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WEAK LOG-MAJORIZATION AND INEQUALITIES OF POWER MEANS*

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Abstract. As noncommutative versions of the quasi-arithmetic mean, we consider the Lim–Pálfia's power mean, Rényi right mean, and Rényi power means. We prove that the Lim–Pálfia's power mean of order $t \in [-1, 0)$ is weakly log-majorized by the log-Euclidean mean and fulfills the Ando–Hiai inequality. We establish the log-majorization relationship between the Rényi relative entropy and the product of square roots of given variables. Furthermore, we show the norm inequalities among power means and provide the boundedness of Rényi power mean in terms of the quasi-arithmetic mean.

Key words. Log-majorization, Cartan mean, Log-Euclidean mean, Lim–Pálfia's power mean, Rényi right mean, Rényi power mean.

AMS subject classifications. 15A45, 15B48.

1. Introduction. Throughout the paper, $\mathbb{C}_{m\times m}$ is the set of all $m \times m$ complex matrices, $\mathbb{H}_m$ is the real vector space of $m \times m$ Hermitian matrices, and $\mathbb{P}_m \subset \mathbb{H}_m$ is the open convex cone of $m \times m$ positive definite matrices. For $A, B \in \mathbb{H}_m$, the Loewner order $A \geq (>)B$ means that $A - B$ is positive semi-definite (resp. positive definite). We denote by $s(X)$ the m-tuple of all singular values of a complex matrix X, and denote by $\lambda(X)$ the m-tuple of all eigenvalues of a Hermitian matrix X in decreasing order: $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_m(X)$.

Let x, y be two m-tuples of positive real numbers. We denote by $x^\downarrow = (x_1^\downarrow, \dots, x_m^\downarrow)$ the rearrangement of x in decreasing order. The notation $x \prec_{\log} y$ represents that x is *log-majorized* by y, that is,

$$\prod_{i=1}^{k} x_i^\downarrow \leq \prod_{i=1}^{k} y_i^\downarrow, \tag{1.1}$$

for $1 \leq k \leq m-1$ and the equality holds for $k = m$. We say that x is *weakly log-majorized* by y, denoted by $x \prec_{w\log} y$, if (1.1) is true for $k = 1, 2, \dots, m$. For simplicity, given $A, B \in \mathbb{P}_m$, we write $A \prec_{\log} B$ if $\lambda(A) \prec_{\log} \lambda(B)$, and $A \prec_{w\log} B$ if $\lambda(A) \prec_{w\log} \lambda(B)$.

For given $A_1, \dots, A_n \in \mathbb{P}_m$, the *quasi-arithmetic mean* (*generalized* or *power mean*) of order $t(\neq 0) \in \mathbb{R}$ is defined by

$$\mathcal{Q}_t(\omega; A_1, \dots, A_n) := \left(\sum_{j=1}^{n} w_j A_j^t \right)^{\frac{1}{t}},$$

where $\omega = (w_1, \dots, w_n)$ is a positive probability vector. Note that

$$\lim_{t\to 0} \mathcal{Q}_t(\omega; A_1, \dots, A_n) = \exp\left(\sum_{j=1}^{n} w_j \log A_j \right),$$

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where the right-hand side is called the *log-Euclidean mean* of $A_1, \dots, A_n$. Log-majorization properties and operator inequalities of the quasi-arithmetic mean have been studied [7, 9, 23]. As noncommutative versions of the quasi-arithmetic mean, we investigate in this paper the Lim–Pálfia's power mean, Rényi right mean and Rényi power mean.

(I) The *Lim–Pálfia's power mean* $P_t(\omega; A_1, \dots, A_n)$ of order $t \in (0, 1]$ is defined as the unique positive definite solution of

$$X = \sum_{i=1}^{n} w_i (X \#_t A_i),$$

where $A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ is known as the weighted geometric mean of $A, B \in \mathbb{P}_m$. For $t \in [-1, 0)$ we define $P_t(\omega; A_1, \dots, A_n) = P_{-t}(\omega; A_1^{-1}, \dots, A_n^{-1})^{-1}$. See [22] for more information. We show in Section 3 that the sequence $P_t(\omega; A_1^p, \dots, A_n^p)^{1/p}$ for $t \in [-1, 0)$ is weakly log-majorized by the log-Euclidean mean for any $p > 0$:

$$P_t(\omega; A_1^p, \dots, A_n^p)^{1/p} \prec_{w \log} \exp\left(\sum_{j=1}^{n} w_j \log A_j\right),$$

and the power mean P_t satisfies the Ando–Hiai inequality: $P_t(\omega; A_1, \dots, A_n) \leq I$ implies $P_t(\omega; A_1^p, \dots, A_n^p)^{1/p} \leq I$. This provides an affirmative answer for the monotone convergence of Lim–Pálfia's power means in terms of the weak log-majorization, but it is an open question:

$$P_t(\omega; A_1^p, \dots, A_n^p)^{1/p} \nearrow_{\prec_{w \log}} \exp\left(\sum_{j=1}^{n} w_j \log A_j\right) \quad \text{as} \quad p \searrow 0.$$

Here, the above symbol $\nearrow_{\prec_{w \log}}$ means that Lim–Pálfia's power means satisfy the following properties: for $0 < p \leq q$

(i) $P_t(\omega; A_1^p, \dots, A_n^p)^{1/p} \prec_{w \log} \exp\left(\sum_{j=1}^{n} w_j \log A_j\right)$,
(ii) $\lim_{p \to 0+} P_t(\omega; A_1^p, \dots, A_n^p)^{1/p} = \exp\left(\sum_{j=1}^{n} w_j \log A_j\right)$,
(iii) $P_t(\omega; A_1^p, \dots, A_n^p)^{1/p} \succ_{w \log} P_t(\omega; A_1^q, \dots, A_n^q)^{1/q}$.

