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NUMERICAL RANGE FOR WEIGHTED MOORE-PENROSE INVERSE OF TENSOR*

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Abstract. This article first introduces the notion of weighted singular value decomposition (WSVD) of a tensor via the Einstein product. The WSVD is then used to compute the weighted Moore-Penrose inverse of an arbitrary-order tensor. We then define the notions of weighted normal tensor for an even-order square tensor and weighted tensor norm. Finally, we apply these to study the theory of numerical range for the weighted Moore-Penrose inverse of an even-order square tensor and exploit its several properties. We also obtain a few new results in matrix setting.

Key word. Tensor, Einstein product, Numerical range, Numerical radius, Weighted Moore-Penrose inverse.

AMS subject classifications. 15A60, 15A69.

1. Introduction. The terms numerical range and numerical radius have drawn significant attention from researchers in the last few decades in the field of matrix and operator theory [8, 9, 45, 47]. These have been widely studied because of their applications in many areas, such as numerical analysis and differential equations [41, 50, 27, 30, 31, 51, 18, 44]. The numerical range (or the field of values) of a square matrix $A \in \mathbb{C}^{n\times n}$ is a subset of complex numbers defined as:

$$W(A) = \{\langle Ax, x\rangle : \ x \in \mathbb{C}^n, \|x\| = 1\}, \tag{1.1}$$

where $\langle x, y\rangle = y^*x$ for $x, y \in \mathbb{C}^n$ and $\|x\| = \langle x, x\rangle^{1/2}$. And, the numerical radius of the matrix A is defined as:

$$w(A) = \max\{|z| : \ z \in W(A)\}. \tag{1.2}$$

One of the main reasons for emphasizing the numerical range concept is its many attractive properties. For example, $W(A)$ is a convex subset of $\mathbb{C}$ (known as Toeplitz-Hausdorff theorem [47]). Further, the numerical range of a matrix contains its spectrum (or the set of all eigenvalues). The numerical radius is frequently employed as a more reliable indicator of the rate of convergence of iterative methods than the spectral radius [41, 27]. In 2016, Ke *et al.* [49] introduced tensor numerical ranges using tensor inner products and tensor norms via the k-mode product, which may not be convex in general (see Example 1, [49]). In 2021, Pakmanesh and Afshin [35] continued the same study for even-order tensors and proved the convexity for the numerical range of an even-order tensor. In 2023, Rout *et al.* [38] introduced tensor numerical ranges using tensor inner products and tensor norms via the Einstein product. The authors [38] studied several fundamental notions of tensor numerical ranges, such as unitary invariance, spectral containment, and convexity. Furthermore, they developed an algorithm to plot the boundary of the numerical range of a tensor, which helps to design faster algorithms for the calculations of its eigenvalues. To understand tensor

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numerical ranges, we first recall some basic facts about tensors. Tensors are generalizations of scalars (that have no index), vectors (that have precisely one index), and matrices (that have precisely two indices) to an arbitrary number of indices. A tensor is represented as a multidimensional array. An N^{th}-order tensor is defined as:

$$\mathcal{A} = (a_{i_1 i_2 \dots i_N}) \in \mathbb{C}^{I_1 \times \cdots \times I_N};\ 1 \leq i_k \leq I_k \text{ for each } k = 1, 2, ..., N,$$

where each mode I_k is a natural number and the notation $a_{i_1 \dots i_N}$ represents the $(i_1, \dots, i_N)^{th}$ element of $\mathcal{A}$. According to the number of modes a tensor is called an even or odd tensor. The transpose of a tensor may not be unique, which is recalled here. Let $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_M}$ be a tensor and let π be a permutation in S_M except the identity permutation, where S_M represents the permutation group over the set $\{1, 2, \dots, M\}$. Then, the transpose [26] of $\mathcal{A}$ associated with π is defined as

$$\mathcal{A}^{T_\pi} = (a_{i_{\pi(1)} i_{\pi(2)} \dots i_{\pi(M)}}) \in \mathbb{C}^{I_{\pi(1)} \times I_{\pi(2)} \times \cdots \times I_{\pi(M)}}. \tag{1.3}$$

Thus, there are $M! - 1$ possible transposes associated with a tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_M}$. In particular, for $\mathcal{A} = (a_{i_1 i_2 \dots i_M j_1 j_2 \dots j_N}) \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $\pi \in S_{M+N}$ such that $\mathcal{A}^{T_\pi} = (b_{j_1 j_2 \dots j_N i_1 i_2 \dots i_M}) = (a_{i_1 i_2 \dots i_M j_1 j_2 \dots j_N}) \in \mathbb{C}^{J_1 \times \cdots \times J_N \times I_1 \times \cdots \times I_M}$, then it is simply written as $\mathcal{A}^T$. To see this numerically, we next produce an example.

Example 1.1. *Let $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times J_1}$ with $I_1 = I_2 = 2, J_1 = 3$ such that*

$\mathcal{A}(:,:,1)$		$\mathcal{A}(:,:,2)$		$\mathcal{A}(:,:,3)$	
1	2	0	3	i	0
0	0	4	0	2	0

Then, the transpose of $\mathcal{A}$, $\mathcal{A}^T \in \mathbb{C}^{J_1 \times I_1 \times I_2}$ is

$\mathcal{A}^T(:,:,1)$		$\mathcal{A}^T(:,:,2)$	
1	0	2	0
0	4	3	0
i	2	0	0

Similarly, the conjugate transpose of $\mathcal{A}$ is denoted by $\mathcal{A}^H$ and defined by $\mathcal{A}^H = (c_{j_1 j_2 \dots j_N i_1 i_2 \dots i_M}) = (\overline{a}_{i_1 i_2 \dots i_M j_1 j_2 \dots j_N}) \in \mathbb{C}^{J_1 \times \cdots \times J_N \times I_1 \times \cdots \times I_M}$, where bar denotes the complex conjugate of a number. Furthermore, if $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_M}$, then $\mathcal{A}^T = (a_{1 i_1 i_2 \dots i_M}) \in \mathbb{C}^{1 \times I_1 \times I_2 \times \cdots \times I_M}$. There are two ways to define a square tensor. One when each mode is of equal size, i.e., $n \times n \times \cdots \times n$ and another when the first N modes are repeated in the same order, i.e., $I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N$. For the tensors with all equal modes, symmetricity can be studied as in [42, 24]. A square tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ is called Hermitian (symmetric) [26, 34] if $\mathcal{A} = \mathcal{A}^H$ ($\mathcal{A} = \mathcal{A}^T$). We next consider an example of a Hermitian tensor of size $2 \times 2 \times 2 \times 2$.

Example 1.2. *$\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times I_1 \times I_2}$ with $I_1 = I_2 = 2$ such that*

$\mathcal{A}(:,:,1,1)$		$\mathcal{A}(:,:,2,1)$		$\mathcal{A}(:,:,1,2)$		$\mathcal{A}(:,:,2,2)$	
1	1	1+i	0	1	2	0	i
1-i	0	0	0	0	-i	0	0

is a Hermitian tensor.

Ke *et al.* [49] extended the notion of the numerical range of a matrix for the former type of square tensors. Further, Rout *et al.* [38] extended the numerical range for the latter type of square tensors. They

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also obtained a few properties of tensor numerical range of the Moore-Penrose inverse via the Einstein product. We aim to study these properties of tensor numerical range for the weighted Moore-Penrose inverse of a tensor. For this purpose, we recall the Einstein product below.

The Einstein product [2] $\mathcal{A} *_N \mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_L}$ of tensors $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times K_1\times\cdots\times K_N}$ and $\mathcal{B} \in \mathbb{C}^{K_1\times\cdots\times K_N\times J_1\times\cdots\times J_L}$ is defined by the operation $*_N$ via

$$(\mathcal{A} *_N \mathcal{B})_{i_1\ldots i_M j_1\ldots j_L} = \sum_{k_1,\ldots,k_N} a_{i_1\ldots i_M k_1\ldots k_N} b_{k_1\ldots k_N j_1\ldots j_L}.$$

In the next example, we compute the Einstein product of the tensors $\mathcal{A}$ and $\mathcal{B}$ because the last two modes of $\mathcal{A}$ are the same as the first two modes of $\mathcal{B}$.

EXAMPLE 1.3. *Let $\mathcal{A} \in \mathbb{R}^{2\times 2\times 3}$ and $\mathcal{B} \in \mathbb{R}^{2\times 3\times 2}$ be such that*

$\mathcal{A}(:,:,1)$		$\mathcal{A}(:,:,2)$		$\mathcal{A}(:,:,3)$	
1	0	2	1	0	2
2	1	0	2	1	1

and

$\mathcal{B}(:,:,1)$			$\mathcal{B}(:,:,2)$		
4	-1	0	1	1	1
1	0	2	0	0	-1

.

Then, their Einstein product $\mathcal{C} = \mathcal{A} *_2 \mathcal{B} \in \mathbb{R}^{2\times 2}$ is

$\mathcal{C}(:,:)$	
6	1
11	2

.

The associative law for the Einstein product holds. In the above formula, if $\mathcal{B} \in \mathbb{C}^{K_1\times\cdots\times K_N}$, then $\mathcal{A} *_N \mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_M}$ and

$$(\mathcal{A} *_N \mathcal{B})_{i_1\ldots i_M} = \sum_{k_1,\ldots,k_N} a_{i_1\ldots i_M k_1\ldots k_N} b_{k_1\ldots k_N}.$$

This product is used in the study of the theory of relativity [2] and in the area of continuum mechanics [54]. Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times l}$. Then, the Einstein product $*_1$ reduces to the standard matrix multiplication as

$$(A *_1 B)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

We refer to [3] for further advantages of studying the theory of tensors via the Einstein product.

In 2005, Lim [23] and Qi [24] independently introduced the notions of eigenvalues and eigenvectors of an m-th order n-dimensional tensor. In this direction, Sturmfels [4] solved two problems on counting the number of eigenvectors and singular vectors of a $3\times 3\times 3$ tensor. The role of the eigenvectors of the third and fourth moment of multivariate distribution is examined by Loperfido [39, 40]. In 2019, Liang and Zheng [34] recalled the definition of eigenvalues of an even-order square tensor via the Einstein product as follows.

DEFINITION 1.4 (Definition 2.3, [34]).
Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Then, a complex number λ is called an eigenvalue of $\mathcal{A}$ if there exists a nonzero tensor $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ such that

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$$\mathcal{A} *_N \mathcal{X} = \lambda \mathcal{X}. \tag{1.4}$$

The tensor $\mathcal{X}$ is called an eigentensor with respect to λ.

The set of all the eigenvalues of $\mathcal{A}$ is denoted by $\sigma(\mathcal{A})$. The spectral radius of the tensor $\mathcal{A}$ is denoted by $\rho(\mathcal{A})$ and defined by $\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}$.
The eigenvalues and their corresponding eigentensors are computed in the next example for a simple even-order square tensor.

Example 1.5. *The set of all eigenvalues of the tensor $\mathcal{A} \in \mathbb{R}^{2\times2\times2\times2}$ such that*

$\mathcal{A}(:,:,1,1)$		$\mathcal{A}(:,:,2,1)$		$\mathcal{A}(:,:,1,2)$		$\mathcal{A}(:,:,2,2)$	
1	0	1	0	1	3	1	1
0	0	2	0	1	0	1	1

is $\sigma(\mathcal{A}) = \{1, 2, 3\}$ and their corresponding eigentensors are

eigenvalue	1		1	
	$\mathcal{X}(:,:)$		$\mathcal{X}(:,:)$	
eigentensor	1	0	0	1
	0	0	1	-2

,

eigenvalue	2		3	
	$\mathcal{X}(:,:)$		$\mathcal{X}(:,:)$	
eigentensor	1	0	1	1
	1	0	1	0

.

Furthermore, the positive square roots of eigenvalues of $\mathcal{A}^H *_N \mathcal{A}$ are called the singular values of $\mathcal{A}$. The maximum singular value of $\mathcal{A}$ is called the spectral norm [13] of the tensor $\mathcal{A}$. Wang and Wei [56] studied the generalized eigenvalue problem via the Einstein product for even-order tensors and showed its applications in multilinear control systems.

In 2013, Brazell *et al.* [26] first introduced the notion of the inverse of a tensor via the Einstein product. For $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$, if there exists a tensor $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ such that $\mathcal{A} *_N \mathcal{X} = \mathcal{I} = \mathcal{X} *_N \mathcal{A}$, then the tensor $\mathcal{X}$ is called the inverse of the tensor $\mathcal{A}$ and it is denoted by $\mathcal{A}^{-1}$. In 2016, Sun *et al.* [25] formally introduced a generalized inverse called the Moore-Penrose inverse of an even-order tensor via the Einstein product. The authors [25] then used the Moore-Penrose inverse to find the minimum-norm least-squares solution of some multilinear systems. Panigrahy and Mishra [20], Stanimirović *et al.* [46], and Liang and Zheng [34] independently improved the definition of the Moore-Penrose inverse of an even-order tensor to a tensor of any order via the same product. In 2017, Ji and Wei [16] defined the notion of Hermitian positive definite tensor and the weighted Moore-Penrose inverse for an even-order square tensor, and then in 2020, Behera *et al.* [48] extended the definition to an arbitrary-order tensor. The definition of the weighted Moore-Penrose inverse of an arbitrary-order tensor and one result are recalled here.

Definition 1.6 (Definition 8, [48]).
Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$, and $\mathcal{M} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$, $\mathcal{N} \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$ be two Hermitian positive definite tensors. Then, the tensor $\mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_N\times I_1\times\cdots\times I_M}$ is called the weighted Moore-Penrose inverse of $\mathcal{A}$ if it satisfies the following four tensor equations:

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 40, pp. 140-171, January 2024.

$$\mathcal{A} *_N \mathcal{X} *_M \mathcal{A} = \mathcal{A}; \tag{1.5}$$
$$\mathcal{X} *_M \mathcal{A} *_N \mathcal{X} = \mathcal{X}; \tag{1.6}$$
$$(\mathcal{M} *_M \mathcal{A} *_N \mathcal{X})^H = \mathcal{M} *_M \mathcal{A} *_N \mathcal{X}; \tag{1.7}$$
$$(\mathcal{N} *_N \mathcal{X} *_M \mathcal{A})^H = \mathcal{N} *_N \mathcal{X} *_M \mathcal{A}. \tag{1.8}$$

It is denoted by $\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}$. In particular, if $\mathcal{M}$ and $\mathcal{N}$ are the identity tensors, then $\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} = \mathcal{A}^{\dagger}$, the Moore-Penrose inverse [20] of $\mathcal{A}$.

Theorem 1.7 (Theorem 4, [48]).
Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$, and $\mathcal{M} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$, $\mathcal{N} \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$ be two Hermitian positive definite tensors. If $\tilde{\mathcal{A}} = \mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{N}^{-1/2}$, then

$$\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} = \mathcal{N}^{-1/2} *_N \tilde{\mathcal{A}}^{\dagger} *_M \mathcal{M}^{1/2}.$$

This article aims to introduce the notions of WSVD of an arbitrary-order tensor, weighted normal tensor, and weighted tensor norm and to establish their various properties. Some of these are utilized to investigate a few properties of the numerical range for the weighted Moore-Penrose inverse of an even-order square tensor. The rest of this article is structured as follows to accomplish our objectives. In Section 2, we recall some preliminaries. Then, we provide the WSVD and some of its applications in Section 3. Section 4 defines the weighted normal tensor and discusses its several features. Section 5 introduces the weighted tensor norm. Finally, we utilize all these notions to collect some properties of the numerical range, which examine different relations between the numerical range of a tensor and its weighted Moore-Penrose inverse in Section 6.

2. Preliminaries. For two tensors $\mathcal{X}, \mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_N}$, an inner product $\langle \mathcal{X}, \mathcal{Y}\rangle$ is defined as $\langle \mathcal{X}, \mathcal{Y}\rangle = \mathcal{Y}^H *_N \mathcal{X}$ and a norm induced by this inner product as $\|\mathcal{X}\| = \langle \mathcal{X}, \mathcal{X}\rangle^{1/2}$. A tensor $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ is called a unit tensor if $\|\mathcal{X}\| = 1$. First, we recall the definition of the numerical range and some results from [38].

