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THE NONNEGATIVE INVERSE EIGENVALUE PROBLEM WITH PRESCRIBED ZERO PATTERNS IN DIMENSION THREE*

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Abstract. The nonnegative inverse eigenvalue problem is considered in this paper with the additional restriction of fixed zero patterns in the matrix. A full analysis of the 3×3 case is given. Some remarks on the four-dimensional case are made.

Key words. Zero patterns, Nonnegative three by three matrices, Spectra.

AMS subject classifications. 15B48, 15A18.

1. Introduction.

The nonnegative inverse eigenvalue problem is a difficult problem that has not been solved in any great generality. In its greatest generality, the problem can be phrased as follows: give necessary and sufficient conditions on a list of n complex numbers to be the eigenvalues of an $n \times n$ nonnegative matrix. There are some obvious necessary conditions resulting from the fact that the matrix must be nonnegative: the list of complex numbers must be symmetric with respect to the real line, and the conditions resulting from the Perron–Frobenius theorem must be satisfied. However, there are many more necessary conditions. For one thing, all of the numbers must satisfy the conditions coming from the *single inverse eigenvalue problem*. This is the following question: give necessary and sufficient conditions on a single complex number to be the eigenvalue of an $n \times n$ nonnegative matrix. The latter problem has been solved by Karpelevič, for a row-stochastic matrix the conditions are given by the fact that the complex number must be in a region in the unit disc bounded by a number of arcs which can be explicitly described. A nice description of this region is given in [6]. We shall call this region the *Karpelevič region*, even though it is a different region for each dimension.

The three-dimensional case of the nonnegative inverse eigenvalue problem was solved by Loewy and London [8] and independently by Oliveira [10]. For that case the following was shown: there is a nonnegative 3×3 matrix with Perron eigenvalue 1 and two (possibly other) eigenvalues λ_2 and λ_3 if and only if $1+\lambda_2+\lambda_3 \geq 0$ and $\lambda_2, \lambda_3 \in [-1, 1]$ are real or λ_2 and $\lambda_3 = \overline{\lambda_2}$ are in the triangle P in the complex plane with vertices at 1 and $-\frac{1}{2} \pm \frac{1}{2}\sqrt{3}i$. Under these conditions, in [8], there is an explicit construction of a matrix with these eigenvalues. When the eigenvalues λ_2 and λ_3 are non-real, the construction involves a circulant matrix.

The four-dimensional case of the nonnegative inverse eigenvalue problem was solved by Meehan [9] and Torre-Mayo et al. [13]. The case $n \geq 5$ remains unsolved in general. The case where the trace is known to be zero (i.e., the case where the diagonal entries are zero) has been solved by Reams [12] for the four-dimensional situation and by Laffey and Meehan [7] for the five-dimensional situation.

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A recent survey of many results on the nonnegative inverse eigenvalue problem and its variants is given in [3].

Not surprisingly, when examples are constructed of nonnegative matrices that have or do not have certain properties, in many cases these examples are chosen to have a certain structure, in particular with respect to a given zero pattern. Cases in point are [1] and [13].

In this paper, we will investigate what happens if we restrict the class of nonnegative matrices by insisting on a given zero pattern in the matrix. The focus will be on the three by three case. It will be shown in this paper that the question is an interesting one, and for all zero patterns in the 3×3 case a complete solution will be given. Notice that we focus on the matrices with up to six zero entries, as reducible cases are fairly easy and a 3×3 matrix with more than six zeros necessarily has a whole column or row equal to zero. In addition, some examples in the 4×4 case will be discussed in part.

The paper has something in common with [13]. That paper focuses on the characteristic polynomial of a nonnegative matrix instead of on the spectrum. It is shown there, among many other things, that if a polynomial of degree 3 or 4 is realizable as the characteristic polynomial of a nonnegative matrix, then it is realizable as the characteristic polynomial of a matrix with a certain zero structure, arising from the underlying graph structure. In this way, a full solution of the four-dimensional nonnegative inverse eigenvalue problem is provided in [13].

