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PERIODIC TWO-DIMENSIONAL DESCRIPTOR SYSTEMS*

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Abstract. In this note, we analyze the compatibility conditions of 2D descriptor systems with periodic coefficients and we derive a special coordinate system in which these conditions reduce to simple matrix commutativity conditions. We also show that the compatibility of the different trajectories in such a periodic 2D descriptor system can elegantly be formulated in terms of so-called matrix relations of regular pencils, which were introduced in [Benner and Byers. An arithmetic for matrix pencils: Theory and new algorithms. *Numer. Math.*, 103(4):539–573, 2006]. We then show that these ideas can be extended to multidimensional periodic descriptor systems and briefly discuss the difference between the case of complex and real coefficient matrices.

Key words. Multidimensional systems, periodic systems, periodic Schur form, commuting systems, simultaneous triangularization.

AMS subject classifications. 15A22, 12A27, 93AXX.

1. Introduction. Two-dimensional systems have received a lot of attention in the 1980s and 1990s because of the many applications that were then emerging in the numerical analysis of partial differential equations [13], in image and signal processing [4], in circuits and systems [3], and because of their use as models for other discrete-time processes [8, 9]. Singular systems, also called descriptor systems or differential-algebraic systems, are known to require special conditions for guaranteeing uniqueness of their trajectories.

In this paper, we combine these two properties with yet another property: we consider a special class of singular two-dimensional systems with linear periodic coefficients. Linear descriptor systems represent a broad class of time evolutionary phenomena and are often the result of the problem formulation in system theory, especially when the variables used are the natural describing variables of the underlying process. Within the general class of linear descriptor systems, periodic systems form an important subclass which are suitable for many natural and man-made phenomena.

We will use the concepts of solvability and conditionability introduced by Luenberger [10, 11] to derive conditions for the existence and uniqueness of solutions of such systems, and this for arbitrary two-point boundary conditions imposed on the 2D system. We then give a characterization of solvability/conditionability in terms of a cyclic matrix pencil and, furthermore, propose a simple test via the periodic Schur decomposition to check for either property. We then derive another coordinate system in which all coefficient matrices are block diagonal, and where each diagonal block corresponds to a single eigenvalue of the periodic system.

Furthermore, we build on an existing formalism of so-called *matrix relations* [1] to rephrase the independence of all trajectories in the 2D system in a very elegant coordinate free condition, expressing that these matrix relations must commute. This formulation is then essentially the same as the commutativity property of nonsingular 2D periodic systems.

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All of the results are derived first for matrices with elements in $\mathbb{C}$ because it is an algebraically closed field. The case of matrices with real coefficients is treated afterward, by just pointing out the few modifications that are needed for this case. We also show that the results developed here for the 2D case, easily extend to the multidimensional case.

The paper is organized as follows. In Sections 2 and 3, we formulate the basic problem and introduce the concepts of solvability and conditionability that play a fundamental role in this paper. In Section 4, we show that periodic subsystems play an important role as well and we recall their properties. In Section 5, we recall the idea of matrix relations that allow us to formulate the solution of our problem in an elegant and compact manner. Section 6 then gives a necessary and sufficient condition for the basic commutativity problem of descriptor systems, and Section 7 shows how this can be interpreted in terms of particular two-point boundary value problems. Sections 8 and 9 then simplify these conditions by linking them to the periodic Schur form and the block diagonal spectral decomposition of the descriptor systems of the two-dimensional basic cell. In Section 10, we discuss a number of extensions, and in Section 11, we conclude with a few final remarks.

2. Problem formulation. We begin by formulating the 2D periodic system that we study in this paper. Let us consider the following system of linear relations

$$\begin{aligned} Bx_{k+1,\ell} &= Ax_{k,\ell}, \quad & Dx_{k+1,\ell+1} &= Cx_{k+1,\ell}, \\ Dx_{k,\ell+1} &= Cx_{k,\ell}, \quad & Bx_{k+1,\ell+1} &= Ax_{k,\ell+1}, \end{aligned} \tag{2.1}$$

on an infinite two-dimensional grid with "basic cell"

$$\begin{array}{ccc} x_{k+1,\ell} & \overset{C}{\underset{D}{\overset{\longrightarrow}{\longleftarrow}}} & x_{k+1,\ell+1} \\ A\uparrow\downarrow B & & A\uparrow\downarrow B \\ x_{k,\ell} & \overset{C}{\underset{D}{\overset{\longrightarrow}{\longleftarrow}}} & x_{k,\ell+1} \end{array} .$$

In this infinite grid, every vertical relation involves the matrix pair (A, B) and every horizontal relation involves the matrix pair (C, D), where the matrices $\{A, B, C, D\}$ are all $n \times n$ with coefficients in $\mathbb{C}$ (the real case requires some adaptations, as will be pointed out later). It is clear that the state transition from $x_{k,\ell}$ to $x_{k+1,\ell+1}$ can be obtained via two different "paths" or "'trajectories" and that we therefore need to impose conditions on the matrices $\{A, B, C, D\}$ to make sure that these trajectories are compatible. If, for instance, B and D were invertible, then

$$x_{k+1,\ell+1} = D^{-1}CB^{-1}Ax_{k,\ell} = B^{-1}AD^{-1}Cx_{k,\ell}, \;\; \forall x_{k,\ell}, \tag{2.2}$$

which implies that the state transition matrices $B^{-1}A$ and $D^{-1}C$ must commute.

We also analyze if such conditions cannot be simplified by transforming the system to an appropriate coordinate system. If we associate a single invertible transformation T to all the state variables :

$$\begin{aligned} T\hat{x}_{k+1,\ell} &= x_{k+1,\ell}, \quad & T\hat{x}_{k+1,\ell+1} &= x_{k+1,\ell+1}, \\ T\hat{x}_{k,\ell} &= x_{k,\ell}, \quad & T\hat{x}_{k,\ell+1} &= x_{k,\ell+1}, \end{aligned}$$

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and multiply the linear relations (2.1) with independent, but invertible transformations S_1 and S_2, we obtain the transformed relations

$$(2.3)\qquad \begin{array}{ll} (S_1BT)\hat{x}_{k+1,\ell} = (S_1AT)\hat{x}_{k,\ell}, & (S_2DT)\hat{x}_{k+1,\ell+1} = (S_2CT)\hat{x}_{k+1,\ell}, \\ (S_2DT)\hat{x}_{k,\ell+1} = (S_2CT)\hat{x}_{k,\ell}, & (S_1BT)\hat{x}_{k+1,\ell+1} = (S_1AT)\hat{x}_{k,\ell+1}, \end{array}$$

describing a system with the same evolution, albeit in a different coordinate system. We will use such a system of constrained transformations in the special periodic Schur form and special block diagonal spectral decomposition that are derived in this paper.

3. Solvability and conditionability. In order to extend commutativity conditions such as in (2.2) to the case of singular systems, we need to recall the notions of solvability and conditionability of descriptor systems that were introduced by Luenberger [10, 11] and extended to periodic systems in [15]. To introduce these concepts, we first consider the following set of 1D time-varying difference equations, over the time interval $i \in [1:m]$

$$(3.4)\qquad F_i y_{i+1} = E_i y_i + u_i,\ i = 1, \ldots, m, \quad (E_i, F_i) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n}.$$

These can be written as a system of equations linking $u_{[1:m]}$ to $y_{[1:m+1]}$ as follows:

$$(3.5)\qquad \left[\begin{array}{c|cccc|c} -E_1 & F_1 & & & & \\ & -E_2 & \ddots & & & \\ & & & \ddots & F_{m-1} & \\ & & & & -E_m & F_m \end{array}\right] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{m+1} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix},$$

where the $mn \times (m+1)n$ matrix on the left is called the *solvability matrix* $S(1:m+1)$, and its center $mn \times (m-1)n$ matrix is called the *conditionability matrix* $C(1:m)$.

DEFINITION 3.1. *The set of difference equations* (3.5) *is said to be* solvable *over the interval* $[1:m+1]$ *if* $S(1:m+1)$ *has full row rank* mn.

DEFINITION 3.2. *The set of difference equations* (3.5) *is said to be* conditionable *over the interval* $[1:m]$ *if* $C(1:m)$ *has full column rank* $(m-1)n$.