Definition 2.1 (Definition 2.1, [38]).
Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Then, the numerical range of $\mathcal{A}$ is denoted by $W(\mathcal{A})$ and defined by

$$W(\mathcal{A}) = \{\langle \mathcal{A} *_N \mathcal{X}, \mathcal{X}\rangle : \ \mathcal{X} \textit{ is a unit tensor in } \mathbb{C}^{I_1\times\cdots\times I_N}\}. \tag{2.9}$$

With some elementary calculations, it can be shown that

$$W(\mathcal{A}) = \left\{ \frac{\langle \mathcal{A} *_N \mathcal{X}, \mathcal{X}\rangle}{\|\mathcal{X}\|^2} : \ \mathcal{O} \neq \mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N} \right\}, \tag{2.10}$$

where $\mathcal{O}$ is the zero tensor having all the entries zero. The numerical radius of $\mathcal{A}$ is defined as:

$$w(\mathcal{A}) = \max\{|z| : \ z \in W(\mathcal{A})\}. \tag{2.11}$$

Note that, in the above Definition 2.1 when $N = 1$, it coincides with the numerical range of a matrix defined in (1.1).

Theorem 2.2 (Theorem 5.1, [38]).
Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Then, $\mathcal{A}$ is normal (resp. Hermitian) if and only if $\mathcal{A}^{\dagger}$ is normal (resp. Hermitian).

We now recall the definition of the weighted conjugate transpose of a tensor proposed by Behera *et al.* [48]

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Volume 40, pp. 140-171, January 2024.

Definition 2.3 (Definition 9, [48]).
Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$. If $\mathcal{M} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ and $\mathcal{N} \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$ are two Hermitian positive definite tensors, then the tensor

$$\mathcal{A}^{\#}_{\mathcal{MN}} = \mathcal{N}^{-1} *_N \mathcal{A}^H *_M \mathcal{M}, \tag{2.12}$$

is called the weighted conjugate transpose of $\mathcal{A}$.

Unfolding (or reshaping or flattening or matricization) is a way to transform a tensor into a matrix. There are several ways of unfolding [26, 12, 28, 29, 52, 53, 34, 22, 46] of a tensor. For the variety of applications of tensors, different unfolding are defined, like n-mode unfoldings [12, 53, 22] give nice relations among the n-mode product of tensor, usual matrix multiplication, and Kronecker product of matrices [52]. Also, the third-mode unfolding is very useful in the computation of c-product of tensors. In the framework of higher order moment of multivariate distributions, Loperfido [40] showed a connection between the star product of matrices and the contraction product of tensors using a tensor unfolding. For the convenience of the present work, unfolding of a tensor is derived from [34, 46]. The reshaping operation transforms an arbitrary tensor $\mathcal{A} = (a_{i_1\ldots i_M j_1\ldots j_N}) \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$ into the matrix $A = (a_{ij}) \in \mathbb{C}^{\mathbf{m}\times\mathbf{n}}$, where $\mathbf{m} = I_1\cdot\cdots\cdot I_M$ and $\mathbf{n} = J_1\cdot\cdots\cdot J_N$, in which the $(i_1\ldots i_M j_1\ldots j_N)^{th}$ element of $\mathcal{A}$ is mapped to $(ij)^{th}$ element of A, where

$$i := i_1 + \sum_{s=2}^{M}(i_s-1)\prod_{u=1}^{s-1} I_u \text{ and } j := j_1 + \sum_{t=2}^{N}(j_t-1)\prod_{v=1}^{t-1} J_v.$$

One can find this reshaping operation using the Matlab function "reshape" [46] as follows:

$$\mathrm{rsh}(\mathcal{A}) = A = \mathrm{reshape}(\mathcal{A}, \mathbf{m}, \mathbf{n}).$$

This reshape map is also bijective [46] and the inverse of the reshaping operation is defined by

$$\mathrm{rsh}^{-1}(A) = \mathcal{A} = \mathrm{reshape}(A, M_1, \ldots, M_m, N_1, \ldots, N_n).$$

For a third-order tensor, different unfoldings are shown and discussed in the following example.

Example 2.4. *Let $\mathcal{A} \in \mathbb{R}^{2\times2\times3}$ such that*

$\mathcal{A}(:,:,1)$		$\mathcal{A}(:,:,2)$		$\mathcal{A}(:,:,3)$	
1	2	5	6	9	10
3	4	7	8	11	12

.

According to Kolda [52], we can find 12 different unfoldings of $\mathcal{A}$. From the 12 unfoldings, particularly, we compute n-mode unfoldings (Kolda and Bader [53]) here. So, first, second, and third-mode unfoldings of $\mathcal{A}$ are

$$\mathcal{A}_{(1)} = \begin{bmatrix}1&2&5&6&9&10\\3&4&7&8&11&12\end{bmatrix},\ \mathcal{A}_{(2)} = \begin{bmatrix}1&3&5&7&9&11\\2&4&6&8&10&12\end{bmatrix},\ \text{and } \mathcal{A}_{(3)} = \begin{bmatrix}1&3&2&4\\5&7&6&8\\9&11&10&12\end{bmatrix}.$$

Kilmer and Martin [28] defined MatVec$(\cdot)$ operator to write a tensor into a matrix. According to them

$$\mathrm{MatVec}(\mathcal{A}) = \begin{bmatrix}1&2\\3&4\\5&6\\7&8\\9&10\\11&12\end{bmatrix}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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The MatVec(·) operator is also defined as unfold(·) in Kilmer et al. [29]. Now, if we consider the tensor $\mathcal{A}$ as $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times J_1}$ with $I_1 = I_2 = 2, J_1 = 3$, then using the reshape function as defined above the unfolding of $\mathcal{A}$ is

$$\mathrm{rsh}(\mathcal{A}) = \begin{bmatrix} 1 & 5 & 9 \\ 3 & 7 & 11 \\ 2 & 6 & 10 \\ 4 & 8 & 12 \end{bmatrix},$$

and if we consider the tensor $\mathcal{A}$ as $\mathcal{A} \in \mathbb{C}^{I_1 \times J_1 \times J_2}$ with $I_1 = 2, J_1 = 2, J_2 = 3$, then the unfolding of $\mathcal{A}$ is

$$\mathrm{rsh}(\mathcal{A}) = \begin{bmatrix} 1 & 2 & 5 & 6 & 9 & 10 \\ 3 & 4 & 7 & 8 & 11 & 12 \end{bmatrix}.$$

Further, rshrank($\mathcal{A}$)=rank(rsh($\mathcal{A}$)) is defined as the rank of the tensor $\mathcal{A}$, in [46]. For the Einstein product of two tensors, reshape map satisfies the following property given as Lemma 3.1 in [46].

Lemma 2.5 ([46]).
Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $\mathcal{B} \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_P}$. Then,

$$\mathrm{rsh}(\mathcal{A} *_N \mathcal{B}) = \mathrm{rsh}(\mathcal{A})\mathrm{rsh}(\mathcal{B}) = AB,$$

where $\mathrm{rsh}(\mathcal{A}) = A \in \mathbb{C}^{\mathbf{m} \times \mathbf{n}}$, $\mathrm{rsh}(\mathcal{B}) = B \in \mathbb{C}^{\mathbf{n} \times \mathbf{l}}$, $\mathbf{m} = I_1 \cdot \cdots \cdot I_M$, $\mathbf{n} = J_1 \cdot \cdots \cdot J_n$, *and* $\mathbf{p} = K_1 \cdot \cdots \cdot K_P$.

Using the same unfolding of tensors several other generalized inverses of tensors are presented in the literature (for example, see [17]). Also, interested readers are referred to [43] for a brief introduction to tensor rank and some decompositions of tensors.

3. Weighted singular value decomposition. This section contains some of the main results of this article. In 2013, Brazell *et al.* first studied the singular value decomposition (SVD) of a tensor via the Einstein product, which is a generalization of the SVD of a matrix [10, 11]. In 2011, Kilmer and Martin [28] defined the T-SVD. Besides applications in image processing [29], the T-SVD was also utilized by Miao *et al.* [55] to define generalized tensor functions based on t-product. In 2015, Loperfido [39] investigated some properties of the SVD of the third multivariate moment and also established that the left singular vectors corresponding to positive singular values of the third multivariate moment are vectorized, symmetric matrices. In this section, we prove that any tensor can be decomposed into the tensor Einstein product of three special tensors. We call it the WSVD of the given tensor as it generalizes the notion of the WSVD of a matrix [5]. After that, we derive a formula to compute the weighted Moore-Penrose inverse of a given arbitrary-order tensor. For applications of the WSVD of a matrix, we refer [32, 14, 19, 15] and references therein. In particular, the WSVD is widely used in solving weighted least-squares solutions [5, 6]. We now propose the WSVD of an arbitrary-order tensor using the reshaping operation that generalizes Theorem 3.17 of [26], Lemma 3.1 of [25], Theorem 3.2 of [34], and Lemma 2 of [48].

Theorem 3.1. *Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ with* rshrank($\mathcal{A}$) $= r$, *and $\mathcal{M} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$, $\mathcal{N} \in \mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N}$ be two Hermitian positive definite tensors. Then, there exist tensors $\mathcal{U} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$ and $\mathcal{V} \in \mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N}$ satisfying $\mathcal{U}^H *_M \mathcal{M} *_M \mathcal{U} = \mathcal{I}_1$ and $\mathcal{V}^H *_N \mathcal{N}^{-1} *_N \mathcal{V} = \mathcal{I}_2$, where $\mathcal{I}_1 \in \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}$ and $\mathcal{I}_2 \in \mathbb{C}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N}$ are the identity tensors, such that*

$$\mathcal{A} = \mathcal{U} *_M \mathcal{S} *_N \mathcal{V}^H, \tag{3.13}$$

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in which the tensor $\mathcal{S} = (\mathcal{S}_{i_1 \ldots i_M j_1 \ldots j_N}) \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ *is defined by*

$$\mathcal{S}_{i_1 \cdots i_M j_1 \cdots j_N} = \begin{cases} \mu_{IJ} > 0, & \text{if } I = J \in \{1,\ 2,\ \ldots,\ r\}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.14}$$

where $I = i_1 + \sum_{s=2}^{M} (i_s - 1) \prod_{u=1}^{s-1} I_u$ *and* $J = j_1 + \sum_{t=2}^{N} (j_t - 1) \prod_{v=1}^{t-1} J_v$.

Proof. Let $A = \mathrm{rsh}(\mathcal{A}) \in \mathbb{C}^{m \times n}$, $M = \mathrm{rsh}(\mathcal{M}) \in \mathbb{C}^{m \times m}$, and $N = \mathrm{rsh}(\mathcal{M}) \in \mathbb{C}^{n \times n}$ be the reshaping of the tensors $\mathcal{A}$, $\mathcal{M}$, and $\mathcal{N}$, respectively, where

$$m = I_1 \cdot I_2 \cdot \cdots \cdot I_M \text{ and } n = J_1 \cdot J_2 \cdot \cdots \cdot J_N. \tag{3.15}$$

The WSVD of the matrix A with respect to the weights M and N is

$$A = USV^*, \tag{3.16}$$

where $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ following $U^* M U = I_1$ and $V^* N^{-1} V = I_2$, $I_1 \in \mathbb{C}^{m \times m}$, $I_2 \in \mathbb{C}^{n \times n}$ are the identity matrices, and $S = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_r, 0, \ldots, 0) \in \mathbb{R}^{m \times n}$, $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_r > 0$ are the nonzero (M, N) singular values of A. Since rsh is a bijection, taking the inverse map rsh^{-1} on both sides of (3.16), we obtain

$$\mathrm{rsh}^{-1}(A) = \mathrm{rsh}^{-1}(USV^*) = \mathrm{rsh}^{-1}(U) *_M \mathrm{rsh}^{-1}(S) *_N \mathrm{rsh}^{-1}(V^*) = \mathrm{rsh}^{-1}(U) *_M \mathrm{rsh}^{-1}(S) *_N (\mathrm{rsh}^{-1}(V))^H,$$

which implies that

$$\mathcal{A} = \mathcal{U} *_M \mathcal{S} *_N \mathcal{V}^H.$$

Now, we have $U^* M U = I_1$ and $V^* N^{-1} V = I_2$. Applying the reverse map rsh^{-1} on both sides in the last two equalities, we get $\mathcal{U}^H *_M \mathcal{M} *_M \mathcal{U} = \mathcal{I}_1$ and $\mathcal{V}^H *_N \mathcal{N}^{-1} *_N \mathcal{V} = \mathcal{I}_2$. From $\mathcal{S} = \mathrm{rsh}^{-1}(S)$,

$$\mathcal{S}_{i_1 \cdots i_M j_1 \cdots j_N} = \begin{cases} \mu_{IJ} > 0, & \text{if } I = J \in \{1,\ 2,\ \ldots,\ r\}, \\ 0, & \text{otherwise.} \end{cases}$$

This completes the proof. ∎

We call (3.13) as the WSVD of the tensor $\mathcal{A}$ and μ_{IJ}*'s* as the $(\mathcal{M}, \mathcal{N})$ singular values of $\mathcal{A}$. We next present an algorithm for computing the WSVD.

The following example demonstrates Algorithm 1.

EXAMPLE 3.2. *Consider the tensor* $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times J_1} = \mathbb{C}^{2 \times 2 \times 2}$ *and the two weights* $\mathcal{M} \in \mathbb{C}^{I_1 \times I_2 \times I_1 \times I_2} = \mathbb{C}^{2 \times 2 \times 2 \times 2}$, $\mathcal{N} \in \mathbb{C}^{J_1 \times J_1} = \mathbb{C}^{2 \times 2}$ *such that*

$\mathcal{A}(:,:,1)$		$\mathcal{A}(:,:,2)$	
1	0	0	1
0	0	0	0

,

$\mathcal{M}(:,:,1,1)$		$\mathcal{M}(:,:,2,1)$		$\mathcal{M}(:,:,1,2)$		$\mathcal{M}(:,:,2,2)$	
1	0	0	0	0	1	0	0
0	0	1	0	0	0	0	4

,

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Algorithm 1 `Computing WSVD of a tensor`

Require: `Positive integers` $M, N, I_1, ..., I_M, J_1, ..., J_N, m, n$ `such that` m `and` n `satisfy` (3.15). $A \in \mathbb{C}^{m\times n}$, `and` $M \in \mathbb{C}^{m\times m}$, $N \in \mathbb{C}^{n\times n}$ `two Hermitian positive definite matrices.`

1: `Compute the WSVD of` A,
$$A = USV^*,$$
`where` $U \in \mathbb{C}^{m\times m}$ `and` $V \in \mathbb{C}^{n\times n}$ `and` $S \in \mathbb{C}^{m\times n}$ `is a diagonal matrix with` (M, N) `singular values of` A `on the main diagonal.`

2: `Perform the reshaping operations`
$\mathrm{rsh}^{-1}(U) = \mathcal{U} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$, $\mathrm{rsh}^{-1}(V^*) = \mathcal{V}^H \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$, $\mathrm{rsh}^{-1}(S) = \mathcal{S} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$.

3: `Compute the output`
$$\mathcal{A} = \mathcal{U} *_M \mathcal{S} *_N \mathcal{V}^H.$$

and

$\mathcal{N}(:,:)$	
4	0
0	1

.