We start with some observation for general $n \times n$ nonnegative matrices. A first reduction is that by scaling, we may assume that the spectral radius of the matrix is 1 or 0. Indeed, if A is a nonnegative matrix, then any positive multiple of it is also nonnegative. This does not change the eigenvectors and merely scales the eigenvalues. So, by dividing A by its spectral radius $\rho(A)$, we may assume without loss of generality that if $\rho(A)$ is positive, then $\rho(A) = 1$. Recall that if a power of a matrix is positive, then the matrix is primitive. Then we can apply the Perron–Frobenius theorem (see, e.g., [2], Chapter 8) to see that $\rho(A)$ is positive. The Perron–Frobenius theorem tells us also that $\rho(A)$ is an eigenvalue and there is a corresponding eigenvector which is nonnegative. In case the matrix is primitive, the eigenvector is actually positive. In case the matrix is reducible, one sees immediately from the zero pattern whether or not the spectral radius is positive.

Second, let us assume that $\rho(A) = 1$. In the case the matrix is irreducible, the corresponding eigenvector does not have a zero coordinate. Let $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$ be a positive eigenvector corresponding to the eigenvalue 1. Then let D be the diagonal matrix with the coordinates x_i on the diagonal, so $D = \operatorname{diag}(x_1, \ldots, x_n)$, and consider $A_1 = D^{-1}AD$. Then A_1 has the same eigenvalues as A, and (hence) the same characteristic polynomial, as well as the same zero pattern. Moreover, denoting by $\mathbf{e}$ the all-ones vector, we have

$$A_1\mathbf{e} = D^{-1}AD\mathbf{e} = D^{-1}A\mathbf{x} = D^{-1}\mathbf{x} = \mathbf{e}.$$

Thus, $\mathbf{e}$ is an eigenvector of A_1 corresponding to the eigenvalue 1. In other words, A_1 is row-stochastic, or, equivalently, the row sums are all one. This will be a standing assumption throughout the paper for the cases where $\rho(A) \neq 0$.

The following proposition will be useful in several cases.

Proposition 1. *Let A be an $n \times n$ row-stochastic matrix with a fixed zero pattern, such that no zero in the pattern is on the diagonal. Suppose the spectrum of A is $\{1, \lambda_2, \ldots, \lambda_n\}$. Then for any $0 < s < 1$, the list $\{1, s\lambda_2 + (1-s), s\lambda_3 + (1-s), \ldots, s\lambda_n + (1-s)\}$ is also the spectrum of an $n \times n$ row-stochastic matrix with the same zero pattern.*

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Proof. Consider for $t > 0$ the matrix $A(t) = (A + tI)/(1 + t)$. The matrix $A(t)$ is row-stochastic and since there is no zero in the pattern that is on the diagonal, $A(t)$ has the same zero pattern as A. Further, if λ is an eigenvalue of A with corresponding eigenvector $\mathbf{x}$, so $A\mathbf{x} = \lambda\mathbf{x}$, then $A(t)\mathbf{x} = (\frac{\lambda}{1+t} + \frac{t}{1+t})\mathbf{x} = (s\lambda + (1-s))\mathbf{x}$ when we take $s = \frac{1}{1+t}$. □

Next, we discuss the region in the complex plane containing the eigenvalues. The following result is one of the so-called Johnson–Loewy–London inequalities (see, e.g., [3]).

Proposition 2. *Let A be an $n \times n$ matrix with nonnegative entries. Then*

$$(\mathrm{trace}\,(A))^2 \le n \cdot \mathrm{trace}\,(A^2).$$

For a 3×3 matrix A with spectrum $\sigma(A) = \{1, \lambda_2, \lambda_3\}$, we have

$$\mathrm{trace}\,(A) = 1 + \lambda_2 + \lambda_3, \qquad \det(A) = \lambda_2\lambda_3,$$

and the sum of the principal two-by-two subdeterminants is equal to $\lambda_2 + \lambda_3 + \lambda_2\lambda_3$, while the characteristic polynomial of a matrix A is given by:

$$\det(A - \lambda I_3) = -\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0,$$

where

$$\begin{aligned} p_0 &= \det(A), \\ p_1 &= \text{minus the sum of the principal two by two subdeterminants}, \\ p_2 &= \mathrm{trace}\,(A). \end{aligned}$$

