertex, where $k \geq 1$ and $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$. In this paper, we present a sharp upper bound for the signless Laplacian spectral radius of all $F_{a_1,\ldots,a_k}$-free graphs and characterize all extremal graphs which attain the bound. The stability methods and structure of graphs associated with the eigenvalue are adapted for the proof.

**Key words.** Brualdi-Solheid-Turán-type problem; Signless Laplacian spectral radius; Intersecting odd cycles free; Extremal graph.

**AMS subject classifications.** 05C50, 05C35.

**1. Introduction.** Let $G$ be an undirected simple graph with vertex set $V(G) = \{v_1,\ldots,v_n\}$ and edge set $E(G)$, where $e(G)$ is the number of edges of $G$. For $v \in V(G)$, the *neighborhood* $N_G(v)$ of $v$ is $\{u : uv \in E(G)\}$ and the *degree* $d_G(v)$ of $v$ is $|N_G(v)|$. We write $N(v)$ and $d(v)$ for $N_G(v)$ and $d_G(v)$, respectively, if there is no ambiguity. Denote by $\triangle(G)$ and $\delta(G)$ the maximum and minimum degree of $G$, respectively. Denote by $P_n$ and $C_n$ the path and the cycle of order $n$, respectively. For $A, B \subseteq V(G)$, $e(A)$ denotes the number of the edges of $G$ with both endvertices in $A$ and $e(A,B)$ denotes the number of the edges of $G$ with one endvertex in $A$ and the other in $B$. For two vertex disjoint graphs $G$ and $H$, we denote by $G \cup H$ and $G\nabla H$ the *union* of $G$ and $H$, and the *join* of $G$ and $H$, that is, joining every vertex of $G$ to every vertex of $H$, respectively. Denote by $kG$ the union of $k$ disjoint copies of $G$. Let $S_{n,t} = K_t\nabla\overline{K}_{n-t}$ denote the join of a complete graph of order $t$ and the independent set of size $n-t$. Let $L_{r,t} = K_1\nabla rK_t$ denote the graph consists of $r$ complete graph $K_{t+1}$ which intersect in exactly one common vertex. For graph notation and terminology undefined here, we refer the readers to [1].

We say a graph $G$ is $H$*-free* if it does not contain $H$ as a subgraph. The Turán number of a graph $H$ is the maximum number of edges in an $H$-free graph of order $n$ and is denoted by $ex(n,H)$. An $H$-free graph of order $n$ with $ex(n,H)$ edges is called an *extremal graph* for $H$. Moreover, denote $Ex(n,H)$ by the set of all extremal graphs of order $n$ for $H$. To determine $ex(n,H)$ and characterize those graphs in $Ex(n,H)$ is a fundamental problem (called Turán-type problem) in extremal graph theory. It will be interesting to look for some nice graphs $H$ such that $ex(n,H)$ and $Ex(n,H)$ will be characterized. The graph $F_{a_1,\ldots,a_k}$ consisting of $k$ cycles of odd length $2a_1+1,\ldots,2a_k+1$, respectively, which intersect in exactly one common vertex may be of interest, where $k \geq 1$ and $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$. If $k=1$, then $F_{a_1,\ldots,a_k}$ is an odd cycle $C_{2a_1+1}$. Simonovits [23] proved that $ex(n, C_{2a_1+1}) = \lfloor \frac{n^2}{4} \rfloor$ for sufficiently large $n$ and $Ex(n, C_{2a_1+1})$ is a balanced complete bipartite graph, that is a complete bipartite graph whose two partite sets have sizes

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differing by at most 1. If $k \geq 2$ and $a_1 = \cdots = a_k = 1$, then $F_{a_1,\ldots,a_k}$ is denoted by $F^{(k)}$ which is called the *friendship graph*. In 1995, Erdős, Füredi, Gould, Gunderson [11] significantly extended Mantel's result and proved the following interesting result.

THEOREM 1.1. *[11] Let $k \geq 1$ and $n \geq 50k^2$. Then,*

$$ex(n, F^{(k)}) = \left\lfloor \frac{n^2}{4} \right\rfloor + \begin{cases} k^2 - k, & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k, & \text{if } k \text{ is even.} \end{cases}$$

*Furthermore, if $k$ is odd, then $Ex(n, F^{(k)})$ consists of graphs which are constructed by taking a complete bipartite graph with two parts of sizes $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$ and embedding two vertex disjoint copies of $K_k$ in one side. If $k$ is even, then $Ex(n, F^{(k)})$ consists of graphs which are constructed by taking a complete bipartite graph with two parts of sizes $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$ and embedding a graph with $2k-1$ vertices, $k^2 - \frac{3}{2}k$ edges with maximum degree $k-1$ in one partite.*

If $k \geq 2$ and $a_1 \geq \cdots \geq a_s \geq 2$, $a_{s+1} = \cdots = a_k = 1$ with $1 \leq s \leq k$, then $F_{a_1,\cdots,a_k}$ is denoted by $H_{k,s}$, that is, $H_{k,s}$ is the graph consisting of $k$ odd cycles and $k-s$ triangles which intersect in exactly one common vertex. In 2018, Hou, Qiu and Liu [14], and Yuan [24] independently proved the following result.