On reshaping these tensors $\mathcal{A}, \mathcal{M}$, and $\mathcal{N}$, we obtain the matrices A, M, and N, respectively, as follows:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},\ \textit{and } N = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now, we compute the WSVD of the matrix A with respect to the weights M and N, we get $A = USV^*$, where

$$U = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix},\ S = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ \textit{and } V = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}.$$

On applying the inverse reshape function on the matrices U, S, and V, we get the tensors $\mathcal{U} \in \mathbb{C}^{I_1\times I_2\times I_1\times I_2} = \mathbb{C}^{2\times2\times2\times2}$, $\mathcal{S} \in \mathbb{C}^{I_1\times I_2\times J_1} = \mathbb{C}^{2\times2\times2}$, and $\mathcal{V} \in \mathbb{C}^{J_1\times J_1} = \mathbb{C}^{2\times2}$, respectively as

$\mathcal{U}(:,:,1,1)$		$\mathcal{U}(:,:,2,1)$		$\mathcal{U}(:,:,1,2)$		$\mathcal{U}(:,:,2,2)$	
0	1	1	0	0	0	0	0
0	0	0	0	1	0	0	$\frac{1}{2}$

,

$\mathcal{S}(:,:,1)$		$\mathcal{S}(:,:,2)$	
1	0	0	0
0	0	$\frac{1}{2}$	0

,

and

$\mathcal{V}(:,:)$	
0	2
1	0

.

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Therefore, $\mathcal{U} *_2 \mathcal{S} *_1 \mathcal{V}^H$ *is*

$\mathcal{U} *_2 \mathcal{S} *_1 \mathcal{V}^H(:,:,1)$		$\mathcal{U} *_2 \mathcal{S} *_1 \mathcal{V}^H(:,:,2)$	
1	0	0	1
0	0	0	0

,

which is same as the tensor $\mathcal{A}$*, i.e.,* $\mathcal{A} = \mathcal{U} *_2 \mathcal{S} *_1 \mathcal{V}^H$.

Next, we provide a result to compute the weighted Moore-Penrose inverse of a tensor $\mathcal{A}$ via the WSVD, $\mathcal{A} = \mathcal{U} *_M \mathcal{S} *_N \mathcal{V}^H$.

THEOREM 3.3. *Let* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$ *with* $\text{rshrank}(\mathcal{A}) = r$*, and* $\mathcal{M} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$, $\mathcal{N} \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$ *be two Hermitian positive definite tensors. If* $\mathcal{A} = \mathcal{U} *_M \mathcal{S} *_N \mathcal{V}^H$ *is the WSVD of the tensor* $\mathcal{A}$*, then*

$$\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} = \mathcal{N}^{-1} *_N \mathcal{V} *_N \mathcal{S}^{\dagger} *_M \mathcal{U}^H *_M \mathcal{M}, \tag{3.17}$$

where $\mathcal{S}^{\dagger} = (\mathcal{S}^{\dagger}_{j_1\ldots j_N i_1\ldots i_M}) \in \mathbb{R}^{J_1\times\cdots\times J_N\times I_1\times\cdots\times I_M}$ *is defined by*

$$\mathcal{S}^{\dagger}_{j_1\cdots j_N i_1\cdots i_M} = \begin{cases} \mathcal{S}^{-1}_{i_1\cdots i_M j_1\cdots j_N}, & \text{if } \mathcal{S}_{i_1\cdots i_M j_1\cdots j_N} \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.18}$$

Proof. It can be easily proved by Definition 1.6. ∎

Here, we present an algorithm for computing the weighted Moore-Penrose inverse via the WSVD.

Algorithm 2 `Computing the weighted Moore-Penrose inverse of a tensor`

Require: `Positive integers` $M, N, I_1, \ldots, I_M, J_1, \ldots, J_N, m, n$ `such that` m `and` n `satisfy` (3.15).
$A \in \mathbb{C}^{m\times n}$`, and` $M \in \mathbb{C}^{m\times m}$, $N \in \mathbb{C}^{n\times n}$ `two Hermitian positive definite matrices.`

1: `Compute the WSVD of` A,

$$A = USV^*,$$

`where` $U \in \mathbb{C}^{m\times m}$ `and` $V \in \mathbb{C}^{n\times n}$ `and` $S \in \mathbb{C}^{m\times n}$ `is a diagonal matrix with` (M, N) `singular values of` A `on the main diagonal.`

2: `Perform the reshaping operations`
$\text{rsh}^{-1}(U) = \mathcal{U} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$, $\text{rsh}^{-1}(V^*) = \mathcal{V}^H \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$, $\text{rsh}^{-1}(S) = \mathcal{S} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$.

3: `Compute the output`

$$\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} = \mathcal{N}^{-1} *_N \mathcal{V} *_N \mathcal{S}^{\dagger} *_M \mathcal{U}^H *_M \mathcal{M}.$$

The next example verifies the above algorithm.

EXAMPLE 3.4. *Consider the same tensors* $\mathcal{A}, \mathcal{M}$*, and* $\mathcal{N}$ *as given in Example 3.2. By the same argument, we have the tensors* $\mathcal{U}, \mathcal{S}$*, and* $\mathcal{V}$*. Using* (3.18)*, we get* $\mathcal{S}^{\dagger} \in \mathbb{C}^{J_1\times I_1\times I_2} = \mathbb{C}^{2\times2\times2}$ *as*

$\mathcal{S}^{\dagger}(:,:,1)$		$\mathcal{S}^{\dagger}(:,:,2)$	
1	0	0	0
0	2	0	0

.

The conjugate transpose $\mathcal{U}^H \in \mathbb{C}^{I_1\times I_2\times I_1\times I_2} = \mathbb{C}^{2\times2\times2\times2}$ *of the tensor* $\mathcal{U}$ *is*

Electronic Journal of Linear Algebra, ISSN 1081-3810
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$\mathcal{U}^H(:,:,1,1)$		$\mathcal{U}^H(:,:,2,1)$		$\mathcal{U}^H(:,:,1,2)$		$\mathcal{U}^H(:,:,2,2)$	
0	0	0	1	1	0	0	0
1	0	0	0	0	0	0	$\frac{1}{2}$

Then, we obtain the tensors $\mathcal{B} = \mathcal{N}^{-1} *_1 \mathcal{V}$, $\mathcal{C} = \mathcal{B} *_1 \mathcal{S}^{\dagger}$, $\mathcal{D} = \mathcal{C} *_2 \mathcal{U}^H$, *and* $\mathcal{E} = \mathcal{D} *_2 \mathcal{M}$ *as follows:*

$\mathcal{B}(:,:)$	
0	$\frac{1}{2}$
1	0

$\mathcal{C}(:,:,1)$		$\mathcal{C}(:,:,2)$	
0	1	0	0
1	0	0	0

$\mathcal{D}(:,:,1)$		$\mathcal{D}(:,:,2)$	
1	0	0	0
0	0	1	0

and

$\mathcal{E}(:,:,1)$		$\mathcal{E}(:,:,2)$	
1	0	0	0
0	0	1	0

It is clear that the tensor $\mathcal{E} = \mathcal{N}^{-1} *_1 \mathcal{V} *_1 \mathcal{S}^{\dagger} *_2 \mathcal{U}^H *_2 \mathcal{M}$, *which is the weighted Moore-Penrose inverse* $\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}$ *of* $\mathcal{A}$.

The next result can be easily verified using the definition of the weighted Moore-Penrose inverse of a tensor.

Lemma 3.5. *Let* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$, *and* $\mathcal{M} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$, $\mathcal{N} \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$ *be two Hermitian positive definite tensors. If for any tensor* $\mathcal{B} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$, $\mathcal{A} = \mathcal{U} *_M \mathcal{B} *_N \mathcal{V}^H$, *where* $\mathcal{U} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ *and* $\mathcal{V} \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$ *satisfying* $\mathcal{U}^H *_M \mathcal{M} *_M \mathcal{U} = \mathcal{I}_1$ *and* $\mathcal{V}^H *_N \mathcal{N}^{-1} *_N \mathcal{V} = \mathcal{I}_2$, *where* $\mathcal{I}_1 \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ *and* $\mathcal{I}_2 \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$ *are the identity tensors, then* $\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} = \mathcal{N}^{-1} *_N \mathcal{V} *_N \mathcal{B}^{\dagger} *_M \mathcal{U}^H *_M \mathcal{M}$.

The above result reduces to the following corollary in the matrix case.

Corollary 3.6. *Let* $A \in \mathbb{C}^{m\times n}$, *and* $M \in \mathbb{C}^{m\times m}$, $N \in \mathbb{C}^{n\times n}$ *be two Hermitian positive definite matrices. If for any matrix* $B \in \mathbb{C}^{m\times n}$, $A = UBV^*$, *where* $U \in \mathbb{C}^{m\times m}$ *and* $V \in \mathbb{C}^{n\times n}$ *satisfying* $U^*MU = I_1$ *and* $V^*N^{-1}V = I_2$, *where* $I_1 \in \mathbb{C}^{m\times m}$ *and* $I_2 \in \mathbb{C}^{n\times n}$ *are the identity matrices, then* $A^{\dagger}_{M,N} = N^{-1}VB^{\dagger}U^*M$.

In the following theorem, we present a representation of the weighted Moore-Penrose inverse $\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}$ of $\mathcal{A}$.

Theorem 3.7. *Let* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$. *If* $\mathcal{M} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ *and* $\mathcal{N} \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$ *are two Hermitian positive definite tensors, then*

$$\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} = \lim_{\lambda\to 0}[(\lambda\mathcal{I}_2 + \mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}} *_M \mathcal{A})^{-1} *_N \mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}}],$$

where $\mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}}$ *is the weighted conjugate transpose of* $\mathcal{A}$, $\lambda \in \mathbb{R}^+$, $\mathbb{R}^+$ *denotes the set of all positive real numbers, and* $\mathcal{I}_2 \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$ *is the identity tensor.*

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Proof. By Theorems 3.1 and 3.3, we have $\mathcal{A} = \mathcal{U} *_M \mathcal{S} *_N \mathcal{V}^H$ and $\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} = \mathcal{N}^{-1} *_N \mathcal{V} *_N \mathcal{S}^{\dagger} *_M \mathcal{U}^H *_M \mathcal{M}$, where

$$\mathcal{S}_{i_1\cdots i_M j_1\cdots j_N} = \begin{cases} \mu_{IJ} > 0, & \text{if } I = J \in \{1,\ 2,\ \ldots,\ r\}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\mathcal{S}^{\dagger}_{j_1\cdots j_N i_1\cdots i_M} = \begin{cases} \mathcal{S}^{-1}_{i_1\cdots i_M j_1\cdots j_N}, & \text{if } \mathcal{S}_{i_1\cdots i_M j_1\cdots j_N} \neq 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $I = i_1 + \sum_{s=2}^{M}(i_s - 1)\prod_{u=1}^{s-1} I_u$ and $J = j_1 + \sum_{t=2}^{N}(j_t - 1)\prod_{v=1}^{t-1} J_v$. Now, using $\mathcal{U}^H *_M \mathcal{M} *_M \mathcal{U} = \mathcal{I}_1$ and $\mathcal{V}^H *_N \mathcal{N}^{-1} *_N \mathcal{V} = \mathcal{I}_2$, we obtain

$$\begin{aligned} \mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}} *_M \mathcal{A} &= \mathcal{N}^{-1} *_N \mathcal{A}^H *_M \mathcal{M} *_M \mathcal{A} \\ &= \mathcal{N}^{-1} *_N \mathcal{V} *_N \mathcal{S}^H *_M \mathcal{U}^H *_M \mathcal{M} *_M \mathcal{U} *_M \mathcal{S} *_N \mathcal{V}^H \\ &= (\mathcal{V}^H)^{-1} *_N \mathcal{S}^H *_M \mathcal{S} *_N \mathcal{V}^H. \end{aligned}$$

Therefore, we get

$$\lambda \mathcal{I}_2 + \mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}} *_M \mathcal{A} = (\mathcal{V}^H)^{-1} *_N (\lambda \mathcal{I}_2 + \mathcal{S}^H *_M \mathcal{S}) *_N \mathcal{V}^H,$$

and

$$\begin{aligned} &(\lambda \mathcal{I}_2 + \mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}} *_M \mathcal{A})^{-1} *_N \mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}} \\ &\qquad = (\mathcal{V}^H)^{-1} *_N (\lambda \mathcal{I}_2 + \mathcal{S}^H *_M \mathcal{S})^{-1} *_N \mathcal{V}^H *_N \mathcal{N}^{-1} *_N \mathcal{V} *_N \mathcal{S}^H *_M \mathcal{U}^H *_M \mathcal{M} \\ &\qquad = \mathcal{N}^{-1} *_N \mathcal{V} *_N (\lambda \mathcal{I}_2 + \mathcal{S}^H *_M \mathcal{S})^{-1} *_N \mathcal{S}^H *_M \mathcal{U}^H *_M \mathcal{M}. \end{aligned}$$

Now,

$$\mathcal{S}^H_{j_1\cdots j_N i_1\cdots i_M} = \overline{\mathcal{S}}_{i_1\cdots i_M j_1\cdots j_N} = \begin{cases} \mu_{IJ} > 0, & \text{if } J = I \in \{1,\ 2,\ \ldots,\ r\}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$(\mathcal{S}^H *_M \mathcal{S})_{j_1\cdots j_N k_1\cdots k_N} = \begin{cases} {\mu_{IJ}}^2, & \text{if } J = K \in \{1,\ 2,\ \ldots,\ r\}, \\ 0, & \text{otherwise}. \end{cases}$$

So, we obtain

$$(\lambda \mathcal{I}_2 + \mathcal{S}^H *_M \mathcal{S})^{-1}_{j_1\cdots j_N k_1\cdots k_N} = \begin{cases} \frac{1}{\lambda + {\mu_{IJ}}^2}, & \text{if } J = K \in \{1,\ 2,\ \ldots,\ r\}, \\ \frac{1}{\lambda}, & \text{if } J = K \notin \{1,\ 2,\ \ldots,\ r\}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$((\lambda \mathcal{I}_2 + \mathcal{S}^H *_M \mathcal{S})^{-1} *_N \mathcal{S}^H)_{j_1\cdots j_N i_1\cdots i_M} = \begin{cases} \frac{\mu_{IJ}}{\lambda + {\mu_{IJ}}^2}, & \text{if } J = I \in \{1,\ 2,\ \ldots,\ r\}, \\ 0, & \text{otherwise}. \end{cases}$$

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The last equation implies that $\lim_{\lambda\to 0}((\lambda\mathcal{I}_2+\mathcal{S}^H *_M \mathcal{S})^{-1} *_N \mathcal{S}^H)=\mathcal{S}^\dagger$. Thus, we get

$$\begin{aligned}\lim_{\lambda\to 0}[(\lambda\mathcal{I}_2+\mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}} *_M \mathcal{A})^{-1} *_N \mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}}] &= \lim_{\lambda\to 0}[\mathcal{N}^{-1} *_N \mathcal{V} *_N (\lambda\mathcal{I}_2+\mathcal{S}^H *_M \mathcal{S})^{-1} *_N \mathcal{S}^H *_M \mathcal{U}^H *_M \mathcal{M}]\\ &= \mathcal{N}^{-1} *_N \mathcal{V} *_N \mathcal{S}^\dagger *_M \mathcal{U}^H *_M \mathcal{M}\\ &= \mathcal{A}^\dagger_{\mathcal{M},\mathcal{N}}.\end{aligned}$$

∎

Same as Theorem 3.7, we can derive the next representation for $\mathcal{A}^\dagger_{\mathcal{M},\mathcal{N}}$.

THEOREM 3.8. *Let $\mathcal{A}\in\mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$. If $\mathcal{M}\in\mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ and $\mathcal{N}\in\mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$ are two Hermitian positive definite tensors, then*

$$\mathcal{A}^\dagger_{\mathcal{M},\mathcal{N}} = \lim_{\lambda\to 0}[\mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}} *_M (\lambda\mathcal{I}_1+\mathcal{A} *_N \mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}})^{-1}],$$

where $\mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}}$ is the weighted conjugate transpose of $\mathcal{A}$, $\lambda\in\mathbb{R}^+$, and $\mathcal{I}_1\in\mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ is the identity tensor.

By Proposition 2.4 of [25] and Definition 1.6, we can prove the following lemma.