The *solvability condition* says that there exists an n-dimensional linear variety of solutions $y_{[1:m+1]}$ for each input sequence $u_{[1:m]}$. The *conditionability condition* says that any two distinct solutions $y_{[1:m+1]}$ of (3.5) must differ in at least one end-point (1 or $m+1$). Together, these conditions imply that there exist $n \times n$ boundary conditions W_1 and W_{m+1} such that

$$(3.6)\qquad \left[\begin{array}{c|cccc|c} -E_1 & F_1 & & & & \\ & -E_2 & \ddots & & & \\ & & & \ddots & F_{m-1} & \\ & & & & -E_m & F_m \\ \hline W_1 & & & & & W_{m+1} \end{array}\right] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{m+1} \end{bmatrix} = \left[\begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_m \\ \hline w \end{array}\right],$$

has a unique solution $y_{[1:m+1]}$ for a given right-hand side $(u_{[1:m]}, w)$ (see [10]). For this, it suffices to choose W_1 and W_{m+1} such that the matrix on the left-hand side of (3.6) is invertible.

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For our 2D periodic descriptor system, we want to apply this to any trajectory going from a starting point (k,ℓ) in the grid to an end point (i,j) that is "forward" in time, i.e., with $k<i$ and $\ell<j$. This means that in the above set of difference equations (3.5), the block rows (E_i,F_i) are either (A,B) or (C,D), and this in any order. Since this also includes the *periodic* alternation of (A,B) and (C,D), corresponding to a path

$$x_{k,\ell} \longrightarrow x_{k+1,\ell} \longrightarrow x_{k+1,\ell+1} \longrightarrow x_{k+2,\ell+1} \longrightarrow x_{k+2,\ell+2} \longrightarrow \ldots,$$

we can use the following theorem that was proven in [15] for a general 1D periodic system of difference equations. To obtain the required result, we only need the case of period 2, and therefore limit the theorem to this case.

THEOREM 3.3 (see [15]). *The following conditions are equivalent :*

1. $S(1{:}2m) = \begin{bmatrix} -A & B & & & & \\ & -C & D & & & \\ & & \ddots & \ddots & & \\ & & & -A & B & \\ & & & & -C & D \end{bmatrix}$ *has full row rank for all* m,
2. $C(1{:}2m) = \begin{bmatrix} D & & & & \\ -A & B & & & \\ & -C & D & & \\ & & \ddots & \ddots & \\ & & & -A & B \\ & & & & -C \end{bmatrix}$ *has full column rank for all* m,
3. *the pencil* $\begin{bmatrix} -A & zB \\ zD & -C \end{bmatrix}$ *is regular,*
4. *the pencil* $\begin{bmatrix} -C & zD \\ zB & -A \end{bmatrix}$ *is regular,*
5. *the polynomial matrix* $\begin{bmatrix} -A & B \\ z^2D & -C \end{bmatrix}$ *is regular,*
6. *the polynomial matrix* $\begin{bmatrix} -C & D \\ z^2B & -A \end{bmatrix}$ *is regular.*

Proof. The equivalence of conditions 1., 2., and 3., and the equivalence of conditions 1., 2., and 4. were given in [15]. Therefore, 3. and 4. are also equivalent. The equivalence of 3. and 4. with 5. and 6., respectively, follows from the fact that scaling the bottom block row with z^{-1} and the last block column with z, transforms the pencils 3. and 4. into the polynomial matrices 5. and 6. Therefore, the determinants of 3. and 5. and of 4. and 6. are equal, which implies that regularity is preserved. ∎

It follows from this theorem that in order to address correctly the problem of compatibility of different paths in our two-dimensional grid, one must impose the condition that the pencils

$$\begin{bmatrix} -A & zB \\ zD & -C \end{bmatrix}, \quad \begin{bmatrix} -C & zD \\ zB & -A \end{bmatrix}, \tag{3.7}$$

are both regular. We show in the next section that this allows us to put the 2D periodic descriptor system in a particular coordinate system that will yield simplified compatibility conditions. Notice that both pencils

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are simple block permutations of each other, which also shows that the regularity of one pencil implies the regularity of the other.

4. 2D periodicity. Let us consider again the 2D system of periodic dynamic equations (2.1). If we assume B and D are invertible, then

$$x_{k+1,\ell+1} = D^{-1}CB^{-1}Ax_{k,\ell} = B^{-1}AD^{-1}Cx_{k,\ell},\ \forall x_{k,\ell},$$

which implies that the state transition matrices $D^{-1}C$ and $B^{-1}A$ must commute. We now show that this implies a basic property of the periodic Schur decomposition. We refer to [2] for the details of this decomposition.

Theorem 4.1. *Let the $n \times n$ complex matrices B and D be invertible and let the matrix product $\Phi := D^{-1}CB^{-1}A$ be a simple matrix, which is also equal to $B^{-1}AD^{-1}C$. Then there exists a periodic Schur form*

$$\begin{bmatrix} -\hat{A} & z\hat{B} \\ z\hat{D} & -\hat{C} \end{bmatrix} := \begin{bmatrix} Z_1 & \\ & Z_2 \end{bmatrix} \begin{bmatrix} -A & zB \\ zD & -C \end{bmatrix} \begin{bmatrix} Q_1^* & \\ & Q_2^* \end{bmatrix}, \tag{4.8}$$

where $Q_1 = Q_2$.

Proof. If B and D are invertible, the coefficient of z in the pencil (4.8) has full rank, and the pencil is then regular, which implies that there exists a periodic Schur form [7, 15]. Let Q_1, Q_2, and Z_1, Z_2 be unitary matrices obtained from the periodic Schur form (4.8). Then the matrices $\hat{B} = Z_1BQ_2^*$ and $\hat{D} = Z_2DQ_1^*$ are also invertible, and the matrices $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ are all upper triangular. Therefore,

$$\begin{aligned} \hat{D}^{-1}\hat{C}\hat{B}^{-1}\hat{A} &= Q_1(D^{-1}C)Q_2^*Q_2(B^{-1}A)Q_1^* = Q_1\Phi Q_1^*, \\ \hat{B}^{-1}\hat{A}\hat{D}^{-1}\hat{C} &= Q_2(B^{-1}A)Q_1^*Q_1(D^{-1}C)Q_2^* = Q_2\Phi Q_2^* \end{aligned} \tag{4.9}$$

are two upper triangular matrices with the same diagonals (namely, the ordered eigenvalues of Φ). They are thus both Schur forms of the same matrix Φ. Since the matrix Φ is simple, the Schur form of Φ (for a particular ordering of the eigenvalues) is unique up to a diagonal unitary scaling matrix. Therefore, we can scale the matrices Q_1 and Q_2 such that $Q := Q_1 = Q_2$, while the matrices $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ remain upper-triangular. ∎

Definition 4.2. *The matrix $\Phi := D^{-1}CB^{-1}A$ of the periodic system*

$$Bx_{k+1,\ell} = Ax_{k,\ell}, \quad Dx_{k+1,\ell+1} = Cx_{k+1,\ell},$$

is called the monodromy matrix. *It describes the state transition over the period of the system (see [14]).*

Corollary 4.3. *Let the $n \times n$ complex matrices B and D be invertible and let the monodromy matrix $\Phi := D^{-1}CB^{-1}A$ be a simple matrix. Then there exists a periodic Schur form*

$$\begin{bmatrix} -\hat{A} & z\hat{B} \\ z\hat{D} & -\hat{C} \end{bmatrix} := \begin{bmatrix} Z_1 & \\ & Z_2 \end{bmatrix} \begin{bmatrix} -A & zB \\ zD & -C \end{bmatrix} \begin{bmatrix} Q_1^* & \\ & Q_2^* \end{bmatrix}, \tag{4.10}$$

where $Q_1 = Q_2$ if and only if the matrices $B^{-1}A$ and $D^{-1}C$ commute.

Proof. The "if" part was shown in Theorem 4.1. The "only if" part follows from (4.9), since $Q_1 = Q_2$ implies that $\hat{B}^{-1}\hat{A}$ and $\hat{D}^{-1}\hat{C}$ commute, and therefore also $B^{-1}A$ and $D^{-1}C$ commute. ∎

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Notice that the periodic Schur form always exists for cyclic regular pencils, even when the matrices B and D are not invertible. The singular case can be seen as a limiting case, where singular matrices B and D can be seen as limits of nonsingular matrices B_ϵ and D_ϵ tending in the limit to B and D. The existence of the periodic Schur form is then guaranteed by the Bolzano–Weierstrass theorem since the set of unitary matrices is compact. Unfortunately, the necessary and sufficient condition of Corollary 4.3 does not hold anymore then. We will return to this point in a later section, and when we discuss the trajectory independence of a periodic 2D descriptor system.

In the case that B and D are the identity matrix, the statement of Corollary 4.3 reduces to the simultaneous triangularization of two commuting matrices, which is a classical result of matrix theory [6, Theorem 2.3.3]. Corollary 4.3 extends that to the product of two quotients $B^{-1}A$ and $D^{-1}C$ that commute.