(II) Recently, a new barycenter minimizing the weighted sum of quantum divergences, called the *t-z Rényi right mean*, has been introduced in [10]. Indeed, for $0 < t \leq z < 1$

$$\Omega_{t,z}(\omega; A_1, \dots, A_n) := \underset{X \in \mathbb{P}_m}{\arg\min} \sum_{j=1}^{n} w_j \Phi_{t,z}(A_j, X),$$

where $\Phi_{t,z}(A, B) = \operatorname{tr}((1-t)A + tB) - \operatorname{tr}(A^{\frac{1-t}{2z}} B^{\frac{t}{z}} A^{\frac{1-t}{2z}})^z$ is the t-z Bures–Wasserstein quantum divergence of $A, B \in \mathbb{P}_m$. Here, $Q_{t,z}(A, B) = (A^{\frac{1-t}{2z}} B^{\frac{t}{z}} A^{\frac{1-t}{2z}})^z$ is known as the t-z Rényi relative entropy of A, B. The t-z Rényi right mean coincides with the unique positive definite solution of the equation

$$X = \sum_{j=1}^{n} w_j \left(X^{\frac{t}{2z}} A_j^{\frac{1-t}{z}} X^{\frac{t}{2z}}\right)^z,$$

which obtained by vanishing the gradient of objective function. For $t = z = 1/2$, the t-z Rényi right mean $\Omega_{t,z}$ coincides with the Wasserstein mean: see [1, 2, 8, 17] for more information. We show in Section 4 the

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log-majorization relationship between the t-z Rényi relative entropy $Q_{t,z}(A,B)$ and $A^{1/2}B^{1/2}$ and establish norm inequalities among the power means.

(III) Dumitru and Franco [12] have defined the *Rényi power mean* $\mathcal{R}_{t,z}(\omega; A_1, \dots, A_n)$ as the unique positive definite solution of the equation

$$X = \sum_{j=1}^{n} w_j \left(A_j^{\frac{1-t}{2z}} X^{\frac{t}{z}} A_j^{\frac{1-t}{2z}} \right)^z,$$

and proved the norm inequality between $\mathcal{R}_{t,z}$ and $\mathcal{Q}_{1-t}$ with respect to the p-norm for $p \geq 2$. Note that for commuting variables

$$\mathcal{R}_{t,z} = \Omega_{t,z} = P_{1-t} = \mathcal{Q}_{1-t}.$$

We show in Section 5 the boundedness of Rényi power mean $\mathcal{R}_{t,z}$ in terms of the quasi-arithmetic mean.

2. Antisymmetric tensor power and homogeneous matrix means. A crucial tool in the theory of log-majorization is the antisymmetric tensor power (or the compound matrix). Note that for $A \geq 0$ and $1 \leq k \leq m$,

$$\prod_{i=1}^{k} \lambda_i(A) = \lambda_1(\Lambda^k A), \tag{2.2}$$

where $\Lambda^k A$ denotes the kth antisymmetric tensor power of A. By the definition of log-majorization, $A \prec_{\log} B$ for $A, B > 0$ if and only if $\lambda_1(\Lambda^k A) \leq \lambda_1(\Lambda^k B)$ for $1 \leq k \leq m-1$, and $\det A = \det B$. We give a list of fundamental properties of the antisymmetric tensor powers by [6] and [14].

LEMMA 2.1. *Let $A, B \in \mathbb{P}_m$, and I the identity matrix with certain dimension.*

(1) $\Lambda^k(cI) = c^k I$ for any constant c
(2) $\Lambda^k(XY) = \Lambda^k(X)\Lambda^k(Y)$ for any $X, Y \in \mathbb{C}_{m\times m}$
(3) $(\Lambda^k(A))^r = \Lambda^k(A^r)$ for any $r \in \mathbb{R}$
(4) $\Lambda^k A \leq \Lambda^k B$ whenever $A \leq B$.

Another interesting property is that the weak log-majorization implies the weak majorization. More precisely, $A \prec_{w\log} B$ implies $A \prec_w B$, where $A \prec_w B$ means that

$$\sum_{i=1}^{k} \lambda_i(A) \leq \sum_{i=1}^{k} \lambda_i(B), \quad 1 \leq k \leq m.$$

Note that $A \prec_w B$ if and only if $|||A||| \leq |||B|||$ for any unitarily invariant norm $|||\cdot|||$. One can easily see from Lemma 2.1 (4) and (2.2) that $A \leq B$ for $A, B \in \mathbb{P}_m$ implies $A \prec_{w\log} B$, so $A \prec_w B$.

Let Δ_n be the simplex of all positive probability vectors in $\mathbb{R}^n$. A (multivariable) matrix mean on the open convex cone $\mathbb{P}_m$ is the map $G : \Delta_n \times \mathbb{P}_m^n \to \mathbb{P}_m$ satisfying the idempotency: $G(\omega; A, \dots, A) = A$ for any $\omega \in \Delta_n$ and $A \in \mathbb{P}_m$. The matrix mean is said to be homogeneous if $G(\omega; c\mathbb{A}) = cG(\omega; \mathbb{A})$ for any $c > 0$, where $\mathbb{A} = (A_1, \dots, A_n) \in \mathbb{P}_m^n$.

LEMMA 2.2. *Let $G_1, G_2 : \Delta_n \times \mathbb{P}_m^n \to \mathbb{P}_m$ be homogeneous matrix means satisfying*

$$G_2(\omega; \mathbb{A}) \leq I \quad \textit{implies} \quad G_1(\omega; \mathbb{A}) \leq I, \tag{2.3}$$

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for any $\omega \in \Delta_n$ and $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}_m^n$. Then $\|G_1(\omega; \mathbb{A})\| \leq \|G_2(\omega; \mathbb{A})\|$, where $\|\cdot\|$ denotes the operator norm. In addition, if such homogeneous matrix means G_i for $i = 1, 2$ are preserved by the antisymmetric tensor power:

$$\Lambda^k G_i(\omega; \mathbb{A}) = G_i(\omega; \Lambda^k \mathbb{A}),$$

where $\Lambda^k \mathbb{A} = (\Lambda^k A_1, \ldots, \Lambda^k A_n)$, then $G_1(\omega; \mathbb{A}) \prec_{w \log} G_2(\omega; \mathbb{A})$.

Proof. Let $\kappa = \|G_2(\omega; \mathbb{A})\|$. Then $G_2(\omega; \mathbb{A}) \leq \kappa I$, and

$$G_2\left(\omega; \frac{1}{\kappa}\mathbb{A}\right) = \frac{1}{\kappa} G_2(\omega; \mathbb{A}) \leq I,$$

since G_2 is homogeneous. By (2.3) and the homogeneity of G_1

$$\frac{1}{\kappa} G_1(\omega; \mathbb{A}) = G_1\left(\omega; \frac{1}{\kappa}\mathbb{A}\right) \leq I.$$

Thus, $G_1(\omega; \mathbb{A}) \leq \kappa I$, which implies $\|G_1(\omega; \mathbb{A})\| \leq \|G_2(\omega; \mathbb{A})\|$.