LEMMA 3.9. *Let $\mathcal{B}\in\mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ be an invertible tensor and let a block tensor $\mathcal{A}$ be defined by*

$$\mathcal{A}=\begin{bmatrix}\mathcal{B} & \mathcal{O}\\ \mathcal{O} & \mathcal{O}\end{bmatrix},$$

where $\mathcal{O}\in\mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ is the zero tensor. Let $\mathcal{M},\mathcal{N}\in\mathbb{C}^{2I_1\times\ldots\times 2I_N\times 2I_1\times\ldots\times 2I_N}$ be two diagonal tensors with positive diagonal entries. Then,

$$\mathcal{A}^\dagger_{\mathcal{M},\mathcal{N}}=\begin{bmatrix}\mathcal{B}^{-1} & \mathcal{O}\\ \mathcal{O} & \mathcal{O}\end{bmatrix}.$$

The above result reduces to the Moore-Penrose inverse case when we consider identity tensors as weights.

4. Weighted normal tensor. In this section, we first introduce the notions of weighted self-conjugate and weighted normal tensor, and then exploit their various properties. Further, we show that Theorem 2.2 does not hold if we replace $\mathcal{A}^\dagger$ by $\mathcal{A}^\dagger_{\mathcal{M},\mathcal{N}}$. We remark that all the definitions and results of this section are also new for matrices and one can state them by taking $N=1$ in tensor case.

DEFINITION 4.1. *Let $\mathcal{A}\in\mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ be an even-order square tensor, and $\mathcal{N}\in\mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ be a Hermitian positive definite tensor. Then, the tensor $\mathcal{A}$ is called weighted self-conjugate tensor if $\mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}}=\mathcal{A}$, i.e.,*

$$\mathcal{N}^{-1} *_N \mathcal{A}^H *_N \mathcal{N}=\mathcal{A}.$$

DEFINITION 4.2. *Let $\mathcal{A}\in\mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ be an even-order square tensor, and $\mathcal{N}\in\mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ be a Hermitian positive definite tensor. Then, the tensor $\mathcal{A}$ is called the weighted normal tensor if*

$$\mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} *_N \mathcal{A}=\mathcal{A} *_N \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}}.$$

In particular, if $\mathcal{N}$ is the identity tensor, then the tensor $\mathcal{A}$ becomes a normal tensor.

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We next provide an example of a tensor which is weighted self-conjugate as well as a weighted normal tensor.

Example 4.3. *Consider* $\mathcal{A},\ \mathcal{N} \in \mathbb{C}^{I_1\times I_2\times I_1\times I_2}$ *with* $I_1 = I_2 = 2$ *such that*

$\mathcal{A}(:,:,1,1)$		$\mathcal{A}(:,:,2,1)$		$\mathcal{A}(:,:,1,2)$		$\mathcal{A}(:,:,2,2)$	
1/2	1	1+i/2	0	1/2	2	0	i
1-i	0	0	0	0	-i/2	0	0

,

$\mathcal{N}(:,:,1,1)$		$\mathcal{N}(:,:,2,1)$		$\mathcal{N}(:,:,1,2)$		$\mathcal{N}(:,:,2,2)$	
2	0	0	0	0	1	0	0
0	0	1	0	0	0	0	2

.

Then, the weighted conjugate transpose of $\mathcal{A}$, $\mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} = \mathcal{A}$. *Therefore,* $\mathcal{A}$ *is a weighted self-conjugate tensor and also a weighted normal tensor.*

Here, we provide an example which shows that $\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}$ is not necessarily normal (or Hermitian) tensor even if $\mathcal{A}$ is normal (or Hermitian) or $\mathcal{A}^{\dagger}$ is normal (or Hermitian).

Example 4.4. *Let* $A = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}^{2\times 2}$. *If* $M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ *and* $N = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ *are two Hermitian positive definite matrices in* $\mathbb{C}^{2\times 2}$, *then* $A^{\dagger} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 0 \end{pmatrix}$ *and* $A^{\dagger}_{M,N} = \begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{1}{8} & 0 \end{pmatrix}$. *Thus, we have*

$$AA^* = A^*A = \begin{pmatrix} 16 & 0 \\ 0 & 0 \end{pmatrix},$$

$$A^{\dagger}(A^{\dagger})^* = (A^{\dagger})^*A^{\dagger} = \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & 0 \end{pmatrix},$$

$$A^{\dagger}_{M,N}(A^{\dagger}_{M,N})^* = \begin{pmatrix} \frac{1}{16} & -\frac{1}{32} \\ -\frac{1}{32} & \frac{1}{64} \end{pmatrix},$$

$$(A^{\dagger}_{M,N})^*A^{\dagger}_{M,N} = \begin{pmatrix} \frac{5}{64} & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly, $A^{\dagger}_{M,N}(A^{\dagger}_{M,N})^* \neq (A^{\dagger}_{M,N})^*A^{\dagger}_{M,N}$, *i.e.,* $A^{\dagger}_{M,N}$ *is not normal.*

The following result gives a necessary and sufficient condition for the weighted Moore-Penrose inverse of a tensor to be weighted normal.

Theorem 4.5. *Let* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ *be an even-order square tensor. If* $\mathcal{N} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ *is a Hermitian positive definite tensor, then*

(i) $\mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} = \mathcal{A}$ *if and only if* $\tilde{\mathcal{A}}^H = \tilde{\mathcal{A}}$;
(ii) $\mathcal{A} *_N \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} = \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} *_N \mathcal{A}$ *if and only if* $\tilde{\mathcal{A}} *_N \tilde{\mathcal{A}}^H = \tilde{\mathcal{A}}^H *_N \tilde{\mathcal{A}}$;
(iii) $\mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}} *_N (\mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}})^{\#}_{\mathcal{N}\mathcal{N}} = (\mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}})^{\#}_{\mathcal{N}\mathcal{N}} *_N \mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}}$ *if and only if* $\tilde{\mathcal{A}}^{\dagger} *_N (\tilde{\mathcal{A}}^{\dagger})^H = (\tilde{\mathcal{A}}^{\dagger})^H *_N \tilde{\mathcal{A}}^{\dagger}$;
(iv) $\mathcal{A} *_N \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} = \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} *_N \mathcal{A}$ *if and only if* $\mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}} *_N (\mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}})^{\#}_{\mathcal{N}\mathcal{N}} = (\mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}})^{\#}_{\mathcal{N}\mathcal{N}} *_N \mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}}$,

where $\mathcal{A}^H$ and $\mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}}$ *are the conjugate transpose and the weighted conjugate transpose of the tensor* $\mathcal{A}$, *respectively, and* $\tilde{\mathcal{A}} = \mathcal{N}^{1/2} *_N \mathcal{A} *_N \mathcal{N}^{-1/2}$.

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Proof. We have

$$\tilde{\mathcal{A}} = \mathcal{N}^{1/2} *_N \mathcal{A} *_N \mathcal{N}^{-1/2}, \tag{4.19}$$

$$\mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} = \mathcal{N}^{-1} *_N \mathcal{A}^H *_N \mathcal{N}, \tag{4.20}$$

$$\mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}} = \mathcal{N}^{-1/2} *_N \tilde{\mathcal{A}}^{\dagger} *_N \mathcal{N}^{1/2}. \tag{4.21}$$

(i) Suppose that $\tilde{\mathcal{A}}^H = \tilde{\mathcal{A}}$. Then,

$$\begin{aligned}
\mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} &= \mathcal{N}^{-1} *_N \mathcal{A}^H *_N \mathcal{N} \\
&= \mathcal{N}^{-1/2} *_N \mathcal{N}^{-1/2} *_N \mathcal{A}^H *_N \mathcal{N}^{1/2} *_N \mathcal{N}^{1/2} \\
&= \mathcal{N}^{-1/2} *_N \tilde{\mathcal{A}}^H *_N \mathcal{N}^{1/2} \\
&= \mathcal{N}^{-1/2} *_N \tilde{\mathcal{A}} *_N \mathcal{N}^{1/2} \quad (\text{since } \tilde{\mathcal{A}}^H = \tilde{\mathcal{A}}) \\
&= \mathcal{A}.
\end{aligned}$$

Conversely, if $\mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} = \mathcal{A}$, then

$$\begin{aligned}
\tilde{\mathcal{A}}^H &= \mathcal{N}^{-1/2} *_N \mathcal{A}^H *_N \mathcal{N}^{1/2} \\
&= \mathcal{N}^{1/2} *_N \mathcal{N}^{-1} *_N \mathcal{A}^H *_N \mathcal{N} *_N \mathcal{N}^{-1/2} \\
&= \mathcal{N}^{1/2} *_N \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} *_N \mathcal{N}^{-1/2} \\
&= \mathcal{N}^{1/2} *_N \mathcal{A} *_N \mathcal{N}^{-1/2} \quad (\text{since } \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} = \mathcal{A}) \\
&= \tilde{\mathcal{A}}.
\end{aligned}$$

(ii) Suppose that $\tilde{\mathcal{A}} *_N \tilde{\mathcal{A}}^H = \tilde{\mathcal{A}}^H *_N \tilde{\mathcal{A}}$. Then,

$$\begin{aligned}
\mathcal{A} *_N \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} &= \mathcal{A} *_N \mathcal{N}^{-1} *_N \mathcal{A}^H *_N \mathcal{N} \\
&= \mathcal{A} *_N \mathcal{N}^{-1/2} *_N \mathcal{N}^{-1/2} *_N \mathcal{A}^H *_N \mathcal{N}^{1/2} *_N \mathcal{N}^{1/2} \\
&= \mathcal{A} *_N \mathcal{N}^{-1/2} *_N \tilde{\mathcal{A}}^H *_N \mathcal{N}^{1/2} \quad (\text{using } (4.19)) \\
&= \mathcal{N}^{-1/2} *_N \tilde{\mathcal{A}} *_N \tilde{\mathcal{A}}^H *_N \mathcal{N}^{1/2} \quad (\text{using } (4.19)) \\
&= \mathcal{N}^{-1/2} *_N \tilde{\mathcal{A}}^H *_N \tilde{\mathcal{A}} *_N \mathcal{N}^{1/2} \quad (\text{since } \tilde{\mathcal{A}} *_N \tilde{\mathcal{A}}^H = \tilde{\mathcal{A}}^H *_N \tilde{\mathcal{A}}) \\
&= \mathcal{N}^{-1/2} *_N \mathcal{N}^{-1/2} *_N \mathcal{A}^H *_N \mathcal{N}^{1/2} *_N \mathcal{N}^{1/2} *_N \mathcal{A} *_N \mathcal{N}^{-1/2} *_N \mathcal{N}^{1/2} \quad (\text{using } (4.19)) \\
&= \mathcal{N}^{-1} *_N \mathcal{A}^H *_N \mathcal{N} *_N \mathcal{A} \\
&= \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} *_N \mathcal{A}.
\end{aligned}$$

Conversely, if $\mathcal{A} *_N \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} = \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} *_N \mathcal{A}$, then

$$\begin{aligned}
\tilde{\mathcal{A}} *_N \tilde{\mathcal{A}}^H &= \mathcal{N}^{1/2} *_N \mathcal{A} *_N \mathcal{N}^{-1/2} *_N \mathcal{N}^{-1/2} *_N \mathcal{A}^H *_N \mathcal{N}^{1/2} \quad (\text{using } (4.19)) \\
&= \mathcal{N}^{1/2} *_N \mathcal{A} *_N \mathcal{N}^{-1} *_N \mathcal{A}^H *_N \mathcal{N} *_N \mathcal{N}^{-1/2} \\
&= \mathcal{N}^{1/2} *_N \mathcal{A} *_N \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} *_N \mathcal{N}^{-1/2} \quad (\text{using } (4.20)) \\
&= \mathcal{N}^{1/2} *_N \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} *_N \mathcal{A} *_N \mathcal{N}^{-1/2} \quad (\text{since } \mathcal{A} *_N \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} = \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} *_N \mathcal{A}) \\
&= \mathcal{N}^{1/2} *_N \mathcal{N}^{-1} *_N \mathcal{A}^H *_N \mathcal{N} *_N \mathcal{A} *_N \mathcal{N}^{-1/2} \quad (\text{using } (4.20)) \\
&= \mathcal{N}^{-1/2} *_N \mathcal{A}^H *_N \mathcal{N}^{1/2} *_N \mathcal{N}^{1/2} *_N \mathcal{A} *_N \mathcal{N}^{-1/2} \\
&= \tilde{\mathcal{A}}^H *_N \tilde{\mathcal{A}}.
\end{aligned}$$

(iii) Similar to part (ii).

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(iv) We have

$$\begin{aligned}\mathcal{A} *_N \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} = \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} *_N \mathcal{A} &\Leftrightarrow \tilde{\mathcal{A}} *_N \tilde{\mathcal{A}}^H = \tilde{\mathcal{A}}^H *_N \tilde{\mathcal{A}} \quad \text{(using part (ii))}\\ &\Leftrightarrow \tilde{\mathcal{A}}^\dagger *_N (\tilde{\mathcal{A}}^\dagger)^H = (\tilde{\mathcal{A}}^\dagger)^H *_N \tilde{\mathcal{A}}^\dagger \quad \text{(since } \tilde{\mathcal{A}} \text{ is normal)}\\ &\Leftrightarrow \mathcal{A}^\dagger_{\mathcal{N},\mathcal{N}} *_N (\mathcal{A}^\dagger_{\mathcal{N},\mathcal{N}})^{\#}_{\mathcal{N}\mathcal{N}} = (\mathcal{A}^\dagger_{\mathcal{N},\mathcal{N}})^{\#}_{\mathcal{N}\mathcal{N}} *_N \mathcal{A}^\dagger_{\mathcal{N},\mathcal{N}} \quad \text{(using part (iii)).}\end{aligned}$$

Thus, the claim. ∎

It is known that if $\lambda \neq 0$ is an eigenvalue of a normal tensor $\mathcal{A}$, then $1/\lambda$ is an eigenvalue of its Moore-Penrose inverse $\mathcal{A}^\dagger$. However, this is not true in the case of the weighted Moore-Penrose inverse $\mathcal{A}^\dagger_{\mathcal{M},\mathcal{N}}$ of $\mathcal{A}$. The next example is in this direction.

Example 4.6. *Let* $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. *If* $M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ *and* $N = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ *are two Hermitian positive definite matrices in* $\mathbb{C}^{2\times 2}$, *then* $A^\dagger_{M,N} = \begin{pmatrix} \frac{1}{12} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$. *Here* A *is normal, and* 2 *is an eigenvalue of* A *but* $1/2$ *is not an eigenvalue of* $A^\dagger_{M,N}$.

The weighted normal tensors fulfill the above requirement, i.e., if $\lambda \neq 0$ is an eigenvalue of $\mathcal{A}$, then $1/\lambda$ is an eigenvalue of its weighted Moore-Penrose inverse, in the case of weighted normal tensor. It is shown in the following theorem.

Theorem 4.7. *Let* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ *be an even-order square tensor. If* $\mathcal{N} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ *is a Hermitian positive definite tensor, then*

(i) $\lambda \in \sigma(\mathcal{A})$ *if and only if* $\lambda \in \sigma(\tilde{\mathcal{A}})$;
(ii) $\lambda \in \sigma(\mathcal{A}^\dagger_{\mathcal{N},\mathcal{N}})$ *if and only if* $\lambda \in \sigma(\tilde{\mathcal{A}}^\dagger)$;
(iii) if $\mathcal{A}$ *is a weighted normal tensor and* $\lambda \neq 0$, *then* $\lambda \in \sigma(\mathcal{A})$ *if and only if* $1/\lambda \in \sigma(\mathcal{A}^\dagger_{\mathcal{N},\mathcal{N}})$.