5. An arithmetic of matrix pencils. We recall in this section the results of [1] regarding arithmetic operations with pencils such as $zB - A$ and $zD - C$ that we consider in this paper. We formulate them for the special case of regular pencils because this is the only case needed here. In [1], the *matrix relation* on $\mathbb{C}^n$ is defined to be

$$(B\backslash A) = \{(y_1, y_2) \in \mathbb{C}^n \times \mathbb{C}^n \mid Ay_1 = By_2\}. \tag{5.11}$$

Such a matrix relation is a vector subspace of $\mathbb{C}^n \times \mathbb{C}^n$. If B is invertible, then this matrix relation is a linear transformation that has the matrix representation $y_2 = (B^{-1}A)y_1$. Notice that such a relation is invariant under left multiplication with an invertible $n \times n$ matrix S :

$$\det(S) \neq 0 \;\Rightarrow\; (B\backslash A) = (SB\backslash SA).$$

We can use this notation to define the *product* of two such matrix relations

$$(D\backslash C)(B\backslash A) = \{(y_1, y_3) \in \mathbb{C}^n \times \mathbb{C}^n \mid \exists y_2 \text{ s.t. } \begin{bmatrix} -A & B & \\ & -C & D \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0\}, \tag{5.12}$$

which represents the *product* $y_3 = (D^{-1}C)(B^{-1}A)y_1$ of the two relations $y_3 = (D^{-1}C)y_2$ and $y_2 = (B^{-1}A)y_1$ when B and D are invertible.

We quote the following theorem from [1] which is specialized here to the case of regular pencils. For the proof, we refer to [1].

THEOREM 5.1. *Consider the relations $(B\backslash A)$ and $(D\backslash C)$ where the assumptions of Theorem 3.3 hold. Then the matrices $\begin{bmatrix} B \\ -C \end{bmatrix}$ and $\begin{bmatrix} D \\ -A \end{bmatrix}$ have full column rank n and have n-dimensional left null spaces satisfying*

$$\begin{bmatrix} C_+ & B_+ \end{bmatrix} \begin{bmatrix} B \\ -C \end{bmatrix} = 0, \quad \begin{bmatrix} C_+ & B_+ \end{bmatrix} \begin{bmatrix} C_+^* \\ B_+^* \end{bmatrix} = I_n,$$

$$\begin{bmatrix} A_+ & D_+ \end{bmatrix} \begin{bmatrix} D \\ -A \end{bmatrix} = 0, \quad \begin{bmatrix} A_+ & D_+ \end{bmatrix} \begin{bmatrix} A_+^* \\ D_+^* \end{bmatrix} = I_n, \tag{5.13}$$

and

$$(D\backslash C)(B\backslash A) = (B_+D\backslash C_+A), \quad (B\backslash A)(D\backslash C) = (D_+B\backslash A_+C).$$

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REMARK 5.2. *A basis-free formulation (used in [1]) of the orthogonal complement used in the above theorem would be*

$$\ker \begin{bmatrix} C_+ & B_+ \end{bmatrix} = \mathrm{Im}\left(\begin{bmatrix} B \\ -C \end{bmatrix}\right), \quad \ker \begin{bmatrix} A_+ & D_+ \end{bmatrix} = \mathrm{Im}\left(\begin{bmatrix} D \\ -A \end{bmatrix}\right),$$

and this does not require any normalization.

6. Trajectory independence in the basic cell. We have now the appropriate tool to analyze the independence of the two paths

$$x_{k,\ell} \longrightarrow x_{k+1,\ell} \longrightarrow x_{k+1,\ell+1} \quad \text{and} \quad x_{k,\ell} \longrightarrow x_{k,\ell+1} \longrightarrow x_{k+1,\ell+1},$$

in the basic cell of a periodic 2D descriptor system. It follows from the periodic decomposition that the pencils must be regular. Since the subsystems are in descriptor form, one cannot compare the trajectories from $x_{k,\ell}$ to $x_{k+1,\ell+1}$ using the commutativity of $B^{-1}A$ and $D^{-1}C$. Instead, we can consider the following two-point boundary value problems

$$\begin{bmatrix} -A & B & 0 \\ 0 & -C & D \\ W_{k,\ell} & 0 & W_{k+1,\ell+1} \end{bmatrix} \begin{bmatrix} x_{k,\ell} \\ x_{k+1,\ell} \\ x_{k+1,\ell+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix}, \tag{6.14}$$

and

$$\begin{bmatrix} -C & D & 0 \\ 0 & -A & B \\ W_{k,\ell} & 0 & W_{k+1,\ell+1} \end{bmatrix} \begin{bmatrix} x_{k,\ell} \\ x_{k,\ell+1} \\ x_{k+1,\ell+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix}, \tag{6.15}$$

where w is an arbitrary n-vector and $W_{k,\ell}$ and $W_{k+1,\ell+1}$ are $n \times n$ matrices that make the systems (6.14) and (6.15) have a unique solution. The following theorem then gives necessary and sufficient conditions for the two trajectories to always give the same solutions for the end points $x_{k,\ell}$ and $x_{k+1,\ell+1}$.

THEOREM 6.1. *Let the 2D periodic system given in (2.1) be conditionable. Then any two trajectories $x_{k,\ell} \to x_{k+1,\ell} \to x_{k+1,\ell+1}$ and $x_{k,\ell} \to x_{k,\ell+1} \to x_{k+1,\ell+1}$ corresponding to the two-point boundary value problems (6.14) and (6.15) have the same end points $x_{k,\ell}$ and $x_{k+1,\ell+1}$ for all conditionable end point conditions $W_{k,\ell}$, $W_{k+1,\ell+1}$ and w, if and only if the orthogonal complements $\begin{bmatrix} C_+ & B_+ \end{bmatrix} \in \mathbb{C}^{n\times 2n}$ and $\begin{bmatrix} A_+ & D_+ \end{bmatrix} \in \mathbb{C}^{n\times 2n}$ defined from*

$$\begin{bmatrix} C_+ & B_+ \end{bmatrix} \begin{bmatrix} B \\ -C \end{bmatrix} = 0, \quad \begin{bmatrix} C_+ & B_+ \end{bmatrix} \begin{bmatrix} C_+^* \\ B_+^* \end{bmatrix} = I_n,$$

$$\begin{bmatrix} A_+ & D_+ \end{bmatrix} \begin{bmatrix} D \\ -A \end{bmatrix} = 0, \quad \begin{bmatrix} A_+ & D_+ \end{bmatrix} \begin{bmatrix} A_+^* \\ D_+^* \end{bmatrix} = I_n,$$

satisfy

$$\mathrm{rank}\left(\begin{bmatrix} -C_+A & B_+D \\ -A_+C & D_+B \end{bmatrix}\right) = n. \tag{6.16}$$

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Proof. Since the system (2.1) is conditionable, the matrices $\begin{bmatrix} B \\ -C \end{bmatrix}$ and $\begin{bmatrix} D \\ -A \end{bmatrix}$ have full column rank n, and hence the QR factorizations

$$\begin{bmatrix} B \\ -C \end{bmatrix} = Q_{bc} \begin{bmatrix} R_{bc} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} D \\ -A \end{bmatrix} = Q_{da} \begin{bmatrix} R_{da} \\ 0 \end{bmatrix}, \tag{6.17}$$

have invertible $n \times n$ factors R_{bc} and R_{da} and the bottom n rows of Q_{bc}^* and Q_{da}^* define the orthogonal complements $\begin{bmatrix} C_+ & B_+ \end{bmatrix} \in \mathbb{C}^{n\times 2n}$ and $\begin{bmatrix} A_+ & D_+ \end{bmatrix} \in \mathbb{C}^{n\times 2n}$ (notice that they are each defined only up to a left unitary factor). Multiplying the two point boundary value problems (6.14) and (6.15) with $Q_{bc}^* \oplus I_n$ and $Q_{da}^* \oplus I_n$, respectively, yields

$$\begin{bmatrix} X_{bc} & R_{bc} & Y_{bc} \\ -C_+A & 0 & B_+D \\ W_{k,\ell} & 0 & W_{k+1,\ell+1} \end{bmatrix} \begin{bmatrix} x_{k,\ell} \\ x_{k+1,\ell} \\ x_{k+1,\ell+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix}, \tag{6.18}$$

$$\begin{bmatrix} X_{da} & R_{da} & Y_{da} \\ -A_+C & 0 & D_+B \\ W_{k,\ell} & 0 & W_{k+1,\ell+1} \end{bmatrix} \begin{bmatrix} x_{k,\ell} \\ x_{k,\ell+1} \\ x_{k+1,\ell+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix}. \tag{6.19}$$