Additionally, assume that G_i for $i = 1, 2$ are preserved by the antisymmetric tensor power. Then using fundamental properties of the antisymmetric tensor powers in Lemma 2.1, (2.3) yields

$$\Lambda^k G_2(\omega; \mathbb{A}) \leq I \quad \Longrightarrow \quad \Lambda^k G_1(\omega; \mathbb{A}) \leq I.$$

So $\lambda_1(\Lambda^k G_1(\omega; \mathbb{A})) \leq \lambda_1(\Lambda^k G_2(\omega; \mathbb{A}))$, equivalently $G_1(\omega; \mathbb{A}) \prec_{w \log} G_2(\omega; \mathbb{A})$. ∎

3. Log-majorization of the Lim–Pálfia's power mean. Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}_m^n$. For convenience, we denote

$$\mathbb{A}^p := (A_1^p, \ldots, A_n^p) \in \mathbb{P}_m^n,$$

for any $p \in \mathbb{R}$.

For $t \in (0, 1]$ we denote by $P_t(\omega; \mathbb{A})$ the unique positive definite solution of

$$X = \sum_{i=1}^n w_i (X \#_t A_i).$$

For $t \in [-1, 0)$ we define $P_t(\omega; \mathbb{A}) = P_{-t}(\omega; \mathbb{A}^{-1})^{-1}$. We call $P_t(\omega; \mathbb{A})$ the *Lim–Pálfia's power mean* of order t for $A_1, \ldots, A_n$. Note that

$$P_1(\omega; \mathbb{A}) = \sum_{j=1}^n w_j A_j = \mathcal{A}(\omega; \mathbb{A}) \quad \text{and} \quad P_{-1}(\omega; \mathbb{A}) = \left(\sum_{j=1}^n w_j A_j^{-1}\right)^{-1} = \mathcal{H}(\omega; \mathbb{A}),$$

where $\mathcal{A}$ and $\mathcal{H}$ denote the arithmetic and harmonic means, respectively. One can easily see that for commuting $A_1, \ldots, A_n$

$$P_t(\omega; \mathbb{A}) = \left(\sum_{i=1}^n w_i A_i^t\right)^{1/t} = \mathcal{Q}_t(\omega; \mathbb{A}),$$

where $\mathcal{Q}_t$ denotes the quasi-arithmetic mean of order t; it can be defined for all $t \in \mathbb{R}$, and

$$\lim_{t \to 0} \mathcal{Q}_t(\omega; \mathbb{A}) = \exp\left(\sum_{i=1}^n w_i \log A_i\right).$$

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The remarkable consequence of power means appeared in [21, 22] is that P_t converges monotonically to the Cartan mean Λ as $t \to 0$ such that

$$P_{-t} \leq P_{-s} \leq \cdots \leq \Lambda = \lim_{t\to 0} P_t \leq \cdots \leq P_s \leq P_t, \tag{3.4}$$

for $0 < s \leq t \leq 1$, where the Cartan mean Λ is the least squares mean for the Riemannian trace metric d_R:

$$\Lambda(\omega; A_1, \ldots, A_n) := \underset{X\in\mathbb{P}_m}{\arg\min} \sum_{j=1}^{n} w_j d_R^2(A_j, X),$$

and $d_R(A, B) = \| \log A^{-1/2} B A^{-1/2} \|_2$.

Remark 3.1. *Note that Lim–Pálfia's power mean and Cartan mean are homogeneous. So applying Lemma 2.2 with the monotonicity (3.4) of Lim–Pálfia's power means yields that*

$$\begin{array}{llll} P_t(\omega; \mathbb{A}) & \searrow_{\succ_{w\log}} \Lambda(\omega; \mathbb{A}) & \text{as} & t \searrow 0, \\ P_t(\omega; \mathbb{A}) & \nearrow_{\prec_{w\log}} \Lambda(\omega; \mathbb{A}) & \text{as} & t \nearrow 0. \end{array}$$

Theorem 3.2. *[26, Theorem 1] Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}_m^n$ and $\omega = (w_1, \ldots, w_n) \in \Delta_n$. Then*

$$\sum_{j=1}^{n} w_j \log A_j \leq 0 \qquad \text{implies} \qquad \Lambda(\omega; \mathbb{A}) \leq I.$$

Proposition 3.3. *Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}_m^n$, $\omega = (w_1, \ldots, w_n) \in \Delta_n$, and $0 < t \leq 1$. Then for any $p > 0$*

$$\|P_{-t}(\omega; \mathbb{A}^p)^{1/p}\| \leq \left\| \exp\left(\sum_{j=1}^{n} w_j \log A_j \right) \right\| \leq \|P_t(\omega; \mathbb{A}^p)^{1/p}\|. \tag{3.5}$$

Furthermore,

$$P_{-t}(\omega; \mathbb{A}^p)^{1/p} \prec_{w\log} \exp\left(\sum_{j=1}^{n} w_j \log A_j \right). \tag{3.6}$$

Proof. Let $p > 0$. Since the Lim–Pálfia's power mean and log-Euclidean mean are homogeneous, by Lemma 2.2 it is enough for the second inequality of (3.5) to show that for $0 < t \leq 1$

$$P_t(\omega; \mathbb{A}^p)^{1/p} \leq I \qquad \text{implies} \qquad \exp\left(\sum_{j=1}^{n} w_j \log A_j \right) \leq I.$$

Assume that $P_t(\omega; \mathbb{A}^p) \leq I$ for $0 < t \leq 1$. By (3.4) $\Lambda(\omega; \mathbb{A}^p) \leq I$, and $\Lambda(\omega; \mathbb{A}^p)^{1/p} \leq I$. Taking the limit as $p \to 0^+$ and applying the Lie–Trotter formula of the Cartan mean [16] imply that

$$\exp\left(\sum_{j=1}^{n} w_j \log A_j \right) \leq I.$$

Now assume that $\exp(\sum_{j=1}^{n} w_j \log A_j) \leq I$. Since the logarithmic map is operator monotone, we have $\sum_{j=1}^{n} w_j \log A_j \leq 0$. Then, $\sum_{j=1}^{n} w_j \log A_j^p = p \sum_{j=1}^{n} w_j \log A_j \leq 0$ for any $p > 0$. By Theorem 3.2

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$$\Lambda(\omega;\mathbb{A}^p) \leq I,$$

and by (3.4) $P_{-t}(\omega;\mathbb{A}^p) \leq I$ for $0 < t \leq 1$. This completes the proof of (3.5).

