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EIGENVALUES FOR STOCHASTIC MATRICES WITH A PRESCRIBED STATIONARY DISTRIBUTION*

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Abstract. Given a vector $0 < w \in \mathbb{R}^n$ whose entries sum to 1, the region $\sigma_{\mathcal{S}}(w)$ in the complex plane consisting of all eigenvalues of all stochastic matrices having $w^\top$ as a left Perron vector is considered. Some general observations about this region are made, it is proven that $\bigcap_{w\in\mathbb{R}^n, w>0, w^\top\mathbf{1}=1} \sigma_{\mathcal{S}}(w) = [0, 1]$, and a characterization is given of the vectors w such that $\sigma_{\mathcal{S}}(w)$ contains an element $\lambda \neq 1$ with $|\lambda| = 1$. The corresponding problem for reversible stochastic matrices with given left Perron vector is also considered, as is the corresponding region $\sigma_{\mathcal{R}}(w)$, which is a subset of $[-1, 1]$. Under a mild hypothesis on w, it is proven that the smallest element of $\sigma_{\mathcal{R}}(w)$ corresponds to a reversible stochastic matrix whose graph is a tree with a loop at one vertex. A general lower bound on the eigenvalues of reversible stochastic matrices with given left Perron vector is also given, as is a complete description of $\sigma_{\mathcal{R}}(w)$ when w has two or three entries.

Key words. Stochastic matrix, Stationary distribution, Eigenvalue, Reversible Markov chain.

AMS subject classifications. 15B51, 15A18, 05C50.

1. Introduction and preliminaries. An $n \times n$ matrix T is stochastic if it is entrywise nonnegative (denoted $T \geq 0$), and each of its rows sums to 1. Stochastic matrices have received enormous attention over the last century. This is due in no small part to the fact that they are central to the study of discrete time, finite state, time-homogeneous Markov chains; Markov chains are, in turn, widely applied throughout science and engineering.

It is well known that for an irreducible stochastic matrix T, there is a corresponding left Perron vector $w^\top$ with all positive entries (denoted $w > 0$) such that $w^\top T = w^\top$ and $w^\top \mathbf{1} = 1$, where $\mathbf{1}$ denotes the all-ones vector of the appropriate order. This vector $w^\top$ is known as the stationary distribution, and in the case that T is primitive (i.e. some power of T has all positive entries), the sequence $T^k, k = 1, 2, \ldots$ converges to $\mathbf{1}w^\top$ as $k \to \infty$. In this case, the entries of the stationary distribution reflect the long-term behavior of the corresponding Markov chain. Further, the nature of convergence of the sequence T^k is dictated by the eigenvalues of T.

In view of the foregoing, it is no surprise that the eigenvalues of stochastic matrices are the subject of intense study, as they carry critical information regarding the convergence properties of Markov chains. A classic result of Karpelevič [4] describes for each $n \in \mathbb{N}$ with $n \geq 2$, the region in the complex plane consisting of all eigenvalues of all stochastic matrices of order n. A sharpening of that description can be found in [7]. In a related direction, Perfect and Mirsky [10] consider the corresponding region for doubly stochastic matrices (i.e. those stochastic matrices for which all rows and columns sum to 1) and formulate a conjecture on the structure of that region. Developments on the Perfect–Mirsky conjecture can be found in [9] and [8].

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How does the stationary distribution of an irreducible stochastic matrix constrain the corresponding eigenvalues? We explore that question in this paper. At first glance, it may not be obvious that any constraints on the eigenvalues are imposed by the stationary distribution. However, the following example illustrates how such constraints arise.

EXAMPLE 1.1. Suppose we have $w \in \mathbb{R}^3$ with $0 < w_1 \le w_2 \le w_3$. A typical stochastic matrix T having $w^\top$ as a left fixed-vector has the form

$$T = \begin{bmatrix} a & b & 1-a-b \\ c & d & 1-c-d \\ \frac{(1-a)w_1 - cw_2}{w_3} & \frac{(1-d)w_2 - bw_1}{w_3} & \frac{w_3 + (a+b-1)w_1 + (c+d-1)w_2}{w_3} \end{bmatrix},$$

where necessarily all entries are nonnegative. Observe that for such a T we have $\text{trace}(T) = a + d + \frac{w_3+(a+b-1)w_1+(c+d-1)w_2}{w_3} \ge \frac{w_3-w_1-w_2}{w_3} = \frac{2w_3-1}{w_3}$. It now follows that if λ is a non-real eigenvalue of T then $Re(\lambda) \ge \frac{w_3-1}{2w_3}$. For example, if $w_3 > \frac{1}{2}$, then any non-real eigenvalue of any stochastic matrix with $w^\top$ as a left Perron vector necessarily has real part strictly greater than $-\frac{1}{2}$.

The following result from [6] further illustrates how the stationary distribution imposes a constraint on eigenvalues.

THEOREM 1.2. *Let T be an irreducible stochastic matrix of order n with eigenvalues $1, \lambda_2, \ldots, \lambda_n$, and stationary distribution $w^\top$, where without loss of generality we assume that $w_1 \le w_2 \le \ldots \le w_n$. Then*

$$\sum_{k=2}^{n} \frac{1}{1-\lambda_k} \ge \sum_{j=1}^{n} (j-1) w_j.$$

Observe that Perfect and Mirsky's conjecture can be viewed in terms of the relationship between the stationary distribution and the eigenvalues of a stochastic matrix. An irreducible $n \times n$ stochastic matrix is doubly stochastic precisely in the case that $\frac{1}{n}\mathbf{1}^\top$ is the stationary vector; hence, the Perfect–Mirsky conjecture can be thought of as an effort to describe eigenvalues of stochastic matrices having $\frac{1}{n}\mathbf{1}^\top$ as a left Perron vector.

Our goal in this paper is to deepen the understanding of how the eigenvalues of an irreducible stochastic matrix are influenced by the structure of its stationary distribution. In order to do so, we introduce a slight relaxation of the problem by considering the family of stochastic matrices having $w^\top$ as a left fixed-vector (i.e. $w^\top T = w^\top$). Evidently, the irreducible stochastic matrices having $w^\top$ as the stationary distribution are a dense subset of that family.

Among the time-homogeneous discrete-time Markov chains, the so-called time-reversible Markov chains constitute an important subfamily. Suppose that T is an $n \times n$ stochastic matrix with positive left Perron vector $w^\top$. The following are equivalent conditions for the corresponding Markov chain to be time-reversible (see [5]):
(i) $w_i t_{i,j} = w_j t_{j,i}, i, j = 1, \ldots, n$;
(ii) $W^{\frac{1}{2}} T W^{-\frac{1}{2}}$ is symmetric, where $W = \text{diag}(w)$;
(iii) $T = W^{-1}A$, where A is a symmetric matrix such that $A\mathbf{1} = w$. If one of those conditions holds, then we say that T is a reversible stochastic matrix. Evidently, any reversible stochastic matrix has all eigenvalues real, as it is (diagonally) similar to a symmetric matrix.

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Suppose that $w \in \mathbb{R}^n$ with $w > 0$ and $w^\top \mathbf{1} = 1$. In this paper, we consider the following sets of stochastic matrices:

$$\mathcal{S}(w) = \{T \in M_n(\mathbb{R}) | T \geq 0, T\mathbf{1} = \mathbf{1}, w^\top T = w^\top\};$$
$$\mathcal{R}(w) = \{T \in M_n(\mathbb{R}) | T \geq 0, T\mathbf{1} = \mathbf{1}, w^\top T = w^\top, w_j t_{j,i} = w_i t_{i,j}, i, j = 1, \ldots, n\}.$$

Evidently, $\mathcal{S}(w)$ consists of the stochastic matrices having $w^\top$ as a left Perron vector, while $\mathcal{R}(w)$ is the subset of $\mathcal{S}(w)$ for which the transition matrices are reversible. Note that both $\mathcal{S}(w)$ and $\mathcal{R}(w)$ are convex polytopes of matrices. We further define

$$\sigma_{\mathcal{S}}(w) = \{\lambda | \lambda \text{ is an eigenvalue of } T \text{ for some } T \in \mathcal{S}(w)\}, \text{ and}$$
$$\sigma_{\mathcal{R}}(w) = \{\lambda | \lambda \text{ is an eigenvalue of } T \text{ for some } T \in \mathcal{R}(w)\},$$

which are the collections of eigenvalues of matrices in $\mathcal{S}(w)$ and $\mathcal{R}(w)$, respectively. We also let $\underline{\lambda}(w) = \min\{\lambda | \lambda \in \sigma_{\mathcal{R}}(w)\}$, and note in passing that the minimum is well-defined since $\mathcal{R}(w)$ is a compact set.

The broad goals of the paper are: i) to pose the challenge of developing a better understanding of $\mathcal{S}(w)$ and ii) to present some results on this topic in the hopes of generating further interest in it. In section 2, we prove some results on the structure of $\mathcal{S}(w)$, in particular we characterize the complex numbers that reside in $\mathcal{S}(w)$ for any positive stationary vector $w^\top$. Section 3 analyzes $\sigma_{\mathcal{R}}(w)$, describes the structure of matrices that yield $\underline{\lambda}(w)$, and provides a lower bound on $\underline{\lambda}(w)$ in terms of w. In section 4, we provide formulas for $\underline{\lambda}(w)$ for $w \in \mathbb{R}^2$ and $w \in \mathbb{R}^3$.