This shows that the subproblems

$$\begin{bmatrix} -C_+A & B_+D \\ W_{k,\ell} & W_{k+1,\ell+1} \end{bmatrix} \begin{bmatrix} x_{k,\ell} \\ x_{k+1,\ell+1} \end{bmatrix} = \begin{bmatrix} 0 \\ w \end{bmatrix},$$

$$\begin{bmatrix} -A_+C & D_+B \\ W_{k,\ell} & W_{k+1,\ell+1} \end{bmatrix} \begin{bmatrix} x_{k,\ell} \\ x_{k+1,\ell+1} \end{bmatrix} = \begin{bmatrix} 0 \\ w \end{bmatrix},$$

have the same solution, provided $\begin{bmatrix} -C_+A & B_+D \end{bmatrix}$ and $\begin{bmatrix} -A_+C & D_+B \end{bmatrix}$ span the same rowspace, and this is equivalent to the rank condition (6.16). ☐

Remark 6.2. *We point out that once the boundary states $x_{k,\ell}$ and $x_{k+1,\ell+1}$ are computed, one can find the intermediate states $x_{k+1,\ell}$ and $x_{k,\ell+1}$ from the top equations in (6.18) and (6.19), since R_{bd} and R_{da} are invertible. These states will in general be different.*

Remark 6.3. *The condition of the above theorem thus generalizes the commutativity condition $B^{-1}AD^{-1}C = D^{-1}CB^{-1}A$ for invertible matrices B and D, which can be retrieved from this theorem by choosing*

$$\begin{bmatrix} C_+ & B_+ \end{bmatrix} = S_{bc} \begin{bmatrix} CB^{-1} & I_n \end{bmatrix}, \quad \begin{bmatrix} A_+ & D_+ \end{bmatrix} = S_{da} \begin{bmatrix} AD^{-1} & I_n \end{bmatrix},$$

for some invertible matrices S_{bc} and S_{da} normalizing the orthogonal complements. We then have

$$\operatorname{rank}\left(\begin{bmatrix} -S_{bc}CB^{-1}A & S_{bc}D \\ -S_{da}AD^{-1}C & S_{da}B \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} -D^{-1}CB^{-1}A & I_n \\ -B^{-1}AD^{-1}C & I_n \end{bmatrix}\right) = n,$$

which implies $D^{-1}CB^{-1}A = B^{-1}AD^{-1}C$.

Remark 6.4. *The rank condition (6.16) can also be expressed as*

$$(B_+D\backslash C_+A) = (D_+B\backslash A_+C),$$

which by Theorem 5.1 also implies

$$(D\backslash C)(B\backslash A) = (B\backslash A)(D\backslash C),$$

and this nicely generalizes the commutativity condition for the case of invertible matrices B and D.

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7. Interpreting the commutativity condition. In order to better understand the rank condition (6.16), we use the equivalent quadratic polynomial matrices

$$\begin{bmatrix} -A & B \\ z^2D & -C \end{bmatrix}, \quad \begin{bmatrix} -C & D \\ z^2B & -A \end{bmatrix},$$

which have the same determinant, and therefore also the same finite zeros, as the pencils (3.7). When B and D are invertible, we can apply the following invertible transformations

$$\begin{bmatrix} I_n & 0 \\ D^{-1}CB^{-1} & D^{-1} \end{bmatrix}\begin{bmatrix} -A & B \\ z^2D & -C \end{bmatrix} = \begin{bmatrix} -A & B \\ z^2I_n - D^{-1}CB^{-1}A & 0 \end{bmatrix},$$

$$\begin{bmatrix} I_n & 0 \\ B^{-1}AD^{-1} & B^{-1} \end{bmatrix}\begin{bmatrix} -C & D \\ z^2B & -A \end{bmatrix} = \begin{bmatrix} -C & D \\ z^2I_n - B^{-1}AD^{-1}C & 0 \end{bmatrix},$$

which show the same dynamical system over two time steps when $B^{-1}A$ and $D^{-1}C$ commute. If, instead, we use the orthogonal transformations from (6.17), we obtain

$$Q_{bc}^*\begin{bmatrix} -A & B \\ z^2D & -C \end{bmatrix} = \begin{bmatrix} \times & R_{bc} \\ z^2B_+D - C_+A & 0 \end{bmatrix},$$

$$Q_{da}^*\begin{bmatrix} -C & D \\ z^2B & -A \end{bmatrix} = \begin{bmatrix} \times & R_{da} \\ z^2D_+B - A_+C & 0 \end{bmatrix},$$

which shows again the same dynamical behavior over two time steps if there exists a constant invertible transformation S such that $S(z^2D_+B - A_+C) = (z^2B_+D - C_+A)$ and that is equivalent to the rank condition (6.16).

REMARK 7.1. *It is shown in [1] that when the system matrices A, B, C, and D are all upper triangular, then the matrices A_+, B_+, C_+, and D_+ of the orthogonal complements, can also be chosen to be upper triangular. The pencils $(\zeta D_+B - A_+C)$ and $(\zeta B_+D - C_+A)$ (where we"redefine" z^2 as a single variable ζ) are then also upper triangular and hence in generalized Schur form. Moreover, when the systems commute, these two pencils have the same deflating subspaces, since they are related by a constant left transformation S.*

We can also link the eigenspaces at a particular eigenvalue of the cyclic pencil $\begin{bmatrix} -A & zB \\ zD & -C \end{bmatrix}$, with those of the polynomial matrices $(z^2B_+D - C_+A)$ and $(z^2D_+B - A_+C)$, seen as a pencil in the variable $\zeta := z^2$.

THEOREM 7.2. *Let $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ be a full rank basis satisfying*

$$\mathrm{Im}\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right) = \ker\begin{bmatrix} -\beta A & \alpha B \\ \alpha D & -\beta C \end{bmatrix}, \tag{7.20}$$

where α and β are supposed to be different from 0. Then $\mathrm{Im}\,(X_1)$ and $\mathrm{Im}\,(X_2)$ are also full rank bases, satisfying

$$\mathrm{Im}\,(X_1) = \ker\left[\alpha^2B_+D - \beta^2C_+A\right], \quad \mathrm{Im}\,(X_2) = \ker\left[\alpha^2D_+B - \beta^2A_+C\right]. \tag{7.21}$$

Also, $\mathrm{Im}\,(X_1) = \mathrm{Im}\,(X_2)$ when the 2D system has compatible trajectories.

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Proof. Let $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ be a full rank basis satisfying

$$\begin{bmatrix} -\beta A & \alpha B \\ \alpha D & -\beta C \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} -\beta C & \alpha D \\ \alpha B & -\beta A \end{bmatrix} \begin{bmatrix} X_2 \\ X_1 \end{bmatrix} = 0. \tag{7.22}$$

It follows from (5.1) that $\begin{bmatrix} \beta C_+ & \alpha B_+ \end{bmatrix}$ and $\begin{bmatrix} \beta A_+ & \alpha D_+ \end{bmatrix}$ are full rank bases for the orthogonal complements of the full rank matrices

$$D_{\alpha\beta} \begin{bmatrix} B \\ -C \end{bmatrix} = \begin{bmatrix} \alpha B \\ -\beta C \end{bmatrix} \quad \text{and} \quad D_{\alpha\beta} \begin{bmatrix} D \\ -A \end{bmatrix} = \begin{bmatrix} \alpha D \\ -\beta A \end{bmatrix}, \tag{7.23}$$

where $D_{\alpha\beta} := \alpha I_n \oplus \beta I_n$ and $\alpha, \beta \neq 0$. Then, left multiplying (7.22) with the invertible matrices $Q_{bc}^* D_{\beta\alpha}$ and $Q_{da}^* D_{\beta\alpha}$ (where Q_{bc} and Q_{da} are defined in (6.17)) yields, respectively, the subsystems of equations

$$\left[\alpha^2 B_+ D - \beta^2 C_+ A\right] X_1 = 0 \quad \text{and} \quad \left[\alpha^2 D_+ B - \beta^2 A_+ BC\right] X_2 = 0.$$

It follows from the invertibility of these transformations that the nullity of these two pencils at the eigenvalue α^2/β^2 is that of the cyclic pencil (7.23) at the eigenvalue α/β. Therefore, X_1 and X_2 will be bases of the corresponding null spaces, provided they have full column rank. The full rank property of X_1 and X_2 follows from the full rank property of the matrices in (7.23). Indeed, suppose that X_1 does not have full column rank. Then there exists a vector x such that $X_1 x = 0$ and $y := X_2 x \neq 0$. But then y must be in the kernel of the first matrix in (7.23), which is a contradiction. The same reasoning can be used to show that X_2 is a full rank basis.