THEOREM 1.2. *[14, 24] Let $k \geq 2$ and $1 \leq s \leq k$. Then*

$$ex(n, H_{k,s}) = \left\lfloor \frac{n^2}{4} \right\rfloor + (k-1)^2,$$

*for sufficiently large $n$. Moreover, $Ex(n, H_{k,s})$ consists of a balanced complete bipartite graph with a complete bipartite graph $K_{k-1,k-1}$ embedded in one part if $(k,s) \neq (4,1)$; a balance complete bipartite graph with a complete bipartite graph $K_{3,3}$ or $3K_3$ embedded in one part if $(k,s) = (4,1)$*

In spectral extremal graph theory, there is an analogy between the Turán-type problem and the Brualdi-Solheid-Turán-type problem which is proposed by Nikiforov [18]. The Brualdi-Solheid-Turán-type problem is to determine the maximum spectral radius of an $H$-free graph of order $n$ and characterize those graphs which attain the maximum spectral radius. The Brualdi-Solheid-Turán-type problem of the spectral radius has been studied for various kinds of $H$ such as the complete graph [17], the complete bipartite graph [19], cycles or paths of specified length [18], the linear forest [3], and star forest [4]. In addition, the Brualdi-Solheid-Turán-type problem of the signless Laplacian spectral radius has also been investigated extensively in the literature. For more details, readers may be referred to [8, 13, 20, 21, 22, 25, 26]. It is of interest to consider this problem for $F_{a_1,\cdots,a_k}$.

The *adjacency matrix* of $G$ is the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$, and 0 otherwise. Moreover, the matrix $Q(G) = D(G) + A(G)$ is known as the *signless Laplacian matrix* of $G$, where $D(G)$ is the degree diagonal matrix of $G$. The largest eigenvalue of $A(G)$ is called the *spectral radius* of $G$. The largest eigenvalue of $Q(G)$ is called the *signless Laplacian spectral radius* of $G$ and denoted by $q(G)$.

In fact, if $k = 1$ and $a_1 = 1$, Nikiforov [17] determined the maximum spectral radius of all $F_1$-free graphs of order $n$ and proved that only the balanced bipartite graph has the maximum spectral radius; while He, Jin and Zhang [13] obtained a sharp bound for the signless Laplacian spectral radius of all $F_1$-free graphs of order $n$ and proved that the corresponding extremal graphs are any complete bipartite graphs $K_{r,s}$ with $r + s = n$. Later, if $k = 1$ and $a_1 \geq 2$, Nikiforov [18] and Yuan [25] determined the maximum spectral radius

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and the signless spectral radius among $F_{a_1}$-free graphs of order $n$, respectively. Recently, Cioabă, Feng, Tait, and Zhang [5] studied the Brualdi-Solheid-Turán-type problem for $F^{(k)}$-free graphs and determined the corresponding spectral extremal graphs, which can be viewed as a spectral analogue of Theorem 1.1.

Theorem 1.3. *[5] Let $G$ be an $F^{(k)}$-free graph of order $n$ with $k \geq 2$. If $G$ has the maximum spectral radius, then*

$$G \in Ex(n, F^{(k)}),$$

*for sufficiently large $n$.*

Inspired by the work of Cioabă, Feng, Tait, and Zhang [5], Zhao, Huang, and Guo [26] focused on the maximum signless Laplacian spectral radius of all $F^{(k)}$-free graphs of order $n$ and proved the following result.

Theorem 1.4. *[26] Let $G$ be an $F^{(k)}$-free graph of order $n$. If $k \geq 2$ and $n \geq 3k^2 - k - 2$, then*

$$q(G) \leq q(S_{n,k}),$$

*with equality if and only if $G = S_{n,k}$.*

Recently, Li and Peng [15] proved the spectral result of all $H_{k,s}$-free graphs of order $n$, which can be viewed as a spectral analogue of Theorem 1.2.

Theorem 1.5. *[15] Let $k \geq 2$ and $1 \leq s \leq k$. If $G$ is an $H_{k,s}$-free graph of order $n$ with the maximum spectral radius, then*

$$G \in Ex(n, H_{k,s}),$$

*for sufficiently large $n$.*

Furthermore, combining with known results, Li and Peng [15] proposed the following conjecture on the signless Laplacian spectral radius of $F_{a_1,\cdots,a_k}$-free graphs of order $n$ with $k \geq 2$ and $a_1 = \cdots = a_k \geq 2$.

Conjecture 1.6. *[15] Let $G$ be an $F_{a_1,\cdots,a_k}$-free graph of order $n$. If $k \geq 2$ and $a_1 = \cdots = a_k = a \geq 2$, then*

$$q(G) \leq q(S_{n,ka}),$$

*for sufficiently large $n$ with equality if and only if $G = S_{n,ka}$.*

Inspired by above known results and Conjecture 1.6, we study the maximum signless Laplacian spectral radius of all $F_{a_1,\cdots,a_k}$-free graphs. The main result of this paper can be stated as follows.

Theorem 1.7. *Let $G$ be an $F_{a_1,\ldots,a_k}$-free graph of order $n \geq 8t^2 - 12t + 9$ with $t = \sum_{i=1}^{k} a_i$.*

*(1) [13] If $k = 1$ and $a_1 = 1$, then $q(G) \leq q(S_{n,t})$ with equality if and only if $G$ is any complete bipartite graphs $K_{r,s}$ with $r + s = n$.*

*(2) [25] If $k = 1$, $a_1 \geq 2$, and $n \geq 110t^2$, then $q(G) \leq q(S_{n,t})$ with equality if and only if $G = S_{n,t}$.*

*(3) [26] If $k \geq 2$ and $a_1 = \cdots = a_k = 1$, then $q(G) \leq q(S_{n,t})$ with equality if and only if $G = S_{n,t}$.*

*(4) If $k \geq 2$ and $a_1 \geq 2$, then $q(G) \leq q(S_{n,t})$ with equality if and only if $G = S_{n,t}$.*

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**Remark 1.** It is easy to see that

$$q(S_{n,t}) = \frac{n+2t-2+\sqrt{(n+2t-2)^2-8(t^2-t)}}{2}.$$

**Remark 2.** It is worth mentioning that the extremal graphs in Theorem 1.7 (4) are not the same as those of Theorems 1.2 and 1.5. The extremal graphs in Theorems 1.2 and 1.5 only depend on the number of intersecting triangles and the number of all intersecting odd cycles, while the extremal graph in Theorem 1.7 (4) not only depends on the number of intersecting odd cycles but also the lengths of all intersecting odd cycles.