Throughout the paper, we rely on basic results in nonnegative matrix theory. For background on that topic and its connection with Markov chains, we refer the interested reader to [11]. We also rely on notions from combinatorial matrix theory; [3] provides an overview of and introduction to that rich subject.

2. The general case.

We begin by recording a few preliminary facts.

Observation 2.1. Suppose that $w \in \mathbb{R}^n, w > 0, w^\top \mathbf{1} = 1$. We have the following simple observations.
(i) $\sigma_{\mathcal{S}}(w)$ is symmetric with respect to the real axis.
(ii) $\sigma_{\mathcal{S}}(w)$ is star-shaped with respect to 1 – i.e., if $\lambda \in \sigma_{\mathcal{S}}(w)$, then $t\lambda + 1 - t \in \sigma_{\mathcal{S}}(w)$ for all $t \in [0, 1]$. This is because if T is stochastic with left Perron vector $w^\top$, then so is $(1 - t)I + tT$ for each $t \in [0, 1]$.
(iii) $\sigma_{\mathcal{S}}(w)$ is star-shaped with respect to the origin – i.e., if $\lambda \in \sigma_{\mathcal{S}}(w)$, then $t\lambda \in \sigma_{\mathcal{S}}(w)$ for all $t \in [0, 1]$. This is because if T is stochastic with left Perron vector $w^\top$, then so is $(1 - t)\mathbf{1}w^\top + tT$ for each $t \in [0, 1]$. In particular, if $\lambda \neq 1$ is an eigenvalue of T, then $t\lambda$ is an eigenvalue of $(1 - t)\mathbf{1}w^\top + tT$ (this is essentially a result of Brauer [1]).
(iv) Suppose that $w_1 = \min\{w_j | j = 1, \ldots, n\}$. If $\lambda \in \sigma_{\mathcal{S}}(w)$, then so is $-\frac{w_1 \lambda}{\sum_{j=2}^n w_j}$. This is because if T is stochastic with left Perron vector $w^\top$, then so is $\frac{1}{1-w_1}(\mathbf{1}w^\top - w_1 T)$. Again referring to Brauer's result, if $\lambda \neq 1$ is an eigenvalue of T, then $-\frac{w_1\lambda}{1-w_1}$ is an eigenvalue of $\frac{1}{1-w_1}(\mathbf{1}w^\top - w_1 T)$. To cover the case that $\lambda = 1$, we note that $\frac{1}{1-w_1}(\mathbf{1}w^\top - w_1 I)$ has $-\frac{w_1}{1-w_1}$ as an eigenvalue.
(v) If $\hat{w}$ is a subvector of w, then $\sigma_{\mathcal{S}}(\frac{1}{\mathbf{1}^\top \hat{w}}\hat{w}) \subseteq \sigma_{\mathcal{S}}(w)$. This is because if $\hat{T} \in \mathcal{S}(\frac{1}{\mathbf{1}^\top \hat{w}}\hat{w})$ has eigenvalue λ, then for the matrix T formed from $\hat{T}$ by taking its direct sum with a suitable identity matrix, we have $T \in \mathcal{S}(w)$, and λ as an eigenvalue of T.

Our first result identifies the numbers that are common to $\mathcal{S}(w)$ for any admissible w.

Theorem 2.1. *Suppose that $n \geq 2$. Then $\bigcap_{w \in \mathbb{R}^n, w>0, w^\top \mathbf{1}=1} \sigma_{\mathcal{S}}(w) = [0, 1]$.*

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Proof. We proceed by induction on n. First suppose that $n = 2$, and that we are given $w \in \mathbb{R}^2$ with $w > 0$ and $w^\top \mathbf{1} = 1$, and without loss of generality we may assume that $w_1 \le w_2$. Then for any $T \in \mathcal{S}(w)$, we may write T as

$$T = \begin{bmatrix} s & 1-s \\ \frac{(1-s)w_1}{w_2} & 1 - \frac{(1-s)w_1}{w_2} \end{bmatrix},$$

for some $s \in [0,1]$. The eigenvalues of T are 1 and $\frac{s-w_1}{w_2}$. Hence, if $\lambda \in \sigma_{\mathcal{S}}(w)$, then necessarily $\lambda \ge -\frac{w_1}{w_2}$. Suppose now that $\lambda \in \bigcap_{w\in\mathbb{R}^2, w>0, w^\top\mathbf{1}=1} \sigma_{\mathcal{S}}(w)$. Then necessarily $\lambda \ge -\frac{w_1}{w_2}$ for any $w \in \mathbb{R}^2$ with $w > 0$ and $w^\top \mathbf{1} = 1$, and it follows readily that $\lambda \ge 0$, and certainly $\lambda \le 1$. Hence, $\bigcap_{w\in\mathbb{R}^2, w>0, w^\top\mathbf{1}=1} \sigma_{\mathcal{S}}(w) \subseteq [0,1]$. Next, consider $a \in [0,1]$ and suppose that we are given $w \in \mathbb{R}^2$ with $w > 0$ and $w^\top \mathbf{1} = 1$. Then $aI + (1-a)\mathbf{1}w^\top \in \sigma_{\mathcal{S}}(w)$ and has a as an eigenvalue, so that $[0,1] \subseteq \bigcap_{w\in\mathbb{R}^2, w>0, w^\top\mathbf{1}=1} \sigma_{\mathcal{S}}(w)$.

Assume now that the statement holds for some $n \ge 2$, and suppose that $\lambda \in \bigcap_{w\in\mathbb{R}^{n+1}, w>0, w^\top\mathbf{1}=1} \sigma_{\mathcal{S}}(w)$. Consider any $\epsilon \in (0,1)$ and any $v \in \mathbb{R}^n$ with $v > 0$ and $v^\top \mathbf{1} = 1$, and let $w(\epsilon)^\top$ be given by $\left[\begin{array}{c|c} \epsilon & (1-\epsilon)v^\top \end{array}\right]$. Then there is a substochastic matrix $T(\epsilon)$ of order n, a nonnegative vector $y(\epsilon) \in \mathbb{R}^n$ with $y(\epsilon)^\top \mathbf{1} \le 1$ and a vector $u(\epsilon) \in \mathbb{C}^n$ with $||u(\epsilon)||_2 = 1$ such that for the matrix

$$S(\epsilon) = \left[\begin{array}{c|c} 1 - y(\epsilon)^\top \mathbf{1} & y(\epsilon)^\top \\ \hline \mathbf{1} - T(\epsilon)\mathbf{1} & T(\epsilon) \end{array}\right], \tag{2.1}$$

we have $w(\epsilon)^\top S(\epsilon) = w(\epsilon)^\top$, and $S(\epsilon)u(\epsilon) = \lambda u(\epsilon)$. Since $w(\epsilon)^\top S(\epsilon) = w(\epsilon)^\top$, we find from (2.1) that

$$\begin{gathered} \epsilon y(\epsilon)^\top + (1-\epsilon)v^\top T(\epsilon) = (1-\epsilon)v^\top, \\ -\epsilon y(\epsilon)^\top \mathbf{1} + (1-\epsilon) - (1-\epsilon)v^\top T(\epsilon)\mathbf{1} = 0. \end{gathered} \tag{2.2}$$

Next we let $\epsilon \to 0^+$. Appealing to compactness, we find that there is a nonnegative vector $y(0)$ with $y(0)^\top \mathbf{1} \le 1$, a substochastic matrix $T(0)$ of order n, and a vector $u(0) \in \mathbb{C}^{n+1}$ of norm 1 such that

$$\left[\begin{array}{c|c} 1 - y(0)^\top \mathbf{1} & y(0)^\top \\ \hline \mathbf{1} - T(0)\mathbf{1} & T(0) \end{array}\right] u(0) = \lambda u(0).$$

Referring to (2.2), we find that necessarily $v^\top T(0) = v^\top$ and $v^\top T(0)\mathbf{1} = 1$. As $T(0)$ is substochastic and v is a positive vector whose entries sum to 1, we deduce that in fact $T(0)$ is stochastic. It now follows that λ is an eigenvalue of the stochastic matrix

$$\left[\begin{array}{c|c} 1 - y(0)^\top \mathbf{1} & y(0)^\top \\ \hline 0 & T(0) \end{array}\right].$$

Hence, either $\lambda = 1 - y(0)^\top \mathbf{1}$ or λ is an eigenvalue of $T(0)$. In the former case, we have $\lambda \in [0,1]$, as desired. Suppose now that λ is an eigenvalue of $T(0)$. Hence, $\lambda \in \sigma_{\mathcal{S}}(v)$, but since v was arbitrary (subject to the constraint on positivity and sum of entries) we deduce that in fact $\lambda \in \bigcap_{v\in\mathbb{R}^n, v>0, v^\top\mathbf{1}=1} \sigma_{\mathcal{S}}(v)$. Hence, $\lambda \in [0,1]$ by the induction hypothesis, as desired. So $\bigcap_{w\in\mathbb{R}^{n+1}, w>0, w^\top\mathbf{1}=1} \sigma_{\mathcal{S}}(w) \subseteq [0,1]$, and to conclude the reverse containment, we appeal to the facts that for any admissible w, $1 \in \sigma_{\mathcal{S}}(w)$, and that $\sigma_{\mathcal{S}}(w)$ is star-shaped with respect to 0 by Observation 2.1 (iii). ∎

We have the following parallel result for eigenvalues of reversible stochastic matrices.