Furthermore, by (3.4) $\Lambda^k P_{-t}(\omega;\mathbb{A}^p) \leq \Lambda^k \Lambda(\omega;\mathbb{A}^p)$ for the kth antisymmetric tensor power Λ^k. So $\lambda_1(\Lambda^k P_{-t}(\omega;\mathbb{A}^p)) \leq \lambda_1(\Lambda^k \Lambda(\omega;\mathbb{A}^p))$, and by Lemma 2.1 (3)

$$\lambda_1(\Lambda^k P_{-t}(\omega;\mathbb{A}^p)^{1/p}) = \lambda_1(\Lambda^k P_{-t}(\omega;\mathbb{A}^p))^{1/p} \leq \lambda_1(\Lambda^k \Lambda(\omega;\mathbb{A}^p))^{1/p} = \lambda_1(\Lambda^k \Lambda(\omega;\mathbb{A}^p)^{1/p}).$$

Since $\Lambda(\omega;\mathbb{A}) \prec_{\log} \exp\left(\sum_{j=1}^n w_j \log A_j\right)$ by [8, Theorem 1], we conclude that

$$P_{-t}(\omega;\mathbb{A}^p)^{1/p} \prec_{w\log} \Lambda(\omega;\mathbb{A}^p)^{1/p} \prec_{\log} \exp\left(\sum_{j=1}^n w_j \log A_j\right).$$

□

REMARK 3.4. *Note from [22, Proposition 3.5] that for $t \in (0,1]$*

$$\det P_{-t}(\omega;\mathbb{A}) \leq \prod_{j=1}^n (\det A_j)^{w_j},$$

so (3.6) must be the weak log-majorization.

A variant of Ando–Hiai inequality for power means has been shown in [23, Corollary 3.2]: for $t \in (0,1]$

$$P_t(\omega;\mathbb{A}) \leq I \quad \text{implies} \quad P_{\frac{t}{p}}(\omega;\mathbb{A}^p) \leq I \quad \text{for all } p \geq 1.$$

We provide different types of Ando–Hiai inequality for power means using Jensen inequalities [13]. Let X be a contraction. For any $A > 0$, we have

$$(XAX^*)^p \leq XA^pX^* \quad \text{if } 1 \leq p \leq 2, \tag{3.7}$$

and

$$(XAX^*)^p \geq XA^pX^* \quad \text{if } 0 \leq p \leq 1. \tag{3.8}$$

THEOREM 3.5. *Let $p \geq 1$. Then*

(i) if $P_t(\omega;\mathbb{A}) \geq I$ then $P_t(\omega;\mathbb{A}) \leq P_t(\omega;\mathbb{A}^p)$ for $0 < t \leq 1$, and
(ii) if $P_t(\omega;\mathbb{A}) \leq I$ then $P_t(\omega;\mathbb{A}) \geq P_t(\omega;\mathbb{A}^p)$ for $-1 \leq t < 0$.

Proof. We first consider $1 \leq p \leq 2$. Assume that $X := P_t(\omega;\mathbb{A}) \geq I$ for $0 < t \leq 1$. Then by taking the congruence transformation

$$I = \sum_{j=1}^n w_j (X^{-1/2}A_jX^{-1/2})^t = \sum_{j=1}^n w_j \left[(X^{-1/2}A_jX^{-1/2})^p\right]^{t/p}.$$

Since $0 < t/p \leq 1$, the above identity reduces to

$$I = P_{t/p}(\omega; (X^{-1/2}A_1X^{-1/2})^p, \dots, (X^{-1/2}A_nX^{-1/2})^p).$$

Since $X^{-1/2} \leq I$, Hansen's inequality (3.7) and the monotonicity of power means yield

$$I \leq P_{t/p}(\omega; X^{-1/2}A_1^pX^{-1/2}, \dots, X^{-1/2}A_n^pX^{-1/2}).$$

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Taking the congruence transformation by $X^{1/2}$ implies that $X \leq P_{t/p}(\omega; \mathbb{A}^p)$. Since $0 < t/p \leq t \leq 1$, we obtain from (3.4)

$$X \leq P_{t/p}(\omega; \mathbb{A}^p) \leq P_t(\omega; \mathbb{A}^p).$$

Replacing A_j by A_j^2 we can extend the interval $[2, 4]$, and successfully for all $p \geq 1$.

Assume that $X := P_t(\omega; \mathbb{A}) \leq I$ for $-1 \leq t < 0$. Then $X^{-1} = P_{-t}(\omega; \mathbb{A}^{-1}) \geq I$. By (i) with $0 < -t \leq 1$

$$X^{-1} \leq P_{-t}(\omega; \mathbb{A}^{-p}),$$

equivalently, $X \geq P_{-t}(\omega; \mathbb{A}^{-p})^{-1} = P_t(\omega; \mathbb{A}^p)$. ∎

Remark 3.6. *We give another proof for Theorem 3.5 (i). Let $1 \leq p \leq 2$. Assume that $X = P_t(\omega; \mathbb{A}) \geq I$ for $0 < t \leq 1$. Since the map $A \in \mathbb{P}_m \mapsto A^p$ is operator convex,*

$$I = \left[\sum_{j=1}^{n} w_j (X^{-1/2} A_j X^{-1/2})^t\right]^p \leq \sum_{j=1}^{n} w_j (X^{-1/2} A_j X^{-1/2})^{pt}.$$

By (3.7) and the monotonicity of the power map $A \in \mathbb{P}_m \mapsto A^t$,

$$I \leq \sum_{j=1}^{n} w_j (X^{-1/2} A_j^p X^{-1/2})^t.$$

Taking congruence transformation by $X^{1/2}$ implies

$$X \leq \sum_{j=1}^{n} w_j X^{1/2} (X^{-1/2} A_j^p X^{-1/2})^t X^{1/2} = \sum_{j=1}^{n} w_j X \#_t A_j^p =: f(X).$$

Since the map f is operator monotone on $\mathbb{P}_m$, we have $X \leq f(X) \leq f^2(X) \leq \cdots \leq f^k(X)$ for all $k \geq 1$. Taking the limit as $k \to \infty$ yields $X \leq P_t(\omega; \mathbb{A}^p)$ for $1 \leq p \leq 2$. Replacing A_j by A_j^2, we can extend the interval $[2, 4]$, and successfully for all $p \geq 1$.