The rest of this paper is organized as follows. In Section 2, some known lemmas are presented. In Section 3, we give the proof of Theorem 1.7.

## 2. Some Lemmas.

Lemma 2.1. *[10] Let $k \geq 3$. If $G$ is a $P_k$-free graph of order $n$, then $e(G) \leq \frac{(k-2)n}{2}$ with equality if and only if $G$ is the union of disjoint copies of $K_{k-1}$.*

Lemma 2.2. *[10] Let $k \geq 2$. If $G$ is a graph of order $n$ with no cycle greater than $k$, then $e(G) \leq \frac{k(n-1)}{2}$ with equality if and only if $G = L_{r,k-1}$ with $n = r(k-1)+1$.*

Lemma 2.3. *[9] If $G$ is a graph with $\delta(G) \geq 2$, then $G$ contains a cycle of length at least $\delta(G)+1$.*

We also need the stability result on the disjoint paths.

Lemma 2.4. *[2] Let $H = \bigcup_{i=1}^{k} P_{2a_i}$ with $k \geq 2$, $a_1 \geq \cdots \geq a_k \geq 1$, and $t = \sum_{i=1}^{k} a_i$. If $\delta(G) \geq t-1$ and $G$ is an $H$-free connected graph of order $n \geq 2t$, then one of the following holds:*
*(1) $G \subseteq S_{n,t-1}$;*
*(2) $H = 2P_{2a_1}$ and $G = L_{r,t-1}$, where $n = r(t-1)+1$.*

Lemma 2.5. *[21] Let $t \geq 2$ and $n > 5t^2$. Then*
*(1) $q(S_{n,t}) > n+2t-2-\frac{2(t^2-t)}{n+2t-3} > n+2t-3$.*
*(2) If $G$ is a graph of order $n$ with $q(G) \geq q(S_{n,t})$, then $e(G) \geq tn-t^2+1$.*

Lemma 2.6. *[7] Let $G$ be a graph on $n \geq 2$ vertices with degree sequence $(d_1, \ldots, d_n)$, where $d_1 \geq \cdots \geq d_n$. For $1 \leq l \leq n$,*

$$q(G) \leq \frac{d_1+2d_l-1+\sqrt{(2d_l-d_1+1)^2+8\sum_{i=1}^{l-1}(d_i-d_l)}}{2}.$$

*Moreover, if $G$ is connected, equality holds if and only if $G$ is either a regular graph or a bidegreed graph in which $d_1 = \cdots = d_{t-1} = n-1 > d_t = \cdots = d_n$ for some $2 \leq t \leq l$.*

Lemma 2.7. *[12, 16] Let $G$ be a graph on $n$ vertices. Then,*

$$q(G) \leq \max_{v \in V(G)} \Big\{ d(v) + \frac{1}{d(v)} \sum_{z \in N(v)} d(z) \Big\}.$$

*If $G$ is connected, then equality holds if and only if either $G$ is a regular graph or $G$ is a semi-regular bipartite graph.*

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Lemma 2.8. *[6] Let $G$ be a graph on $n$ vertices and $m$ edges. Then,*

$$\max_{v\in V(G)}\left\{d(v)+\frac{1}{d(v)}\sum_{z\in N(v)}d(z)\right\}\leq\frac{2m}{n-1}+n-2.$$

*with equality if and only if $G$ is an $S_n$ graph $(K_{1,n-1}\subseteq S_n\subseteq K_n)$ or a complete graph of order $n-1$ with one isolated vertex.*

Lemma 2.9. *If $n=r(t-1)+2$ with $r\geq 1$ and $t\geq 3$, then $q(K_1\nabla L_{r,t-1})<n+2t-3$.*

**Proof.** Let $G=K_1\nabla L_{r,t-1}$. Note that the nonincreasing degree sequence $(d_1,\ldots,d_n)$ of $G$ is $(n-1,n-1,t,\ldots,t)$. From Lemma 2.6, we have

$$\begin{aligned} q(G) &= \frac{d_1+2d_3-1+\sqrt{(2d_3-d_1+1)^2+8\sum_{i=1}^{2}(d_i-d_3)}}{2}\\ &= \frac{n+2t-2+\sqrt{(n-2t-2)^2+16(n-2t-2)+16t+16}}{2}\\ &\leq \frac{n+2t-2+(n-2t-2)+\frac{3t+7}{2}}{2}\\ &< n+2t-3. \end{aligned}$$

This completes the proof. □

**3. Proof of Theorem 1.7.** In this section, we only need to prove that Theorem 1.7 holds for $k\geq 2$ and $a_1\geq 2$. Hence, we always assume that $F_{a_1,\cdots,a_k}$ is a graph with $k\geq 2$ and $a_1\geq 2$ throughout the section. Firstly, we present some preliminary results.

Lemma 3.1. *Let $G$ be an $F_{a_1,\ldots,a_k}$-free graph of order $n\geq 8t^2-12t+9$ with $t=\sum_{i=1}^{k}a_i$. If $q(G)\geq q(S_{n,t})$, then $\Delta(G)=n-1$.*

**Proof.** From Lemma 2.7, there exists a vertex $u$ such that

$$q(G)\leq\max_{v\in V(G)}\left\{d(v)+\frac{1}{d(v)}\sum_{z\in N(v)}d(z)\right\}=d(u)+\frac{1}{d(u)}\sum_{z\in N(u)}d(z).$$

Let $A=N(u)$ and $B=V(G)\backslash(N(u)\cup\{u\})$. Then $|A|+|B|+1=n$ and

$$q(S_{n,t})\leq q(G)\leq d(u)+\frac{1}{d(u)}\sum_{z\in N(u)}d(z)=|A|+1+\frac{2e(A)+e(A,B)}{|A|}. \tag{1}$$

Next we show that $d(u)=n-1$. Assume for a contradiction that $d(u)<n-1$. We prove the following claims.