Corollary 2.2. $\bigcap_{w\in\mathbb{R}^n, w>0, w^\top\mathbf{1}=1} \sigma_{\mathcal{R}}(w) = [0,1]$.

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Proof. Since $\sigma_{\mathcal{R}}(w) \subseteq \sigma_{\mathcal{S}}(w)$ for any admissible w, we find from Theorem 2.1 that

$$\bigcap_{w\in\mathbb{R}^n, w>0, w^\top \mathbf{1}=1} \sigma_{\mathcal{R}}(w) \subseteq [0,1].$$

Further, for any admissible w and any $a \in [0,1]$ observe that $aI + (1-a)\mathbf{1}w^\top \in \mathcal{R}(w)$ and has a as an eigenvalue. ∎

The next result characterizes the structure of the vector w when $\mathcal{S}(w)$ contains a root of unity distinct from 1.

THEOREM 2.3. *Suppose that $n \geq 2$, that $2 \leq k \leq n$, and that $w \in \mathbb{R}^n$ with $w > 0, w^\top \mathbf{1} = 1$. We have $e^{\frac{2\pi i j}{k}} \in \sigma_{\mathcal{S}}(w)$ for some $j = 1, \ldots, k-1$ that is relatively prime to k, if and only if there is a collection of non-empty disjoint subsets $S_1, \ldots, S_k \subseteq \{1, \ldots, n\}$ such that the values $\sum_{l\in S_i} w_l, i = 1, \ldots, k$, are all equal.*

Further, $-1 \in \sigma_{\mathcal{R}}(w)$ if and only if there is a pair of non-empty disjoint subsets S_1, S_2 of $\{1, \ldots, n\}$ such that $\sum_{l\in S_1} w_l = \sum_{l\in S_2} w_l$.

Proof. First suppose that $e^{\frac{2\pi j}{k}} \in \sigma_{\mathcal{S}}(w)$ for some $j = 1, \ldots, k-1$. Then there is a stochastic matrix T such that $w^\top T = w^\top$ and having $e^{\frac{2\pi j}{k}}$ as an eigenvalue. It follows now that by performing a suitable simultaneous permutation of the rows and columns of T, we may write T as

$$T = \left[\begin{array}{c|c} T_1 & 0 \\ \hline X & T_2 \end{array}\right],$$

where T_1 is irreducible, stochastic, and has period k. (We take $T_1 = T$ in the case that T itself is irreducible.) Permute and partition w conformally, so that $w^\top = \left[\begin{array}{c|c} w_1^\top & w_2^\top \end{array}\right]$. Observe that $w_1^\top T_1 + w_2^\top X = w_1^\top$, from which we find that $w_2^\top X\mathbf{1} = w_1^\top \mathbf{1} - w_1^\top T_1 \mathbf{1} = 0$. Hence $X = 0$.

So, $w_1^\top = w_1^\top T_1$. Since T_1 is irreducible and periodic with period k, we may, after a suitable permutation similarity of T_1 (and corresponding reordering of the entries in w_1), rewrite the eigenequation as

$$\left[\begin{array}{c|c|c|c} w_1(1)^T & w_1(2)^T & \ldots & w_1(k)^T \end{array}\right]$$
$$= \left[\begin{array}{c|c|c|c} w_1(1)^\top & w_1(2)^\top & \ldots & w_1(k)^\top \end{array}\right] \left[\begin{array}{c|c|c|c|c} 0 & T_1(2) & 0 & \ldots & 0 \\ \hline 0 & 0 & T_1(3) & \ldots & 0 \\ \hline \vdots & & \ddots & \ddots & \vdots \\ \hline 0 & 0 & \ldots & 0 & T_1(k) \\ \hline T_1(1) & 0 & 0 & \ldots & 0 \end{array}\right].$$

Hence, we have $w_1^\top(1) = w_1(k)^T T_1(1)$, and $w_1^\top(j) = w_1(j-1)^T T_1(j), j = 2, \ldots, k$. But each $T_1(j)$ has row sums equal to 1, and it follows readily that $w_1^\top(1)\mathbf{1} = w_1(k)^T \mathbf{1}$, and $w_1^\top(j)\mathbf{1} = w_1(j-1)^T \mathbf{1}, j = 2, \ldots, k$. The desired conclusion now follows. Conversely, if there are subsets $S_1, \ldots, S_k$ satisfying the stated condition, we may write w (after suitable reordering) as $w^\top = \left[\begin{array}{c|c|c|c|c} w_1^\top & w_2^\top & \ldots & w_k^\top & \overline{w}^\top \end{array}\right]$, where $\overline{w}$ corresponds to the indices falling outside of $\cup_{j=1}^k S_j$. Observe then that the matrix

$$\left[\begin{array}{c|c|c|c|c} 0 & \mathbf{1}w_2^\top & 0 & \ldots & 0 \\ \hline 0 & 0 & \mathbf{1}w_3^\top & \ldots & 0 \\ \hline \vdots & & \ddots & & \vdots \\ \hline 0 & 0 & \ldots & 0 & \mathbf{1}w_k^\top \\ \hline \mathbf{1}w_1^\top & 0 & 0 & \ldots & 0 \end{array}\right] \oplus I,$$

has w as a fixed vector and the k-th roots of unity as eigenvalues.

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The statements regarding -1 being in $\sigma_{\mathcal{R}}(w)$ are specializations of the arguments above, along with the observation that for the converse, the matrix we produce corresponds to a reversible Markov chain. ∎

We have the following immediate consequence.

COROLLARY 2.4. *Consider a vector $w \in \mathbb{R}^n$ with $w > 0, w^\top \mathbf{1} = 1$. Suppose that for any pair of non-empty disjoint subsets $S_1, S_2 \subset \{1, \ldots, n\}, \sum_{j \in S_1} w_j \neq \sum_{k \in S_2} w_k$. Then for any stochastic matrix T such that $w^\top T = w^\top$, the only eigenvalue of T of unit modulus is 1. In particular, if such a T is irreducible, it is necessary primitive.*

3. The reversible case. Suppose that $w \in \mathbb{R}^n$ with $w > 0, w^\top \mathbf{1} = 1$. From Theorem 2.3, we find that $\underline{\lambda}(w) > -1$ if and only if for each pair of non-empty disjoint subsets S_1, S_2 of $\{1, \ldots, n\}$, we have $\sum_{l \in S_1} w_l \neq \sum_{l \in S_2} w_l$. In order to ease the exposition going forward, henceforth we assume that our vector w satisfies the following properties, to which we collectively refer as Assumption A:
Assumption A:
(i) $w \in \mathbb{R}^n$;
(ii) $w > 0$;
(iii) $w^\top \mathbf{1} = 1$;
(iv) for each pair of non-empty disjoint subsets S_1, S_2 of $\{1, \ldots, n\}$, we have $\sum_{l \in S_1} w_l \neq \sum_{l \in S_2} w_l$.

THEOREM 3.1. *Suppose that $n \geq 2$, and that $w \in \mathbb{R}^n$ with $w > 0, w^\top \mathbf{1} = 1$. There is an extreme point T of $\mathcal{R}(w)$ such that $\underline{\lambda}(w)$ is an eigenvalue of T.*

Proof. Suppose that $T_1, T_2 \in \mathcal{R}(w)$, and that $t \in [0, 1]$. Set $W = \operatorname{diag}(w)$ and note that both $W^{\frac{1}{2}} T_1 W^{\frac{-1}{2}}$ and $W^{\frac{1}{2}} T_2 W^{\frac{-1}{2}}$ are symmetric matrices. Observe that

$$\begin{aligned}
&\lambda_{\min}(tT_1 + (1-t)T_2) \\
&\quad = \lambda_{\min}(tW^{\frac{1}{2}} T_1 W^{\frac{-1}{2}} + (1-t) W^{\frac{1}{2}} T_2 W^{\frac{-1}{2}}) \\
&\quad \geq \lambda_{\min}(tW^{\frac{1}{2}} T_1 W^{\frac{-1}{2}}) + \lambda_{\min}((1-t) W^{\frac{1}{2}} T_2 W^{\frac{-1}{2}}) \\
&\quad = t\lambda_{\min}(T_1) + (1-t)\lambda_{\min}(T_2).
\end{aligned}$$

Hence, we find that $\lambda_{\min}(\bullet)$ is a concave function on the convex polytope $\mathcal{R}(w)$. The conclusion follows readily. ∎

Suppose that $w > 0, w^\top \mathbf{1} = 1$. Observe that any matrix in $\mathcal{R}(w)$ can be constructed as follows: start with a symmetric nonnegative matrix A such that $A\mathbf{1} = w$, then produce the stochastic matrix $\operatorname{diag}(w)^{-1} A$. Consequently, the matrices in $\mathcal{R}(w)$ are in one-to-one correspondence with the matrices in the polytope $P = \{A | A = A^\top, A \geq 0, A\mathbf{1} = w\}$. Recall that for a combinatorially symmetric $n \times n$ matrix A the graph associated with A, which we denote by $G(A)$, has an edge between vertices j and k if and only if $a_{jk} \neq 0, j, k = 1, \ldots, n$. Brualdi [2] has characterized the extreme points of P as those $A \in P$ such that each connected component of $G(A)$ is either a tree or a unicyclic graph containing an odd cycle. (We recall that a graph is unicyclic if it is connected and contains just one cycle.) That characterization immediately yields the following.

