Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 322-332, February 2024.

THE VERTEX CONNECTIVITY AND THE THIRD LARGEST EIGENVALUE IN REGULAR (MULTI-)GRAPHS*

TINGYAN MA†, LIGONG WANG†, AND YANG HU†

Abstract. Let G be a simple graph or a multigraph. The vertex connectivity $\kappa(G)$ of G is the minimum size of a vertex set S such that $G-S$ is disconnected or has only one vertex. We denote by $\lambda_3(G)$ the third largest eigenvalue of the adjacency matrix of G. In this paper, we present an upper bound for $\lambda_3(G)$ in a d-regular (multi-)graph G which guarantees that $\kappa(G)\geq t+1$, which is based on the result of Abiad et al. [Spectral bounds for the connectivity of regular graphs with given order. *Electron. J. Linear Algebra* 34:428–443, 2018]. Furthermore, we improve the upper bound for $\lambda_3(G)$ in a d-regular multigraph which assures that $\kappa(G)\geq 2$.

Key words. Eigenvalue, Vertex connectivity, Regular graph, Multigraph.

AMS subject classifications. 05C50, 05C40, 05C05.

1. Introduction. Throughout this paper, we consider finite undirected (multi-)graphs. A simple graph is a graph without multiple edges or loops. A multigraph is a graph with multiple edges but no loops. Notice that a simple graph is a special case of a multigraph. Let $G=(V(G),E(G))$ be a (multi-)graph with vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$ and edge set $E(G)=\{e_1,e_2,\ldots,e_m\}$. We use $n=|V(G)|$ and $m=|E(G)|$ to denote the order and size of G, respectively. The degree $d_G(v)$ of a vertex v in G is the number of edges of G incident with v. A (multi-)graph G is d-regular if $d_G(v)=d$ for each $v\in V(G)$. For two disjoint vertex subsets V_1 and V_2 of G, we denote $[V_1,V_2]$ to be the number of edges each of which has one vertex in V_1 and the other vertex in V_2. Let G_1 and G_2 be two vertex-disjoint graphs, and we denote by $G_1\cup G_2$ the disjoint union of G_1 and G_2. Let $G[S]$ be the induced subgraph of G whose vertex set is S and whose edge set consists of all edges of G which have both end points in S. A vertex cut of G is a subset S of $V(G)$ such that $G-S$ is disconnected or has only one vertex. A k-vertex cut is a vertex cut of k elements. For a (multi-)graph G, the vertex connectivity $\kappa(G)$ is the minimum k for which G has a k-vertex cut. G is said to be k-vertex-connected if $\kappa(G)\geq k$. Similarly, the edge connectivity $\kappa'(G)$ is the minimum k for which G has a k-edge cut. For undefined terms and notions, one can refer to [3] and [4].

The adjacency matrix $A(G)$ of a (multi-)graph G is an $n\times n$ matrix in which the (i,j)-entry equal to the number of edges joining v_i and v_j. $D(G)=\mathrm{diag}(d_1,d_2,\ldots,d_n)$ is the diagonal matrix of vertex degrees of G. Let $L(G)=D(G)-A(G)$ and $Q(G)=D(G)+A(G)$ be the Laplacian matrix and the signless Laplacian matrix of G, respectively. We denote the eigenvalues of $A(G)$, $L(G)$, and $Q(G)$ by $\lambda_1(G)\geq\lambda_2(G)\geq\cdots\geq\lambda_n(G)$, $0=\mu_1(G)\leq\mu_2(G)\leq\cdots\leq\mu_n(G)$, and $q_1(G)\geq q_2(G)\geq\cdots\geq q_n(G)$, respectively. In general, the largest eigenvalues of $A(G)$, $L(G)$, and $Q(G)$ are named as the spectral radius, the Laplacian spectral radius, and the signless Laplacian spectral radius of G, respectively. For a d-regular (multi-)graph G, we know that $\lambda_1(G)=d$ and $\lambda_i(G)=d-\mu_i(G)$ for $i=1,2,\ldots,n$. Thus, we can use

*Received by the editors on June 2, 2023. Accepted for publication on February 15, 2024. Handling Editor: Sebastian M. Cioaba. Corresponding Author: Ligong Wang.

†School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an, Shaanxi 710129, P.R. China and Xi'an-Budapest Joint Research Center for Combinatorics, Northwestern Polytechnical University, Xi'an, Shaanxi 710129, P.R. China (matingylw@163.com, lgwangmath@163.com, yhumath@163.com).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 322-332, February 2024.

the Laplacian eigenvalues to express the adjacency eigenvalues for a d-regular (multi-)graph G. The second smallest Laplacian eigenvalue $\mu_2(G)$ of $L(G)$ is called as the algebraic connectivity of G.

In 1973, Fiedler [11] first obtained the following relation between the vertex connectivity $\kappa(G)$ and the algebraic connectivity $\mu_2(G)$ of a simple non-complete graph G.