Proof. (i) Suppose that $\lambda \in \sigma(\mathcal{A})$. Then, there exists a nonzero tensor $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ such that

$$\mathcal{A} *_N \mathcal{X} = \lambda\mathcal{X}.$$

Using the expression $\tilde{\mathcal{A}} = \mathcal{N}^{1/2} *_N \mathcal{A} *_N \mathcal{N}^{-1/2}$ in the last equation, we obtain $\tilde{\mathcal{A}} *_N (\mathcal{N}^{1/2} *_N \mathcal{X}) = \lambda(\mathcal{N}^{1/2} *_N \mathcal{X})$, which implies that $\lambda \in \sigma(\tilde{\mathcal{A}})$. Conversely, if $\lambda \in \sigma(\tilde{\mathcal{A}})$, then there exists a nonzero tensor $\mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ such that

$$\tilde{\mathcal{A}} *_N \mathcal{Y} = \lambda\mathcal{Y}.$$

Again, using the expression $\tilde{\mathcal{A}} = \mathcal{N}^{1/2} *_N \mathcal{A} *_N \mathcal{N}^{-1/2}$ in the last equation, we obtain $\mathcal{A} *_N (\mathcal{N}^{-1/2} *_N \mathcal{Y}) = \lambda(\mathcal{N}^{-1/2} *_N \mathcal{Y})$, which implies that $\lambda \in \sigma(\mathcal{A})$.

(ii) This part can be proved by following the steps as in part (i).

(iii) Suppose that $\mathcal{A}$ is a weighted normal tensor, i.e., $\mathcal{A} *_N \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} = \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} *_N \mathcal{A}$, and $\lambda \neq 0$. Then,

$$\begin{aligned}\lambda \in \sigma(\mathcal{A}) &\Leftrightarrow \lambda \in \sigma(\tilde{\mathcal{A}}) \quad \text{(using part (i))}\\ &\Leftrightarrow \frac{1}{\lambda} \in \sigma(\tilde{\mathcal{A}}^\dagger) \quad \text{(since } \tilde{\mathcal{A}} \text{ is normal using Theorem 4.5(ii))}\\ &\Leftrightarrow \frac{1}{\lambda} \in \sigma(\mathcal{A}^\dagger_{\mathcal{N},\mathcal{N}}) \quad \text{(using part (ii)).}\end{aligned}$$

Hence the proof. ∎

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The next result provides a necessary and sufficient condition for a tensor to commute with its weighted Moore-Penrose inverse.

THEOREM 4.8. *Let* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ *be an even-order square tensor. If* $\mathcal{N} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ *is a Hermitian positive definite tensor, then* $\mathcal{A}*_N\mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}} = \mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}}*_N\mathcal{A}$ if and only if $\tilde{\mathcal{A}} *_N \tilde{\mathcal{A}}^{\dagger} = \tilde{\mathcal{A}}^{\dagger} *_N \tilde{\mathcal{A}}$.

Proof. We have $\tilde{\mathcal{A}} = \mathcal{N}^{1/2} *_N \mathcal{A} *_N \mathcal{N}^{-1/2}$ and $\mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}} = \mathcal{N}^{-1/2} *_N \tilde{\mathcal{A}}^{\dagger} *_N \mathcal{N}^{1/2}$. So, $\tilde{\mathcal{A}} *_N \tilde{\mathcal{A}}^{\dagger} = \mathcal{N}^{1/2} *_N \mathcal{A} *_N \mathcal{N}^{-1/2} *_N \mathcal{N}^{1/2} *_N \mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}} *_N \mathcal{N}^{-1/2} = \mathcal{N}^{1/2} *_N \mathcal{A} *_N \mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}} *_N \mathcal{N}^{-1/2}$. Similarly, we obtain $\tilde{\mathcal{A}}^{\dagger} *_N \tilde{\mathcal{A}} = \mathcal{N}^{1/2} *_N \mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}} *_N \mathcal{A} *_N \mathcal{N}^{-1/2}$. Thus, the result follows. ◻

Weighted normality is a sufficient condition for the commutativity of a tensor with its weighted Moore-Penrose inverse. The following theorem is in this direction.

THEOREM 4.9. *Let* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ *be an even-order square tensor, and* $\mathcal{N} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ *be a Hermitian positive definite tensor. If* $\mathcal{A}$ *is weighted normal, then* $\mathcal{A}*_N\mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}} = \mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}} *_N \mathcal{A}$.

Proof. We have $\tilde{\mathcal{A}} = \mathcal{N}^{1/2} *_N \mathcal{A} *_N \mathcal{N}^{-1/2}$ and $\mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}} = \mathcal{N}^{-1/2} *_N \tilde{\mathcal{A}}^{\dagger} *_N \mathcal{N}^{1/2}$. If $\mathcal{A}$ is weighted normal, then $\mathcal{A} *_N \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} = \mathcal{A}^{\#}_{\mathcal{N}\mathcal{N}} *_N \mathcal{A}$. So, by Theorem 4.5(ii), we find $\tilde{\mathcal{A}} *_N \tilde{\mathcal{A}}^{H} = \tilde{\mathcal{A}}^{H} *_N \tilde{\mathcal{A}}$, which implies that $\tilde{\mathcal{A}} *_N \tilde{\mathcal{A}}^{\dagger} = \tilde{\mathcal{A}}^{\dagger} *_N \tilde{\mathcal{A}}$, by Theorem 3.4 of [21]. Using Theorem 4.9, we get $\mathcal{A} *_N \mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}} = \mathcal{A}^{\dagger}_{\mathcal{N},\mathcal{N}} *_N \mathcal{A}$. ◻

5. Weighted tensor norm. For two Hermitian positive definite tensors $\mathcal{M} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ and $\mathcal{N} \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$, we define the weighted inner product and their induced weighted tensor norms here. The weighted inner products in $\mathbb{C}^{I_1\times\cdots\times I_M}$ and $\mathbb{C}^{J_1\times\cdots\times J_N}$ are

$$\langle \mathcal{X}, \mathcal{Y}\rangle_{\mathcal{M}} = \langle \mathcal{M} *_M \mathcal{X}, \mathcal{Y}\rangle,\ \mathcal{X}, \mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_M}$$

and

$$\langle \mathcal{X}, \mathcal{Y}\rangle_{\mathcal{N}} = \langle \mathcal{N} *_N \mathcal{X}, \mathcal{Y}\rangle,\ \mathcal{X}, \mathcal{Y} \in \mathbb{C}^{J_1\times\cdots\times J_N},$$

respectively. Then, their induced weighted tensor norms are

$$\|\mathcal{X}\|_{\mathcal{M}} = \sqrt{\langle \mathcal{X}, \mathcal{X}\rangle_{\mathcal{M}}},\ \mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_M}$$

and

$$\|\mathcal{X}\|_{\mathcal{N}} = \sqrt{\langle \mathcal{X}, \mathcal{X}\rangle_{\mathcal{N}}},\ \mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_N},$$

respectively.

LEMMA 5.1. *For* $\mathcal{W} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$, $\mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_N}$, *and* $\mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_M}$, *we have*

$$\langle \mathcal{W} *_N \mathcal{X}, \mathcal{Y}\rangle = \left\langle \mathcal{X}, \mathcal{W}^{H} *_M \mathcal{Y}\right\rangle.$$

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Proof.

$$\begin{aligned}
\langle \mathcal{W} *_N \mathcal{X}, \mathcal{Y}\rangle &= \mathcal{Y}^H *_M \mathcal{W} *_N \mathcal{X} \\
&= \sum_{i_1,i_2,\dots,i_M} \bar{y}_{i_1 i_2 \cdots i_M} \Big(\sum_{j_1,j_2,\dots,j_N} \mathcal{W}_{i_1 i_2 \cdots i_M j_1 j_2 \dots j_N} x_{j_1 j_2 \cdots j_N} \Big) \\
&= \sum_{i_1,i_2,\dots,i_M} \sum_{j_1,j_2,\dots,j_N} \bar{y}_{i_1 i_2 \cdots i_M} \mathcal{W}_{i_1 i_2 \cdots i_M j_1 j_2 \dots j_N} x_{j_1 j_2 \cdots j_N} \\
&= \sum_{j_1,j_2,\dots,j_N} \Big(\sum_{i_1,i_2,\dots,i_M} \bar{y}_{i_1 i_2 \cdots i_M} \mathcal{W}_{i_1 i_2 \cdots i_M j_1 j_2 \dots j_N} x_{j_1 j_2 \cdots j_N} \Big) \\
&= \sum_{j_1,j_2,\dots,j_N} \big(\mathcal{W}^H *_M \mathcal{Y}\big)^H_{j_1 j_2 \cdots j_N} x_{j_1 j_2 \cdots j_N} \\
&= \big(\mathcal{W}^H *_M \mathcal{Y}\big)^H *_N \mathcal{X} \\
&= \big\langle \mathcal{X}, \mathcal{W}^H *_M \mathcal{Y} \big\rangle .
\end{aligned}$$

From Lemma 5.1, the next result is easy to deduce.

LEMMA 5.2. *Let $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_M}$ and $\mathcal{Y} \in \mathbb{C}^{J_1\times\cdots\times J_N}$. If $\mathcal{M} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ and $\mathcal{N} \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$ are two Hermitian positive definite tensors, then*

(i) $\|\mathcal{X}\|_{\mathcal{M}} = \|\mathcal{M}^{1/2} *_M \mathcal{X}\|$;
(ii) $\|\mathcal{X}\|_{\mathcal{N}} = \|\mathcal{N}^{1/2} *_N \mathcal{X}\|$.

Proof. (i) We can write

$$\begin{aligned}
\|\mathcal{X}\|_{\mathcal{M}} &= \sqrt{\langle \mathcal{X}, \mathcal{X}\rangle_{\mathcal{M}}} \\
&= \sqrt{\langle \mathcal{M} *_M \mathcal{X}, \mathcal{X}\rangle} \\
&= \sqrt{\big\langle \mathcal{M}^{1/2} *_M \mathcal{M}^{1/2} *_M \mathcal{X}, \mathcal{X}\big\rangle} \\
&= \sqrt{\big\langle \mathcal{M}^{1/2} *_M \mathcal{X}, \mathcal{M}^{1/2} *_M \mathcal{X}\big\rangle} \\
&= \|\mathcal{M}^{1/2} *_M \mathcal{X}\|.
\end{aligned}$$

Similarly, we can prove (ii).

Let $\mathcal{X}, \mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_M}$ with a Hermitian positive definite tensor $\mathcal{M} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$. Then, $\mathcal{X}$ and $\mathcal{Y}$ are called $\mathcal{M}$-orthogonal if $\langle \mathcal{X}, \mathcal{Y}\rangle_M = 0$. Next, we prove the weighted Pythagorean theorem for tensors.

THEOREM 5.3. *Let $\mathcal{X}, \mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_M}$ be $\mathcal{M}$-orthogonal. Then,*

$$\|\mathcal{X}+\mathcal{Y}\|^2_{\mathcal{M}} = \|\mathcal{X}\|^2_{\mathcal{M}} + \|\mathcal{Y}\|^2_{\mathcal{M}}.$$

Proof. Using Lemma 5.2, we can write

$$\begin{aligned}
\|\mathcal{X}+\mathcal{Y}\|^2_{\mathcal{M}} &= \|\mathcal{M}^{1/2} *_M (\mathcal{X}+\mathcal{Y})\|^2 \\
&= \|\mathcal{M}^{1/2} *_M \mathcal{X}\|^2 + \|\mathcal{M}^{1/2} *_M \mathcal{Y}\|^2 \quad \text{(since } \mathcal{M}^{1/2} *_M \mathcal{X} \text{ and } \mathcal{M}^{1/2} *_M \mathcal{Y} \text{ are orthogonal)} \\
&= \|\mathcal{X}\|^2_{\mathcal{M}} + \|\mathcal{Y}\|^2_{\mathcal{M}}.
\end{aligned}$$

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For a tensor $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$, we define the tensor norm as:

$$\|\mathcal{A}\| = \sup\{\|\mathcal{A} *_N \mathcal{X}\| \ : \ \|\mathcal{X}\| = 1,\ \mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_N}\}. \tag{5.22}$$

By performing the steps of the existing proofs for matrices, we can verify the following.

(i) $\|\mathcal{A}\| = \|\mathcal{A}\|_2$, where $\|.\|_2$ is the spectral norm of $\mathcal{A}$.
(ii) $\|\mathcal{A} *_N \mathcal{X}\| \leq \|\mathcal{A}\|\|\mathcal{X}\|$.
(iii) $\|\mathcal{A} *_N \mathcal{B}\| \leq \|\mathcal{A}\|\|\mathcal{B}\|$, where $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$ and $\mathcal{B} \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times I_2\times\cdots\times I_L}$.
(iv) $\|\mathcal{A}\| = \|\mathcal{A}^H\|$.
(v) $\|\mathcal{A}^H *_N \mathcal{A}\| = \|\mathcal{A}\|^2$.

Now, for tensors $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$ and $\mathcal{B} \in \mathbb{C}^{J_1\times\cdots\times J_N\times I_1\times\cdots\times I_M}$ with two Hermitian positive definite tensors $\mathcal{M} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ and $\mathcal{N} \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$, we define the weighted tensor norms as:

$$\|\mathcal{A}\|_{\mathcal{M}\mathcal{N}} = \sup\{\|\mathcal{A} *_N \mathcal{X}\|_{\mathcal{M}} \ : \ \|\mathcal{X}\|_{\mathcal{N}} = 1,\ \mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_N}\}, \tag{5.23}$$

and

$$\|\mathcal{B}\|_{\mathcal{N}\mathcal{M}} = \sup\{\|\mathcal{B} *_M \mathcal{X}\|_{\mathcal{N}} \ : \ \|\mathcal{X}\|_{\mathcal{M}} = 1,\ \mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_M}\}. \tag{5.24}$$

The following result provides a relation between the weighted tensor norm and the tensor norm.

LEMMA 5.4. *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$ and $\mathcal{B} \in \mathbb{C}^{J_1\times\cdots\times J_N\times I_1\times\cdots\times I_M}$. If $\mathcal{M} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ and $\mathcal{N} \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$ are two Hermitian positive definite tensors, then*

(i) $\|\mathcal{A}\|_{\mathcal{M}\mathcal{N}} = \|\mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{N}^{-1/2}\|$;
(ii) $\|\mathcal{B}\|_{\mathcal{N}\mathcal{M}} = \|\mathcal{N}^{1/2} *_N \mathcal{B} *_M \mathcal{M}^{-1/2}\|$.

Proof. From (5.23), we obtain

$$\begin{aligned}
\|\mathcal{A}\|_{\mathcal{M}\mathcal{N}} &= \sup\ \{\|\mathcal{A} *_N \mathcal{X}\|_{\mathcal{M}} \ : \ \|\mathcal{X}\|_{\mathcal{N}} = 1,\ \mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_N}\}\\
&= \sup\ \{\|\mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{X}\| \ : \ \|\mathcal{N}^{1/2} *_N \mathcal{X}\| = 1,\ \mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_N}\} \text{ (using Lemma 5.2)}\\
&= \sup\ \{\|\mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{N}^{-1/2} *_N \mathcal{N}^{1/2} *_N \mathcal{X}\| \ : \ \|\mathcal{N}^{1/2} *_N \mathcal{X}\| = 1,\ \mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_N}\}\\
&= \sup\ \{\|\mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{N}^{-1/2} *_N \mathcal{Y}\| \ : \ \|\mathcal{Y}\| = 1,\ \mathcal{Y} \in \mathbb{C}^{J_1\times\cdots\times J_N}\}\\
&= \|\mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{N}^{-1/2}\|.
\end{aligned}$$

Thus, the assertion (i) follows. The assertion (ii) can be proved similarly. ∎

The following lemma shows the consistent property of the weighted tensor norm.