Finally, if the systems commute, then $\operatorname{Im}(X_1)$ and $\operatorname{Im}(X_2)$ are bases for the null space of the pencils $(\zeta B_+ D - C_+ A)$ and $(\zeta D_+ B - A_+ C)$ evaluated at the eigenvalue $\zeta = \alpha^2/\beta^2$. Since these pencils are related by an invertible left transformation S, the null spaces must be equal. ∎

Corollary 7.3. *Let*

$$\begin{bmatrix} -\beta A & \alpha B \\ \alpha D & -\beta C \end{bmatrix},$$

be singular for $\alpha, \beta \neq 0$. Then if the subsystems $zB - A$ and $zD - C$ of the periodic descriptor system commute, they have a common eigenvector x, and

$$\begin{bmatrix} -\beta A & \alpha B \\ \alpha D & -\beta C \end{bmatrix} \begin{bmatrix} \gamma x \\ \delta x \end{bmatrix} = 0,$$

for $\gamma, \delta \neq 0$.

Proof. It follows from Theorem 7.2 that the null-space bases X_1 and X_2 then satisfy $X_2 = X_1 T$ for some invertible matrix T. Now choose any eigenvector y of T, then $Ty = \lambda y$, for $\lambda \neq 0$. It then suffices to choose $\gamma x = X_1 y$ and $\delta x = X_2 y = X_1 T y = \lambda X_1 y$. ∎

Remark 7.4. *Theorem 7.2 and Corollary 7.3 do not extend to the eigenvalues $z = 0$ and $z = \infty$ (i.e. when α or β are 0), as the following example shows. Let $B = D = I_2$, $A = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, and $C = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then the equations (7.22) become*

$$\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right] = 0, \qquad \left[\begin{array}{cc|cc} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right] \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \\ \hline 1 & 0 \\ 0 & 0 \end{array}\right] = 0.$$

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Clearly, X_1 and X_2 are not full rank and their images are not equal, while they commute since $B^{-1}AD^{-1}C = D^{-1}CB^{-1}A = 0$.

Theorem 7.5. *Let $\begin{bmatrix} -A & zB \\ zD & -C \end{bmatrix}$ be a regular cyclic pencil with eigenvalue $z = \alpha/\beta$, normalized using $\beta \in \mathbb{R}$ and $|\alpha|^2 + \beta^2 = 1$. If the subsystems $zB - A$ and $zD - C$ commute, then there exists a corresponding eigenvector*

$$\begin{bmatrix} -\beta A & \alpha B \\ \alpha D & -\beta C \end{bmatrix} \begin{bmatrix} \gamma x \\ \delta x \end{bmatrix} = 0,$$

with $x \neq 0$ and $(\gamma, \delta) \neq (0,0)$.

Proof. For the case $\alpha \neq 0$ and $\beta \neq 0$, this was essentially shown in Corollary 7.3. For $\alpha = 0$, the eigenvalue is $z = 0$, and we consider two cases. One of the two matrices A and C must be singular. If A is singular, we choose a nonzero vector $x \in \ker(A)$ and $(\gamma, \delta) = (1, 0)$. If C is singular, we choose a nonzero vector $x \in \ker(C)$ and $(\gamma, \delta) = (0, 1)$. For $\beta = 0$, the eigenvalue is $z = \infty$, and we again consider two cases. One of the two matrices B and D must then be singular. If B is singular, we choose a nonzero vector $x \in \ker(B)$ and $(\gamma, \delta) = (0, 1)$. If D is singular, we choose a nonzero vector $x \in \ker(D)$ and $(\gamma, \delta) = (1, 0)$. ∎

8. Periodic Schur form of commuting descriptor systems. The results of the previous section can now be used to extend the periodic Schur decomposition of Theorems 4.1 and 4.3 to 2D periodic descriptor systems with compatible basic trajectories.

Theorem 8.1. *Let the $2n \times 2n$ pencil*

$$\begin{bmatrix} -A & zB \\ zD & -C \end{bmatrix}, \tag{8.24}$$

be regular. Then there exists a periodic Schur form

$$\begin{bmatrix} -\hat{A} & z\hat{B} \\ z\hat{D} & -\hat{C} \end{bmatrix} := \begin{bmatrix} Z_1 & \\ & Z_2 \end{bmatrix} \begin{bmatrix} -A & zB \\ zD & -C \end{bmatrix} \begin{bmatrix} Q_1^* & \\ & Q_2^* \end{bmatrix}, \tag{8.25}$$

with $Q_1 = Q_2$ if the two basic cell trajectories of the corresponding 2D periodic descriptor system (2.1) are compatible.

Proof. Assume that α/β is a normalized eigenvalue of the cyclic pencil (8.24), then there exists a corresponding eigenvector satisfying

$$\begin{bmatrix} -\beta A & \alpha B \\ \alpha D & -\beta C \end{bmatrix} \begin{bmatrix} \gamma x \\ \delta x \end{bmatrix} = 0,$$

with $(\gamma, \delta) \neq (0,0)$, and there exists a matrix Q such that Qx is proportional to $\mathbf{e}_1$. Therefore, $AQ^*\mathbf{e}_1$ and $BQ^*\mathbf{e}_1$ are parallel[1] and there exists a unitary matrix Z_1 such that $Z_1AQ\mathbf{e}_1 = a\mathbf{e}_1$ and $Z_1BQ\mathbf{e}_1 = b\mathbf{e}_1$ for some $(a, b) \neq (0, 0)$. Similarly, $DQ^*\mathbf{e}_1$ and $CQ^*\mathbf{e}_1$ are also parallel and again this implies that there exists a unitary matrix Z_2 such that $Z_2DQ^*\mathbf{e}_1 = d\mathbf{e}_1$ and $Z_2CQ^*\mathbf{e}_1 = c\mathbf{e}_1$ for some $(c, d) \neq (0, 0)$. We have thus constructed unitary matrices Q, Z_1, and Z_2 giving the block decomposition

$$\left[\begin{array}{cc|cc} -a & \times & zb & z\times \\ 0 & -\tilde{A} & 0 & z\tilde{B} \\ \hline zd & z\times & -c & \times \\ 0 & z\tilde{D} & 0 & -\tilde{C} \end{array}\right] := \begin{bmatrix} Z_1 & \\ & Z_2 \end{bmatrix} \left[\begin{array}{c|c} -A & zB \\ \hline zD & -C \end{array}\right] \begin{bmatrix} Q^* & \\ & Q^* \end{bmatrix},$$

[1] We say x and y are parallel when $ax = by$ with $(a, b) \neq (0, 0)$.

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which is the first stage of the recursive construction of a periodic Schur form. Notice also that there are two roots of equal modulus associated with the vector x. It is easy to see that the compatibility conditions are preserved by these transformations and that they also hold for the subpencil

$$\left[\begin{array}{c|c} -\tilde{A} & z\tilde{B} \\ \hline z\tilde{D} & -\tilde{C} \end{array}\right].$$

The recursive construction can therefore be repeated on that subsystem, which then proves the result. ∎

In the case of commuting trajectories, the existence of a periodic Schur form (8.25) with $Q_1 = Q_2$ also implies that the subsystems $zB - A$ and $zD - C$ of the 2D periodic system are regular as well. This is shown in the next theorem.

THEOREM 8.2. *Let the regular pencil* (8.24) *have a periodic Schur form* (8.25) *with* $Q_1 = Q_2$*, then the pencils* $zB - A$ *and* $zD - C$ *of the corresponding* 2D *basic cell are also regular.*

Proof. Since the regular pencil (8.25) is in periodic Schur form, the four blocks $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ are upper triangular [7, 15]. Reordering this pencil in a block upper triangular form of 2×2 blocks yields as diagonal blocks the 2×2 pencils

$$\begin{bmatrix} -\hat{a}_{i,i} & z\hat{b}_{i,i} \\ z\hat{d}_{i,i} & -\hat{c}_{i,i} \end{bmatrix},$$

that must also be regular. Therefore, none of the pairs $(\hat{a}_{i,i}, \hat{b}_{i,i})$ or $(\hat{c}_{i,i}, \hat{d}_{i,i})$ can be simultaneously zero. Since $Q_1 = Q_2$, this implies that the pencils

$$z\hat{B} - \hat{A} = Z_1(zB - A)Q^*, \quad z\hat{D} - \hat{C} = Z_2(zD - C)Q^*, \quad \text{where} \quad Q := Q_1 = Q_2,$$

are the generalized Schur forms of $zB - A$ and $zD - C$, respectively. Therefore, $z\hat{B} - \hat{A}$ and $z\hat{D} - \hat{C}$ are regular because their diagonal elements are not simultaneously zero. ∎

9. A block diagonal decomposition.

The periodic Schur decomposition can be further updated in a block diagonal decomposition as indicated in the following theorem, inspired by [14].