As the trace of a nonnegative matrix is nonnegative, it follows that the pair (λ_2, λ_3) always satisfies $1 + \lambda_2 + \lambda_3 \ge 0$. This is particularly interesting when λ_2 and λ_3 are non-real, say $\lambda_{2,3} = a \pm ib$, as in that case it implies that $a \ge -\frac{1}{2}$. The following proposition describes the location of the spectrum of a nonnegative row-stochastic 3×3 matrix. As the result is well known, a proof will be omitted (see [11] for details if necessary).

Proposition 3. *Let A be a nonnegative row-stochastic 3×3 matrix A, and let $\sigma(A) = \{1, \lambda_2, \lambda_3\}$. Then the following hold:*

i. *If $\lambda_{2,3} = a \pm ib$ with $b \ne 0$, then $a \ge -\frac{1}{2}$ and $|b| \le \frac{\sqrt{3}}{3}|1 - a|$.*
ii. *If $\lambda_3 \le \lambda_2$ are real, then $-1 \le \lambda_2 + \lambda_3 \le 2$, and the point (λ_2, λ_3) lies in the closed region in the plane bounded by the lines $\lambda_2 = \lambda_3$, $\lambda_2 = 1$, $\lambda_3 = -1$ and $\lambda_2 + \lambda_3 = -1$.*

Conversely, when the pair (λ_2, λ_3) satisfies these conditions, then there is a nonnegative row-stochastic 3×3 matrix A with $\sigma(A) = \{1, \lambda_2, \lambda_3\}$.

The region for the complex eigenvalues, see Figure 1, is precisely the region bounded by the so-called Karpelevič arcs in the three-dimensional case (see, e.g., [4, 6]).

2. Zeros on the diagonal and corresponding restrictions on the eigenvalue location. It turns out that the presence of zeros on the diagonal further restricts the region in the complex plane in which there can be eigenvalues. That is not a big surprise: the trace is the sum of the eigenvalues, so limiting the trace gives conditions on the sum of the eigenvalues. However, the situation is more intricate than just this. First

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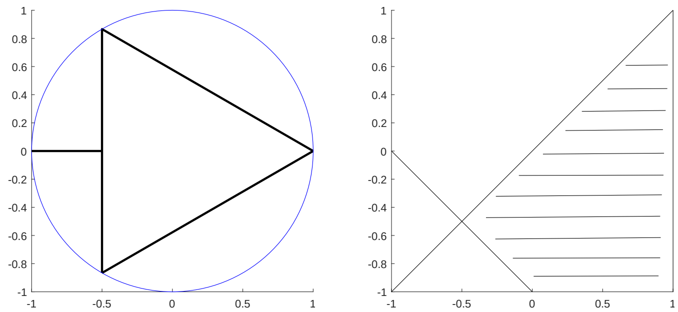

FIGURE 1. *Left: The Karpelevič region for 3×3 matrices. Right: The possible region for the real eigenvalues $\lambda_2 \geq \lambda_3$.*

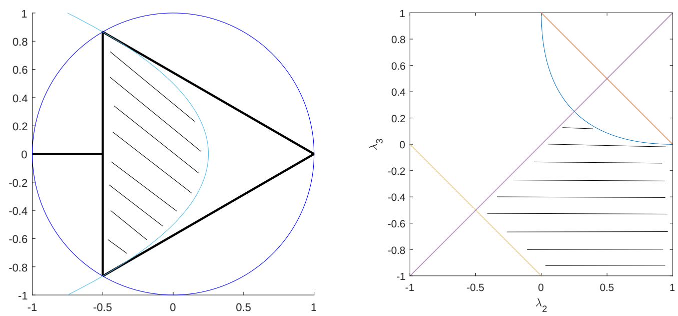

FIGURE 2. *Restrictions on the spectrum when there is one zero on the diagonal.*

note that if A is a 3×3 matrix with one zero on the diagonal, similarity with an appropriate permutation matrix will put that zero in the $(1,1)$ position. Likewise, when there are two zeros on the diagonal, we may assume that the zeros are in the $(1,1)$ and $(2,2)$ positions.