Applying Lemma 2.2 to Theorem 3.5 (ii) we obtain

Corollary 3.7. *Let $-1 \leq t < 0$. Then*

$$\|P_t(\omega; \mathbb{A}^p)^{1/p}\| \leq \|P_t(\omega; \mathbb{A})\|,$$

for $p \geq 1$, where $\|\cdot\|$ denotes the operator norm.

Remark 3.8. *The following is the unique characterization of the Cartan mean among other multivariable geometric means satisfying the Ando–Li–Mathias axioms:*

$$\Lambda(\omega; \mathbb{A}) \leq I \qquad \textit{implies} \qquad \Lambda(\omega; \mathbb{A}^p) \leq I, \tag{3.9}$$

for all $p \geq 1$. This is known as the Ando–Hiai inequality; see [26, Theorem 3, Corollary 6]. We can derive it by using Theorem 3.5 (ii). Indeed, assume that $\Lambda(\omega; \mathbb{A}) \leq I$. Then by (3.4) $P_t(\omega; \mathbb{A}) \leq I$ for $-1 \leq t < 0$, and by Theorem 3.5 (ii) $P_t(\omega; \mathbb{A}^p) \leq I$. Taking the limit as $t \to 0^-$ yields $\Lambda(\omega; \mathbb{A}^p) \leq I$.

Theorem 3.9. *Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}_m^n$, and $\omega = (w_1, \ldots, w_n) \in \Delta_n$. Then*

$$\Lambda(\omega; \mathbb{A}^p)^{1/p} \nearrow_{\prec_{\log}} \exp\left(\sum_{j=1}^{n} w_j \log A_j\right) \quad \textit{as} \quad p \searrow 0.$$

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Proof. Note from [8] that

$$\lim_{p\to 0} \Lambda(\omega; \mathbb{A}^p)^{1/p} = \exp\left(\sum_{j=1}^{n} w_j \log A_j\right),$$

and

$$\Lambda(\omega; \mathbb{A}^p)^{1/p} \prec_{\log} \exp\left(\sum_{j=1}^{n} w_j \log A_j\right).$$

So it is enough to show that $\Lambda(\omega; \mathbb{A}^q)^{1/q} \prec_{\log} \Lambda(\omega; \mathbb{A}^p)^{1/p}$ for $0 < p \leq q$. By (3.9), if $\Lambda(\omega; \mathbb{A}) \leq I$, then $\Lambda(\omega; \mathbb{A}^r) \leq I$ for any $r \geq 1$ so $\Lambda(\omega; \mathbb{A}^r)^{1/r} \leq I$.

Since the Cartan mean and $\Lambda(\omega; \mathbb{A}^r)^{1/r}$ are preserved by the antisymmetric tensor power and homogeneous, from Lemma 2.2 we have $\Lambda(\omega; \mathbb{A}^r)^{1/r} \prec_{\log} \Lambda(\omega; \mathbb{A})$ for all $r \geq 1$. Letting $r = q/p$ for $0 < p \leq q$ and replacing A_j by A_j^p, we obtain $\Lambda(\omega; \mathbb{A}^q)^{1/q} \prec_{\log} \Lambda(\omega; \mathbb{A}^p)^{1/p}$. □

REMARK 3.10. *Ando and Hiai [3, 4] have shown that* $(A^p \#_t B^p)^{1/p}$ *converges increasingly to the log-Euclidean mean as* $p \to 0^+$ *with respect to the log-majorization:*

$$(A^p \#_t B^p)^{1/p} \nearrow_{\prec_{\log}} \exp((1-t)\log A + t \log B) \quad \text{as} \quad p \searrow 0.$$

Theorem 3.9 is a generalization of the Ando–Hiai's log-majorization result to multivariable geometric mean, which is the Cartan mean.

REMARK 3.11. *Since the Lim–Pálfia's power mean satisfies the arithmetic-power-harmonic mean inequalities:*

$$\mathcal{H}(\omega; \mathbb{A}) = \left(\sum_{j=1}^{n} w_j A_j^{-1}\right)^{-1} \leq P_t(\omega; \mathbb{A}) \leq \sum_{j=1}^{n} w_j A_j = \mathcal{A}(\omega; \mathbb{A}),$$

for any nonzero $t \in [-1, 1]$*, it satisfies from [16, Theorem 4.2]*

$$\lim_{p\to 0} P_t(\omega; \mathbb{A}^p)^{1/p} = \exp\left(\sum_{j=1}^{n} w_j \log A_j\right).$$

Moreover, $P_t(\omega; \mathbb{A}^p)^{1/p} \prec_{w\log} \exp\left(\sum_{j=1}^{n} w_j \log A_j\right)$ *for* $t \in [-1, 0)$ *by Proposition 3.3. One can naturally ask that the Lim–Pálfia's power mean* $P_t(\omega; \mathbb{A}^p)^{1/p}$ *for* $t \in [-1, 0)$ *converges increasingly to the log-Euclidean mean as* $p \to 0^+$ *with respect to the weak log-majorization. In order to show this, it remains an open problem as follows: for* $0 < p \leq q$

$$P_t(\omega; \mathbb{A}^q)^{1/q} \prec_{w\log} P_t(\omega; \mathbb{A}^p)^{1/p}.$$

4. Log-majorization of the t-z Rényi right mean. Let $A, B \in \mathbb{P}_m$. For $0 \leq t \leq 1$ and $z > 0$

$$Q_{t,z}(A, B) = \left(A^{\frac{1-t}{2z}} B^{\frac{t}{z}} A^{\frac{1-t}{2z}}\right)^z$$

is the matrix version of the t-z Rényi relative entropy [5, 24]. Especially, $Q_{t,t}(A, B)$ is known as the sandwiched Rényi relative entropy [25]. This can be considered as a noncommutative version of geometric mean in the sense that $Q_{t,z}(A, B) = A^{1-t}B^t$ for commuting A and B. From this point of view, it is interesting to find a log-majorization relation between $Q_{t,z}(A, B)$ and $A^{1/2}B^{1/2}$.