Theorem 1.1. ([11]) *If G is a simple non-complete graph, then $\kappa(G) \geq \mu_2(G)$.*

Stimulated by Fiedler's above work, many scholars studied a lot of research about the relationships between the connectivity and the eigenvalues of a graph G over the past 50 years. The results on algebraic connectivity can be found in the survey [2]. In 2016, Cioabă and Gu [7] established the following theorem that relates the vertex connectivity and the second largest eigenvalue.

Theorem 1.2. ([7]) *For any connected d-regular simple graph G with $d \geq 3$, if*

$$\lambda_2(G) < \begin{cases} \frac{d-2+\sqrt{d^2+12}}{2} & \text{if } d \text{ is even,} \\ \frac{d-2+\sqrt{d^2+8}}{2} & \text{if } d \text{ is odd,} \end{cases}$$

then $\kappa(G) \geq 2$.

In 2018, Abiad et al. [1] raised and addressed the following research problem.

Problem 1.3. ([1]) *For a d-regular (multi-)graph G of a given order and for $1 \leq t \leq d-1$, what is the best upper bound for $\lambda_2(G)$ which guarantees that $\kappa'(G) \geq t+1$ or that $\kappa(G) \geq t+1$?*

Theorem 1.4. ([1]) *Let G be an n-vertex d-regular (multi-)graph, which is not obtained by duplicating edges in a complete graph on at most $t+1$ vertices. Let*

$$\phi(d,t) = \begin{cases} 2 & \text{if } G \text{ is a multigraph and } t=1, \\ 1 & \text{if } G \text{ is a multigraph and } t \geq 2, \\ d+1 & \text{if } G \text{ is a simple graph and } t=1, \\ d+1-t & \text{if } G \text{ is a simple graph and } t \geq 2, \end{cases}$$

where $1 \leq t \leq d-1$. If $\lambda_2(G) < d - \frac{dt}{2\phi(d,t)} - \frac{dt}{2(n-\phi(d,t))}$, then $\kappa(G) \geq t+1$.

For the special case where the graph G is a multigraph and $t=1$, Abiad et al. [1] got the following result.

Theorem 1.5. ([1]) *Let G be an n-vertex d-regular multigraph with $n \geq 5$ and $d \geq 3$. If $\lambda_2(G) < \frac{8n-25}{9n-25}d$, then $\kappa(G) \geq 2$.*

Recently, Liu et al. [20] characterized the following bounds about $\lambda_2(G)$, $\mu_2(G)$, or $q_2(G)$ which guarantee that $\kappa(G) \geq k$ for a simple graph G.

Theorem 1.6. ([20]) *Let G be a simple graph of order n with maximum degree Δ and minimum degree $\delta \geq k \geq 2$. Define $\alpha = \lceil \frac{1}{2}(\delta+1+\sqrt{(\delta+1)^2-2(k-1)\Delta}) \rceil$ and*

$$\phi(\delta,\Delta,k) = \begin{cases} (\delta-k+2)(n-\delta+k-2) & \text{if } \Delta \geq 2(\delta-k+2), \\ \alpha(n-\alpha) & \text{if } \delta \leq \Delta < 2(\delta-k+2). \end{cases}$$

If $\lambda_2(G) < \delta - \frac{(k-1)\Delta n}{2\phi(\delta,\Delta,k)}$, or $\mu_2(G) > \frac{(k-1)\Delta n}{2\phi(\delta,\Delta,k)}$, or $q_2(G) < 2\delta - \frac{(k-1)\Delta n}{2\phi(\delta,\Delta,k)}$, then $\kappa(G) \geq k$.

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THEOREM 1.7. ([20]) *Let G be d-regular simple graph of order n with $d \geq k \geq 2$. Define $\beta = \lceil \frac{1}{2}(d+1+\sqrt{(d+1)^2-2(k-1)d})\rceil$ and*

$$\varphi(d,k) = \begin{cases} (d+1)(n-d-1) & \text{if } k=2, \\ (d-k+2)(n-d+k-2) & \text{if } k \geq 3 \text{ and } d \leq 2k-4, \\ \beta(n-\beta) & \text{if } k \geq 3 \text{ and } d > 2k-4. \end{cases}$$

If $\lambda_2(G) < d - \frac{(k-1)dn}{2\varphi(d,k)}$, then $\kappa(G) \geq k$.

THEOREM 1.8. ([20]) *Let G be a simple bipartite graph of order n with maximum degree Δ and minimum degree $\delta \geq k \geq 2$. Define $\gamma = \lceil(\delta + \sqrt{\delta^2-(k-1)\Delta})\rceil$ and*

$$\psi(\delta,\Delta,k) = \begin{cases} (2\delta-k+1)(n-2\delta+k-1) & \text{if } \Delta \geq 2\delta-k+1, \\ \gamma(n-\gamma) & \text{if } \delta \leq \Delta < 2\delta-k+1. \end{cases}$$

If $\lambda_2(G) < \delta - \frac{(k-1)\Delta n}{2\psi(\delta,\Delta,k)}$, or $\mu_2(G) > \frac{(k-1)\Delta n}{2\psi(\delta,\Delta,k)}$, or $q_2(G) < 2\delta - \frac{(k-1)\Delta n}{2\psi(\delta,\Delta,k)}$, then $\kappa(G) \geq k$.