**Claim 1.** No vertex in $B$ is adjacent to every vertex in $A$.

Suppose that there exists a vertex $v\in B$ which is adjacent to every vertex in $A$. Since $G$ is $F_{a_1,\ldots,a_k}$-free, we claim that $G[A]$ is $P_{2a_1-1}\bigcup(\bigcup_{i=2}^{k}P_{2a_i})$-free. Otherwise, we assume that $G[A]$ contains a copy of $P_{2a_1-1}\bigcup(\bigcup_{i=2}^{k}P_{2a_i})$. Then $G$ contains a copy of the cycle of length $2a_1+1$ which is constructed from a path $P_{2a_1-1}$ and the two vertices $v$ and $u$ (since both $u$ and $v$ are adjacent to all vertices in $A$). Moreover, $G$ contains a copy of a cycle of length $2a_i+1$ which is constructed from a path $P_{2a_i}$ and $u$ for $2\leq i\leq k$. Hence,

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$G$ contains a copy of $F_{a_1,\ldots,a_k}$, which is a contradiction. Hence, $G[A]$ is $P_{2t-1}$-free, where $t=\sum_{i=1}^{k} a_i$. From Lemma 2.1, we have $2e(A)\le(2t-3)|A|$. It follows from (1) and $e(A,B)\le|A||B|$ that

$$\begin{aligned} q(S_{n,t}) &\le |A|+1+\frac{2e(A)+e(A,B)}{|A|}\\ &\le |A|+1+\frac{(2t-3)|A|+|A||B|}{|A|}\\ &= n+2t-3<q(S_{n,t}), \end{aligned}$$

where the last inequality is from Lemma 2.5 (1) with $8t^2-12t+9\ge 5t^2$, which is a contraction. This proves Claim 1.

Since $G$ is $F_{a_1,\ldots,a_k}$-free, the subgraph $G[A]$ induced by $A$ is $\bigcup_{i=1}^{k} P_{2a_i}$-free (otherwise, the subgraph $G[A\cup\{u\}]$ induced by $A\cup\{u\}$ contains a copy of $F_{a_1,\ldots,a_k}$, which is a contradiction). Hence, $G[A]$ is $P_{2t}$-free since $t=\sum_{i=1}^{k} a_i$. Therefore, applying Lemma 2.1 to the induced subgraph $G[A]$ yields

$$e(A)=e(G[A]) \le \frac{(2t-2)|A|}{2}. \tag{2}$$

**Claim 2.** $|B|\le 2t^2-2t$.

It follows from Claim 1 that $e(A,B)\le(|A|-1)|B|$. Together with (1) and (2), we have

$$\begin{aligned} q(S_{n,t}) &\le |A|+1+\frac{2e(A)+e(A,B)}{|A|}\\ &\le |A|+1+\frac{(2t-2)|A|+(|A|-1)|B|}{|A|}\\ &= n+2t-2-\frac{|B|}{n-1-|B|}\\ &< n+2t-2-\frac{|B|}{n+2t-3}. \end{aligned}$$

Hence, $|B|\le 2t^2-2t$ follows from Lemma 2.5 (1). This proves Claim 2.

Let $A'$ be the set of all vertices in $A$ which are adjacent to every vertex in $B$.

**Claim 3.** $|A'|\ge|A|-2t^2+2t$.

Note that

$$e(A,B)\le|A'||B|+(|A|-|A'|)(|B|-1)=|A||B|-|A|+|A'|.$$

Together with (2), we have

$$\begin{aligned} q(S_{n,t}) &\le |A|+1+\frac{2e(A)+e(A,B)}{|A|}\\ &\le |A|+1+\frac{(2t-2)|A|+|A||B|-|A|+|A'|}{|A|}\\ &= n+2t-2-\frac{|A|-|A'|}{|A|}\\ &\le n+2t-2-\frac{|A|-|A'|}{n+2t-3}. \end{aligned}$$

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Hence, $|A'| \geq |A| - 2t^2 + 2t$ follows from Lemma 2.5 (1). This proves Claim 3.

Let $G_1$ be the union of all components of $G[A]$ each of which contains at least a vertex in $A'$, and let $G_2$ be the union of the remaining components of $G[A]$. Set $n_1 = |V(G_1)|$ and $n_2 = |V(G_2)|$. Note that $G_2$ is also $\bigcup_{i=1}^{k} P_{2a_i}$-free which implies that $G_2$ is $P_{2t}$-free. From Lemma 2.1 again, $e(G_2) \leq (t-1)n_2$.

**Claim 4.** $G_1$ does not contain any cycle of length greater than $2t-3$.

Suppose that $G_1$ contains a cycle $C_p$ with $p \geq 2t-2$. Since $G$ is $F_{a_1,\ldots,a_k}$-free, $G[A]$ must be $\bigcup_{i=1}^{k} P_{2a_i}$-free, which implies $p \leq 2t-1$. Hence $2t-2 \leq p \leq 2t-1$. By the definition of $G_1$, there must be a vertex $z \in A'$ that either belongs to $C_p$ or can be joined to a vertex of $C_p$ by a path $Q$ contained in $G_1$. From Claims 2-3,

$$|A'| \geq |A| - 2t^2 + 2t = n - 1 - |B| - 2t^2 + 2t \geq n - 4t^2 + 4t - 1 \geq 2t > 2.$$

We consider the following two cases.