THEOREM 3.2. *Suppose that $w \in \mathbb{R}^n, w > 0, w^\top \mathbf{1} = 1$. A matrix $T \in \mathcal{R}(w)$ is an extreme point of $\mathcal{R}(w)$ if and only if each connected component of $G(T)$ is either a tree or a unicyclic graph whose unique cycle has odd length (possibly a loop).*

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The following technical result will be useful in the sequel.

Lemma 3.3. *Suppose that $x \in \mathbb{R}^{2k+1}$, and for each $j = 0, 1, \ldots, 2k$, let $y_j = (x_{j+1} - x_{j+2k})(x_{j+2k-1} - x_{j+2k+1})$, where the subscripts are considered modulo $2k + 1$. Then $\Pi_{j=0}^{2k} y_j \le 0$. In particular, for some $j_0 \in \{0, 1, \ldots, 2k\}, y_{j_0} \le 0$.*

Proof. Observe that for each $j = 0, 1, \ldots, 2k - 1$, the left factor of y_j is $(x_{j+1} - x_{j+2k})$, while the right factor of y_{j+1} is $-(x_{j+1} - x_{j+2k})$. Similarly, the left factor of y_{2k} is $(x_{2k+1} - x_{2k-1})$ while the right factor of y_0 is $-(x_{2k+1} - x_{2k-1})$. Consequently, $\Pi_{j=0}^{2k} y_j = (-1)^{2k+1}\Pi(x_{j+1} - x_{j+2k})^2$. ∎

Theorem 3.4. *Suppose that $w \in \mathbb{R}^n$ satisfies Assumption A. Suppose further that $T \in \mathcal{R}(w)$ and that $G(T)$ is unicyclic, with cycle length $2k + 1$ for some $k \in \mathbb{N}$. Then there is a matrix $\hat{T} \in \mathcal{R}(w)$ such that $\lambda_{\min}(\hat{T}) \le \lambda_{\min}(T)$, and the graph of $\hat{T}$ is a tree with a loop at one vertex.*

Proof. Set $W = \mathrm{diag}(w)$, and note that $A \equiv W^{\frac{1}{2}} T W^{-\frac{1}{2}}$ is symmetric, similar to T, and has the same graph as T. Without loss of generality, we assume that the cycle of length $2k + 1$ in the graph of A is $1 \sim 2 \sim \ldots \sim 2k + 1 \sim 1$. Let u be an eigenvector of A corresponding to $\lambda_{\min}(A)$, and set $x = W^{-\frac{1}{2}} u$. From Lemma 3.3, there is a $j_0 \in \{0, 1, \ldots, 2k\}$ such that $(x_{j_0+1} - x_{j_0+2k})(x_{j_0+2k-1} - x_{j_0+2k+1}) \le 0$, where the subscripts are considered modulo $2k + 1$. We note that the cycle of length $2k + 1$ contains the subpath $j_0 + 2k - 1 \sim j_0 + 2k \sim j_0 + 2k + 1 \sim j_0 + 1$.

Suppose first that $k \ge 2$. Let E be the matrix whose principal submatrix on lines $j_0+1, 2k+j_0-1, 2k+j_0, 2k+j_0+1$ is

$$\begin{bmatrix} 0 & \frac{1}{w_{j_0+1}} & 0 & -\frac{1}{w_{j_0+1}} \\ \frac{1}{w_{2k+j_0-1}} & 0 & -\frac{1}{w_{2k+j_0-1}} & 0 \\ 0 & -\frac{1}{w_{2k+j_0}} & 0 & \frac{1}{w_{2k+j_0}} \\ -\frac{1}{w_{2k+j_0+1}} & 0 & \frac{1}{w_{2k+j_0+1}} & 0 \end{bmatrix},$$

and whose remaining entries are zero. Observe that $T + cE \in \mathcal{R}(w)$ for all

$$0 \le c \le \min\{w_{j_0+1} t_{j_0+1,2k+j_0+1}, w_{2k+j_0-1} t_{2k+j_0-1,2k+j_0}\} \equiv c_0.$$

Let $\tilde{T} = T + c_0 E$, $\tilde{A} = W^{\frac{1}{2}} \tilde{T} W^{-\frac{1}{2}}$, and note that $u^\top \tilde{A} u = u^\top A u + 2c_0(x_{j_0+1} - x_{j_0+2k})(x_{j_0+2k-1} - x_{j_0+2k+1}) \le \lambda_{\min}(A) u^\top u$. Hence, $\lambda_{\min}(\tilde{T}) \le \lambda_{\min}(T)$. Note that $G(\tilde{T})$ is formed from $G(T)$ by either deleting the edge $j_0 \sim j_0 + 2k + 1$ and adding the edge $j_0 + 1 \sim j_0 + 2k - 1$ (thus creating a unicyclic graph with a cycle of length $2k - 1$), or deleting the edge $j_0 + 2k - 1 \sim j_0 + 2k$ and adding the edge $j_0 + 1 \sim j_0 + 2k - 1$ (again creating a unicyclic graph with a cycle of length $2k - 1$); note that it cannot be the case that both $j_0 \sim j_0 + 2k + 1$ and $j_0 + 2k - 1 \sim j_0 + 2k$ are deleted from $G(T)$, otherwise $\tilde{T}$ would be disconnected, and the connected component containing vertices $j_0 + 2k$ and $j_0 + 2k + 1$ would bipartite, contrary to our hypothesis that w satisfies Assumption A. Hence, $G(\tilde{T})$ is unicyclic with a cycle of length $2k - 1$.

For the case that $k = 1$, we consider the matrix E whose principal submatrix on lines j_0+1, j_0+2, j_0+3 is

$$\begin{bmatrix} \frac{2}{w_{j_0+1}} & -\frac{1}{w_{j_0+1}} & -\frac{1}{w_{j_0+1}} \\ -\frac{1}{w_{j_0+2}} & 0 & \frac{1}{w_{j_0+2}} \\ -\frac{1}{w_{j_0+3}} & \frac{1}{w_{j_0+3}} & 0 \end{bmatrix},$$

and whose remaining entries are zero. Arguing as above, we find that there is a matrix $\tilde{T} \in \mathcal{R}(w)$ whose graph is unicyclic with a cycle of length $2k - 1 (= 1)$, and such that $\lambda_{\min}(\tilde{T}) \le \lambda_{\min}(T)$.

The conclusion now follows by a straightforward argument by induction on k. ∎

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LEMMA 3.5. *Suppose that $w \in \mathbb{R}^n$ satisfies Assumption A and that $T \in \mathcal{R}(w)$. Suppose further that $G(T)$ is a tree with a loop, and that x is a right eigenvector of T corresponding to the eigenvalue λ. If x has a zero entry, then $\lambda > \underline{\lambda}(w)$.*

Proof. Since x has a zero entry, there are indices j, k, ℓ such that $x_j = 0, x_k < 0, x_\ell > 0$ and $G(T)$ contains the edges $j \sim k, j \sim \ell$. Without loss of generality, we assume that $j = 1, k = 2, \ell = 3$. Let E be the matrix whose leading 3×3 principal submatrix is

$$\begin{bmatrix} \frac{2}{w_1} & -\frac{1}{w_1} & -\frac{1}{w_1} \\ -\frac{1}{w_2} & 0 & \frac{1}{w_2} \\ -\frac{1}{w_3} & \frac{1}{w_3} & 0 \end{bmatrix},$$

and whose remaining entries are zero. Observe that for all sufficiently small $\epsilon > 0, T + \epsilon E \in \mathcal{R}(w)$. Set $W = \mathrm{diag}(w)$, let $A_\epsilon \equiv W^{\frac{1}{2}}(T + \epsilon E)W^{-\frac{1}{2}}$, and let $u = W^{\frac{1}{2}}x$. Note that $u^\top A_\epsilon u = x^\top WTx + \epsilon x^\top WEx = \lambda x^\top Wx + \epsilon x^\top W(\frac{2x_1 - x_2 - x_3}{w_1}e_1 + \frac{x_3 - x_1}{w_2}e_2 + \frac{x_2 - x_1}{w_3}e_3) = \lambda u^\top u + 2\epsilon x_2 x_3 < \lambda u^\top u$. The conclusion now follows. ∎

The following result identifies a family of extreme points that includes a matrix realizing $\underline{\lambda}(w)$ as an eigenvalue.

THEOREM 3.6. *Suppose that $w \in \mathbb{R}^n$ satisfies Assumption A. There is a $T \in \mathcal{R}(w)$ such that i) $\lambda_{\min}(T) = \underline{\lambda}(w)$ and ii) $G(T)$ is a tree with a loop.*

Proof. From Theorems 3.2 and 3.4, it follows that there is a $T \in \mathcal{R}(w)$ such that $\lambda_{\min}(T) = \underline{\lambda}(w)$ and each connected component of $G(T)$ is either a tree or a tree with a loop. If some connected component of $G(T)$ were a tree, then part (iv) of Assumption A would be violated, and so we conclude that each connected component of $G(T)$ is a tree with a loop. If $G(T)$ is connected, we are done.

Suppose now that $G(T)$ is not connected, say with connected components $C_1, \ldots, C_m$ for some $m \geq 2$. For each $\ell = 1, \ldots, m$, let i_ℓ denote the vertex in C_ℓ at which there is a loop. Without loss of generality, assume that the principal submatrix of T corresponding to C_1 has $\underline{\lambda}(w)$ as an eigenvalue, with corresponding eigenvector x. By Lemma 3.5, x has no zero entries.