Motivated by the results of Liu et al. [20]. Hong et al. [13] found better bounds on $\lambda_2(G)$, $\mu_2(G)$, or $q_2(G)$ which guarantee that $\kappa(G) \geq k$.

THEOREM 1.9. ([13]) *Let G be a simple graph of order n with maximum degree Δ and minimum degree $\delta \geq k$ for any integer $k > 0$. Let*

$$\mathcal{H}(\delta,\Delta,k) = \frac{(k-1)\Delta n}{(n-k+1)(k-1)+4(\delta-k+2)(n-\delta-1)}.$$

If $\lambda_2(G) < \delta - \mathcal{H}(\delta,\Delta,k)$, or $\mu_2(G) > \mathcal{H}(\delta,\Delta,k)$, or $q_2(G) < 2\delta - \mathcal{H}(\delta,\Delta,k)$, then $\kappa(G) \geq k$.

In the following, O [21] extended Fiedler's result [11] to multigraphs.

THEOREM 1.10. ([21]) *For any multigraph G whose underlying graph is not a complete graph, we have $\mu_2(G) \leq \kappa(G)m(G)$, where $m(G) = max_{u,v\in E(G)} m(u,v)$ and denote $m(u,v)$ by the number of edges between any two vertices u and v.*

THEOREM 1.11. ([21]) *If G is a d-regular multigraph with $\mu_2(G) > \frac{d}{4}$, except for the d-regular multigraph of order 2, then $\kappa(G) \geq 2$.*

The problem for involving relationships between the graph parameters and eigenvalues of simple graphs was earlier intensively investigated in [5, 6, 8, 9, 16, 17, 20, 23, 24] by many researchers. Nevertheless, the relationships between the vertex connectivity and the third (or fourth) largest eigenvalue in (multi-)graphs was less studied. The results about the third largest eigenvalue of graphs are found in [18, 22]. Later, Duan et al. [10] gave the relationships between $\lambda_3(G)$, $\mu_3(G)$, or $q_3(G)$ and edge connectivity $\kappa'(G) \geq k$ or spanning tree packing number $\tau(G) \geq k$ of a (multi-)graph G. Moreover, Hu et al. [15] investigated the bounds on $\lambda_4(G)$, $\mu_4(G)$, and $q_4(G)$ which guarantee that edge connectivity $\kappa'(G) \geq k$ or spanning tree packing number $\tau(G) \geq k$ for a (multi-)graph G, respectively.

These former results motivated the current research. According to the above discussion, in this paper, we extend earlier results of Abiad et al. [1] and determine the bounds of third largest eigenvalue of a connected d-regular (multi-)graph G which guarantees that $\kappa(G) \geq t+1$ as follows.

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Theorem 1.12. *For a connected d-regular (multi-)graph G with a vertex cut C, we partition the vertex set $V(G)$ into three parts A, B, and C with $a := |A|$, $b := |B|$, and $c := |C|$.*
(i) *If $n \geq 2a + c$ (that is $a \leq b$) and*

$$\lambda_3(G) < \begin{cases} \frac{1-d}{2} & \text{if } G \text{ is a multigraph and } t = 1, \\ 1 - dt & \text{if } G \text{ is a multigraph and } t \geq 2, \\ \frac{1-d}{1+d} & \text{if } G \text{ is a simple graph and } t = 1, \\ \frac{1-dt}{2+d-t} & \text{if } G \text{ is a simple graph and } t \geq 2, \end{cases}$$

where $1 \leq t \leq d-1$, then $\kappa(G) \geq t+1$.
(ii) *If $2b + c < n < 2a + c$ (that is $b < a$) and*

$$\lambda_3(G) \leq \begin{cases} \frac{2-3d}{4} & \text{if } G \text{ is a multigraph and } t = 1, \\ 1 - \frac{3dt}{2} & \text{if } G \text{ is a multigraph and } t \geq 2, \\ \frac{2-3d}{2(d+1)} & \text{if } G \text{ is a simple graph and } t = 1, \\ \frac{2-3dt}{2(d+2-t)} & \text{if } G \text{ is a simple graph and } t \geq 2, \end{cases}$$

where $1 \leq t \leq d-1$, then $\kappa(G) \geq t+1$.

We improve Theorem 1.12 for the case when G is a multigraph and $t = 1$ into Theorem 1.13.

Theorem 1.13. *Let G be a connected d-regular multigraph with order $n \geq 5$ and $d \geq 3$. If $\lambda_3(G) < -\frac{dn}{9n-25}$, then $\kappa(G) \geq 2$.*

The rest of this paper is organized as follows. In Section 2, we introduce some known lemmas and results which will be used in the rest of the statement. In Section 3, we investigate the relationships between the vertex connectivity and the third largest eigenvalue of a connected d-regular (multi-)graph.