THEOREM 9.1. *Let the pencil* (4.10) *be regular and in periodic Schur form with upper triangular matrices* $\{\hat{A}, \hat{B}, \hat{C}, \hat{D}\}$ *and ordered diagonal elements* $\hat{d}_{i,i}^{-1}\hat{c}_{i,i}\hat{b}_{i,i}^{-1}\hat{a}_{i,i}$. *Then there exist invertible upper triangular matrices* $\{S_1, S_2\}$ *and* $\{T_1, T_2\}$ *such that*

$$\begin{bmatrix} -\tilde{A} & z\tilde{B} \\ z\tilde{D} & -\tilde{C} \end{bmatrix} := \begin{bmatrix} S_1 & \\ & S_2 \end{bmatrix} \begin{bmatrix} -\hat{A} & z\hat{B} \\ z\hat{D} & -\hat{C} \end{bmatrix} \begin{bmatrix} T_1 & \\ & T_2 \end{bmatrix}, \tag{9.26}$$

where

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \tilde{A}_k \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \tilde{B}_k \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} \tilde{C}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \tilde{C}_k \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} \tilde{D}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \tilde{D}_k \end{bmatrix},$$

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and where the quadruples $\{\tilde{A}_j, \tilde{B}_j, \tilde{C}_j, \tilde{D}_j\} \in \mathbb{C}^{n_j \times n_j}$, $j \in [1:k]$ *have a repeated squared root in each block* j *and* $\sum_{j=1}^{k} n_j = n$. *Moreover, if the system is 2D periodic with compatible trajectories in each basic cell, then the matrices* T_1 *and* T_2 *can be chosen equal.*

Proof. The proof is based on [14] and on the solution of generalized Sylvester equations as described in Lemma A.1 of the Appendix. We start from the periodic Schur form with the nonincreasing ordering of the diagonal quadruples. Let $\{\tilde{A}_1, \tilde{B}_1, \tilde{C}_1, \tilde{D}_1\}$ be the leading quadruple containing all the eigenvalues of largest modulus (which is possibly ∞). We show in Lemma A.1 that this block can be decoupled from the rest of the periodic Schur form, using a block transformation of the form (9.26). Moreover, in the case of compatible trajectories, $T_1 = T_2$. The same reduction procedure is then repeated on the bottom subpencil, which proves the result. □

The conditions of Theorem 6.1 are still valid after applying periodic transformations on the above two pencils, and this as well for unitary transformations as invertible ones. The reason for this is that such transformations correspond to block diagonal invertible transformations applied to the two-point boundary value problems (6.14) and (6.15). We can thus recast the formulation of the theorem in the coordinate system of the above block diagonal decomposition. This then leads to the following theorem.

THEOREM 9.2. *Consider the* 2*D periodic system* (2.1) *and its corresponding block diagonal decomposition. Then the states corresponding to the different trajectories are compatible provided the following conditions are satisfied*

- *the pencil* $\begin{bmatrix} -\tilde{A} & z\tilde{B} \\ z\tilde{D} & -\tilde{C} \end{bmatrix}$ *is regular*
- *the quadruples* $\{\tilde{A}_j, \tilde{B}_j, \tilde{C}_j, \tilde{D}_j\} \in \mathbb{C}^{n_j \times n_j}$, $j \in [1:k]$, *commute in the sense that*

$$\tilde{A}_j^{-1}\tilde{B}_j\tilde{C}_j^{-1}\tilde{D}_j = \tilde{C}_j^{-1}\tilde{D}_j\tilde{A}_j^{-1}\tilde{B}_j \quad \text{or} \quad \tilde{D}_j^{-1}\tilde{C}_j\tilde{B}_j^{-1}\tilde{A}_j = \tilde{B}_j^{-1}\tilde{A}_j\tilde{D}_j^{-1}\tilde{C}_j,$$

depending on which matrices are invertible.

Proof. We can rephrase the conditions by looking at the separate blocks, since in that coordinate system, these blocks are decoupled from each other. We then retrieve the standard commutativity conditions as explained in Remark 6.3. □

REMARK 9.3. *The decomposition described in Theorem 9.1 can be updated so that the diagonal invertible blocks are transformed to the identity. Such transformations can all be absorbed in the left factors* S_i *and are thus compatible with that theorem. In such a case, the commutativity conditions reduce to the commutativity of the product of two matrices. Such conditions are described, e.g., in* [5].

10. Extensions.

10.1. Arbitrary trajectories. Once we put ourselves in the coordinate system where the spectral blocks are decoupled, we can analyze the two point boundary value problem between any two points (k,ℓ) and (m,n) provided $m \geq k$ and $n \geq \ell$. In the language of matrix relations, it is clear that since

$$(D\backslash C)(B\backslash A) = (B\backslash A)(D\backslash C),$$

all transitions between (k,ℓ) and (m,n) are described by the matrix relation

$$(D\backslash C)^{n-\ell}(B\backslash A)^{m-k},$$

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and this can be specialized to the individual blocks of the diagonal decomposition as

$$(\tilde{D}_i^{-1}\tilde{C}_i)^{n-\ell}(\tilde{B}_i^{-1}\tilde{A}_i)^{m-k} \quad \text{or} \quad (\tilde{A}_j^{-1}\tilde{B}_j)^{m-k}(\tilde{C}_j^{-1}\tilde{D}_j)^{n-\ell},$$

depending on which ones are invertible.

10.2. Real coefficients. All results derived in this paper were obtained with the assumption that the matrices A, B, C, and D were complex because the eigenvalue problems are simpler when considering complex arithmetic. The major difference is that the periodic Schur form is now block triangular, rather than just triangular. This also affects the results of the ordered block diagonal spectral decomposition since it requires that complex conjugate eigenvalues (which are of equal modulus) are kept in the same diagonal blocks. The results of this paper therefore essentially carry over to the real case as well, provided we take into account these modifications.

10.3. The multidimensional case. The extension of the problem to several dimensions easily follows from the two-dimensional case. For simplicity, we will only treat the three-dimensional case since the more general case is obtained using essentially the same arguments. We are thus considering the basic cell

$$\begin{array}{rlrl} Bx_{k+1,\ell,m} &= Ax_{k,\ell,m}, & Dx_{k+1,\ell+1,m} &= Cx_{k+1,\ell,m}, \\ Dx_{k,\ell+1,m} &= Cx_{k,\ell,m}, & Bx_{k+1,\ell+1,m} &= Ax_{k,\ell+1,m}, \\[6pt] Fx_{k,\ell,m+1} &= Ex_{k,\ell,m}, & Fx_{k+1,\ell+1,m+1} &= Ex_{k+1,\ell+1,m}, \\ Fx_{k+1,\ell,m+1} &= Ex_{k+1,\ell,m}, & Fx_{k,\ell+1,m+1} &= Ex_{k,\ell+1,m}, \\[6pt] Bx_{k+1,\ell,m+1} &= Ax_{k,\ell,m+1}, & Dx_{k+1,\ell+1,m+1} &= Cx_{k+1,\ell,m+1}, \\ Dx_{k,\ell+1,m+1} &= Cx_{k,\ell,m+1}, & Bx_{k+1,\ell+1,m+1} &= Ax_{k,\ell+1,m+1}. \end{array} \tag{10.27}$$

This involves three different descriptor systems $zB-A$, $zD-C$, and $zF-E$, which can all be transformed on the left with independent transformations S_1, S_2, and S_3, but with a unique state transformation T on the right. These systems describe the evolution in each direction of the 3D space. It is again clear that the state transition from $x_{k,\ell,m}$ to $x_{k+1,\ell+1,m+1}$ can be reached now via six different paths and the corresponding two-point boundary value problems should again have the same solutions when imposing the same boundary conditions at the two end points. If we, e.g., consider the periodic evolution

$$zB - A \quad \longrightarrow \quad zD - C \quad \longrightarrow \quad zF - E,$$

then the following theorem can be proven in a similar way as Theorem 3.3.