LEMMA 5.5. *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$, $\mathcal{B} \in \mathbb{C}^{J_1\times\cdots\times J_N\times I_1\times\cdots\times I_M}$, $\mathcal{X} \in \mathbb{C}^{J_1\times\cdots\times J_N}$, and $\mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_M}$. If $\mathcal{M} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ and $\mathcal{N} \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$ are two Hermitian positive definite tensors, then*

(i) $\|\mathcal{A} *_N \mathcal{X}\|_{\mathcal{M}} \leq \|\mathcal{A}\|_{\mathcal{M}\mathcal{N}}\ \|\mathcal{X}\|_N$;
(ii) $\|\mathcal{B} *_M \mathcal{Y}\|_{\mathcal{N}} \leq \|\mathcal{B}\|_{\mathcal{N}\mathcal{M}}\ \|\mathcal{Y}\|_{\mathcal{M}}$;
(iii) $\|\mathcal{A} *_N \mathcal{B}\|_{\mathcal{M}\mathcal{M}} \leq \|\mathcal{A}\|_{\mathcal{M}\mathcal{N}}\ \|\mathcal{B}\|_{\mathcal{N}\mathcal{M}}$.

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Proof. (i)

$$\begin{aligned}\|\mathcal{A} *_N \mathcal{X}\|_{\mathcal{M}} &= \|\mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{X}\| \text{ (using Lemma 5.2(i))}\\ &= \|\mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{N}^{-1/2} *_N \mathcal{N}^{1/2} *_N \mathcal{X}\|\\ &\le \|\mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{N}^{-1/2}\| \, \|\mathcal{N}^{1/2} *_N \mathcal{X}\|\\ &= \|\mathcal{A}\|_{\mathcal{M}\mathcal{N}} \, \|\mathcal{X}\|_{\mathcal{N}} \text{ (using Lemma 5.4(i))}\end{aligned}$$

(ii) Similar to part (i).

(iii)

$$\begin{aligned}\|\mathcal{A} *_N \mathcal{B}\|_{\mathcal{M}\mathcal{M}} &= \|\mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{B} *_M \mathcal{M}^{-1/2}\| \text{ (using Lemma 5.4(i))}\\ &= \|\mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{N}^{-1/2} *_N \mathcal{N}^{1/2} *_N \mathcal{B} *_M \mathcal{M}^{-1/2}\|\\ &\le \|\mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{N}^{-1/2}\| \, \|\mathcal{N}^{1/2} *_N \mathcal{B} *_M \mathcal{M}^{-1/2}\| \text{ (using Lemma 5.4(i))}\\ &= \|\mathcal{A}\|_{\mathcal{M}\mathcal{N}} \, \|\mathcal{B}\|_{\mathcal{N}\mathcal{M}} \text{ (using Lemma 5.4).}\end{aligned}$$

The following lemma comprises some properties of the weighted conjugate transpose with the weighted tensor norm.

LEMMA 5.6. *Let* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$. *If* $\mathcal{M} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ *and* $\mathcal{N} \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$ *are two Hermitian positive definite tensors, then*

(i) $\|\mathcal{A}\|_{\mathcal{M}\mathcal{N}} = \|\mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}}\|_{\mathcal{N}\mathcal{M}}$;
(ii) $\|\mathcal{A}\|^2_{\mathcal{M}\mathcal{N}} = \|\mathcal{A} *_N \mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}}\|_{\mathcal{M}\mathcal{M}} = \|\mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}} *_M \mathcal{A}\|_{\mathcal{N}\mathcal{N}}$.

Proof. (i)

$$\begin{aligned}\|\mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}}\|_{\mathcal{N}\mathcal{M}} &= \|\mathcal{N}^{1/2} *_N \mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}} *_M \mathcal{M}^{-1/2}\| \text{ (using Lemma 5.4(ii))}\\ &= \|\mathcal{N}^{1/2} *_N \mathcal{N}^{-1} *_N \mathcal{A}^{H} *_M \mathcal{M} *_M \mathcal{M}^{-1/2}\|\\ &= \|(\mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{N}^{-1/2})^{H}\|\\ &= \|\mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{N}^{-1/2}\|\\ &= \|\mathcal{A}\|_{\mathcal{M}\mathcal{N}} \text{ (using Lemma 5.4(i)).}\end{aligned}$$

(ii)

$$\begin{aligned}\|\mathcal{A} *_N \mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}}\|_{\mathcal{M}\mathcal{M}} &= \|\mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}} *_M \mathcal{M}^{-1/2}\| \text{ (using Lemma 5.4)}\\ &= \|\mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{N}^{-1} *_N \mathcal{A}^{H} *_M \mathcal{M} *_M \mathcal{M}^{-1/2}\|\\ &= \|(\mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{N}^{-1/2}) *_N (\mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{N}^{-1/2})^{H}\|\\ &= \|\mathcal{M}^{1/2} *_M \mathcal{A} *_N \mathcal{N}^{-1/2}\|^2\\ &= \|\mathcal{A}\|^2_{\mathcal{M}\mathcal{N}} \text{ (using Lemma 5.4(i)).}\end{aligned}$$

The next result defines the weighted tensor norm as the maximum $(\mathcal{M},\mathcal{N})$ singular value of $\mathcal{A}$.

THEOREM 5.7. *Let* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_M\times J_1\times\cdots\times J_N}$. *If* $\mathcal{M} \in \mathbb{C}^{I_1\times\cdots\times I_M\times I_1\times\cdots\times I_M}$ *and* $\mathcal{N} \in \mathbb{C}^{J_1\times\cdots\times J_N\times J_1\times\cdots\times J_N}$ *are two Hermitian positive definite tensors, then*

$$\|\mathcal{A}\|_{\mathcal{M}\mathcal{N}} = \mu_{max} \text{ and } \|\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}\|_{\mathcal{N}\mathcal{M}} = \frac{1}{\mu_{min}},$$

where μ_{max} and μ_{min} *are the maximum and minimum* $(\mathcal{M},\mathcal{N})$ *singular values of* $\mathcal{A}$.

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Proof. By Theorems 3.1 and 3.3, we have $\mathcal{A}=\mathcal{U}*_M\mathcal{S}*_N\mathcal{V}^H$ and $\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}=\mathcal{N}^{-1}*_N\mathcal{V}*_N\mathcal{S}^{\dagger}*_M\mathcal{U}^H*_M\mathcal{M}$, where

$$\mathcal{S}_{i_1\cdots i_M j_1\cdots j_N}=\begin{cases}\mu_{IJ}>0, & \text{if } I=J\in\{1,\ 2,\ \ldots,\ r\},\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\mathcal{S}^{\dagger}_{j_1\cdots j_N i_1\cdots i_M}=\begin{cases}\mathcal{S}^{-1}_{i_1\cdots i_M j_1\cdots j_N}, & \text{if } \mathcal{S}_{i_1\cdots i_M j_1\cdots j_N}\neq 0,\\ 0, & \text{otherwise,}\end{cases}$$

where $I=i_1+\sum_{s=2}^{M}(i_s-1)\prod_{u=1}^{s-1}I_u$ and $J=j_1+\sum_{t=2}^{N}(j_t-1)\prod_{v=1}^{t-1}J_v$. Now,

$$\begin{aligned}\|\mathcal{A}\|^2_{\mathcal{M}\mathcal{N}}&=\|\mathcal{A}*_N\mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}}\|_{\mathcal{M}\mathcal{M}}\quad\text{(using Lemma 5.6 (ii))}\\&=\|\mathcal{U}*_M\mathcal{S}*_N\mathcal{V}^H*_N\mathcal{N}^{-1}*_N\mathcal{V}*_N\mathcal{S}^H*_M\mathcal{U}^H*_M\mathcal{M}\|_{\mathcal{M}\mathcal{M}}\\&=\|\mathcal{U}*_M\mathcal{S}*_N\mathcal{S}^H*_M\mathcal{U}^H*_M\mathcal{M}\|_{\mathcal{M}\mathcal{M}}\quad(\text{since } \mathcal{V}^H*_N\mathcal{N}^{-1}*_N\mathcal{V}=\mathcal{I}_2)\\&=\|\mathcal{M}^{1/2}*_M\mathcal{U}*_M\mathcal{S}*_N\mathcal{S}^H*_M\mathcal{U}^H*_M\mathcal{M}*_M\mathcal{M}^{-1/2}\|\ \text{(using Lemma 5.4 (i))}\\&=\|\mathcal{M}^{1/2}*_M\mathcal{U}*_M\mathcal{S}*_N\mathcal{S}^H*_M\mathcal{U}^H*_M\mathcal{M}^{1/2}\|\\&=\|(\mathcal{M}^{1/2}*_M\mathcal{U})*_M\mathcal{S}*_N\mathcal{S}^H*_M(\mathcal{M}^{1/2}*_M\mathcal{U})^H\|\\&=\mu^2_{max}\end{aligned}$$

Thus, $\|\mathcal{A}\|_{\mathcal{M}\mathcal{N}}=\mu_{max}$, since $\mathcal{M}^{1/2}*_M\mathcal{U}$ is unitary tensor and

$$(\mathcal{S}*_N\mathcal{S}^H)_{i_1\cdots i_M k_1\cdots k_M}=\begin{cases}{\mu_{IJ}}^2, & \text{if } I=K\in\{1,\ 2,\ \ldots,\ r\},\\ 0, & \text{otherwise,}\end{cases}$$

where $I=i_1+\sum_{s=2}^{M}(i_s-1)\prod_{u=1}^{s-1}I_u$ and $K=k_1+\sum_{t=2}^{M}(k_t-1)\prod_{v=1}^{t-1}K_v$. Since the nonzero $(\mathcal{M},\mathcal{N})$ singular values of $\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}$ are the reciprocals of the nonzero $(\mathcal{M},\mathcal{N})$ singular values of $\mathcal{A}$, therefore, we get $\|\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}\|_{\mathcal{N}\mathcal{M}}=\frac{1}{\mu_{min}}$. ∎

6. Numerical range for the weighted Moore-Penrose inverse of an even-order square tensor.

In this section, we establish several properties of the numerical ranges of a tensor and its weighted Moore-Penrose inverse. The first result conveys that the spectra of $\mathcal{A}$ and $\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}$ as well as their numerical ranges simultaneously contain the origin. Here, Definition 3.12, [33] is used for the determinant of a square tensor.

THEOREM 6.1. *Let $\mathcal{A}\in\mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. If $\mathcal{M},\mathcal{N}\in\mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ are two Hermitian positive definite tensors, then*

(i) $0\in\sigma(\mathcal{A})$ if and only if $0\in\sigma(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})$;
(ii) $0\in W(\mathcal{A})$ if and only if $0\in W(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})$.

Proof. (i) By the properties $\mathcal{A}*_N\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}*_N\mathcal{A}=\mathcal{A}$ and $\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}*_N\mathcal{A}*_N\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}=\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}$, we have that $\det^2(\mathcal{A})\det(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})=\det(\mathcal{A})$ and $\det(\mathcal{A})\det^2(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})=\det(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})$. Thus, if $\det(\mathcal{A})=0$, then $\det(\mathcal{A})\det^2(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})=\det(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})$ yields $\det(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})=0$ and if $\det(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})=0$, then $\det^2(\mathcal{A})\det(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})=\det(\mathcal{A})$ yields $\det(\mathcal{A})=0$. Hence, the assertion (i) follows.

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(ii) Here, two cases are possible.
Case 1: Suppose that $\mathcal{A}$ is singular. Then, $0 \in \sigma(\mathcal{A}) \subseteq W(\mathcal{A})$ and it is possible if and only if $0 \in \sigma(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}) \subseteq W(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})$.
Case 2: Suppose that $\mathcal{A}$ is non-singular. Then, $\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} = \mathcal{A}^{-1}$. Now, using (2.10), we have

$$W(\mathcal{A}) = \left\{ \frac{\langle \mathcal{A} *_N \mathcal{Y}, \mathcal{Y}\rangle}{\|\mathcal{Y}\|^2} : \mathcal{O} \neq \mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_N} \right\}, \tag{6.25}$$

$$\begin{aligned} W(\mathcal{A}^{-1}) &= \left\{ \frac{\langle \mathcal{A}^{-1} *_N \mathcal{X}, \mathcal{X}\rangle}{\|\mathcal{X}\|^2} : \mathcal{O} \neq \mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N} \right\} \\ &= \left\{ \frac{\langle \mathcal{A}^{-1} *_N \mathcal{A} *_N \mathcal{Y}, \mathcal{A} *_N \mathcal{Y}\rangle}{\|\mathcal{A} *_N \mathcal{Y}\|^2} : \mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N}, \mathcal{A} *_N \mathcal{Y} = \mathcal{X} \right\} \\ &= \left\{ \frac{\langle \mathcal{A}^{H} *_N \mathcal{A}^{-1} *_N \mathcal{A} *_N \mathcal{Y}, \mathcal{Y}\rangle}{\|\mathcal{A} *_N \mathcal{Y}\|^2} : \mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N}, \mathcal{A} *_N \mathcal{Y} = \mathcal{X} \right\} \\ &= \left\{ \frac{\langle \mathcal{A}^{H} *_N \mathcal{Y}, \mathcal{Y}\rangle}{\|\mathcal{A} *_N \mathcal{Y}\|^2} : \mathcal{O} \neq \mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_N} \right\}. \end{aligned} \tag{6.26}$$

If $0 \in W(\mathcal{A})$, then there exists a nonzero tensor $\mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ such that $\langle \mathcal{A} *_N \mathcal{Y}, \mathcal{Y}\rangle = 0$ by (6.25), which implies that $\langle \mathcal{A}^H *_N \mathcal{Y}, \mathcal{Y}\rangle = 0$. Thus, $0 \in W(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})$ due to (6.26). Conversely, if $0 \in W(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})$, then there exists a nonzero tensor $\mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ such that $\langle \mathcal{A}^H *_N \mathcal{Y}, \mathcal{Y}\rangle = 0$, which gives $\langle \mathcal{A} *_N \mathcal{Y}, \mathcal{Y}\rangle = 0$. Thus, $0 \in W(\mathcal{A})$. This completes the proof. ∎

Theorems 5.2(i) and 5.4 of [38] are immediate consequences of the above result. Also, the following corollary generalizes Theorem 2 in [37].

COROLLARY 6.2. *Let $\mathcal{A} \in \mathbb{C}^{n\times n}$. If $M, N \in \mathbb{C}^{n\times n}$ are two Hermitian positive definite matrices, then*

(i) $0 \in \sigma(A)$ if and only if $0 \in \sigma(A^{\dagger}_{M,N})$;
(ii) $0 \in W(A)$ if and only if $0 \in W(A^{\dagger}_{M,N})$.

As an application of Theorem 6.1, we have the following result.