2. Preliminaries. We will present some important results in this section that will be used in our subsequent arguments.

Theorem 2.1. ([12]) *If A is a real symmetric $n \times n$ matrix and B is a principal submatrix of A of order $m \times m$ with $m < n$, then for $1 \leq i \leq m$, $\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{n-m+i}(A)$, that is, the eigenvalues of B interlace the eigenvalues of A.*

Let M be the following $n \times n$ matrix

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,1} & M_{m,2} & \cdots & M_{m,m} \end{pmatrix},$$

whose rows and columns are partitioned into subsets $X_1, X_2, \ldots, X_m$ of $\{1, 2, \ldots, n\}$. The quotient matrix $R(M)$ of the matrix M (with respect to the given partition) is the $m \times m$ matrix whose entries are the average row sums of the blocks $M_{i,j}$ of M. The above partition is called equitable if each block $M_{i,j}$ of M has constant row (and column) sum.

Lemma 2.2. ([12]) *The eigenvalues of any quotient matrix $R(M)$ interlace the eigenvalues of G.*

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3. Proofs of Theorems 1.12 and 1.13.

3.1. Proof of Theorem 1.12. For a connected d-regular (multi-)graph G with a vertex cut C, we partition the vertex set $V(G)$ into three parts $\{A, B, C\}$ with $a := |A|$, $b := |B|$, and $c := |C|$, respectively. Then the connected d-regular (multi-)graph G of order n is shown in Fig. 1. Suppose that $1 \le \kappa(G) \le t$,

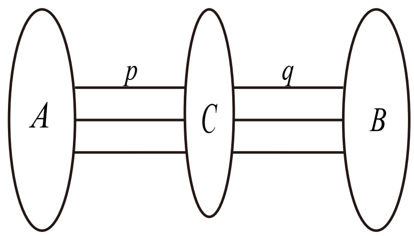

FIGURE 1. *The d-regular (multi-)graph G with vertex partition* $V(G) = \{A, B, C\}$.

then there exists a vertex cut C of G with $1 \le c \le t$. Let $p := [A, C]$, $q := [B, C]$, and without loss of generality, we assume that $p \ge q$ (the roles of p and q can be reversed). Then

$$1 \le q = [B, C] \le \frac{dc}{2} \le \frac{dt}{2}, \tag{3.1}$$

and

$$p + q \le dc \le dt. \tag{3.2}$$

For the numbers of edges between A and B, we have

$$2[A, A] = ad - p, \quad 2[B, B] = bd - q.$$

According to the vertex partition $V(G) = \{A, B, C\}$, we can get that the quotient matrix Q for the connected d-regular (multi-)graph G as follows:

$$Q = \begin{pmatrix} d - \frac{p}{a} & \frac{p}{a} & 0 \\ \frac{p}{c} & d - \frac{p+q}{c} & \frac{q}{c} \\ 0 & \frac{q}{b} & d - \frac{q}{b} \end{pmatrix}.$$

Furthermore, by direct calculation using Mathematica, we obtain the eigenvalues of Q: $\lambda_1(Q) = d$,

$$\lambda_2(Q) = d - \frac{p}{2a} - \frac{q}{2b} - \frac{p+q}{2c} + \frac{1}{2}\sqrt{\frac{p^2}{a^2} + \frac{2p^2}{ac} + \frac{q^2}{a^2} + \frac{2q^2}{bc} + \frac{p^2}{c^2} + \frac{q^2}{b^2} + \frac{2pq}{c^2} - \frac{2pq}{ab} - \frac{2pq}{bc} - \frac{2pq}{ac}},$$

and

$$\lambda_3(Q) = d - \frac{p}{2a} - \frac{q}{2b} - \frac{p+q}{2c} - \frac{1}{2}\sqrt{\frac{p^2}{a^2} + \frac{2p^2}{ac} + \frac{q^2}{a^2} + \frac{2q^2}{bc} + \frac{p^2}{c^2} + \frac{q^2}{b^2} + \frac{2pq}{c^2} - \frac{2pq}{ab} - \frac{2pq}{bc} - \frac{2pq}{ac}}.$$

When $n \ge 2a + c$, that is $b \ge a$, we have

$$\frac{\frac{2q^2}{bc}}{\frac{2pq}{ac}} = \frac{aq}{bp} = \frac{aq}{p(n-a-c)} = \frac{q}{p} \cdot \frac{a}{n-a-c}.$$