THEOREM 10.1. *The following conditions are equivalent :*

1. $S(1{:}3m) = \begin{bmatrix} -A & B & & & & \\ & -C & D & & & \\ & & -E & F & & \\ & & & -A & B & \\ & & & & \ddots & \ddots \end{bmatrix}$ *has full row rank for all* m,

2. $C(1{:}3m) = \begin{bmatrix} D & & & & \\ -A & B & & & \\ & -C & D & & \\ & & -E & F & \\ & & & -A & B \\ & & & & \ddots \end{bmatrix}$ *has full column rank for all* m,

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3. *the pencils* $\begin{bmatrix} -A & zB & \\ & -C & zD \\ zF & & -E \end{bmatrix}\begin{bmatrix} -C & zD & \\ & -E & zF \\ zB & & -A \end{bmatrix}, \begin{bmatrix} -E & zF & \\ & -A & zB \\ zD & & -C \end{bmatrix}$

are regular, and so are the polynomial matrices

$$\begin{bmatrix} -A & B & \\ & -C & D \\ z^3F & & -E \end{bmatrix}\begin{bmatrix} -C & D & \\ & -E & F \\ z^3B & & -A \end{bmatrix}, \begin{bmatrix} -E & F & \\ & -A & B \\ z^3D & & -C \end{bmatrix}.$$

REMARK 10.2. *We could have derived similar conditions for other ordering of the pencils $zB-A$, $zD-C$, and $zF-E$ or also have considered periodic orderings of any two pencils. Since all of these must yield the same trajectories, it follows that all pairs of subsystems must commute. The commutativity conditions using matrix relations can therefore be expressed as follows:*

$$(D\backslash C)(B\backslash A) = (B\backslash A)(D\backslash C), \quad (F\backslash E)(B\backslash A) = (B\backslash A)(F\backslash E),$$

$$(F\backslash E)(D\backslash C) = (D\backslash C)(F\backslash E),$$

which generalizes the commutativity condition for the case of invertible matrices B, D, and F.

Rather than checking the commutativity of all possible pairs, it may be more practical to use the properties of the periodic Schur form of a single cyclic pencil. The analog of the relevant theorems would be as follows.

THEOREM 10.3. *Let the $3n \times 3n$ pencil*

$$\begin{bmatrix} -A & zB & \\ & -C & zD \\ zF & & -E \end{bmatrix}, \tag{10.28}$$

be regular. Then there exists a periodic Schur form

$$\begin{bmatrix} -\hat{A} & z\hat{B} & \\ & -\hat{C} & z\hat{D} \\ z\hat{F} & & \hat{E} \end{bmatrix} := \begin{bmatrix} Z_1 & & \\ & Z_2 & \\ & & Z_3 \end{bmatrix}\begin{bmatrix} -A & zB & \\ & -C & zD \\ zF & & -E \end{bmatrix}\begin{bmatrix} Q_1^* & & \\ & Q_2^* & \\ & & Q_3^* \end{bmatrix}, \tag{10.29}$$

with $Q_1 = Q_2 = Q_3$ if the basic cell trajectories of the corresponding $3D$ periodic descriptor system (10.27)*, are compatible.*

The following analog of Theorem 8.2 can then be derived as well. The proof is omitted since it is completely analogous to that of Theorem 8.2.

THEOREM 10.4. *Let the regular pencil* (10.28) *have a periodic Schur form* (10.29) *with $Q_1 = Q_2 = Q_3$, then the pencils $zB - A$, $zD - C$, and $zF - E$ of the corresponding 3D basic cell are regular as well.*

THEOREM 10.5. *Let the pencil* (10.28) *be regular and in periodic Schur form with upper triangular matrices $\{\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}, \hat{F}\}$ and ordered diagonal elements. Then there exist invertible upper triangular matrices $\{S_1, S_2, S_3\}$ and $\{T_1, T_2, T_3\}$ such that*

$$\begin{bmatrix} -\tilde{A} & z\tilde{B} & \\ & -\tilde{C} & z\tilde{D} \\ z\tilde{F} & & -\tilde{E} \end{bmatrix} := \begin{bmatrix} S_1 & & \\ & S_2 & \\ & & S_3 \end{bmatrix}\begin{bmatrix} -\hat{A} & z\hat{B} & \\ & -\hat{C} & z\hat{D} \\ z\hat{F} & & -\hat{E} \end{bmatrix}\begin{bmatrix} T_1 & & \\ & T_2 & \\ & & T_3 \end{bmatrix}, \tag{10.30}$$

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where

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \tilde{A}_k \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} \tilde{B}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \tilde{B}_k \end{bmatrix}, \ \tilde{C} = \begin{bmatrix} \tilde{C}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \tilde{C}_k \end{bmatrix},$$

$$\tilde{D} = \begin{bmatrix} \tilde{D}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \tilde{D}_k \end{bmatrix}, \ \tilde{E} = \begin{bmatrix} \tilde{E}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \tilde{E}_k \end{bmatrix}, \ \tilde{F} = \begin{bmatrix} \tilde{F}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \tilde{F}_k \end{bmatrix},$$

and where the sixtuples $\{\tilde{A}_j, \tilde{B}_j, \tilde{C}_j, \tilde{D}_j, \tilde{E}_j, \tilde{F}_j\} \in \mathbb{C}^{n_j \times n_j}$, $j \in [1:k]$ *have a repeated cubed root in each block* j *and* $\sum_{j=1}^{k} n_j = n$. *Moreover, if the system is 3D periodic with compatible trajectories in each basic cell, then the matrices* T_1, T_2, *and* T_3 *can be chosen equal.*

Again this can then be used to test commutativity on much smaller subsystems.

11. Concluding remarks. In this paper, we showed that the commutativity of two regular descriptor systems can be expressed in terms of a simple matrix rank condition. Moreover, it can also be rewritten in terms of matrix relations that are a natural extension of the quotient of two matrices to the singular case. We also showed that the periodic Schur form, based on unitary equivalence transformations, and the subsequent block diagonal spectral decomposition, based on upper-triangular updating of the periodic Schur form, play an important role in this problem. They allow us to reformulate the commutativity properties in terms of smaller subblocks of the given pencils. Finally, we also showed that these results extend to the multidimensional case as well.

Appendix A. Appendix: Block diagonalization.

Lemma A.1. *Let the regular cyclic pencil*

$$\begin{bmatrix} -\hat{A}_1 & -\hat{A}_3 & z\hat{B}_1 & z\hat{B}_3 \\ & -\hat{A}_2 & & z\hat{B}_2 \\ z\hat{D}_1 & z\hat{D}_3 & -\hat{C}_1 & -\hat{C}_3 \\ & z\hat{D}_2 & & -\hat{C}_2 \end{bmatrix}, \tag{A.31}$$

be in its periodic Schur form where the subpencils

$$\begin{bmatrix} -\hat{A}_1 & z\hat{B}_1 \\ z\hat{D}_1 & -\hat{C}_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\hat{A}_2 & z\hat{B}_2 \\ z\hat{D}_2 & -\hat{C}_2 \end{bmatrix},$$

have disjoint spectra Λ_1 *and* Λ_2 *and every eigenvalue in* Λ_2 *is smaller or equal, in modulus, than the eigenvalues in* Λ_1. *Then the matrices* $\{\hat{A}_3, \hat{B}_3, \hat{C}_3, \hat{D}_3\}$ *can be eliminated using the following upper-triangular equivalence transformation*

$$\begin{bmatrix} I_{n_1} & Y_1 & 0 & 0 \\ & I_{n_2} & 0 & 0 \\ & & I_{n_1} & Y_2 \\ & & & I_{n_2} \end{bmatrix} \begin{bmatrix} -\hat{A}_1 & -\hat{A}_3 & z\hat{B}_1 & z\hat{B}_3 \\ & -\hat{A}_2 & & z\hat{B}_2 \\ z\hat{D}_1 & z\hat{D}_3 & -\hat{C}_1 & -\hat{C}_3 \\ & z\hat{D}_2 & & -\hat{C}_2 \end{bmatrix} \begin{bmatrix} I_{n_1} & X_1 & 0 & 0 \\ & I_{n_2} & 0 & 0 \\ & & I_{n_1} & X_2 \\ & & & I_{n_2} \end{bmatrix}$$

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$$= \begin{bmatrix} -\hat{A}_1 & & z\hat{B}_1 & \\ & -\hat{A}_2 & & z\hat{B}_2 \\ z\hat{D}_1 & & -\hat{C}_1 & 0 \\ & z\hat{D}_2 & & -\hat{C}_2 \end{bmatrix}.$$

Moreover, if the trajectories of the $2D$ *system with pencils* $zB-A$ *and* $zD-C$ *commute, then we can choose* $X_1 = X_2$.