For all sufficiently small $\epsilon > 0$, note that $T - \epsilon(e_{i_1} - \frac{w_{i_1}}{w_{i_2}}e_{i_2})(e_{i_1} - e_{i_2})^\top \in \mathcal{R}(w)$. Let y denote the vector formed from x by appending zeros in the positions corresponding to vertices in $C_2, \ldots, C_m$. Let $W = \mathrm{diag}(w), u = W^{\frac{1}{2}}y$ and $A_\epsilon \equiv W^{\frac{1}{2}}(T - \epsilon(e_{i_1} - \frac{w_{i_1}}{w_{i_2}}e_{i_2})(e_{i_1} - e_{i_2})^\top)W^{-\frac{1}{2}}$. Then $u^\top A_\epsilon u = \lambda_{\min}(T)y^\top Wy - \epsilon y^\top W(e_{i_1} - \frac{w_{i_1}}{w_{i_2}}e_{i_2})(e_{i_1} - e_{i_2})^\top y = \lambda_{\min}(T)u^\top u - \epsilon w_{i_1}x_{i_1}^2 < \lambda_{\min}(T)u^\top u$. This last is a contradiction, and we conclude that $G(T)$ must be connected. ∎

The following result shows how the graphs arising in Theorem 3.6 generate the corresponding matrices in $\mathcal{R}(w)$ (see also [2]).

THEOREM 3.7. *Suppose that $w \in \mathbb{R}^n$ satisfies Assumption A. Let H be a tree on vertices labeled $1, \ldots, n$, with a loop at vertex j. For each $\ell = 1, \ldots, n, \ell \neq j$, let $\tilde{H}_\ell$ denote the subtree formed by deleting the branch at ℓ that contains j. Define S_ℓ^0 to be the set of vertices in $\tilde{H}_\ell$ that are at even distance from ℓ, and S_ℓ^1 to be the set of vertices in $\tilde{H}_\ell$ that are at odd distance from ℓ. Finally, let S_j^0 (respectively S_j^1) be the set of vertices in H that are at even (respectively odd) distance from j. There is a $T \in \mathcal{R}(w)$ such that $G(T) = H$ if and only if for each $\ell = 1, \ldots, n, \sum_{p \in S_\ell^0} w_p - \sum_{q \in S_\ell^1} w_q > 0$. When that condition holds, for each edge $k \sim \ell$ of H that is not a loop, and where ℓ is farther from j than k is, we have $t_{k,\ell} = \frac{1}{w_k}(\sum_{p \in S_\ell^0} w_p - \sum_{q \in S_\ell^1} w_q), t_{\ell,k} = \frac{1}{w_\ell}(\sum_{p \in S_\ell^0} w_p - \sum_{q \in S_\ell^1} w_q)$. Finally, $t_{j,j} = \frac{1}{w_j}(\sum_{p \in S_j^0} w_p - \sum_{q \in S_j^1} w_q)$.*

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Proof. Suppose that there is an $T \in \mathcal{R}(w)$ such that $G(T) = H$. Using the fact that $w_k t_{k,\ell} = w_\ell t_{\ell,k}$ for $k, \ell = 1, \ldots, n$, we deduce from the equation $w^\top T = w^\top$ that the following linear system (in unknowns that are indexed according to the edges of H) has a positive solution:

$$\sum_{\ell \sim k, \ell \geq k} w_k x_{k,\ell} + \sum_{\ell \sim k, \ell < k} w_\ell x_{\ell,k} = w_k, k = 1, \ldots, n. \tag{3.3}$$

Indeed, letting B denote the vertex-edge incidence matrix of H, the coefficient matrix of the linear system (3.3) can be written as B diag(u), where u is the vector with entries indexed by the edges of H such that for each edge $k \sim \ell$, the corresponding entry of u is given by $w_{\min\{k,\ell\}}$.

Since H is a tree with a loop, there is an ordering of the rows and columns of B diag(u) such that it is square and lower triangular with all diagonal entries positive. Hence, (3.3) has a unique solution, and it is straightforward to check that the stated values of the nonzero entries in T are a solution (and hence the unique solution) to (3.3). The conclusion follows readily. ∎

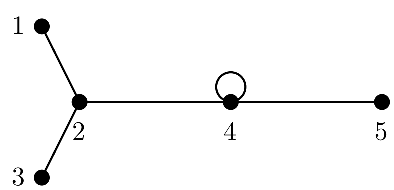

Figure 1. *The graph H_0 for Example 3.8.*

Example 3.8. In this example, we briefly illustrate the technique used in the proof of Theorem 3.7. Consider the graph H_0 that is depicted in Fig. 1. We have the following unknowns corresponding to the edges: $x_{1,2}, x_{2,3}, x_{2,4}, x_{4,4}, x_{4,5}$. For the present example, the linear system (3.3) is

$$\begin{aligned} w_1 x_{1,2} &= w_1; \\ w_2 x_{2,3} + w_2 x_{2,4} + w_1 x_{1,2} &= w_2; \\ w_2 x_{2,3} &= w_3; \\ w_4 x_{4,4} + w_4 x_{4,5} + w_2 x_{2,4} &= w_4; \\ w_4 x_{4,5} &= w_5. \end{aligned} \tag{3.4}$$

This system can be conveniently re-written as

$$\begin{bmatrix} w_1 & 0 & 0 & 0 & 0 \\ 0 & w_2 & 0 & 0 & 0 \\ 0 & 0 & w_4 & 0 & 0 \\ w_1 & w_2 & 0 & w_2 & 0 \\ 0 & 0 & w_4 & w_2 & w_4 \end{bmatrix} \begin{bmatrix} x_{1,2} \\ x_{2,3} \\ x_{4,5} \\ x_{2,4} \\ x_{4,4} \end{bmatrix} = \begin{bmatrix} w_1 \\ w_3 \\ w_5 \\ w_2 \\ w_4 \end{bmatrix}.$$

According to the conclusion of Theorem 3.7, we have

$$t_{1,2} = \frac{w_1}{w_1}, t_{2,3} = \frac{w_3}{w_2}, t_{4,5} = \frac{w_5}{w_4}, t_{2,4} = \frac{w_2 - w_1 - w_3}{w_2}, t_{4,4} = \frac{w_1 + w_3 + w_4 - w_2 - w_5}{w_4}.$$

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Computing the matrix-vector product with those values now yields

$$\begin{bmatrix} w_1 & 0 & 0 & 0 & 0 \\ 0 & w_2 & 0 & 0 & 0 \\ 0 & 0 & w_4 & 0 & 0 \\ w_1 & w_2 & 0 & w_2 & 0 \\ 0 & 0 & w_4 & w_2 & w_4 \end{bmatrix} \begin{bmatrix} (\frac{w_1}{w_1}) \\ (\frac{w_3}{w_2}) \\ (\frac{w_5}{w_4}) \\ (\frac{w_2-w_1-w_3}{w_2}) \\ (\frac{w_1+w_3+w_4-w_2-w_5}{w_4}) \end{bmatrix} = \begin{bmatrix} w_1 \\ w_3 \\ w_5 \\ w_2 \\ w_4 \end{bmatrix},$$

thus verifying that those values yield the unique solution to (3.4).

Assuming that w is a positive vector with $w_2 > w_1 + w_3$ and $w_1 + w_3 + w_4 > w_2 + w_5$, our solution to (3.4) is positive. In that case, we assemble the above information to produce the stochastic matrix

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{w_1}{w_2} & 0 & \frac{w_3}{w_2} & \frac{w_2-w_1-w_3}{w_2} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{w_2-w_1-w_3}{w_4} & 0 & \frac{w_1+w_3+w_4-w_2-w_5}{w_4} & \frac{w_5}{w_4} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

which is in $\mathcal{R}(w)$ with $G(T) = H_0$.

Lemma 3.9. *Suppose that $w \in \mathbb{R}^n$ satisfies Assumption A, and that $w_n > \frac{1}{2}$. If $T \in \mathcal{R}(w)$ and $G(T)$ is a tree with a loop, then necessarily the loop is at vertex n.*

Proof. Suppose that in $G(T)$, the loop is at vertex j. By Theorem 3.7, the weight of the loop is given by

$$\frac{1}{w_j}\left(\sum_{p\in S_j^0} w_p - \sum_{p\in S_j^1} w_p\right) = \frac{1}{w_j}\left(1 - 2\sum_{p\in S_j^1} w_p\right).$$

If $n \in S_j^1$, then the weight of the loop would be negative, so we conclude that $n \in S_j^0$.

Suppose that $j \neq n$. Then there is a vertex k adjacent to j such that $n \in \widetilde{G(T)}_k$, and necessarily $n \in S_k^1$ since k is adjacent to j and the distance between j and n is even. But then we have $\sum_{\ell\in S_k^0} w_\ell - \sum_{\ell\in S_k^1} w_\ell \leq 1 - w_j - 2\sum_{\ell\in S_k^1} w_\ell \leq 1 - w_j - 2w_n < 0$, a contradiction. We deduce that $j = n$, as desired. ∎

Next we present an asymptotic result.