THEOREM 6.3. *Let $\{z_{i_1 i_2\cdots i_N}\}_{i_j=1}^{I_j}$, for $j = 1, 2, ..., N$ be nonzero complex numbers. If*

$$0 = \sum_{i_1,i_2,\ldots,i_N} \alpha_{i_1 i_2\cdots i_N} z_{i_1 i_2\cdots i_N},$$

for some nonnegative scalars $\{\alpha_{i_1 i_2\cdots i_N}\}_{i_j=1}^{I_j}$ for $j = 1, 2, ..., N$ with $\sum_{i_1,i_2,\ldots,i_N} \alpha_{i_1 i_2\cdots i_N} = 1$, then there exist nonnegative scalars $\{\beta_{i_1 i_2\cdots i_N}\}_{i_j=1}^{I_j}$ for $j = 1, 2, ..., N$ with $\sum_{i_1,i_2,\ldots,i_N} \beta_{i_1 i_2\cdots i_N} = 1$ such that

$$0 = \sum_{i_1,i_2,\ldots,i_N} \beta_{i_1 i_2\cdots i_N} \frac{1}{z_{i_1 i_2\cdots i_N}}.$$

Proof. Consider the tensor $\mathcal{A} = (a_{i_1 i_2\cdots i_N j_1 j_2\cdots j_N}) \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$, where

$$a_{i_1 i_2\cdots i_N j_1 j_2\cdots j_N} = \begin{cases} z_{i_1 i_2\cdots i_N}, & \text{if } (i_1, i_2, ..., i_N) = (j_1, j_2, ..., j_N), \\ 0, & \text{otherwise.} \end{cases}$$

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Then, $W(\mathcal{A}) = \left\{ \sum\limits_{i_1,i_2,\dots,i_N} z_{i_1 i_2 \cdots i_N} |x_{i_1 i_2 \cdots i_N}|^2 : \sum\limits_{i_1,i_2,\dots,i_N} |x_{i_1 i_2 \cdots i_N}|^2 = 1 \right\}$, which is a convex polygon with vertices $\{z_{i_1 i_2 \cdots i_N}\}_{i_j=1}^{I_j}$. If $0 = \sum\limits_{i_1,i_2,\dots,i_N} \alpha_{i_1 i_2 \cdots i_N} z_{i_1 i_2 \cdots i_N}$ with $\sum\limits_{i_1,i_2,\dots,i_N} \alpha_{i_1 i_2 \cdots i_N} = 1$, then $0 \in W(\mathcal{A})$. So, by Theorem 6.1(ii), $0 \in W(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})$, where $\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} = (b_{i_1 i_2 \cdots i_N j_1 j_2 \cdots j_N}) \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ such that

$$b_{i_1 i_2 \cdots i_N j_1 j_2 \cdots j_N} = \begin{cases} \frac{1}{z_{i_1 i_2 \cdots i_N}}, & \text{if } (i_1, i_2, \dots, i_N) = (j_1, j_2, \dots, j_N), \\ 0, & \text{otherwise}, \end{cases}$$

a convex polygon with vertices $\frac{1}{z_{i_1 i_2 \cdots i_N}}$. Therefore, there exist nonnegative scalars $\{\beta_{i_1 i_2 \cdots i_N}\}_{i_j=1}^{I_j}$ with $\sum\limits_{i_1,i_2,\dots,i_N} \beta_{i_1 i_2 \cdots i_N} = 1$ such that $0 = \sum\limits_{i_1,i_2,\dots,i_N} \beta_{i_1 i_2 \cdots i_N} \dfrac{1}{z_{i_1 i_2 \cdots i_N}}$. ∎

Next theorem establishes a relation among $\sigma(\mathcal{A})$, $W(\mathcal{A})$, and $\dfrac{1}{W(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})}$. Here,

$$\frac{1}{W(\mathcal{A})} := \{z^{\dagger} : \ z \in W(\mathcal{A})\}, \ \text{where } z^{\dagger} = \begin{cases} \frac{1}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases} \text{(see page 43, [1]).}$$

THEOREM 6.4. *Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$, and $\mathcal{M}, \mathcal{N} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ be two Hermitian positive definite tensors. If $\mathcal{A} *_N \mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} = \mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} *_N \mathcal{A}$, then*

$$\sigma(\mathcal{A}) \subset W(\mathcal{A}) \bigcap \frac{1}{W(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})}. \tag{6.27}$$

Proof. Let $\lambda \in \sigma(\mathcal{A})$. It is well-known that $\sigma(\mathcal{A}) \subseteq W(\mathcal{A})$. If $\lambda = 0$, then $0 \in W(\mathcal{A})$ and by Theorem 6.1(ii), $0 \in W(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})$. Thus, the inclusion in (6.27) holds. If $\lambda \neq 0$, then there exists a unit eigentensor $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ such that

$$\mathcal{A} *_N \mathcal{X} = \lambda \mathcal{X}, \tag{6.28}$$

which implies that

$$\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} *_N \mathcal{X} = \frac{1}{\lambda} \mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} *_N \mathcal{A} *_N \mathcal{X}. \tag{6.29}$$

Using (6.28) and the property $\mathcal{A} *_N \mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} *_N \mathcal{A} = \mathcal{A}$, we obtain

$$\mathcal{A} *_N \mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} *_N \mathcal{X} = \mathcal{X}. \tag{6.30}$$

Using (6.29), (6.30), and the condition $\mathcal{A} *_N \mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} = \mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} *_N \mathcal{A}$, we conclude

$$\begin{aligned} \mathcal{X}^H *_N \mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} *_N \mathcal{X} &= \frac{1}{\lambda} \mathcal{X}^H *_N \mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} *_N \mathcal{A} *_N \mathcal{X} \\ &= \frac{1}{\lambda} \mathcal{X}^H *_N \mathcal{A} *_N \mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} *_N \mathcal{X} \\ &= \frac{1}{\lambda} \mathcal{X}^H *_N \mathcal{X} = \frac{1}{\lambda}. \end{aligned}$$

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Thus,

$$\lambda = \frac{1}{\mathcal{X}^H *_N \mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} *_N \mathcal{X}} \in \frac{1}{W(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})}.$$

Hence proved. ∎

Our next example shows that the assumption $\mathcal{A} *_N \mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} = \mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} *_N \mathcal{A}$ in Theorem 6.4 is essential.

Example 6.5. *Let* $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}^{2\times 2}$. *If* $M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $N = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ *are two Hermitian positive definite matrices in* $\mathbb{C}^{2\times 2}$, *then* $A^{\dagger}_{M,N} = \begin{pmatrix} \frac{1}{4} & 0 \\ \frac{3}{4} & 0 \end{pmatrix}$, $AA^{\dagger}_{M,N} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $A^{\dagger}_{M,N}A = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{3}{4} \end{pmatrix}$. *So,* $AA^{\dagger}_{M,N} \neq A^{\dagger}_{M,N}A$. *Now,*

$$W(A^{\dagger}_{M,N}) = \left\{ \frac{1}{4}\bar{z}_1 z_1 + \frac{3}{4}\bar{z}_2 z_1 : \; z = (z_1, z_2)^T \in \mathbb{C}^2, \; |z_1|^2 + |z_2|^2 = 1 \right\},$$
$$\frac{1}{W(A^{\dagger}_{M,N})} = \left\{ z^{\dagger} : \; z \in W(A^{\dagger}_{M,N}) \right\}.$$

Let $0 \neq \alpha \in W(A^{\dagger}_{M,N})$. *Then,*

$$\begin{aligned} \alpha &= \frac{1}{4}\bar{z}_1 z_1 + \frac{3}{4}\bar{z}_2 z_1, \quad \text{for some } z = (z_1, z_2)^T \in \mathbb{C}^2, \; |z_1|^2 + |z_2|^2 = 1, \\ |\alpha| &\leq \frac{1}{4}|z_1|^2 + \frac{3}{4}|z_2||z_1| \\ &\leq \frac{1}{4}(|z_1|^2 + |z_2|^2) + \frac{3}{4}\left(\frac{|z_1|^2 + |z_2|^2}{2}\right) = \frac{1}{4} \times 1 + \frac{3}{8} \times 1 = \frac{5}{8} < 1. \end{aligned}$$

For any $0 \neq \beta \in \frac{1}{W(A^{\dagger}_{M,N})}$, *we now have* $\beta = \frac{1}{\alpha}$ for some $0 \neq \alpha \in W(A^{\dagger}_{M,N})$. *So,* $|\beta| > 1$. *Thus, the eigenvalue* 1 *of* A *is not in* $\frac{1}{W(A^{\dagger}_{M,N})}$.

If $\mathcal{M}$ and $\mathcal{N}$ are identity tensors, then the above theorem coincides with Theorem 5.14, [38]. The following result generalizes Theorem 5, [37].

Corollary 6.6. *Let* $A \in \mathbb{C}^{n\times n}$ *be a square matrix, and* $M, N \in \mathbb{C}^{n\times n}$ *be two Hermitian positive definite matrices. If* A *is weighted EP-matrix, i.e.,* $AA^{\dagger}_{M,N} = A^{\dagger}_{M,N}A$, *then*

$$\sigma(A) \subset W(A) \bigcap \frac{1}{W(A^{\dagger}_{M,N})}.$$

The next theorem is an application of the WSVD.

Theorem 6.7. *Let* $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ *such that* $W(\mathcal{A}) = W(\mathcal{A}^H)$. *If*

$\mathcal{M} = \mathcal{N} = \beta\mathcal{I} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ *are two Hermitian positive definite tensors, then*

$$W(\mathcal{A}) \bigcap \mu^2 W(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}) \neq \emptyset,$$

for every $(\mathcal{M}, \mathcal{N})$ *singular value* μ *of* $\mathcal{A}$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Proof. Let $\mathcal{A} = \mathcal{U} *_N \mathcal{S} *_N \mathcal{V}^H$ be the WSVD of $\mathcal{A}$, where $\mathcal{S} = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_n)$, $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq 0$ and $n = I_1 \cdot I_2 \cdot \cdots \cdot I_N$. If $\mu = 0$, then $\mathcal{A}$ is singular. Hence, $0 \in \sigma(\mathcal{A}) \subseteq W(\mathcal{A})$. Then, by Theorem 6.1(ii), we have $0 \in W(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})$. Thus, $W(\mathcal{A}) \bigcap \mu^2 W(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}) \neq \emptyset$.
Now, let $\mu \neq 0$. Assume that $\mu = \mu_1$. Since μ is an $(\mathcal{M},\mathcal{N})$ singular value of $\mathcal{A}$, therefore, 1 is an $(\mathcal{M},\mathcal{N})$ singular value of $\mathcal{A}/\mu$. Since $\mathcal{V}^H *_N \mathcal{N}^{-1} *_N \mathcal{V} = \mathcal{I}$, so we can choose a unit tensor $\mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N}$ such that $\mathcal{V}^H *_N \mathcal{X} \in \mathbb{C}^{I_1\times\cdots\times I_N}$, whose all entries are zero except $(\mathcal{V}^H *_N \mathcal{X})_{11\ldots1}$.

$$\begin{aligned}
\mathcal{X}^H *_N (\mathcal{A}/\mu) *_N \mathcal{X} &= \mathcal{X}^H *_N \mathcal{U} *_N (\mathcal{S}/\mu) *_N \mathcal{V}^H *_N \mathcal{X} \\
&= (\mathcal{U}^H *_N \mathcal{X})^H *_N (\mathcal{S}/\mu) *_N (\mathcal{V}^H *_N \mathcal{X}) \\
&= \overline{(\mathcal{U}^H *_N \mathcal{X})}_{11\ldots1} (\mathcal{V}^H *_N \mathcal{X})_{11\ldots1},
\end{aligned}$$

which implies that $\overline{(\mathcal{U}^H *_N \mathcal{X})}_{11\ldots1}\, (\mathcal{V}^H *_N \mathcal{X})_{11\ldots1} \in W(\mathcal{A}/\mu)$. Since $W(\mathcal{A}) = W(\mathcal{A}^H)$, therefore, we have $\overline{(\mathcal{V}^H *_N \mathcal{X})}_{11\ldots1}\, (\mathcal{U}^H *_N \mathcal{X})_{11\ldots1} \in W(\mathcal{A}/\mu)$. Now,

$$\begin{aligned}
\mathcal{X}^H *_N (\mu\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}) *_N \mathcal{X} &= \mathcal{X}^H *_N \mathcal{N}^{-1} *_N \mathcal{V} *_N (\mu\mathcal{S}^{\dagger}) *_N \mathcal{U}^H *_N \mathcal{M} *_N \mathcal{X} \\
&= \mathcal{X}^H *_N (\mathcal{I}/\beta) *_N \mathcal{V} *_N (\mu\mathcal{S}^{\dagger}) *_N \mathcal{U}^H *_N (\beta\mathcal{I}) *_N \mathcal{X} \\
&= \mathcal{X}^H *_N \mathcal{V} *_N (\mu\mathcal{S}^{\dagger}) *_N \mathcal{U}^H *_N \mathcal{X} \\
&= (\mathcal{V}^H *_N \mathcal{X})^H *_N (\mu\mathcal{S}^{\dagger}) *_N (\mathcal{U}^H *_N \mathcal{X}) \\
&= \overline{(\mathcal{V}^H *_N \mathcal{X})}_{11\ldots1}\, (\mathcal{U}^H *_N \mathcal{X})_{11\ldots1},
\end{aligned}$$

which implies that $\overline{(\mathcal{V}^H *_N \mathcal{X})}_{11\ldots1}\, (\mathcal{U}^H *_N \mathcal{X})_{11\ldots1} \in W(\mu\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}})$. Hence,

$$W(\mathcal{A}/\mu) \bigcap W(\mu\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}) \neq \emptyset,$$

i.e.,

$$W(\mathcal{A}) \bigcap \mu^2 W(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}) \neq \emptyset.$$

Similarly, we can prove $W(\mathcal{A}) \bigcap \mu^2 W(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}) \neq \emptyset$ for the other $(\mathcal{M},\mathcal{N})$ singular values of $\mathcal{A}$. ∎

Theorem 5.5 [38] is a particular case of the above theorem. It can be verified that Corollary 6.8 generalizes Theorem 4 in [37].

COROLLARY 6.8. *Let $A \in \mathbb{C}^{n\times n}$ such that $W(A)$ is symmetric with respect to x-axis, i.e., $W(A) = W(A^*)$. If $M = N = \beta I \in \mathbb{C}^{n\times n}$ are two Hermitian positive definite matrices, then*

$$W(A) \bigcap \mu^2 W(A^{\dagger}_{M,N}) \neq \emptyset,$$

for every (M,N) singular value μ of A.

In the following result, we derive that the numerical ranges for the weighted Moore-Penrose inverse and the weighted conjugate transpose are equal for a particular type of tensor.

THEOREM 6.9. *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$, and $\mathcal{M},\mathcal{N} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ be two Hermitian positive definite tensors. Let $\{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_r\}$ be an $\mathcal{M}$-orthonormal and $\{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_r\}$ be an $\mathcal{N}^{-1}$-orthonormal subsets of $\mathbb{C}^{I_1\times\cdots\times I_N}$. If $\mathcal{A} = \mathcal{U}_1 *_1 \mathcal{V}_1^H + \mathcal{U}_2 *_1 \mathcal{V}_2^H + \cdots + \mathcal{U}_r *_1 \mathcal{V}_r^H$, then $\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}} = \mathcal{N}^{-1} *_N (\mathcal{V}_1 *_1 \mathcal{U}_1^H + \mathcal{V}_2 *_1 \mathcal{U}_2^H + \cdots + \mathcal{V}_r *_1 \mathcal{U}_r^H) *_N \mathcal{M}$ and $W(\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}) = W(\mathcal{A}^{\#}_{\mathcal{M}\mathcal{N}})$.*

Proof. It can be verified by considering $\mathcal{X} = \mathcal{N}^{-1} *_N (\mathcal{V}_1 *_1 \mathcal{U}_1^H + \cdots + \mathcal{V}_r *_1 \mathcal{U}_r^H) *_N \mathcal{M}$ and then using Definition 1.6. ∎

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The above result reduces to the following corollary when we consider identity tensor as weights.

Corollary 6.10. *Let $\mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. Let $\{\mathcal{U}_1,\mathcal{U}_2,\ldots,\mathcal{U}_r\}$ and $\{\mathcal{V}_1,\mathcal{V}_2,\ldots,\mathcal{V}_r\}$ be two orthonormal subsets of $\mathbb{C}^{I_1\times\cdots\times I_N}$. If $\mathcal{A} = \mathcal{U}_1 *_1 \mathcal{V}_1^H + \mathcal{U}_2 *_1 \mathcal{V}_2^H + \cdots + \mathcal{U}_r *_1 \mathcal{V}_r^H$, then $\mathcal{A}^\dagger = \mathcal{V}_1 *_1 \mathcal{U}_1^H + \mathcal{V}_2 *_1 \mathcal{U}_2^H + \cdots + \mathcal{V}_r *_1 \mathcal{U}_r^H$ and $W(\mathcal{A}^\dagger) = W(\mathcal{A}^H)$.*

The following corollary is a generalization of Theorem 6 in [37]. We provide its proof as it is different than the Theorem 6.9.