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A publication of the International Linear Algebra Society
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Combining with $p \geq q$, we can get $\frac{\frac{2q^2}{bc}}{\frac{2pq}{ac}} \leq 1$, and hence, $-\frac{2q^2}{bc} \geq -\frac{2pq}{ac}$. Then by Lemma 2.2, we have

$$\begin{aligned}
\lambda_3(G) &\geq d - \frac{p}{2a} - \frac{q}{2b} - \frac{p+q}{2c} - \frac{1}{2}\sqrt{\frac{p^2}{a^2} + \frac{2p^2}{ac} + \frac{q^2}{a^2} + \frac{2q^2}{bc} + \frac{p^2}{c^2} + \frac{q^2}{b^2} + \frac{2pq}{c^2} - \frac{2pq}{ab} - \frac{2pq}{bc} - \frac{2pq}{ac}} \\
&\geq d - \frac{p}{2a} - \frac{q}{2b} - \frac{p+q}{2c} - \frac{1}{2}\sqrt{\frac{p^2}{a^2} + \frac{2p^2}{ac} + \frac{q^2}{c^2} + \frac{2pq}{ac} + \frac{p^2}{c^2} + \frac{q^2}{b^2} + \frac{2pq}{c^2} - \frac{2pq}{ab} - \frac{2pq}{bc} - \frac{2q^2}{bc}} \\
&= d - \frac{p}{2a} - \frac{q}{2b} - \frac{p+q}{2c} - \frac{1}{2}\sqrt{\left(\frac{p}{a} + \frac{q}{c} + \frac{p}{c} - \frac{q}{b}\right)^2} \\
&\geq -\frac{p}{a}.
\end{aligned}$$

Now, we consider the following two cases based on whether G is a connected d-regular simple graph or multigraph with order $n \geq 2a + c$.

Case 1: *When G is a simple graph, we discuss the following two subcases.*

Subcase 1.1. $t = 1$.

Note that $1 \leq c \leq t$, which implies that $c = 1$. Since the degree of each vertex in A is d, we have $a \geq d$. If $a = d$, then $G[A \cup C]$ is a complete subgraph of G, while the vertex in C has degree greater than d because $q \geq 1$, which contradicts with $d_G(v) = d$ for the vertex $v \in C$. Thus, $a \geq d + 1$. Combining $1 \leq q \leq \frac{d}{2}$ with $p + q = d$, we have $\frac{d}{2} \leq p \leq d - 1$. Then

$$\lambda_3(G) \geq -\frac{p}{a} \geq \frac{1-d}{1+d}.$$

Subcase 1.2. $t \geq 2$.

Since the degree of each vertex in A is d, we have $a \geq d + 1 - c$. If $a = d + 1 - c$, then $G[A \cup C]$ is a complete subgraph of G, while each vertex in C has degree greater than d because $q \geq 1$, which contradicts with $d_G(v) = d$ for any vertex $v \in C$. Thus, $a \geq d + 2 - c \geq d + 2 - t$ $(1 \leq c \leq t)$. Combining Inequalities (3.1) with (3.2), we have $p \leq dc - 1 \leq dt - 1$. Then

$$\lambda_3(G) \geq -\frac{p}{a} \geq \frac{1-dt}{2+d-t}.$$

Case 2: *When G is a multigraph, we discuss the following two subcases.*

Subcase 2.1. $t = 1$.

In this subcase, we have $c = 1$ and $a \geq 2$. Combining $1 \leq q \leq \frac{d}{2}$ with $p + q = d$, we obtain $\frac{d}{2} \leq p \leq d - 1$. Then

$$\lambda_3(G) \geq -\frac{p}{a} \geq \frac{1-d}{2}.$$

Subcase 2.2. $t \geq 2$.

In this subcase, we have $a \geq 1$. Combining Inequalities (3.1) with (3.2), we obtain $p \leq dt - 1$. Then

$$\lambda_3(G) \geq -\frac{p}{a} \geq 1 - dt.$$

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Next, we consider the case $2b+c<n<2a+c$, that is $b<a$. We have

$$\begin{aligned}\lambda_3(G) &\geq d-\frac{p}{2a}-\frac{q}{2b}-\frac{p+q}{2c}-\frac{1}{2}\sqrt{\frac{p^2}{a^2}+\frac{2p^2}{ac}+\frac{q^2}{c^2}+\frac{2q^2}{bc}+\frac{p^2}{c^2}+\frac{q^2}{b^2}+\frac{2pq}{c^2}-\frac{2pq}{ab}-\frac{2pq}{bc}-\frac{2pq}{ac}}\\ &> d-\frac{p}{2a}-\frac{q}{2b}-\frac{p+q}{2c}-\frac{1}{2}\sqrt{\frac{p^2}{a^2}+\frac{2p^2}{ac}+\frac{q^2}{c^2}+\frac{2q^2}{bc}+\frac{p^2}{c^2}+\frac{q^2}{b^2}+\frac{2pq}{c^2}+\frac{2pq}{ab}+\frac{2pq}{bc}+\frac{2pq}{ac}}\\ &= d-\frac{p}{2a}-\frac{q}{2b}-\frac{p+q}{2c}-\frac{1}{2}\sqrt{\left(\frac{p}{a}+\frac{q}{c}+\frac{p}{c}+\frac{q}{b}\right)^2}\\ &= d-\frac{p}{a}-\frac{q}{b}-\frac{p+q}{c}\quad (p+q\leq dc)\\ &\geq -\frac{p}{a}-\frac{q}{b}.\end{aligned}$$

We also divide the rest of the proof into two cases based on whether G is a simple graph or multigraph.
Case 3: *When G is a simple graph, we discuss the following two subcases.*
Subcase 3.1. $t=1$.