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Theorem 4.1. *Let $A, B \in \mathbb{P}_m$. For $0 \le t \le 1/2$ and $t \le z \le 1$,*

(i) $\lambda(Q_{t,z}(A,B)) \prec_{\log} s(A^{1/2}B^{1/2})$*, and*
(ii) $s(A^{t-\frac{1}{2}}Q_{t,z}(A,B)B^{\frac{1}{2}-t}) \prec_{\log} s(A^{1/2}B^{1/2})$.

Proof. Note that $s(A^{1/2}B^{1/2}) = \lambda((A^{1/2}BA^{1/2})^{1/2})$. Since $Q_{t,z}(A,B)$ and $(A^{1/2}BA^{1/2})^{1/2}$ are invariant under the antisymmetric tensor product and homogeneous, it is enough from Lemma 2.2 to show that

(i) $A^{1/2}BA^{1/2} \le I$ implies $Q_{t,z}(A,B) \le I$,
(ii) $A^{1/2}BA^{1/2} \le I$ implies $A^{t-\frac{1}{2}}Q_{t,z}(A,B)B^{1-2t}Q_{t,z}(A,B)A^{t-\frac{1}{2}} \le I$.

Let $0 \le t \le 1/2$ and $t \le z \le 1$.

(i) We first prove it when $B \ge I$. Assuming that $A^{1/2}BA^{1/2} \le I$, we have $B \le A^{-1}$ so $B^{\frac{t}{z}} \le A^{-\frac{t}{z}}$ by the Loewner–Heinz inequality with $0 < t \le z \le 1$. Then

$$Q_{t,z}(A,B) \le \left(A^{\frac{1-t}{2z}}A^{-\frac{t}{z}}A^{\frac{1-t}{2z}}\right)^z = A^{1-2t} \le I,$$

since $A \le B^{-1} \le I$ and $1-2t \ge 0$. So (i) holds when $B \ge I$.

Let $\lambda_m := \min\{\lambda_i(B) : 1 \le i \le m\}$. Then, $\lambda_m^{-1}B \ge I$. By the preceding argument

$$\begin{aligned}\lambda_m^{-1}Q_{t,z}(A,B) &= Q_{t,z}(\lambda_m^{-1}A, \lambda_m^{-1}B)\\ &\prec_{\log} ((\lambda_m^{-1}A)^{1/2}(\lambda_m^{-1}B)(\lambda_m^{-1}A)^{1/2})^{1/2} = \lambda_m^{-1}(A^{1/2}BA^{1/2})^{1/2},\end{aligned}$$

which completes the proof of (i).

(ii) Assume that $A^{1/2}BA^{1/2} \le I$. Then $B \le A^{-1}$, and $B^{1-2t} \le A^{2t-1}$ by the Loewner–Heinz inequality since $2t \in [0,1]$. Therefore, we have

$$\begin{aligned}A^{t-\frac{1}{2}}Q_{t,z}(A,B)B^{1-2t}Q_{t,z}(A,B)A^{t-\frac{1}{2}} &\le A^{t-\frac{1}{2}}Q_{t,z}(A,B)A^{2t-1}Q_{t,z}(A,B)A^{t-\frac{1}{2}}\\ &= \left(A^{t-\frac{1}{2}}Q_{t,z}(A,B)A^{t-\frac{1}{2}}\right)^2.\end{aligned}$$

Since $B \le A^{-1}$ and $0 \le t \le z \le 1$, we obtain $Q_{t,z}(A,B) \le A^{1-2t}$ by the Loewner–Heinz inequality. So

$$A^{t-\frac{1}{2}}Q_{t,z}(A,B)A^{t-\frac{1}{2}} \le I,$$

and thus, $A^{t-\frac{1}{2}}Q_{t,z}(A,B)B^{1-2t}Q_{t,z}(A,B)A^{t-\frac{1}{2}} \le I$. Moreover,

$$\det\left[A^{t-\frac{1}{2}}Q_{t,z}(A,B)B^{1-2t}Q_{t,z}(A,B)A^{t-\frac{1}{2}}\right] = \det(AB) = \det(A^{1/2}BA^{1/2}),$$

and hence, (ii) holds for $t \in [0,1/2]$. ∎

The t-z Rényi right mean $\Omega_{t,z}$ is defined as

$$\Omega_{t,z}(\omega;\mathbb{A}) = \underset{X\in\mathbb{P}_m}{\arg\min}\sum_{j=1}^{n} w_j\Phi_{t,z}(A_j,X).$$

Since the map $A \in \mathbb{P}_m \mapsto \operatorname{tr} A^t$ for $t \in (0,1)$ is strictly concave, the map $X \in \mathbb{P}_m \mapsto \Phi_{t,z}(A,X)$ is strictly convex for $0 < t \le z < 1$. So one can see that $\Omega_{t,z}(\omega;\mathbb{A})$ coincides with the unique positive definite solution of the matrix nonlinear equation

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$$X = \sum_{j=1}^{n} w_j Q_{1-t,z}(X, A_j). \tag{4.10}$$

Note that (4.10) is equivalent to

$$X^{1-\frac{t}{z}} = \sum_{j=1}^{n} w_j X^{-\frac{t}{z}} \#_z A_j^{\frac{1-t}{z}}.$$

See [10, 15, 18] for more details.

Theorem 4.2. *[18, Theorem 3.2] Let $0 < t \leq z < 1$. If $\Omega_{t,z}(\omega; \mathbb{A}) \leq I$ then*

$$\Omega_{t,z}(\omega; \mathbb{A})^{1-\frac{t}{z}} \geq \mathcal{A}(\omega; \mathbb{A}^{1-t}).$$

If $\Omega_{t,z}(\omega; \mathbb{A}) \geq I$, then the reverse inequality holds.

Theorem 4.3. *[11, Theorem 13] Let $0 < t \leq z < 1$. Then we have*

$$\frac{1+z-t}{1-t} I - \frac{z}{1-t} \sum_{j=1}^{n} w_j A_j^{-\frac{1-t}{z}} \leq \Omega_{t,z}(\omega; \mathbb{A}) \leq \left(\frac{1+z-t}{1-t} I - \frac{z}{1-t} \sum_{j=1}^{n} w_j A_j^{\frac{1-t}{z}} \right)^{-1},$$

where the second inequality holds when $(1+z-t)I - z\sum_{j=1}^{n} w_j A_j^{\frac{1-t}{z}}$ is invertible.