**Case 1.** $z \in V(C_p)$.

There exists a vertex $v \in A' \backslash V(C_p)$ because $|A'| \geq 2t > p$. Choose a vertex $w \in B$ and construct a path $P$ of order $2t$ whose first three successive vertices are $v, w, z$, respectively, and all vertices of $P$ except $w$ are in $A$. As a result, $G[A \cup \{w\}]$ contains a copy of $\bigcup_{i=1}^{k} P_{2a_i}$, where $\bigcup_{i=1}^{k} V(P_{2a_i}) = V(P)$ and the first three successive vertices of $P_{2a_1}$ are $v, w, z$ respectively. Thus, $G$ contains a copy of $F_{a_1,\ldots,a_k}$, which is a contradiction.

**Case 2.** $z$ is joined to a vertex of $C_p$ by a path $Q$ contained in $G_1$.

Choose a vertex $w \in B$. If $A' \backslash (V(C_p) \cup V(Q)) \neq \emptyset$, then there exists a vertex $v \in A' \backslash (V(C_p) \cup V(Q))$, and we can get a path $P$ of order $2t$ whose first three successive vertices are $v, w, z$, respectively, and all vertices of $P$ except $w$ are in $A$. If $A' \subseteq (V(Q) \cup V(C_p))$, then by $|A'| \geq 2t > p$, there exists a vertex $v \in A' \cap (V(Q) \backslash V(C_p))$. Hence, we can also construct a path $P$ of order $2t$ whose first two successive vertices are $w, v$ and all vertices of $P$ except $w$ are in $V(P) \cup V(Q)$. Therefore, $G[A \cup \{w\}]$ contains a copy of $\bigcup_{i=1}^{k} P_{2a_i}$, implying that $G$ contains a copy of $F_{a_1,\ldots,a_k}$. This is a contradiction. Hence, Claim 4 holds.

From Claim 4 and Lemma 2.2, we have

$$2e(G_1) \leq (2t-3)(n_1 - 1) < (2t-3)n_1.$$

Furthermore,

$$\begin{aligned} 2e(A) &= 2e(G_1) + 2e(G_2) \\ &\leq (2t-3)n_1 + (2t-2)n_2 = (2t-3)(n_1+n_2) + n_2 \\ &\leq (2t-3)|A| + |A| - |A'| \leq (2t-3)|A| + 2t^2 - 2t. \end{aligned}$$

From (1),

$$\begin{aligned} q(S_{n,t}) &\leq |A| + 1 + \frac{2e(A) + e(A,B)}{|A|} \\ &\leq |A| + 1 + \frac{(2t-3)|A| + 2t^2 - 2t + |A||B|}{|A|} \\ &= n + 2t - 3 + \frac{2t^2 - 2t}{|A|}. \end{aligned}$$

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Then, it follows from Lemma 2.5 (1) that

$$n+2t-2-\frac{2t^2-2t}{|A|}\leq n+2t-2-\frac{2t^2-2t}{n+2t-3}\leq n+2t-2-1+\frac{2t^2-2t}{|A|}$$

which implies that $|A|\leq 4t^2-4t$. Hence from Claim 3,

$$n=1+|A|+|B|\leq 1+4t^2-4t+2t^2-2t<8t^2-12t+9,$$

which is a contradiction. Hence, $d(u)=n-1$, and it completes the proof. □

LEMMA 3.2. *Let $G$ be a $\bigcup_{i=1}^{k}P_{2a_i}$-free graph of order $n>t^2-t-1$ with $k\geq 2$, $a_1\geq\cdots\geq a_k\geq 1$, and $t=\sum_{i=1}^{k}a_i$. If $e(G)\geq(t-1)n-(t^2-t-1)$, then there exists an induced subgraph $H$ of $G$ which satisfies the following conditions: (1). $|V(H)|\geq n-(t^2-t-1)$; (2). $\delta(H)\geq t-1$; (3). $d_H(v)\leq t-2$ for any vertex $v\in V(G)\backslash V(H)$, where $d_H(v)=|N_G(v)\cap V(H)|$.*

**Proof.** We will construct the desired induced subgraph $H$ by iteratively deleting vertices. Starting step: if $\delta(G)\geq t-1$, then $G$ clearly satisfies the conditions (1)-(3) and let $H=G$, and we are done. Iterative process: if $\delta(G_{i-1})<t-1$, then let the induced subgraph $G_i$ of $G_{i-1}$ obtained from $G_{i-1}$ by deleting a vertex of minimum degree $\delta(G_{i-1})\leq t-2$ for $1\leq i\leq n$, where $G_0=G$. Suppose that the constructive process stops after $r$ steps when $\delta(G_r)\geq t-1$. Now we prove that $G_r$ satisfies conditions (1)-(3). Note that $G_r$ satisfies the following conditions: (1). $|V(G_r)|=n-r$; (2). $\delta(G_r)\geq t-1$; (3). $d_{G_r}(v)\leq t-2$ for any vertex $v\in V(G)\backslash V(G_r)$. Hence, we only need to prove that $r\leq t^2-t-1$. In fact,

$$e(G_r)\geq e(G_{r-1})-(t-2)\geq\cdots\geq e(G)-r(t-2)\geq(t-1)n-(t^2-t-1)-r(t-2).$$