Note that $1\leq c\leq t$, which implies that $c=1$. Since the degree of each vertex in B is d, we have $b\geq d$. If $b=d$, then $G[B\cup C]$ is a complete subgraph of G, while the vertex in C has degree greater than d because $p\geq 1$, which contradicts with $d_G(v)=d$ for the vertex $v\in C$. Thus, $a>b\geq d+1$, and thus, $n>b+d+2$. Combining $1\leq q\leq \frac{d}{2}$ with $p+q=d$, we have $\frac{d}{2}\leq p\leq d-1$. Then

$$\lambda_3(G)\geq -\frac{p}{a}-\frac{q}{b}=-\frac{p}{n-b-1}-\frac{q}{b}>\frac{2-3d}{2(d+1)}.$$

Subcase 3.2. $t\geq 2$.

Since the degree of each vertex in B is d, we have $b\geq d+1-c$. If $b=d+1-c$, then $G[B\cup C]$ is a complete subgraph of G, while each vertex in C has degree greater than d because $p\geq 1$, which contradicts with $d_G(v)=d$ for any vertex $v\in C$. Thus, $a>b\geq d+2-c\geq d+2-t$ $(1\leq c\leq t)$, and thus, $n>b+d+2$. Combining Inequalities (3.1) with (3.2), we have $p\leq dc-1\leq dt-1$. Then

$$\lambda_3(G)\geq -\frac{p}{a}-\frac{q}{b}=-\frac{p}{n-b-c}-\frac{q}{b}>\frac{2-3dt}{2(d+2-t)}.$$

Case 4: *When G is a multigraph, we discuss the following two subcases.*
Subcase 4.1. $t=1$.

In this subcase, we have $c=1$ and $a>b\geq 2$. Thus, $n>b+3$. Combining $1\leq q\leq \frac{d}{2}$ with $p+q=d$, we have $\frac{d}{2}\leq p\leq d-1$. Then

$$\lambda_3(G)\geq -\frac{p}{a}-\frac{q}{b}=-\frac{p}{n-b-1}-\frac{q}{b}>\frac{2-3d}{4}.$$

Subcase 4.2. $t\geq 2$.

In this subcase, we have $a>b\geq 1$ and $n>b+1+c$. Combining Inequalities (3.1) with (3.2), we have $p\leq dt-1$. Then

$$\lambda_3(G)\geq -\frac{p}{a}-\frac{q}{b}=-\frac{p}{n-b-c}-\frac{q}{b}>\frac{2-3dt}{2}.$$

By the above arguments, we have the results as desired. ☐

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A publication of the International Linear Algebra Society
Volume 40, pp. 322-332, February 2024.

3.2. Proof of Theorem 1.13. For a connected d-regular multigraph G with a cut vertex v, we suppose that $\kappa(G) = 1$ and consider the vertex partition $V(G) = \{A, \{v\}, B\}$. Let $a := |A|$ and $b := |B|$ be two partitions of $G - v$, respectively. Here, the multigraph G is shown in Fig. 2.

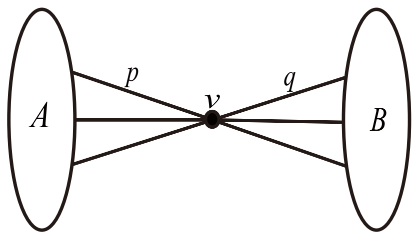

FIGURE 2. *The d-regular multigraph G with vertex partition* $V(G) = \{A, \{v\}, B\}$.

Let $p := [A, v]$, $q := [B, v]$, and without loss of generality, we assume that $p \geq q$ (the roles of p and q can be reversed). Then, $1 \leq q \leq \frac{d}{2}$ and $p + q = d$. Notice that $d \geq 3$, we have $2 \leq a \leq n - 3$. Let

$$Q^{'} = \begin{pmatrix} d - \frac{p}{a} & \frac{p}{a} & 0 \\ p & 0 & q \\ 0 & \frac{q}{b} & d - \frac{q}{b} \end{pmatrix}$$

be the quotient matrix of the vertex partition $V(G) = \{A, \{v\}, B\}$. Then, its characteristic polynomial with respect to λ of the above matrix is

$$(\lambda - d)\left[\lambda^2 - \Big(d - \frac{p}{a} - \frac{q}{b}\Big)\lambda - \frac{p^2}{a} - \frac{q^2}{b} + \frac{pq}{ab}\right].$$