Proof. The following reordered generalized Sylvester equation

$$\left[\begin{array}{cc|cc} I_{n_1} & 0 & Y_1 & Z_1 \\ & I_{n_1} & Z_2 & Y_2 \\ \hline & & I_{n_2} & 0 \\ & & & I_{n_2} \end{array}\right] \left[\begin{array}{cc|cc} -\hat{A}_1 & z\hat{B}_1 & -\hat{A}_3 & z\hat{B}_3 \\ z\hat{D}_1 & -\hat{C}_1 & z\hat{D}_3 & -\hat{C}_3 \\ \hline & & -\hat{A}_2 & z\hat{B}_2 \\ & & z\hat{D}_2 & -\hat{C}_2 \end{array}\right] \left[\begin{array}{cc|cc} I_{n_1} & 0 & X_1 & W_1 \\ & I_{n_1} & W_2 & X_2 \\ \hline & & I_{n_2} & 0 \\ & & & I_{n_2} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} -\hat{A}_1 & z\hat{B}_1 & & \\ z\hat{D}_1 & -\hat{C}_1 & & \\ \hline & & -\hat{A}_2 & z\hat{B}_2 \\ & & z\hat{D}_2 & -\hat{C}_2 \end{array}\right]. \tag{A.32}$$

is known to have a solution because the spectra of the diagonal blocks are disjoint. We first derive from this that the matrices W_1, W_2, Z_1 and Z_2 are 0. The system of equations (A.32) is equivalent to the coupled equations

$$\begin{bmatrix} Y_1 & Z_1 \\ Z_2 & Y_2 \end{bmatrix} \begin{bmatrix} \hat{A}_2 & 0 \\ 0 & \hat{C}_2 \end{bmatrix} + \begin{bmatrix} \hat{A}_1 & 0 \\ 0 & \hat{C}_1 \end{bmatrix} \begin{bmatrix} X_1 & W_1 \\ W_2 & X_2 \end{bmatrix} = -\begin{bmatrix} \hat{A}_3 & 0 \\ 0 & \hat{C}_3 \end{bmatrix},$$

and

$$\begin{bmatrix} Y_1 & Z_1 \\ Z_2 & Y_2 \end{bmatrix} \begin{bmatrix} 0 & \hat{B}_2 \\ \hat{D}_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \hat{B}_1 \\ \hat{D}_1 & 0 \end{bmatrix} \begin{bmatrix} X_1 & W_1 \\ W_2 & X_2 \end{bmatrix} = -\begin{bmatrix} 0 & \hat{B}_3 \\ \hat{D}_3 & 0 \end{bmatrix}.$$

From this, we can extract the following generalized Sylvester equation involving only W_1, W_2, Z_1 and Z_2 :

$$\begin{bmatrix} 0 & Z_1 \\ Z_2 & 0 \end{bmatrix} \begin{bmatrix} -\hat{A}_2 & z\hat{B}_2 \\ z\hat{D}_2 & -\hat{C}_2 \end{bmatrix} + \begin{bmatrix} -\hat{A}_1 & z\hat{B}_1 \\ z\hat{D}_1 & -\hat{C}_1 \end{bmatrix} \begin{bmatrix} 0 & W_1 \\ W_2 & 0 \end{bmatrix} = 0.$$

It follows then from the spectral properties of the subpencils, that W_1, W_2, Z_1 and Z_2 must be zero. We are then left over with the following equations in X_1, X_2, Y_1 and Y_2

$$\begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix} \begin{bmatrix} \hat{A}_2 & 0 \\ 0 & \hat{C}_2 \end{bmatrix} + \begin{bmatrix} \hat{A}_1 & 0 \\ 0 & \hat{C}_1 \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} = -\begin{bmatrix} \hat{A}_3 & 0 \\ 0 & \hat{C}_3 \end{bmatrix},$$

and

$$\begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix} \begin{bmatrix} 0 & \hat{B}_2 \\ \hat{D}_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \hat{B}_1 \\ \hat{D}_1 & 0 \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} = -\begin{bmatrix} 0 & \hat{B}_3 \\ \hat{D}_3 & 0 \end{bmatrix}.$$

Moreover, $\hat{B}_2$ and $\hat{D}_2$ are invertible because otherwise, the corresponding eigenvalues would be infinite and they would belong to the first block. We can then write

$$\begin{bmatrix} -\hat{A}_1 & -\hat{A}_3 & z\hat{B}_1 & z\hat{B}_3 \\ & -\hat{A}_2 & & z\hat{B}_2 \\ z\hat{D}_1 & z\hat{D}_3 & -\hat{C}_1 & -\hat{C}_3 \\ & z\hat{D}_2 & & -\hat{C}_2 \end{bmatrix} \begin{bmatrix} X_1 & \\ I_{n_2} & \\ & X_2 \\ & I_{n_2} \end{bmatrix} = \begin{bmatrix} -Y_1 & \\ I_{n_2} & \\ & -Y_2 \\ & I_{n_2} \end{bmatrix} \begin{bmatrix} -\hat{A}_2 & z\hat{B}_2 \\ z\hat{D}_2 & -\hat{C}_2 \end{bmatrix}$$

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which is an equation for a pair of deflating subspaces with spectrum Λ_2. We now link these to deflating subspaces of the pencils $\zeta B_+ D - C_+ A$ and $\zeta D_+ B - A_+ C$, where $\zeta := z^2$. For this we first transform the above equation by diagonal scalings to

$$\begin{bmatrix} -\hat{A}_1 & -\hat{A}_3 & \hat{B}_1 & \hat{B}_3 \\ & -\hat{A}_2 & & \hat{B}_2 \\ \zeta\hat{D}_1 & \zeta\hat{D}_3 & -\hat{C}_1 & -\hat{C}_3 \\ & \zeta\hat{D}_2 & & -\hat{C}_2 \end{bmatrix} \begin{bmatrix} X_1 & \\ I_{n_2} & \\ & X_2 \\ & I_{n_2} \end{bmatrix} = \begin{bmatrix} -Y_1 & \\ I_{n_2} & \\ & -Y_2 \\ & I_{n_2} \end{bmatrix} \begin{bmatrix} -\hat{A}_2 & \hat{B}_2 \\ \zeta\hat{D}_2 & -\hat{C}_2 \end{bmatrix}.$$

We now multiply this equation to the left by the orthogonal complement $\begin{bmatrix} C_+ & B_+ \end{bmatrix}$ of $\begin{bmatrix} B \\ -C \end{bmatrix}$ (described in Sections 5) :

$$\begin{bmatrix} \hat{C}_{+1} & \hat{C}_{+3} & \hat{B}_{+1} & \hat{B}_{+3} \\ & \hat{C}_{+2} & & \hat{B}_{+2} \end{bmatrix} \begin{bmatrix} \hat{B}_1 & \hat{B}_3 \\ & \hat{B}_2 \\ -\hat{C}_1 & -\hat{C}_3 \\ & -\hat{C}_2 \end{bmatrix} = 0$$

and by a basis $\begin{bmatrix} I_{n_2} \\ \hat{B}_2^{-1}\hat{A}_2 \end{bmatrix}$ of the kernel of $\begin{bmatrix} -\hat{A}_2 & \hat{B}_2 \end{bmatrix}$, applied to the right. Together, this yields

$$\begin{bmatrix} \zeta\hat{B}_{+1}\hat{D}_1 - \hat{C}_{+1}\hat{A}_1 & \zeta(\hat{B}_{+1}\hat{D}_3 + \hat{B}_{+3}\hat{D}_2) - (\hat{C}_{+1}\hat{A}_3 + \hat{C}_{+3}\hat{A}_2) \\ & \zeta\hat{B}_{+2}\hat{D}_2 - \hat{C}_{+2}\hat{A}_2 \end{bmatrix} \begin{bmatrix} X_1 \\ I_{n_2} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{B}_{+3} - \hat{B}_{+1}Y_2 \\ \hat{B}_{+2} \end{bmatrix} \left[\zeta\hat{D}_2 - \hat{C}_2\hat{B}_2^{-1}\hat{A}_2\right],$$

which indicates that $\operatorname{Im}\left(\begin{bmatrix} X_1 \\ I_{n_2} \end{bmatrix}\right)$ is a deflating subspace of $\zeta\hat{B}_+\hat{D} - \hat{C}_+\hat{A}$ with spectrum Λ_2. A similar derivation will indicate that $\operatorname{Im}\left(\begin{bmatrix} X_2 \\ I_{n_2} \end{bmatrix}\right)$ is a deflating subspace of the pencil $\zeta\hat{D}_+\hat{B} - \hat{A}_+\hat{C}$ with spectrum Λ_2 as well. But since $\zeta\hat{B}_+\hat{D} - \hat{C}_+\hat{A} = S(\zeta\hat{D}_+\hat{B} - \hat{A}_+\hat{C})$, the deflating subspaces are equal, and hence $X_1 = X_2$. $\square$

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