Theorem 4.4. *For $0 < t \leq z < 1$,*

$$\|P_{1-t}(\omega; \mathbb{A})\| \leq \|\Omega_{t,z}(\omega; \mathbb{A})\| \leq \|\mathcal{Q}_{\frac{1-t}{z}}(\omega; \mathbb{A})\|.$$

Furthermore, $\|\mathcal{Q}_{\frac{t-1}{z}}(\omega; \mathbb{A})\| \leq \|\Omega_{t,z}(\omega; \mathbb{A})\|$.

Proof. Let $0 < t \leq z < 1$. Since the Rényi right mean $\Omega_{t,z}$, power mean P_{1-t}, and quasi-arithmetic mean $\mathcal{Q}_{\frac{1-t}{z}}$ are all homogeneous, it is enough from Lemma 2.2 to show that for each cases

$$\begin{aligned} \Omega_{t,z}(\omega; \mathbb{A}) \leq I \quad &\text{implies} \quad P_{1-t}(\omega; \mathbb{A}) \leq I, \\ \mathcal{Q}_{\frac{1-t}{z}}(\omega; \mathbb{A}) \leq I \quad &\text{implies} \quad \Omega_{t,z}(\omega; \mathbb{A}) \leq I. \end{aligned}$$

By Theorem 4.2, $\Omega_{t,z}(\omega; \mathbb{A}) \leq I$ implies that

$$\sum_{j=1}^{n} w_j A_j^{1-t} \leq \Omega_{t,z}(\omega; \mathbb{A})^{1-\frac{t}{z}} \leq I,$$

and hence, $\mathcal{Q}_{1-t}(\omega; \mathbb{A}) = \left(\sum_{j=1}^{n} w_j A_j^{1-t}\right)^{\frac{1}{1-t}} \leq I$. By [23, Theorem 3.1] $P_{1-t}(\omega; \mathbb{A}) \leq I$. Next, we assume $\mathcal{Q}_{\frac{1-t}{z}}(\omega; \mathbb{A}) \leq I$. Then $\sum_{j=1}^{n} w_j A_j^{\frac{1-t}{z}} \leq I$ so one can see that

$$\frac{1+z-t}{1-t} I - \frac{z}{1-t} \sum_{j=1}^{n} w_j A_j^{\frac{1-t}{z}} \geq I.$$

By assumption, the second inequality in Theorem 4.3 holds, and hence, we have

$$\Omega_{t,z}(\omega; \mathbb{A})^{\frac{1-t}{z}} \leq \left(\frac{1+z-t}{1-t} I - \frac{z}{1-t} \sum_{j=1}^{n} w_j A_j^{\frac{1-t}{z}} \right)^{-1} \leq I.$$

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Moreover, assuming that $\Omega_{t,z}(\omega;\mathbb{A}) \leq I$ yields

$$\frac{1+z-t}{1-t}I - \frac{z}{1-t}\sum_{j=1}^{n} w_j A_j^{-\frac{1-t}{z}} \leq I,$$

by Theorem 4.3. Then $\sum_{j=1}^{n} w_j A_j^{\frac{t-1}{z}} \geq I$, and hence, $\mathcal{Q}_{\frac{t-1}{z}}(\omega;\mathbb{A}) \leq I$ since $t \in (0,1)$. This completes the proof. ∎

5. Boundedness of the Rényi power mean. Another type of the Rényi power mean has been introduced in [12], as a unique positive definite solution of the equation

$$X = \sum_{j=1}^{n} w_j Q_{t,z}(A_j, X) = \sum_{j=1}^{n} w_j \left(A_j^{\frac{1-t}{2z}} X^{\frac{t}{z}} A_j^{\frac{1-t}{2z}}\right)^z. \tag{5.11}$$

We denote it as $\mathcal{R}_{t,z}(\omega;\mathbb{A})$. We see the inequalities between the Rényi power mean and quasi-arithmetic mean by using Jensen-type inequalities.

Theorem 5.1. *Let $0 < t \leq z < 1$. If $\mathcal{R}_{t,z}(\omega;\mathbb{A}) \leq I$ then*

$$\mathcal{R}_{t,z}(\omega;\mathbb{A}) \leq \mathcal{Q}_{\frac{1}{p}}(\omega;\mathbb{A}^{1-t}) = \left(\sum_{j=1}^{n} w_j A_j^{\frac{1-t}{p}}\right)^p,$$

for all p such that $p \leq z$.

Proof. Let $X = \mathcal{R}_{t,z}(\omega;\mathbb{A}) \leq I$ for $0 < t \leq z < 1$. Since $X^{\frac{t}{z}} \leq I$, we have $A_j^{\frac{1-t}{2z}} X^{\frac{t}{z}} A_j^{\frac{1-t}{2z}} \leq A_j^{\frac{1-t}{z}}$ for each $j = 1,\ldots,n$. Then from the equation (5.12)

$$X = \sum_{j=1}^{n} w_j \left(A_j^{\frac{1-t}{2z}} X^{\frac{t}{z}} A_j^{\frac{1-t}{2z}}\right)^z \leq \sum_{j=1}^{n} w_j A_j^{1-t}.$$

Since the map $\mathbb{P}_m \ni A \mapsto A^z$ is concave, we obtain

$$X \leq \sum_{j=1}^{n} w_j A_j^{1-t} \leq \left[\sum_{j=1}^{n} w_j A_j^{\frac{1-t}{z}}\right]^z = \mathcal{Q}_{\frac{1}{z}}(\omega;\mathbb{A}^{1-t}).$$

Moreover, $\mathcal{Q}_p$ is monotone on $p \in (-\infty,-1] \cup [1,\infty)$ from [19, Theorem 5.1] so

$$\mathcal{Q}_{\frac{1}{z}}(\omega;\mathbb{A}^{1-t}) \leq \mathcal{Q}_{\frac{1}{p}}(\omega;\mathbb{A}^{1-t}),$$

for $0 < p \leq z < 1$. Hence, we completes the proof. ∎

Lemma 5.2. *Let $0 < t \leq z < 1$.*

(1) If $A_j \leq I$ for all j, then $\mathcal{R}_{t,z}(\omega;\mathbb{A}) \leq I$.
(2) If $A_j \geq I$ for all j, then $\mathcal{R}_{t,z}(\omega;\mathbb{A}) \geq I$.