Then by Lemma 2.2, we have

$$\lambda_3(G) \geq \lambda_3(Q^{'}) = \frac{1}{2}\left[d - \frac{p}{a} - \frac{q}{b} - \sqrt{\Big(d - \frac{p}{a} - \frac{q}{b}\Big)^2 + 4\Big(\frac{p^2}{a} + \frac{q^2}{b} - \frac{pq}{ab}\Big)}\right]. \tag{3.3}$$

We get the derivative of the above Equation (3.3) with respect to q

$$\frac{d\lambda_3(Q^{'})}{dq} = \frac{1}{2}\left(-\frac{1}{b} - \frac{-\frac{2\left(d - \frac{p}{a} - \frac{q}{b}\right)}{b} + 4\left(-\frac{p}{ab} + \frac{2q}{b}\right)}{\sqrt{\Big(d - \frac{p}{a} - \frac{q}{b}\Big)^2 + 4\Big(\frac{p^2}{a} + \frac{q^2}{b} - \frac{pq}{ab}\Big)}}\right).$$

In the following, we substitute $d - q$ for p. Let $\theta(q) = \frac{d\lambda_3(Q^{'})}{dq}$. The roots of $\theta(q) = 0$ are

$$q_1 = 0$$

and

$$q_2 = \frac{(1 + 2a)bd}{a + b + 4ab}.$$

Substituting $n - a - 1$ for b and q_2 for q into Equation (3.3), and simplifying, we obtain

$$\lambda_3(G) \geq \frac{dn}{1 + 4a(1 + a - n) - n}.$$

Finally, the result of (3.3) has a minimum value $-\frac{dn}{9n-25}$ for $n \geq 5$, $d \geq 3$, and $2 \leq a \leq n - 3$. $\square$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 322-332, February 2024.

In order to illustrate that the upper bound of the third largest eigenvalue $\lambda_3(G)$ in Theorem 1.13 is tight, we give the following example.

EXAMPLE 3.1. Let G be the d-regular multigraph of order 5 with a cut vertex v as shown in Fig. 3. We have the following adjacency matrix:

$$\begin{pmatrix} 0 & \frac{3d}{4} & \frac{d}{4} & 0 & 0 \\ \frac{3d}{4} & 0 & \frac{d}{4} & 0 & 0 \\ \frac{d}{4} & \frac{d}{4} & 0 & \frac{d}{4} & \frac{d}{4} \\ 0 & 0 & \frac{d}{4} & 0 & \frac{3d}{4} \\ 0 & 0 & \frac{d}{4} & \frac{3d}{4} & 0 \end{pmatrix},$$

where $d = 4k$ and $k \geq 1$. By directly calculation, we have $\lambda_3(G) = -\frac{d}{4}$. Moreover, G is a d-regular multigraph with five vertices, $-\frac{d}{4} = -\frac{dn}{9n-25}$, and $\kappa(G) = 1$. Thus, the bound of $\lambda_3(G)$ in Theorem 1.13 is the best possible for this infinite family of multigraphs.

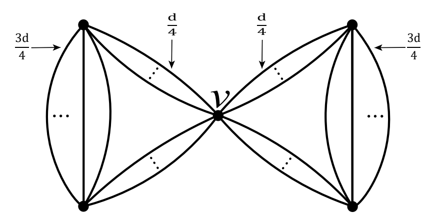

FIGURE 3. *d-regular multigraph of order 5 with a cut vertex v.*

4. Conclusion. We have studied the upper bound for $\lambda_3(G)$ in a d-regular (multi-)graph which assures that $\kappa(G) \geq t + 1$. However, the relationship between the vertex connectivity and the k-th ($k \geq 4$) largest eigenvalue of a connected (multi-)graph can be a direction for future work.

Acknowledgment. The authors thank the anonymous referee for careful review and helpful comments.

Declaration of competing interest. The authors declare that they have no conflict of interest.

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Appendix. In the following, we give the Mathematica code which be used to calculate the minimum of the third largest eigenvalue in the Proof of Theorem 1.13.
First, we calculate the derivative $\frac{d\lambda_3(Q^{'})}{dq}$ with respect to q as follows:

$$\partial_{\mathbf{q}}\left(\frac{\mathbf{1}}{\mathbf{2}}\left(\mathbf{d}-\frac{\mathbf{p}}{\mathbf{a}}-\frac{\mathbf{q}}{\mathbf{b}}-\sqrt{\left(\mathbf{d}-\frac{\mathbf{p}}{\mathbf{a}}-\frac{\mathbf{q}}{\mathbf{b}}\right)^{\mathbf{2}}+\mathbf{4}\left(\frac{\mathbf{p^2}}{\mathbf{a}}+\frac{\mathbf{q^2}}{\mathbf{b}}-\frac{\mathbf{pq}}{\mathbf{ab}}\right)}\right)\right).$$