Theorem 3.10. *Suppose that $\tilde{w} \in \mathbb{R}^{n-1}$ satisfies Assumption A. For all $0 < \epsilon < \frac{1}{2}$, let $w(\epsilon)^\top = \left[\ (\frac{1}{2}-\epsilon)\tilde{w}^\top \ \middle|\ \frac{1}{2}+\epsilon\ \right]$. For all sufficiently small $\epsilon > 0, \underline{\lambda}(w(\epsilon)) = -\left(\frac{\frac{1}{2}-\epsilon}{\frac{1}{2}+\epsilon}\right)$.*

Proof. Suppose that $0 < \epsilon < \frac{1}{2}$. Observe that the matrix

$$T = \left[\begin{array}{c|c} 0 & \mathbf{1} \\ \hline \left(\frac{\frac{1}{2}-\epsilon}{\frac{1}{2}+\epsilon}\right)\tilde{w}^\top & \frac{2\epsilon}{\frac{1}{2}+\epsilon} \end{array}\right],$$

is in $\mathcal{R}(w(\epsilon))$; it is straightforward to determine that $\lambda_{\min}(T) = -\left(\frac{\frac{1}{2}-\epsilon}{\frac{1}{2}+\epsilon}\right)$. Evidently, $G(T)$ is a star with a loop at vertex n.

Suppose that $T(\epsilon) \in \mathcal{R}(w(\epsilon))$, and that $G(T(\epsilon))$ is a tree with a loop. From Lemma 3.9, necessarily that loop is at vertex n. Suppose further that $G(T(\epsilon))$ is not a star with a loop at vertex n. Then S_n^0 contains at least two vertices. Observe that the weight of the loop at vertex n is given by

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$$\frac{2\sum_{j\in S_n^0} w(\epsilon)_j - 1}{\frac{1}{2}+\epsilon} = \frac{4\epsilon + 4\sum_{j\in S_n^0, j\neq n} w(\epsilon)_j}{1+2\epsilon} = \frac{4\epsilon + (2-4\epsilon)\sum_{j\in S_n^0, j\neq n} \tilde{w}_j}{1+2\epsilon},$$

which is bounded away from 0 as $\epsilon \to 0^+$. Similarly, if $k \sim n$ in $G(T(\epsilon))$, then the (n,k) entry in $T(\epsilon)$ is given by $\left(\frac{\frac{1}{2}-\epsilon}{\frac{1}{2}+\epsilon}\right)(\sum_{p\in S^0(k)} \tilde{w}_p - \sum_{p\in S^1(k)} \tilde{w}_p)$, which is bounded away from 0 as $\epsilon \to 0^+$. Finally, we note that for any $j \le n-1$ and any $k \le n$, the (j,k) entry of $T(\epsilon)$ is independent of ϵ. It now follows that as $\epsilon \to 0^+$, $T(\epsilon)$ converges to a primitive stochastic matrix, and hence that as $\epsilon \to 0^+$, the eigenvalues of $T(\epsilon)$ are all bounded away from -1. Since $-\left(\frac{\frac{1}{2}-\epsilon}{\frac{1}{2}+\epsilon}\right) \to -1$ as $\epsilon \to 0^+$, we find that for all sufficiently small positive values of ϵ, $\lambda_{\min}(T(\epsilon)) > -\left(\frac{\frac{1}{2}-\epsilon}{\frac{1}{2}+\epsilon}\right)$. As there is a finite number of trees on n vertices, the conclusion follows.□

Using the results above, we can bound the elements of $\sigma_{\mathcal{R}}(w)$ away from -1, as the following theorem establishes.

Theorem 3.11. *Suppose that $w \in \mathbb{R}^n$ satisfies Assumption A. Set*

$$\gamma = \frac{1}{w_n}\min\left\{\sum_{p\in S_1} w_p - \sum_{q\in S_2} w_q | S_1, S_2 \in \{1,\ldots,n\}, S_1\cap S_2 = \emptyset, \sum_{p\in S_1} w_p > \sum_{q\in S_2} w_q\right\}.$$

Then $\underline{\lambda}(w) \ge -(1-\gamma^{n-1})^{\frac{1}{n-1}}$.

Proof. From a result of Brauer [1], it follows that if a stochastic matrix T has a column with all entries bounded below by $c > 0$, then for any eigenvalue $\lambda \neq 1$ of T, we have $|\lambda| \le 1-c$.

Now consider $T \in \mathcal{R}(w)$ having $\underline{\lambda}(w)$ as an eigenvalue. (Note that $\underline{\lambda}(w)$ necessarily has absolute value less than 1.) Without loss of generality, we may assume that $G(T)$ is a tree with a loop, say at vertex j. From Theorem 3.7, it follows that every positive entry in T is bounded below by γ. The diameter of $G(T)$ is at most $n-1$, and hence for every vertex k in $G(T)$ there is a path from k to j of length at most $n-1$. Further, since there is a loop at vertex j, we find that every vertex k in $G(T)$ there is a path from k to j of length exactly $n-1$. Hence, the j-th column of T^{n-1} has all positive entries, and indeed $T^{n-1}e_j \ge \gamma^{n-1}\mathbf{1}$. Since $\underline{\lambda}(w)$ is an eigenvalue of T, $\underline{\lambda}(w)^{n-1}$ is an eigenvalue of T^{n-1}. Consequently, $|\underline{\lambda}(w)^{n-1}| \le 1-\gamma^{n-1}$, from which the desired inequality follows. □

4. Reversible low order cases.

In this section, we explicitly describe $\sigma_{\mathcal{R}}(w)$ for $w \in \mathbb{R}^2$ and $w \in \mathbb{R}^3$. As in section 3, the interesting case is that w satisfies Assumption A.

4.1. $\sigma_{\mathcal{R}}(w)$ for $n = 2$.

Theorem 4.1. *Suppose that $w \in \mathbb{R}^2$ with $w > 0, w^\top \mathbf{1} = 1$, and $w_1 < w_2$. Then $\underline{\lambda}(w) = -\frac{w_1}{w_2}$.*

Proof. From Theorem 3.6, the only extreme point to consider is $\begin{bmatrix} 0 & 1 \\ \frac{w_1}{w_2} & \frac{w_2-w_1}{w_2}\end{bmatrix}$, which has eigenvalues 1 and $-\frac{w_1}{w_2}$. The conclusion follows immediately. □

We note in passing that any irreducible stochastic matrix of order 2 is reversible. Hence, Theorem 4.1 yields the fact that when $n = 2$ and $w_1 \le w_2, \sigma_{\mathcal{S}}(w) = \left[-\frac{w_1}{w_2}, 1\right]$.

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4.2. $\sigma_{\mathcal{R}}(w)$ for $n = 3, w_3 > \frac{1}{2}$.

THEOREM 4.2. *Suppose that $w \in \mathbb{R}^3$ satisfies Assumption A, and that $w_3 > \frac{1}{2}$. Then*

$$\underline{\lambda}(w) = \begin{cases} \frac{w_3-1}{w_3} \text{ if } w_1 + 3w_2 \geq 1, \\ \frac{-1}{2}\left(\frac{w_2-w_1}{w_3} + \sqrt{\frac{(w_2-w_1)^2}{w_3^2} + \frac{4w_1(1-2w_2)}{w_2w_3}}\right) \text{ if } w_1 + 3w_2 < 1. \end{cases}$$

Proof. Suppose that $T \in \mathcal{R}(w)$ and has $\underline{\lambda}(w)$ as an eigenvalue. From Theorem 3.6, we may assume that $G(T)$ is a tree with a loop at one vertex, and from Lemma 3.9, the loop is at vertex 3. It follows that there are two candidate matrices to consider:

$$T_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{w_1}{w_3} & \frac{w_2}{w_3} & \frac{2w_3-1}{w_3} \end{bmatrix} \text{ and } T_2 = \begin{bmatrix} 0 & 1 & 0 \\ \frac{w_1}{w_2} & 0 & \frac{w_2-w_1}{w_2} \\ 0 & \frac{w_2-w_1}{w_3} & \frac{w_1+w_3-w_2}{w_3} \end{bmatrix}.$$

The smallest eigenvalue of T_1 is $\frac{w_3-1}{w_3}$ and the smallest eigenvalue of T_2 is

$$\frac{-1}{2}\left(\frac{w_2 - w_1}{w_3} + \sqrt{\frac{(w_2 - w_1)^2}{w_3^2} + \frac{4w_1(1 - 2w_2)}{w_2w_3}}\right).$$

It remains only to determine the circumstances under which one of these two values is smaller than the other.

In order to make that determination, we consider

$$T_2 - \frac{w_3 - 1}{w_3}I = \begin{bmatrix} \frac{1-w_3}{w_3} & 1 & 0 \\ \frac{w_1}{w_2} & \frac{1-w_3}{w_3} & \frac{w_2-w_1}{w_2} \\ 0 & \frac{w_2-w_1}{w_3} & \frac{1+w_1-w_2}{w_3} \end{bmatrix},$$

and recall that it is diagonally similar to a symmetric matrix. We find that $T_2 - \frac{w_3-1}{w_3}I$ has at least two positive eigenvalues, since the submatrix formed by deleting the second row and columns does (this follows from interlacing on the diagonally similar symmetric matrix). Hence, the two largest eigenvalues of T_2 exceed $\frac{w_3-1}{w_3}$; we deduce that $\lambda_{\min}(T_2)$ is less than, larger than, or equal to $\frac{w_3-1}{w_3}$ according as $\det(T_2 - \frac{w_3-1}{w_3}I)$ is negative, positive, or zero. An uninteresting computation reveals that $\det(T_2 - \frac{w_3-1}{w_3}I) = \frac{w_1(w_1+3w_2-1)}{w_2w_3^2}$. The conclusion follows. ∎

4.3. $\sigma_{\mathcal{R}}(w)$ for $n = 3, w_3 < \frac{1}{2}$.