On the other hand, since $G$ is a $\bigcup_{i=1}^{k}P_{2a_i}$-free graph, the induced subgraph $G_r$ is also $\bigcup_{i=1}^{k}P_{2a_i}$-free. Hence, $G_r$ is $P_{2t}$-free, where $t=\sum_{i=1}^{k}a_i$. Applying Lemma 2.1 to $G_r$ yields

$$e(G_r)\leq\frac{(2t-2)(n-r)}{2}=(t-1)(n-r).$$

Hence,

$$(t-1)n-(t^2-t-1)-r(t-2)\leq e(G_r)\leq(t-1)(n-r),$$

which implies that $r\leq t^2-t-1$. So let $H=G_r$ and this completes the proof. □

LEMMA 3.3. *Let $G_1$ be a $\bigcup_{i=1}^{k}P_{2a_i}$-free graph of order $n_1$ with $t=\sum_{i=1}^{k}a_i\geq 3$ and $G_2$ be a graph of order $n_2$ with $1\leq n_2\leq t^2+t-2$. If $G=K_1\nabla(G_1\cup G_2)$ is a graph of order $n$ with $n=n_1+n_2+1\geq 8t^2-12t+9$, then $q(G)<q(S_{n,t})$.*

**Proof.** By contradiction assume that $q(G)\geq q(S_{n,t})$. Note that adding edges to $G$ will increase $q(G)$ by the Perron-Frobenius Theorem. Without loss of generality, we assume that $G_2=K_{n_2}$. Since $G_1$ is $\bigcup_{i=1}^{k}P_{2a_i}$-free, we have $G_1$ is $P_{2t}$-free. From Lemma 2.1,

$$e(G_1)\leq(t-1)(n-n_2-1),$$

Thus,

$$e(G-V(G_2))=e(G_1)+n-n_2-1\leq t(n-n_2-1).$$

From Lemmas 2.7 and 2.8,

$$q(G-V(G_2))\quad\leq\quad\frac{2e(G-V(G_2))}{|V(G-V(G_2))|-1}+|V(G-V(G_2))|-2$$

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$$\begin{aligned} &\leq \frac{2t(n-n_2-1)}{n-n_2-1}+n-n_2-2 \\ &= n+2t-n_2-2. \end{aligned}$$

Let $\mathbf{x}=(x_1,\ldots,x_n)^\top$ be a unit positive eigenvector of $Q(G)$ corresponding to the signless Laplacian spectral radius $q(G)$. By symmetry, all entries of $x$ corresponding to the vertices of $G_2=K_{n_2}$ are the same, say $a$. From the eigenvalue-eigenvector equations of $Q(G)$ on $u$ with maximum degree $n-1$ and one vertex of $G_2$ with degree $n_2$, we have

$$\begin{aligned} (q(G)-n+1)x_u &= \sum_{v\in V(G)\setminus\{u\}} x_v \leq \sqrt{(n-1)\sum_{v\in V(G)\setminus\{u\}} x_v^2} = \sqrt{(n-1)(1-x_u^2)}, \\ (q(G)-n_2)a &= (n_2-1)a+x_u. \end{aligned}$$

In addition, $n_2\leq t^2+t-2$ and $q(G)\geq q(S_{n,t})>n+2t-3$ from Lemma 2.5. Hence,

$$x_u^2 \leq \frac{n-1}{(q(G)-n+1)^2+n-1} \leq \frac{n-1}{n-1+4(t-1)^2} < 1-\frac{4(t-1)^2}{n+4t^2},$$

and

$$a=\frac{x_u}{q(G)-2n_2+1}\leq \frac{x_u}{n+2t-2n_2-2}\leq \frac{x_u}{n-2t^2+2}.$$

Therefore,

$$\begin{aligned} q(G) &= \sum_{ij\in E(G)}(x_i+x_j)^2 = \sum_{ij\in E(G)\setminus E(G_2)}(x_i+x_j)^2+n_2(a+x_u)^2+\sum_{ij\in E(G_2)}(x_i+x_j)^2 \\ &< q(G-V(G_2))+n_2(a+x_u)^2+2n_2(n_2-1)a^2 \\ &\leq n+2t-n_2-2+n_2\left(1+\frac{1}{(n-2t^2+2)^2}+\frac{2}{n-2t^2+2}+\frac{2(n_2-1)}{(n-2t^2+2)^2}\right)x_u^2 \\ &< n+2t-n_2-2+n_2\left(1+\frac{3}{n-2t^2+2}\right)\left(1-\frac{4(t-1)^2}{n+4t^2}\right) \\ &< n+2t-2-\left(\frac{4(t-1)^2}{n+4t^2}-\frac{3}{n-2t^2+2}\right) \\ &< n+2t-2-\frac{2t(t-1)}{n-2t^2+2} \\ &< n+2t-2-\frac{2t(t-1)}{n+2t-3} \\ &< q(S_{n,t}), \end{aligned}$$

which is a contradiction. This completes the proof. □

Now we are ready to prove Theorem 1.7.

**Proof of Theorem 1.7.** Suppose that $q(G)\geq q(S_{n,t})$. We will show that $G=S_{n,t}$. From Lemma 3.1, $\Delta(G)=n-1$. Let $u\in V(G)$ be a vertex with maximum degree $\Delta(G)$, that is, $d(u)=n-1$. From

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Lemma 2.5 (2), $e(G) \geq tn - t^2 + 1$, which implies that

$$\begin{aligned} e(G-u) &= e(G) - n + 1 \\ &\geq (t-1)n - t^2 + 2 \\ &= (t-1)(n-1) - (t^2 - t - 1). \end{aligned}$$

Since $G$ is $F_{a_1,\ldots,a_k}$-free, $G-u$ is $\bigcup_{i=1}^{k} P_{2a_i}$-free. From Lemma 3.2, there exists an induced subgraph $H$ of $G-u$ such that $\delta(H) \geq t-1$, $|V(H)| = n_1 \geq n - (t^2 - t)$, and $d_H(v) \leq t-2$ for every vertex $v \in V(G) \backslash (V(H) \cup \{u\})$. Let $H = \bigcup_{i=1}^{s} H_i$ and $|V(H_i)| = h_i$, where $H_i$ is a component of $H$ for $i = 1, \ldots, s$.