Corollary 6.11. *Let $A \in \mathbb{C}^{n\times n}$, and $M, N \in \mathbb{C}^{n\times n}$ be two Hermitian positive definite matrices. Let $\{u_1, u_2, \ldots, u_r\}$ be a subset of M-orthonormal vectors and $\{v_1, v_2, \ldots, v_r\}$ be a subset of N^{-1}-orthonormal vectors of $\mathbb{C}^n$, respectively. If $A = u_1v_1^* + u_2v_2^* + \cdots + u_rv_r^*$, then $A^\dagger_{M,N} = N^{-1}(v_1^*u_1 + v_2^*u_2 + \cdots + v_r^*u_r)M$ and $W(A^\dagger_{M,N}) = W(A^\#)$, where $A^\#$ is the weighted conjugate transpose of the matrix A.*

Proof. We have that $\{u_1, u_2, \ldots, u_r\}$ is a subset of M-orthonormal vectors and $\{v_1, v_2, \ldots, v_r\}$ is a subset of N^{-1}-orthonormal vectors of $\mathbb{C}^n$. So, $\{M^{1/2}u_1, M^{1/2}u_2, \ldots, M^{1/2}u_r\}$ and $\{N^{-1/2}v_1, N^{-1/2}v_2, \ldots, N^{-1/2}v_r\}$ are subsets of orthonormal vectors of $\mathbb{C}^n$. Extend $\{M^{1/2}u_1, M^{1/2}u_2, \ldots, M^{1/2}u_r\}$ and $\{N^{-1/2}v_1, N^{-1/2}v_2, \ldots, N^{-1/2}v_r\}$ to orthonormal bases $\{M^{1/2}u_1, M^{1/2}u_2, \ldots, M^{1/2}u_r, \ldots, M^{1/2}u_n\}$ and $\{N^{-1/2}v_1, N^{-1/2}v_2, \ldots, N^{-1/2}v_r, \ldots, N^{-1/2}v_n\}$ of $\mathbb{C}^n$, respectively. Let $U = [u_1\ u_2\ \cdots\ u_n]$ and $V = [v_1\ v_2\ \cdots\ v_n]$. Then, $U^*MU = I$, $V^*N^{-1}V = I$, and $A = u_1v_1^* + u_2v_2^* + \cdots + u_rv_r^* = U(I_r \oplus 0_{n-r})V^*$. Thus, by Corollary 3.6, we have

$$\begin{aligned} A^\dagger_{M,N} &= N^{-1}V(I_r \oplus 0_{n-r})^\dagger U^*M \\ &= [N^{-1}v_1\ N^{-1}v_2\ \cdots\ N^{-1}v_n](I_r \oplus 0_{n-r})[Mu_1\ Mu_2\ \cdots\ Mu_n]^* \\ &= N^{-1}(v_1u_1^* + v_2u_2^* + \cdots + v_ru_r^*)M = N^{-1}A^*M = A^\#. \end{aligned}$$

Hence, $W(A^\dagger_{M,N}) = W(A^\#)$. ∎

As an application of the weighted tensor norm, the following result provides a bound for the product of the weighted tensor norms of a tensor $\mathcal{A}$ and its weighted Moore-Penrose inverse $\mathcal{A}^\dagger_{\mathcal{M},\mathcal{N}}$ in terms of the product of numerical radii of the tensor $\tilde{\mathcal{A}}$ and its Moore-Penrose inverse $\tilde{\mathcal{A}}^\dagger$.

Theorem 6.12. *Let $\mathcal{O} \neq \mathcal{A} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$. If $\mathcal{M},\mathcal{N} \in \mathbb{C}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ are two Hermitian positive definite tensors, then for the weighted tensor norm $\|\mathcal{A}\|_{\mathcal{M}\mathcal{N}}$,*

$$1 \leq \|\mathcal{A}\|_{\mathcal{M}\mathcal{N}}\|\mathcal{A}^\dagger_{\mathcal{M},\mathcal{N}}\|_{\mathcal{N}\mathcal{M}} \leq 4w(\tilde{\mathcal{A}})w(\tilde{\mathcal{A}}^\dagger),$$

where $\tilde{\mathcal{A}} = \mathcal{M}^{1/2} *_N \mathcal{A} *_N \mathcal{N}^{-1/2}$ and $\|.\|$ is the spectral norm of a tensor.

Proof. From Lemma 5.4(i), we have $\|\mathcal{A}\|_{\mathcal{M}\mathcal{N}} = \|\tilde{\mathcal{A}}\|$. Now,

$$\|\mathcal{A}^\dagger_{\mathcal{M},\mathcal{N}}\|_{\mathcal{N}\mathcal{M}} = \|\mathcal{N}^{1/2} *_N \mathcal{A}^\dagger_{\mathcal{M},\mathcal{N}} *_N \mathcal{M}^{-1/2}\| = \|\mathcal{N}^{1/2} *_N \mathcal{N}^{-1/2} *_N \tilde{\mathcal{A}}^\dagger *_N \mathcal{M}^{1/2} *_N \mathcal{M}^{-1/2}\| = \|\tilde{\mathcal{A}}^\dagger\|.$$

From Theorem 5.16 of [38] for the norm $\|.\|$, we have

$$1 \leq \|\tilde{\mathcal{A}}\|\|\tilde{\mathcal{A}}^\dagger\| \leq 4w(\tilde{\mathcal{A}})w(\tilde{\mathcal{A}}^\dagger).$$

Therefore, we obtain

$$1 \leq \|\mathcal{A}\|_{\mathcal{M}\mathcal{N}}\|\mathcal{A}^\dagger_{\mathcal{M},\mathcal{N}}\|_{\mathcal{N}\mathcal{M}} \leq 4w(\tilde{\mathcal{A}})w(\tilde{\mathcal{A}}^\dagger),$$

this completes the proof. ∎

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In particular, the identity weights give Theorem 5.16, [38]. The next corollary is a generalization of Theorem 7, [37].

COROLLARY 6.13. *Let $0 \neq A \in \mathbb{C}^{n\times n}$, and $M, N \in \mathbb{C}^{n\times n}$ be two Hermitian positive definite matrices. Then, for the weighted matrix norm $\|A\|_{MN} = \|M^{1/2}AN^{-1/2}\|$,*

$$1 \leq \|A\|_{MN}\|A^{\dagger}_{M,N}\|_{NM} \leq 4\omega(\tilde{A})\omega(\tilde{A}^{\dagger}),$$

where $\tilde{A} = M^{1/2}AN^{-1/2}$, $A^{\dagger}_{M,N} = N^{-1/2}\tilde{A}^{\dagger}M^{1/2}$ and $\|.\|$ is the spectral norm.

With respect to the diagonal weights, the weighted Moore-Penrose inverse of a weighted shift matrix is again a weighted shift matrix; this is shown in the following theorem. Also, for their numerical radii, some upper bounds are established.

THEOREM 6.14. *Let $A \in \mathbb{C}^{n\times n}$ be a weighted shift matrix*

$$A = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & a_{n-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{6.31}$$

If $M, N \in \mathbb{C}^{n\times n}$ are two positive diagonal matrices, then

$$A^{\dagger}_{M,N} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1/a_1 & 0 & \cdots & 0 & 0 \\ 0 & 1/a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1/a_{n-1} & 0 \end{pmatrix}. \tag{6.32}$$

Furthermore,

(i) $W(A)$, $W(A^{\dagger}_{M,N})$ are circular disks centered at the origin, and

$$\omega(A)\ \omega(A^{\dagger}_{M,N}) \leq \frac{\max|a_k|}{\min|a_k|}\cos^2\left(\frac{\pi}{n+1}\right),$$

where minimum is taken over those k with $a_k \neq 0$.

(ii) If $a_k a_{n-k} = 1$ for all $k = 1, 2, \ldots, [n/2]$, then $W(A) = W(A^{\dagger}_{M,N})$, and

$$\omega(A) = \omega(A^{\dagger}_{M,N}) \leq \max\{|a_k|, 1/|a_k|\}\cos\left(\frac{\pi}{n+1}\right).$$

Proof. By the definition of the weighted Moore-Penrose inverse of a matrix A, it is easy to compute the representation of $A^{\dagger}_{M,N}$ in (6.32). By Theorem 3 of [36], $W(A)$ and $W(A^{\dagger}_{M,N})$ are circular disks centered at origin. Again by the same theorem, we have

$$\omega(A) \leq \max\{|a_k|\}\cos\left(\frac{\pi}{n+1}\right), \tag{6.33}$$

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and

$$\omega(A^{\dagger}_{M,N}) \leq \max\{1/|a_k|\} \cos\left(\frac{\pi}{n+1}\right) = \frac{1}{\min\{|a_k|\}} \cos\left(\frac{\pi}{n+1}\right). \tag{6.34}$$

Therefore, from (6.33) and (6.34), the assertion (i) follows.
If $a_k a_{n-k} = 1$ for all $k = 1, 2, \ldots, [n/2]$, then $A^{\dagger}_{M,N} = P^* A P$, where

$$P = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Thus, $W(A) = W(A^{\dagger}_{M,N})$, since P is unitary. Let $a = \max\{|a_k|, 1/|a_k|\}$, then $1/a = \min\{|a_k|, 1/|a_k|\}$. Therefore, the numerical radius inequality (ii) follows from (i). □

Instead of diagonal matrices, if we take M and N as identity matrices, then Theorem 6.14 coincides with Theorem 8, [37].

We end this section with an example in which we plot the numerical ranges of a tensor, and its Moore-Penrose inverse and weighted Moore-Penrose inverse using Algorithm 3.1 of [38]. To compute the Moore-Penrose inverse and the weighted Moore-Penrose inverse, we apply Algorithms 1 and 2 here.

EXAMPLE 6.15. *Consider $\mathcal{A} \in \mathbb{C}^{2\times3\times2\times3}$ and the two weights $\mathcal{M}, \mathcal{N}$ in $\mathbb{C}^{2\times3\times2\times3}$ such that*

$\mathcal{A}(:,:,1,1)$			$\mathcal{A}(:,:,2,1)$			$\mathcal{A}(:,:,1,2)$			$\mathcal{A}(:,:,2,2)$			$\mathcal{A}(:,:,1,3)$			$\mathcal{A}(:,:,2,3)$		
1	1	2	1	1	1	2	2	1	3	3	2	1	1	2	3	3	3
1	1	2	2	2	1	2	2	1	4	4	2	1	1	2	3	3	3

,

$\mathcal{M}(:,:,1,1)$			$\mathcal{M}(:,:,2,1)$			$\mathcal{M}(:,:,1,2)$			$\mathcal{M}(:,:,2,2)$			$\mathcal{M}(:,:,1,3)$			$\mathcal{M}(:,:,2,3)$		
1	0	0	0	0	0	0	3	0	0	0	0	0	0	2	0	0	0
0	0	0	2	0	0	0	0	0	0	1	0	0	0	0	0	0	3

,

and

$\mathcal{N}(:,:,1,1)$			$\mathcal{N}(:,:,2,1)$			$\mathcal{N}(:,:,1,2)$			$\mathcal{N}(:,:,2,2)$			$\mathcal{N}(:,:,1,3)$			$\mathcal{N}(:,:,2,3)$		
3	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0
0	0	0	2	0	0	0	0	0	0	1	0	0	0	0	0	0	1

.

By Algorithms 1 and 2, the Moore-Penrose inverse and the weighted Moore-Penrose inverse of $\mathcal{A}$ are given as

$\mathcal{A}^{\dagger}(:,:,1,1)$			$\mathcal{A}^{\dagger}(:,:,2,1)$			$\mathcal{A}^{\dagger}(:,:,1,2)$		
-3/26	9/26	-3/26	0	-1/6	0	-3/26	9/26	-3/26
-11/26	-1/13	3/13	1/3	1/6	-1/6	-11/26	-1/13	3/13

$\mathcal{A}^{\dagger}(:,:,2,2)$			$\mathcal{A}^{\dagger}(:,:,1,3)$			$\mathcal{A}^{\dagger}(:,:,2,3)$		
0	-1/6	0	2/13	-5/39	2/13	2/13	-5/39	2/13
1/3	1/6	-1/6	5/78	-5/78	1/39	5/78	-5/78	1/39

and

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$\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}(:,:,1,1)$			$\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}(:,:,2,1)$			$\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}(:,:,1,2)$		
-1/32	7/32	-3/32	1/90	-3/10	1/30	-3/32	21/32	-9/32
-5/32	-3/32	1/8	29/90	31/90	-4/15	-15/32	-9/32	3/8

$\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}(:,:,2,2)$			$\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}(:,:,1,3)$			$\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}(:,:,2,3)$		
1/180	-3/20	1/60	17/300	-13/100	17/100	17/200	-39/200	51/200
29/180	31/180	-2/15	13/300	-13/300	1/25	13/200	-13/200	3/50

,

respectively. Now, applying Algorithm 3.1 of [38] to the tensors $\mathcal{A}$, $\mathcal{A}^{\dagger}$, and $\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}$ for 500 different choices of θ, we obtain Fig. 1, and the colored doted points inside the plotted region represent the eigenvalues of the corresponding tensor.

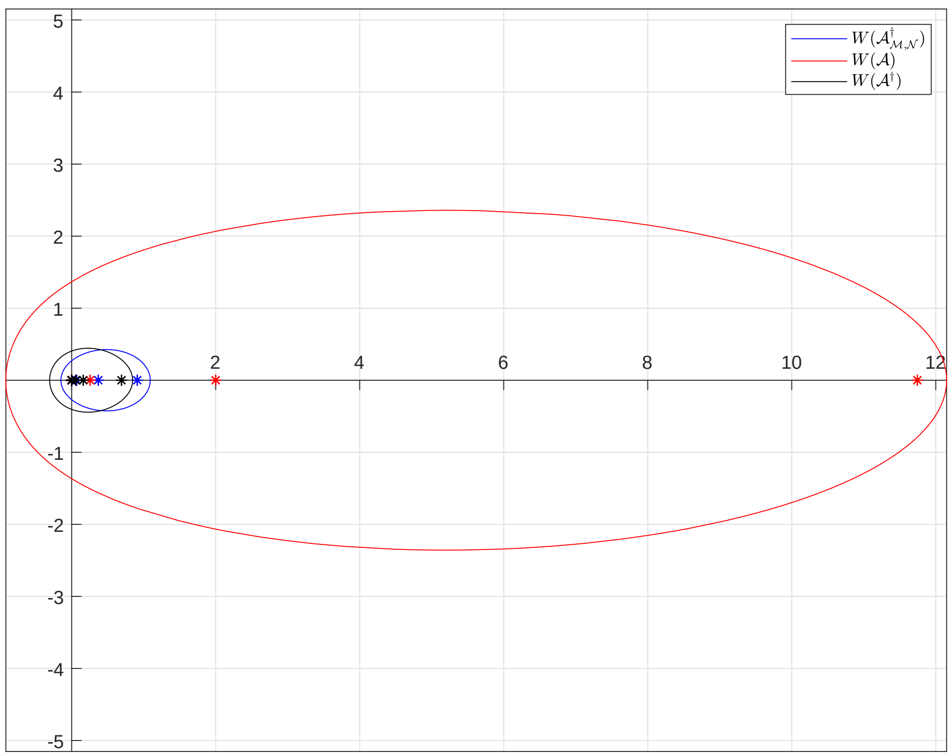

Figure 1: Numerical ranges of the tensors $\mathcal{A}$, $\mathcal{A}^{\dagger}$, and $\mathcal{A}^{\dagger}_{\mathcal{M},\mathcal{N}}$.

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7. Conclusions. In this article, we have introduced the notion of the WSVD and derived the formula for computing the weighted Moore-Penrose inverse of an arbitrary-order tensor using the WSVD. After that, we have defined the notions of weighted normal tensor and weighted tensor norm. Further, we have established several properties that examine some relationship between a tensor's numerical range and its weighted Moore-Penrose inverse. An upper bound for the product of the numerical radii of a weighted shift matrix and its weighted Moore-Penrose inverse with diagonal weights has been established. An equality between the numerical ranges of the weighted Moore-Penrose inverse and the weighted conjugate transpose for a special tensor has been given. Our work on numerical ranges and numerical radii will also be beneficial in finding the iterative solution to tensor equations. These theories add new contributions to the theory of tensors and will be crucial for future research on tensors.

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Conflict of interest. The authors declare that there is no conflict of interest.

Data availability statement. Data sharing is not applicable to this article as no new data is analyzed in this study.

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