THEOREM 4.3. *Suppose that $w \in \mathbb{R}^3$ satisfies Assumption A, and that $w_3 < \frac{1}{2}$. Let*

$$x_1^- = \frac{-1}{2}\left(\frac{w_2 - w_1}{w_3} + \sqrt{\frac{(w_2 - w_1)^2}{w_3^2} + \frac{4w_1(1 - 2w_2)}{w_2w_3}}\right),$$

$$x_2^- = \frac{-1}{2}\left(\frac{w_3 - w_2}{w_1} + \sqrt{\frac{(w_3 - w_2)^2}{w_1^2} + \frac{4w_2(1 - 2w_3)}{w_1w_3}}\right).$$

Then $\underline{\lambda}(w) = \min\{x_1^-, x_2^-\}$. More specifically, for each $w_2 \in (\frac{1}{4}, \frac{1}{2})$ with $w_2 \neq \frac{1}{3}$, there is a unique $w_1^* \in (\frac{1}{2} - w_2, w_2)$ such that

$$(1 - 2w_2){w_1^*}^4 + (8w_2^2 - w_2 - 1){w_1^*}^3 + (24w_2^2 - 27w_2 + 7)w_2{w_1^*}^2$$
$$+w_2(2w_2 - 1)(13w_2^2 - 15w_2 + 4)w_1^* + w_2(1 - 2w_2)^4 = 0.$$

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We then have

$$\underline{\lambda}(w) = \begin{cases} x_2^-, & \textit{if } w_2 \in (\frac{1}{4}, \frac{1}{3}), w_1 \in (\frac{1}{2} - w_2, w_1^*) \\ x_1^- & \textit{if } w_2 \in (\frac{1}{4}, \frac{1}{3}), w_1 \in [w_1^*, w_2) \\ x_2^- & \textit{if } w_2 \in [\frac{1}{3}, \frac{1}{2}), w_1 \in (\frac{1}{2} - w_2, 1 - 2w_2). \end{cases}$$

Proof. There are three matrices in $\mathcal{R}(w)$ having a graph that is a tree with a loop. They are

$$T_1 = \begin{bmatrix} 0 & 1 & 0 \\ \frac{w_1}{w_2} & 0 & \frac{w_2-w_1}{w_2} \\ 0 & \frac{w_2-w_1}{w_3} & \frac{w_1+w_3-w_2}{w_3} \end{bmatrix}, T_2 = \begin{bmatrix} \frac{w_1+w_2-w_3}{w_1} & 0 & \frac{w_3-w_2}{w_1} \\ 0 & 0 & 1 \\ \frac{w_3-w_2}{w_3} & \frac{w_2}{w_3} & 0 \end{bmatrix}, \text{and}$$
$$T_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{w_1+w_2-w_3}{w_2} & \frac{w_3-w_1}{w_2} \\ \frac{w_1}{w_3} & \frac{w_3-w_1}{w_3} & 0 \end{bmatrix}.$$

We have the following characteristic polynomials for these three matrices:

$$\det(xI - T_1) = (x-1)\left(x^2 + \frac{w_2 - w_1}{w_3}x - \frac{w_1(1-2w_2)}{w_2 w_3}\right) \equiv (x-1)p_1(x),$$
$$\det(xI - T_2) = (x-1)\left(x^2 + \frac{w_3 - w_2}{w_1}x - \frac{w_2(1-2w_3)}{w_1 w_3}\right) \equiv (x-1)p_2(x),$$
$$\det(xI - T_3) = (x-1)\left(x^2 + \frac{w_3 - w_1}{w_2}x - \frac{w_1(1-2w_3)}{w_2 w_3}\right) \equiv (x-1)p_3(x).$$

Observe that $p_3(x) - p_2(x) = \frac{(w_2-w_1)(1-2w_3)}{w_1 w_2 w_3}(w_3 x + w_1 + w_2)$. Hence, for any $x > -1, p_3(x) - p_2(x) > \frac{(w_2-w_1)(1-2w_3)}{w_1 w_2 w_3}(-w_3 + w_1 + w_2) = \frac{(w_2-w_1)(1-2w_3)^2}{w_1 w_2 w_3} > 0$. We deduce that the negative root of p_2 is smaller than the negative root of p_3. Hence, we find that $\underline{\lambda}(w)$ is the minimum of the negative root of p_1 and the negative root of p_2.

The remainder of the proof is dedicated to determining the circumstances under which one of those negative roots is smaller than the other. First we claim that p_1 and p_2 cannot have two roots in common. To see the claim, suppose to the contrary that p_1 and p_3 have the same roots. Then necessarily a) $\frac{w_3-w_2}{w_1} = \frac{w_2-w_1}{w_3}$ and b) $\frac{w_2(1-2w_3)}{w_1 w_3} = \frac{w_1(1-2w_2)}{w_2 w_3}$. Substituting $w_3 = 1 - w_1 - w_2$ and simplifying, we find that a) is equivalent to $2w_1^2 + 2w_2^2 = 2w_1(1-w_2) + 3w_2 - 1$. Similarly, substituting $w_3 = 1 - w_1 - w_2$ and simplifying, we find that b) is equivalent to $w_1^2 + w_2^2 = \frac{2w_1 w_2^2}{1-2w_2}$. Hence, if both a) and b) hold, it must be the case that $\frac{2w_1 w_2^2}{1-2w_2} = w_1(1-w_2) + \frac{3w_2-1}{2}$, i.e. $\frac{w_1(3w_2-1)}{1-2w_2} = \frac{3w_2-1}{2}$. If $w_2 \neq \frac{1}{3}$, then we would have $w_1 + w_2 = \frac{1}{2}$, contrary to our hypothesis. Hence, $w_2 = \frac{1}{3}$ but substituting that value into the equation $w_1^2 + w_2^2 = \frac{2w_1 w_2^2}{1-2w_2}$ now yields $w_1 = \frac{1}{3}$, again contrary to our hypothesis. We conclude that p_1 and p_2 can share at most one common root.

Next, we establish some constraints on w_1, w_2, w_3. Since $\frac{1}{2} > w_3 = 1 - w_1 - w_2$, we have $w_1 > \frac{1}{2} - w_2$. Also, $1 = w_1 + w_2 + w_3 > w_1 + 2w_2$, so that $w_1 < 1 - 2w_2$. We have $w_2 < w_3 < \frac{1}{2}$, and since $2w_2 > w_1 + w_2 > \frac{1}{2}$, we see that $w_2 > \frac{1}{4}$. Summarizing, we find that

$$\frac{1}{2} - w_2 < w_1 < w_2 \text{ if } \frac{1}{4} < w_2 \leq \frac{1}{3}, \text{ and}$$
$$\frac{1}{2} - w_2 < w_1 < 1 - 2w_2 \text{ if } \frac{1}{3} \leq w_2 < \frac{1}{2}.$$

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Note that p_1 has a positive root and a negative root, and we denote these by x_1^+, x_1^-, respectively. Similarly x_2^+ and x_2^- denote the positive and negative roots of p_2, respectively. For a fixed $w_2 \in (\frac{1}{4}, \frac{1}{3}]$, we can think of $x_1^+, x_1^-, x_2^+, x_2^-$ as continuous functions of w_1 on the interval $[\frac{1}{2} - w_2, w_2]$; similarly for fixed $w_2 \in [\frac{1}{3}, \frac{1}{2})$, $x_1^+, x_1^-, x_2^+, x_2^-$ are continuous functions of w_1 on the interval $[\frac{1}{2} - w_2, 1 - 2w_2]$.

Suppose that $w_1 = \frac{1}{2} - w_2$. It then follows that $p_1(x) = x^2 + (4w_2 - 1)x - \frac{(1-2w_2)^2}{w_2}$ and $p_2(x) = x^2 + x$. For this value of w_1, we then have $x_2^- = -1 < x_1^-, x_2^+ = 0 < x_1^+$. If $w_2 < \frac{1}{3}$ and $w_1 = w_2$, we have $p_1(x) = x^2 - 1, p_2(x) = x^2 + \frac{1-3w_2}{w_2}x - \frac{4w_2-1}{1-2w_2}$, so that $x_1^- = -1 < x_2^+$. From the intermediate value theorem, we deduce that for each $w_2 \in (\frac{1}{4}, \frac{1}{3}]$ there is a $w_1 \in (\frac{1}{2} - w_2, w_2)$ such that $x_1^- = x_2^-$. Similarly, if $w_2 > \frac{1}{3}$ and $w_1 = 1 - 2w_2$, then $p_1(x) = x^2 + \frac{3w_2-1}{w_2}x - \frac{(1-2w_2)^2}{w_2^2}, p_(x) = x^2 - 1$. For this value of w_1, we have $x_2^+ = 1 > x_1^+$. Hence, for each $w_2 \in [\frac{1}{3}, \frac{1}{2})$ there is a $w_1 \in (\frac{1}{2} - w_2, 1 - 2w_2)$ such that $x_1^+ = x_2^+$.