LEMMA 4. *Let* $A = \begin{bmatrix} 0 & \alpha & 1-\alpha \\ \beta & \gamma & 1-\beta-\gamma \\ \delta & \phi & 1-\delta-\phi \end{bmatrix}$ *be a nonnegative primitive row-stochastic matrix, with* $\gamma > 0$ *and* $\delta + \phi < 1$. *Let the spectrum of* A *be* $\{1, \lambda_2, \lambda_3\}$. *Then the following hold*

i. *If* $\lambda_{2,3} = a \pm ib$ *with* $b \neq 0$, *then* $a + b^2 < \frac{1}{4}$.
ii. *If* $\lambda_3 \leq \lambda_2$ *are real, then* $-1 < \lambda_2 + \lambda_3 < 1$, *and the point* (λ_2, λ_3) *lies in the region in the real plane bounded by the lines* $\lambda_2 = \lambda_3$, $\lambda_2 + \lambda_3 = -1$, $\lambda_3 = -1$, $\lambda_2 = 1$ *and the parabola* $1 - 2\lambda_2 - 2\lambda_3 - 2\lambda_2\lambda_3 + \lambda_2^2 + \lambda_3^2 = 0$.

Figure 2 gives the position of possible complex eigenvalues in the complex plane when there is one zero on the diagonal as dictated by part (i) in the above lemma on the left and the position of the possible pairs (λ_2, λ_3) in the real plane as dictated by part (ii) on the right.

Proof. The proof is based on the fact that the sum of the two-by-two principal subdeterminants is equal to $\lambda_2 + \lambda_3 + \lambda_2\lambda_3$, combined with the fact that the trace is equal to $1 + \lambda_2 + \lambda_3$. For the matrix A, the sum

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of the two-by-two principal subdeterminants becomes

$$\begin{aligned}&\begin{vmatrix}0 & \alpha\\ \beta & \gamma\end{vmatrix}+\begin{vmatrix}0 & 1-\alpha\\ \delta & 1-\delta-\phi\end{vmatrix}+\begin{vmatrix}\gamma & 1-\beta-\gamma\\ \phi & 1-\delta-\phi\end{vmatrix}\\ =&-\alpha\beta-\delta(1-\alpha)+\gamma(1-\delta-\phi)-\phi(1-\beta-\gamma)\\ =&\lambda_2+\lambda_3+\lambda_2\lambda_3.\end{aligned}$$

Take $\gamma(1-\delta-\phi)$ to the other side of the equation, then it follows that

$$\lambda_2+\lambda_3+\lambda_2\lambda_3-\gamma(1-\delta-\phi)<0.$$

Now use that

$$1+\lambda_2+\lambda_3=\gamma+(1-\delta-\phi).$$

It follows that

$$\begin{aligned}&\lambda_2+\lambda_3+\lambda_2\lambda_3-\gamma(1-\delta-\phi)\\ =&\lambda_2+\lambda_3+\lambda_2\lambda_3-\gamma(1+\lambda_2+\lambda_3-\gamma)\\ =&\gamma^2-\gamma(1+\lambda_2+\lambda_3)+(\lambda_2+\lambda_3+\lambda_2\lambda_3).\end{aligned}$$

View this as a quadratic expression in γ. As this must be negative, the discriminant must be positive. Hence,

$$(1+\lambda_2+\lambda_3)^2-4(\lambda_2+\lambda_3+\lambda_2\lambda_3)>0. \tag{1}$$

If A has three real eigenvalues, we number them so that $-1<\lambda_3\leq\lambda_2<1=\lambda_1$. From the fact that $0<\operatorname{trace}(A)=\gamma+(1-\delta-\phi)<2$, we have that $-1<\lambda_2+\lambda_3<1$. The condition (1), after working out the square and grouping terms becomes

$$1-2\lambda_2-2\lambda_3-2\lambda_2\lambda_3+\lambda_2^2+\lambda_3^2>0.$$

The points where $1-2\lambda_2-2\lambda_3-2\lambda_2\lambda_3+\lambda_2^2+\lambda_3^2=0$ lie on a parabola in the plane with central axis $\lambda_2=\lambda_3$, vertex at $(\frac{1}{4},\frac{1}{4})$ and going through the points $(1,0)$ and $(0,1)$. This proves part (ii).