Proof. Assume that $A_j \leq I$ for all j. Let $X = \mathcal{R}_{t,z}(\omega;\mathbb{A})$ for $0 < t \leq z < 1$. Suppose that $\lambda_1(X) > 1$. Since $X \leq \lambda_1(X)I$,

$$X = \sum_{j=1}^{n} w_j \left(A_j^{\frac{1-t}{2z}} X^{\frac{t}{z}} A_j^{\frac{1-t}{2z}}\right)^z \leq \lambda_1(X)^t \sum_{j=1}^{n} w_j A_j^{1-t} \leq \lambda_1(X)^t I.$$

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This inequality implies that $\lambda_1(X) \leq \lambda_1(X)^t$, which is a contradiction because $\lambda_1(X) > 1$ and $0 < t < 1$. So $\lambda_1(X) \leq 1$, equivalently $X \leq I$.

In order to prove (2), suppose that $\lambda_m(X) < 1$. Since $X \geq \lambda_m(X)I$, the similar argument as above yields $\lambda_m(X) \geq \lambda_m(X)^t$, but it is a contradiction. Thus, $\lambda_m(X) \geq 1$, equivalently $X \geq I$. □

In the following, we denote as $\lambda_M := \max\{\lambda_1(A_j) : 1 \leq j \leq n\}$.

Corollary 5.3. *Let $0 < t \leq z < 1$. Then for all p such that $p \leq z$*

$$\mathcal{R}_{t,z}(\omega; \mathbb{A}) \leq \lambda_M^t \mathcal{Q}_{\frac{1}{p}}(\omega; \mathbb{A}^{1-t}).$$

Proof. Since $\lambda_M^{-1} A_j \leq I$ for all j, we have $\mathcal{R}_{t,z}\left(\omega; \lambda_M^{-1}\mathbb{A}\right) \leq I$ by Lemma 5.2 (1). From Theorem 5.1 together with the homogeneity of the Rényi power mean,

$$\lambda_M^{-1}\mathcal{R}_{t,z}(\omega; \mathbb{A}) = \mathcal{R}_{t,z}\left(\omega; \lambda_M^{-1}\mathbb{A}\right) \leq \left(\sum_{j=1}^n w_j (\lambda_M^{-1} A_j)^{\frac{1-t}{p}}\right)^p = \lambda_M^{t-1}\left(\sum_{j=1}^n w_j A_j^{\frac{1-t}{p}}\right)^p.$$

By simplifying the terms of λ_M, we complete the proof. □

Theorem 5.4. *Let $0 < t \leq z < 1$. Then*

$$\mathcal{R}_{t,z}(\omega; \mathbb{A})^{\frac{1-t}{2}} \geq \lambda_M^{-\frac{(1-t)(1-z)}{2z}} \sum_{j=1}^n w_j A_j^{\frac{1-t}{2z}}.$$

Proof. We first assume that $A_j \leq I$ for all j. Let $X = \mathcal{R}_{t,z}(\omega; \mathbb{A})$ for $0 < t \leq z < 1$. By (3.8)

$$X = \sum_{j=1}^n w_j \left(A_j^{\frac{1-t}{2z}} X^{\frac{t}{z}} A_j^{\frac{1-t}{2z}}\right)^z \geq \sum_{j=1}^n w_j A_j^{\frac{1-t}{2z}} X^t A_j^{\frac{1-t}{2z}}.$$

Taking the congruence transformation by $X^{\frac{t}{2}}$ and applying the convexity of a square map yield

$$X^{1+t} \geq \sum_{j=1}^n w_j \left(X^{\frac{t}{2}} A_j^{\frac{1-t}{2z}} X^{\frac{t}{2}}\right)^2 \geq \left(\sum_{j=1}^n w_j X^{\frac{t}{2}} A_j^{\frac{1-t}{2z}} X^{\frac{t}{2}}\right)^2.$$

Since the square root map is operator monotone, we have $X^{\frac{1+t}{2}} \geq \sum_{j=1}^n w_j X^{\frac{t}{2}} A_j^{\frac{1-t}{2z}} X^{\frac{t}{2}}$. Taking the congruence transformation by $X^{-t/2}$, we obtain

$$X^{\frac{1-t}{2}} \geq \sum_{j=1}^n w_j A_j^{\frac{1-t}{2z}}. \tag{5.12}$$

Now, replacing A_j by $\lambda_M^{-1} A_j (\leq I)$ for all j in (5.13), we have

$$\mathcal{R}_{t,z}\left(\omega; \lambda_M^{-1}\mathbb{A}\right)^{\frac{1-t}{2}} \geq \sum_{j=1}^n w_j \left(\lambda_M^{-1} A_j\right)^{\frac{1-t}{2z}}.$$

Since the Rényi power mean $\mathcal{R}_{t,z}$ is homogeneous, it reduces to

$$\lambda_M^{\frac{t-1}{2}} \mathcal{R}_{t,z}\left(\omega; \mathbb{A}\right)^{\frac{1-t}{2}} \geq \lambda_M^{\frac{t-1}{2z}} \sum_{j=1}^n w_j A_j^{\frac{1-t}{2z}}.$$

By simplifying the terms of λ_M, we obtain the desired inequality. □

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REMARK 5.5. *The multivariable matrix mean on the open convex cone $\mathbb{P}_m$ can be defined as a map $G : \Delta_n \times \mathbb{P}_m^n \to \mathbb{P}_m$ satisfying the idempotency: $G(\omega; A, \dots, A) = A$ for any $\omega \in \Delta_n$ and $A \in \mathbb{P}_m$. Boundedness of the multivariable matrix mean plays an important role in operator inequality and majorization. Especially, the multivariable matrix mean G satisfying the arithmetic-G-harmonic mean inequalities*

$$\left(\sum_{j=1}^{n} w_j A_j^{-1} \right)^{-1} \leq G(\omega; A_1, \dots, A_n) \leq \sum_{j=1}^{n} w_j A_j,$$

fulfills the extended version of Lie–Trotter formula [16]:

$$\lim_{s \to 0} G(\omega; A_1^s, \dots, A_n^s)^{1/s} = \exp\left(\sum_{j=1}^{n} w_j \log A_j \right). \tag{5.13}$$

See [17, 20] for more information. We here have established boundedness of the Rényi power mean, but it is still open whether (5.14) *holds for the Rényi power mean.*

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