**Claim 1.** Every component of $G-u$ contains at most one graph of $H_1, \ldots, H_s$ as an induced subgraph.

Note that $\delta(H_i) \geq t - 1 \geq 2$ for $i = 1, \ldots, s$. From Lemma 2.3, $H_i$ contains a cycle of length at least $t$ for $i = 1, \ldots, s$. In fact, if there is a component of $G-u$ containing at least two graphs $H_i$ and $H_j$ as an induced subgraph for $1 \leq i \neq j \leq s$, then $G-u$ contains a copy of $P_{2t+1}$ and thus $G$ contains a copy of $F_{a_1,\ldots,a_k}$, which is a contradiction. This proves Claim 1.

From Claim 1, let $T_i$ be the component of $G-u$ containing $H_i$ as an induced subgraph, and $G - u = (\bigcup_{i=1}^{s} T_i) \bigcup T_0$, where $T_0$ is the union of the remaining components of $G-u$. Since $G-u$ is $\bigcup_{i=1}^{k} P_{2a_i}$-free, we have $T_i$ is $P_{2t}$-free for $i = 0, \ldots, s$.

**Claim 2.** $T_0 = \emptyset$.

In fact, if $T_0 \neq \emptyset$, then

$$\begin{aligned} 1 \leq |V(T_0)| &= n - 1 - \sum_{i=1}^{s} |V(T_i)| \leq n - 1 - \sum_{i=1}^{s} |V(H_i)| \\ &= n - 1 - n_1 \leq t^2 - t - 1. \end{aligned}$$

From Lemma 3.3, $q(G) < q(S_{n,t})$, which is a contradiction. This proves Claim 2.

**Claim 3.** $h_i \geq 2t$ for $i = 1, \ldots, s$.

Suppose there exists an $h_i$ such that $h_i \leq 2t - 1$. Since $\delta(H_i) \geq \delta(H) \geq t - 1$, we have $h_i \geq t$. Then,

$$t \leq h_i \leq |V(T_i)| \leq h_i + |V(G-u) \setminus V(H)| \leq 2t - 1 + t^2 - t - 1 = t^2 + t - 2.$$

From Lemma 3.3, $q(G) < q(S_{n,t})$, which is a contradiction. This proves Claim 3.

By the definition of $L_{r,t}$, we have the following claim directly.

**Claim 4.** For any fixed $1 \leq i \leq s$, if $H_i = L_{r_i,t-1}$ with $h_i = r_i(t-1) + 1$, then $H_i$ contains a copy of $P_{2t-1}$.

**Claim 5.** For any fixed $1 \leq i \leq s$, if $H_i$ is a subgraph of $S_{h_i,t-1}$, then $H_i$ contains a copy of $P_{2t-1}$. Moreover, $T_i$ is subgraph of $S_{|V(T_i)|,t-1}$.

If $H_i$ is a subgraph of $S_{h_i,t-1}$, then there exists $I_i \subseteq V(H_i)$ of size $h_i - t + 1$ such that $I_i$ induces an independent set of $H_i$. Since $\delta(H_i) \geq t-1$, every vertex in $I_i$ is adjacent to every vertex in $V(H_i) \backslash I_i$. Then,

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$H_i$ contains a copy of a path $P$ of order $2t-1$ with both endvertices in $I_i$. If $(V(T_i)\backslash V(H_i))\cup I_i$ induces at least an edge, then we can get a path of order $2t$ from $P$. Hence, $G$ contains a copy of $F_{a_1,\ldots,a_k}$, which is a contradiction. This implies that $(V(T_i)\backslash V(H_i))\cup I_i$ is an independent set, and thus, $T_i$ is subgraph of $S_{|V(T_i)|,t-1}$. This proves Claim 5.

Note that $q(G)\geq q(S_{n,t})$. Since $H_i$ is $\bigcup_{i=1}^{k}P_{2a_i}$-free and $\delta(H_i)\geq\delta(H)\geq t-1$, it follows from Lemma 2.4 that $H_i$ is a subgraph of $S_{h_i,t-1}$ or $H_i=L_{r_i,t-1}$ with $h_i=r_i(t-1)+1$ and $k=2, a_1=a_2$ for $i=1,\ldots,s$. If $s\geq 2$, then it follows from Claims 4 and 5 that $G-u$ contains a copy of $2P_{2t-1}$, and thus, $G$ contains a copy of $F_{a_1,\ldots,a_k}$, which is a contradiction. So $s=1$. This implies that

$$G-u=T_1,$$

and $H_1$ is an induced graph of $T_1$, where $H_1=S_{h_1,t-1}$ or $H_1=L_{r_1,t-1}$ with $h_1=r_1(t-1)+1$ and $k=2, a_1=a_2$.

First suppose that $H_1=S_{h_1,t-1}$. From Claim 5, $T_1$ is a subgraph of $S_{|V(T_1)|,t-1}$, that is, $G$ is a subgraph of $S_{n,t}$. If $G$ is a proper subgraph of $S_{n,t}$, then it follows from the Perron-Frobenius theorem that $q(G)<q(S_{n,t})$, which is also a contradiction. Hence, $G=S_{n,t}$.