If A has two non-real eigenvalues $\lambda_{2,3}=a\pm ib$, then $\lambda_2+\lambda_3=2a$ and $\lambda_2\lambda_3=a^2+b^2$. So (1) becomes

$$(1+2a)^2-4(2a+a^2+b^2)>0,$$

which, after working out the square, becomes $1-4a-4b^2>0$. This proves part (i). □

The next lemma describes the situation where there are two zeros on the diagonal.

Lemma 5. *Let* $A=\begin{bmatrix}0 & \alpha & 1-\alpha\\ \beta & 0 & 1-\beta\\ \gamma & \delta & 1-\gamma-\delta\end{bmatrix}$ *be a nonnegative primitive matrix with* $1-\delta-\gamma>0$. *Let the spectrum of* A *be* $\{1,\lambda_2,\lambda_3\}$. *Then the following hold*

i. *If* $\lambda_{2,3}=a\pm ib$ *with* $b\neq 0$, *then* $(a+1)^2+b^2<1$.
ii. *If* $\lambda_3\leq\lambda_2$ *are real, then* $-1<\lambda_2+\lambda_3<0$.

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Proof. Part (ii) is fairly immediate: the trace of A is less than 1, so $0 < 1+\lambda_2+\lambda_3 < 1$, which means $-1 < \lambda_2+\lambda_3 < 0$.

Part (i) again follows by considering the sum of the two-by-two principal subdeterminants, which in this case is equal to

$$\begin{vmatrix} 0 & \alpha \\ \beta & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1-\alpha \\ \gamma & 1-\delta-\gamma \end{vmatrix} + \begin{vmatrix} 0 & 1-\beta \\ \delta & 1-\delta-\gamma \end{vmatrix}$$
$$= -\alpha\beta - \gamma(1-\alpha) - \delta(1-\beta) = \lambda_2+\lambda_3+\lambda_2\lambda_3.$$

It follows that $\lambda_2+\lambda_3+\lambda_2\lambda_3 < 0$. Since $\lambda_{2,3} = a \pm ib$ in this case, we have $2a+a^2+b^2 < 0$, in other words, $(a+1)^2+b^2 < 1$. □

Note that in this case when the eigenvalues are real, the condition $\lambda_2+\lambda_3+\lambda_2\lambda_3 < 0$ holds automatically for all pairs (λ_2, λ_3) in the real plane which satisfy $-1 < \lambda_2+\lambda_3 < 0$ as the hyperbola $\lambda_2+\lambda_3+\lambda_2\lambda_3 = 0$ lies outside that region.

Figure 3 gives the position of possible complex eigenvalues when there are two zeros on the diagonal as dictated by part (i) in the above lemma.

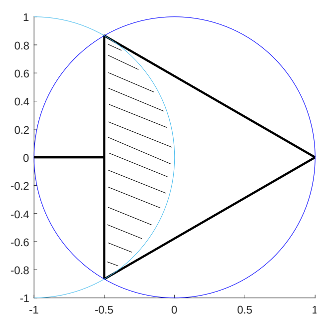

FIGURE 3. *Restrictions imposed on the location of complex eigenvalues when there are two zeros on the diagonal.*

Next, we discuss the trace zero case.