We have

$$\frac{1}{2}\left(-\frac{1}{b}-\frac{-\frac{2\left(d-\frac{p}{a}-\frac{q}{b}\right)}{b}+4\left(-\frac{p}{ab}+\frac{2q}{b}\right)}{\sqrt{\left(d-\frac{p}{a}-\frac{q}{b}\right)^2+4\left(\frac{p^2}{a}+\frac{q^2}{b}-\frac{pq}{ab}\right)}}\right).$$

Let

$$\mathbf{f}=\frac{\mathbf{1}}{\mathbf{2}}\left(-\frac{\mathbf{1}}{\mathbf{b}}-\frac{-\frac{\mathbf{2}\left(\mathbf{d}-\frac{\mathbf{p}}{\mathbf{a}}-\frac{\mathbf{q}}{\mathbf{b}}\right)}{\mathbf{b}}+\mathbf{4}\left(-\frac{\mathbf{p}}{\mathbf{ab}}+\frac{\mathbf{2q}}{\mathbf{b}}\right)}{\sqrt{\left(\mathbf{d}-\frac{\mathbf{p}}{\mathbf{a}}-\frac{\mathbf{q}}{\mathbf{b}}\right)^{\mathbf{2}}+\mathbf{4}\left(\frac{\mathbf{p^2}}{\mathbf{a}}+\frac{\mathbf{q^2}}{\mathbf{b}}-\frac{\mathbf{pq}}{\mathbf{ab}}\right)}}\right);$$

We substitute $d-q$ for p:
$\mathbf{p}=\mathbf{d}-\mathbf{q}$;
We solve the equation $\frac{1}{2}\left(-\frac{1}{b}-\frac{-\frac{2\left(d-\frac{p}{a}-\frac{q}{b}\right)}{b}+4\left(-\frac{p}{ab}+\frac{2q}{b}\right)}{\sqrt{\left(d-\frac{p}{a}-\frac{q}{b}\right)^2+4\left(\frac{p^2}{a}+\frac{q^2}{b}-\frac{pq}{ab}\right)}}\right)$ about q:

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
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Solve[f==0, q];
The following results are obtained:
$\{q \to 0\}$, $\{q \to \frac{(1+2a)bd}{a+b+4ab}\}$
Next, we substitute $n-1-a$ for b and $\frac{(1+2a)bd}{a+b+4ab}$ for q:
b=n - 1 - a;
$\mathbf{q} = \frac{\mathbf{(1+2a)bd}}{\mathbf{a+b+4ab}}$;
We continue to calculate

$$\mathbf{\frac{1}{2}\left(d - \frac{p}{a} - \frac{q}{b} - \sqrt{\left(d - \frac{p}{a} - \frac{q}{b}\right)^2 + 4\left(\frac{p^2}{a} + \frac{q^2}{b} - \frac{pq}{ab}\right)}\right)};$$

Then, we get

$$\frac{1}{2}\left(d - \frac{(1+2a)d}{-1+n+4a(-1-a+n)} - \frac{d - \frac{(1+2a)d(-1-a+n)}{-1+n+4a(-1-a+n)}}{a} - \left(\left(d - \frac{(1+2a)d}{-1+n+4a(-1-a+n)} - \frac{d - \frac{(1+2a)d(-1-a+n)}{-1+n+4a(-1-a+n)}}{a}\right)^2 + 4\left(\frac{(1+2a)^2d^2(-1-a+n)}{(-1+n+4a(-1-a+n))^2} - \frac{(1+2a)d\left(d - \frac{(1+2a)d(-1-a+n)}{-1+n+4a(-1-a+n)}\right)}{a(-1+n+4a(-1-a+n))} + \frac{\left(d - \frac{(1+2a)d(-1-a+n)}{-1+n+4a(-1-a+n)}\right)^2}{a}\right)\right)^{\frac{1}{2}}\right)$$

Under the condition of $d \geq 3$, we simplify the above equation:
Fullsimplify[%**7** && **d** $\geq$ **3**];
Hence, we obtain
$\frac{dn}{1+4a(1+a-n)-n}$ && $d \geq 3$.
Finally, under the condition of $n \geq 5, d \geq 3$, and $2 \leq a \leq n-3$, we calculate the minimum of $\frac{dn}{1+4a(1+a-n)-n}$ as follows:
$\mathbf{Minimize}\left[\{\frac{\mathbf{dn}}{\mathbf{1+4a(1+a-n)-n}}, \mathbf{n \geq 5, d \geq 3, 2 \leq a \leq n-3}\}, \mathbf{a}\right]$;
Hence, we obtain the result:

$$\begin{cases} -\frac{dn}{9n-25} & d \geq 3 \text{ \&\& } n > 5 \\ \infty & \text{True} \end{cases}, \ a \to \begin{cases} \frac{1}{2}(-1+n) - \frac{1}{2}\sqrt{-\frac{(-25+9n)\left(dn + \frac{dn^2}{-25+9n} - \frac{dn^3}{-25+9n}\right)}{dn}} & d \geq 3 \text{ \&\& } n \geq 5 \\ \mathit{Indeterminate} & \text{True} \end{cases}.$$