Next, suppose that $H_1=L_{r_1,t-1}$ with $h_1=r_1(t-1)+1$ and $k=2, a_1=a_2$. If $H_1=T_1$, then $G=K_1\nabla L_{r_1,t-1}$ with $n=r_1(t-1)+2$. From Lemma 2.9, $q(G)<n+2t-3<q(S_{n,t})$, which is a contradiction. Thus, $H_1$ is a proper subgraph of $T_1$. Let $H'=T_1-V(H_1)$ and $|V(H')|=n_2$. Since $d_{H_1}(v)\leq t-2$ for every vertex $v\in V(H')$, we have

$$e(V(H'),V(H_1))\leq (t-2)n_2\leq (t-2)(t^2-t-1).$$

Then,

$$\begin{aligned}
e(H') &= e(T_1)-e(H_1)-e(V(H'),V(H_1))\\
&> (t-1)n-t^2+2-\frac{t(n_1-1)}{2}-(t-2)(t^2-t-1)\\
&= (t-1)n-t^2+2-\frac{t(n-n_2-2)}{2}-(t-2)(t^2-t-1)\\
&= \frac{(t-2)n-(2t^3-4t^2)}{2}+\frac{tn_2}{2}>\frac{tn_2}{2}.
\end{aligned}$$

From Lemma 2.1, $H'$ contains a copy of $P_{t+2}$. Together with Claim 4, $T_1$ contains a copy of $P_{2t-1}\cup P_{t+2}$. This implies that $G$ contains a copy of $F_{a_1,a_2}$, which is a contradiction. This completes the proof. $\square$

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REFERENCES

[1] J.A. Bondy and U.S.R. Murty. *Graph Theory*. Springer, New York, 2007.
[2] M.-Z. Chen and X.-D. Zhang. Erdős–Gallai stability theorem for linear forests. *Discrete Math.,* 342:904–916, 2019.
[3] M.-Z. Chen, A.-M. Liu, and X.-D. Zhang. Spectral extremal results with forbidding linear forests. *Graphs Combin.,* 35:335–351, 2019.
[4] M.-Z. Chen, A.-M. Liu, and X.-D. Zhang. On the spectral radius of graphs without a star forest. *Discrete Math.,* 344:112269, 2021.
[5] S. Cioabă, L. Feng, M. Tait, and X.-D. Zhang. The maximum spectral radius of graphs without friendship subgraphs. *Electron. J. Combin.,* 27(4):P4.22, 2020.
[6] K. Das. Maximizing the sum of the squares of the degrees of a graph. *Discrete Math.,* 285:57–66, 2004.
[7] X. Duan and B. Zhou. Sharp bounds on the spectral radius of a nonnegative matrix. *Linear Algebra Appl.,* 439:2961–2970, 2013.
[8] M.A.A. de Freitas, V. Nikiforov, and L. Patuzzi. Maxima of the $Q$-index: Graphs with no $K_{s,t}$. *Linear Algebra Appl.,* 496:381–391, 2016.
[9] G.A. Dirac. Some theorems on abstract graphs. *Proc. London Math. Soc.,* 2:69–81, 1952.
[10] P. Erdős and T. Gallai. On maximal paths and circuits of graphs. *Acta Math. Acad. Sci. Hungar.,* 10:337–356, 1959.
[11] P. Erdős, Z. Füredi, R.J. Gould, and D.S. Gunderson. Extremal graphs for intersecting triangles. *J. Combin. Theory. Ser. B,* 64:89–100, 1995.
[12] L. Feng and G. Yu. On three conjectures involving the signless Laplacian spectral radius of graphs. *Publ. Inst. Math.,* 85(99):35–38, 2009.
[13] B. He, Y.-L. Jin, and X.-D. Zhang. Sharp bounds for the signless Laplacian spectral radius in terms of clique number. *Linear Algebra Appl.,* 438:3851–3861, 2013.
[14] X. Hou, Y. Qiu, and B. Liu. Turán number and decomposition number of intersecting odd cycles. *Discrete Math.,* 341:126–137, 2018.
[15] Y. Li and Y. Peng. The spectral radius of graphs with no intersecting odd cycles. *Discrete Math.,* 345, Paper No. 112907:16 pp., 2022.
[16] R. Merris. A note on Laplacian graph eigenvalues. *Linear Algebra Appl.,* 285:33–35, 1998.
[17] V. Nikiforov. Bounds on graph eigenvalues II. *Linear Algebra Appl.,* 427:183–189, 2007.
[18] V. Nikiforov. The spectral radius of graphs without paths and cycles of specified length. *Linear Algebra Appl.,* 432:2243–2256, 2010.
[19] V. Nikiforov. A contribution to the Zarankiewicz problem. *Linear Algebra Appl.,* 432:1405–1411, 2010.
[20] V. Nikiforov. Some new results in extremal graph theory. In: *Surveys in Combinatorics* 2011, 141–181, London Math. Soc. Lecture Note Series, vol. 392, Cambridge University Press, Cambridge, 2011.
[21] V. Nikiforov and X. Yuan. Maxima of the $Q$-index: Graphs without long paths. *Electron. J. Linear Algebra*, 27:504–514, 2014.
[22] V. Nikiforov and X. Yuan. Maxima of the $Q$-index: Forbidden even cycles. *Linear Algebra Appl.,* 471:636–653, 2015.
[23] M. Simonovits. A method for solving extremal problems in graph theory, stability problems. *Theory Graphs (Proc. Colloq., Tihany, 1966):*279–319, 1968.
[24] L.-T. Yuan. Extremal graphs for the $k$-flower. *J. Graph Theory,* 89:26–39, 2018.
[25] X. Yuan. Maxima of the $Q$-index: Forbidden odd cycles. *Linear Algebra Appl.,* 458:207–216, 2014.
[26] Y. Zhao, X. Huang, and H. Guo. The signless Laplacian spectral radius of graphs with no intersecting triangles. *Linear Algebra Appl.,* 618:12–21, 2021.