PROPOSITION 6. *Let A be a 3×3 row-stochastic matrix with zeros only on the diagonal. Then the possible spectra are of the form $\sigma(A) = \{1, -\frac{1}{2} \pm a\}$ with $0 \le a < \frac{1}{2}$ or of the form $\sigma(A) = \{1, -\frac{1}{2} \pm ib\}$ with $0 < b < \frac{1}{2}\sqrt{3}$, and all such sets can be achieved by a row-stochastic matrix with zeros on the diagonal (and only there).*

Proof. Necessity. Suppose the trace of the matrix is zero. Note that A^2 is a positive matrix, by the assumption that A has zeros only on the diagonal. Hence, A is primitive. Obviously, if $\Lambda = \{1, \lambda_2, \lambda_3\}$ is the spectrum of a nonnegative row-stochastic matrix with trace zero, then $\lambda_2+\lambda_3 = -1$. If the eigenvalues λ_2 and λ_3 are non-real, they must therefore be of the form $\lambda_{2,3} = -\frac{1}{2} + bi$ with $0 < b < \frac{1}{2}\sqrt{3}$ (because 1 is the spectral radius and A is primitive). If λ_2 and λ_3 are real and $\lambda_2 \ge \lambda_3$, then $-1 < \lambda_3 \le -\frac{1}{2}$ and $-\frac{1}{2} \le \lambda_2 < 0$ (again, because 1 is the spectral radius and A is primitive).

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Sufficiency. Conversely, consider two cases separately: the case where $\lambda_{2,3} = -\frac{1}{2} \pm bi$ with $0 < b < \frac{1}{2}\sqrt{3}$ and the case where λ_2 and λ_3 are real. In the first case, consider

$$A = \begin{bmatrix} 0 & \varepsilon & 1-\varepsilon \\ c & 0 & 1-c \\ \varepsilon & 1-\varepsilon & 0 \end{bmatrix}.$$

One checks that the eigenvalues of A are 1 and the two roots of

$$0 = \lambda^2 + \lambda + (c(1-2\varepsilon) + \varepsilon^2) = (\lambda + \frac{1}{2})^2 + (c(1-2\varepsilon) + \varepsilon^2 - \frac{1}{4}).$$

So the eigenvalues are 1 and

$$\lambda_{2,3} = -\frac{1}{2} \pm i\sqrt{-\frac{1}{4} + c(1-2\varepsilon) + \varepsilon^2},$$

when $\frac{1}{4} - c(1-2\varepsilon) - \varepsilon^2 < 0$. Take $\frac{1}{2} < \varepsilon < 1$ so that $\frac{1}{4} + b^2 < \varepsilon^2 - 2\varepsilon + 1 = (1-\varepsilon)^2$, which is possible as $0 < b < \frac{1}{2}\sqrt{3}$, and take

$$c = \frac{\frac{1}{4} + b^2 - \varepsilon^2}{1 - 2\varepsilon}.$$

Then $0 < c < 1$ and $\lambda_{2,3} = -\frac{1}{2} \pm bi$ as desired.

In case $\lambda_{2,3}$ are real, they are of the form:

$$\lambda_2 = -\frac{1}{2} + a, \quad \lambda_3 = -\frac{1}{2} - a, \quad 0 < a < \frac{1}{2}.$$

In this case, take $\varepsilon > \frac{1}{2}$ so that $\frac{1}{4} - a^2 > (1-\varepsilon)^2$ and take

$$c = \frac{\varepsilon^2 + a^2 - \frac{1}{4}}{2\varepsilon - 1}.$$

Then $0 < c < 1$ and $\lambda_{2,3} = -\frac{1}{2} \pm a$ as desired.

Finally, the remaining case is where $\lambda_{2,3} = -\frac{1}{2}$. Then take

$$A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix},$$

which has eigenvalues 1 and $\frac{1}{2}$, the latter with multiplicity 2. □

Note that this also means that if a 3×3 row-stochastic matrix has eigenvalues 1 and two eigenvalues on the line segment between $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$, then (since the trace is zero) it must have zeroes on the diagonal.

In general, a lot is known about matrices realizing eigenvalues on the boundary of the Karpelevič region (see e.g., [4]). For our purpose, we cite Theorem 4.1 in [5] which shows the following when applied to the three-dimensional case. We provide a proof for completeness.