The condition for p_1 and p_2 to have precisely one common root is

$$\begin{aligned} &\left(\frac{w_2(1-2w_3)}{w_1 w_3} - \frac{w_1(1-2w_2)}{w_2 w_3}\right)^2 \\ &= \left(\frac{w_3 - w_2}{w_1}\frac{w_1(1-2w_2)}{w_2 w_3} - \frac{w_2 - w_1}{w_3}\frac{w_2(1-2w_3)}{w_1 w_3}\right)\left(\frac{w_3 - w_2}{w_1} - \frac{w_2 - w_1}{w_3}\right). \end{aligned} \tag{4.5}$$

Substituting $w_3 = 1 - w_1 - w_2$ and simplifying, we find that (4.5) is equivalent to the following:

$$\begin{aligned} &g(w_1) \equiv \\ &(1-2w_2)w_1^4 + (8w_2^2 - w_2 - 1)w_1^3 + (24w_2^2 - 27w_2 + 7)w_2 w_1^2 \\ &+ w_2(2w_2 - 1)(13w_2^2 - 15w_2 + 4)w_1 + w_2(1-2w_2)^4 = 0. \end{aligned} \tag{4.6}$$

Our next goal is to understand, for fixed $w_2 \in (\frac{1}{4}, \frac{1}{2})$, the number and nature of the roots w_1 of (4.6). We begin by considering the special case that $w_2 = \frac{1}{3}$. For this value of w_2, we have $g(w_1) = \frac{1}{3}(w_1 - \frac{1}{3})^4$, so that the only root of (4.6) is $w_1 = \frac{1}{3}$, of multiplicity 4. Henceforth, we will assume that $w_2 \neq \frac{1}{3}$.

We continue with the following general observation. Suppose that $w_2 \in (\frac{1}{4}, \frac{1}{2}), w_2 \neq \frac{1}{3}$. We have $g(\frac{1}{2} - w_2) = (1-2w_2)^4(4w_2 - 1)/16 > 0, g(w_2) = 8w_2(3w_2-1)^2(w_2 - \frac{1}{2})(w_2 - \frac{1}{4}) < 0, g(1-2w_2) = -w_2(6(w_2 - \frac{1}{3})(w_2 - \frac{1}{2}))^2 < 0$. It now follows that for $w_2 \in (\frac{1}{4}, \frac{1}{2}), w_2 \neq \frac{1}{3}$, $g(w_1)$ has a root in $(\frac{1}{2} - w_2, \min\{w_2, 1 - 2w_2\})$, and a root greater than $\min\{w_2, 1 - 2w_2\}$.

Next we consider two cases.

Case 1: $\frac{1}{2} > w_2 \geq \frac{27-\sqrt{57}}{48} \approx 0.4052$.
We begin by noting that in (4.6), the coefficient of w_1^2 changes sign at $w_2 = \frac{27-\sqrt{57}}{48}$, the coefficient of w_1 changes sign at $w_2 = \frac{15-\sqrt{17}}{26} \approx 0.4138$, and the coefficient of w_1^3 changes sign at $w_2 = \frac{1+\sqrt{33}}{16} \approx 0.4215$. It now follows that for $\frac{1}{2} > w_2 \geq \frac{27-\sqrt{57}}{48}$, the coefficients of g have one of the following sign patterns:

$$\begin{array}{ccccc} + & - & 0 & - & +\,; \\ + & - & - & - & +\,; \\ + & - & - & 0 & +\,; \\ + & - & - & + & +\,; \\ + & 0 & - & + & +\,; \\ + & + & - & + & +\,. \end{array}$$

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Thus, g has two changes of signs in the coefficients, and hence it follows from Descartes' rule of signs, at most two positive roots. We deduce that in this case, g has precisely two positive roots.

Case 2: $\frac{1}{4} < w_2 < \frac{27-\sqrt{57}}{48}, w_2 \neq \frac{1}{3}$.
Then, $g'(w_1) = 4(1-2w_2)w_1^3 + 3(8w_2^2 - w_2 - 1)w_1^2 + 2(24w_2^2 - 27w_2 + 7)w_2w_1 + w_2(2w_2-1)(13w_2^2 - 15w_2 + 4)$. This cubic has discriminant equal to

$$-\frac{w_2(3w_2-1)^4}{32(2w_2-1)^4}(3168w_2^5 - 4416w_2^4 + 1312w_2^3 + 565w_2^2 - 369w_2 + 54)$$
$$\equiv -\frac{w_2(3w_2-1)^4}{32(2w_2-1)^4}h(w_2).$$

We find that $h(-1) = -7908, h(0) = 54, h(\frac{41}{100}) = \frac{121}{4247}, h(\frac{42}{100}) = -\frac{337}{2807}, h(\frac{7}{10}) = -\frac{332}{63}, h(\frac{8}{10}) = \frac{9391}{438}$. Hence, h has roots in the intervals $(-1,0), (0.41, 0.42), (0.7, 0.8)$. Further, the roots of h' are approximately $-0.2624, 0.6459, 0.3659 \pm 0.0598i$ so h has no critical points in the interval $[0, \frac{1}{2}]$. We deduce that h has just one root in $[0, \frac{1}{2}]$, which is necessarily in $(0.41, 0.42)$. We thus conclude that $h(w_2) > 0$ for $\frac{1}{4} < w_2 < \frac{27-\sqrt{57}}{48}$.

Hence, when $\frac{1}{4} < w_2 < \frac{27-\sqrt{57}}{48}$, and $w_2 \neq \frac{1}{3}$, g' has one real root and one complex conjugate pair of roots. Consequently, for these values of w_2, g itself has exactly two positive real roots.

From cases 1 and 2, we find that for each $w_2 \in (\frac{1}{4}, \frac{1}{3}) \cup (\frac{1}{3}, \frac{1}{2})$, there is a unique value $w_1^* \in (\frac{1}{2} - w_2, \min\{w_2, 1-2w_2\})$ such that p_1 and p_2 have a common root. For $w_2 \in (\frac{1}{4}, \frac{1}{3}), w_1^*$ yields $x_1^- = x_2^-$, while for $w_2 \in (\frac{1}{3}, \frac{1}{2}), w_1^*$ yields $x_1^+ = x_2^+$.

It now follows that if $w_2 \in (\frac{1}{4}, \frac{1}{3})$ and $\frac{1}{2} - w_2 < w_1 < w_1^*$, we have $x_2^- < x_1^-$, and if $w_1^* < w_1 < w_2$, we have $x_1^- < x_2^-$. If $w_2 \in (\frac{1}{3}, \frac{1}{2})$ and $\frac{1}{2} - w_2 < w_1 < 1 - 2w_2$, we have $x_2^- < x_1^-$. In the case that $w_2 = \frac{1}{3}$, then for $w_1 \in (\frac{1}{2} - w_2, w_2) = (\frac{1}{6}, \frac{1}{3}), g(w_1) > 0$ and hence for all such w_1, p_1 and p_2 do not have a common root. It follows then that $x_2^- < x_1^-$ when $w_2 = \frac{1}{3}$ and $w_1 \in (\frac{1}{2} - w_2, w_2)$. ∎

Figure 2 depicts $\underline{\lambda}(w)$ for $w^\top = \begin{bmatrix} w_1 & \frac{7}{24} & w_3 \end{bmatrix}$, where $w_1 \in [\frac{1}{2} - w_2, w_2], w_3 = \frac{17}{24} - w_1$. Here, the value of w_1^* is approximately 0.2409. We note that, despite the appearance of this figure to the naked eye, $\underline{\lambda}(w)$ is not a linear function of w_1 for $w_1 \in [w_1^*, w_2]$.

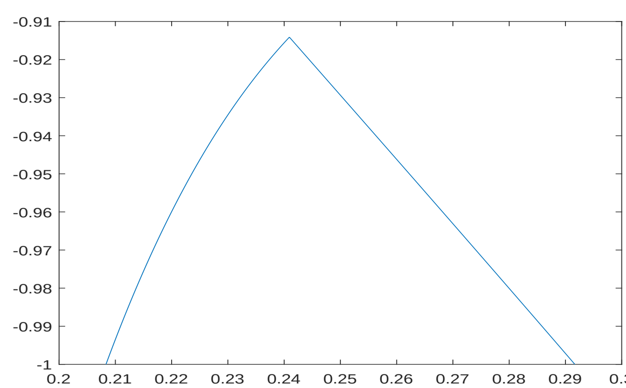

FIGURE 2. *Plot of* $\underline{\lambda}(w)$ *for* $w_2 = \frac{7}{24}, w_1 \in [\frac{1}{2} - w_2, w_2]$.

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Figure 3 compares w_1^* to $\frac{1}{2} - w_2$ and w_2 as w_2 ranges over the interval $(\frac{1}{4}, \frac{1}{3})$.

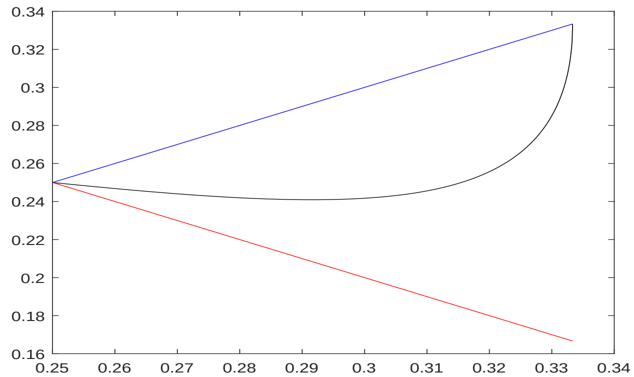

FIGURE 3. *Plots of* w_1^* *(black),* $\frac{1}{2} - w_2$ *(red),* w_2 *(blue) for* $w_2 \in (\frac{1}{4}, \frac{1}{3})$.

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