Proposition 7. *Suppose A is a row-stochastic matrix with eigenvalues 1 and $\lambda_{1,2} = a \pm ib$ with $b = \frac{\sqrt{3}}{3}(1-a)$ (so on the line segments connecting 1 and $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$). Then there is a permutation matrix P such that*

$$A = P \begin{bmatrix} t & 0 & 1-t \\ 1-t & t & 0 \\ 0 & 1-t & t \end{bmatrix} P^{-1}.$$

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Proof. Consider $\Lambda = \{1, a \pm bi\}$ where $b = \frac{\sqrt{3}}{3}(1-a)$. From Proposition 2 , and the discussion following it for the particular case $n = 3$, we see that Λ can only be the spectrum of a 3×3 nonnegative matrix A when $(\text{trace}\,(A))^2 = 3\text{trace}\,(A^2)$. Writing the traces in terms of the entries of A, it follows that this means that if Λ is the spectrum of a nonnegative A, then

$$(a_{11} + a_{22} + a_{33})^2 = 3(a_{11}^2 + a_{22}^2 + a_{33}^2) + 6(a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32}).$$

Working out the square and collecting terms together, it may be seen that this is equivalent to

$$(a_{11} - a_{22})^2 + (a_{11} - a_{33})^2 + (a_{22} - a_{33})^2 + 6(a_{12}a_{21} + a_{13}a_{31} + a_{23}a_{32}) = 0.$$

In turn, that is equivalent to $a_{11} = a_{22} = a_{33}$ and $a_{12}a_{21} = 0$, $a_{13}a_{31} = 0$, $a_{23}a_{32} = 0$. If A has Λ as its spectrum, then A cannot be reducible (a reducible 3×3 nonnegative matrix has real eigenvalues). One then checks that up to permutation or transpose, if Λ is equal to the spectrum of a nonnegative row-stochastic matrix A, then

$$A = \begin{bmatrix} t & 1-t & 0 \\ 0 & t & 1-t \\ 1-t & 0 & t \end{bmatrix}.$$

So A must then have three zeros in a circulant pattern. □

Together the two propositions shows that eigenvalue locations can force interesting zero patterns on the matrix. Our goal in this paper is to consider the converse: what do zero patterns tell us about eigenvalue locations.

3. Zero patterns with one zero.

In case A has one zero entry, there are two subcases: the zero is on the diagonal, or it is not. In the former case, Lemma 4 gives a necessary condition that should be satisfied for the spectrum. We shall prove that the necessary condition is also sufficient, that is, for any list of three points satisfying the conditions of Lemma 4, there is a nonnegative matrix with precisely one zero, which is on the diagonal, and with spectrum equal to the given list of points (repetitions included).

In the case there is one zero, but it is not on the diagonal, we shall show that the spectrum can be any list $\{1, \lambda_2, \lambda_3\}$ that is feasible subject to the conditions that the matrix is nonnegative, as long as λ_2 and λ_3 are not on the boundary of the triangle, and in case they are real, the smallest one is not equal to minus one, the largest one is not equal to one, and their sum is not equal to minus one.

For the case where there is one zero on the diagonal, we show that the conditions given in 4 is also sufficient.

Proposition 8. *Let $\Lambda = \{1, \lambda_2, \lambda_3\}$ be a list of complex numbers for which the conditions* i *and* ii *of Lemma 4 are satisfied. Then there is a nonnegative matrix A of the form given in Lemma 4 for which $\sigma(A) = \Lambda$.*

Proof. Let $\Lambda = \{1, \lambda_2, \lambda_3\}$ be a list (possibly with repetitions) where λ_2 and λ_3 satisfy the conditions of Lemma 4. We have to show that there is a matrix A as in Lemma 4, depending on $\alpha, \beta, \gamma, \delta$ and ϕ, such that Λ is the spectrum of A. The strategy for the proof is as follows: $\Lambda = \sigma(A)$ if and only if the characteristic polynomial of A equals $(\lambda - 1)(\lambda^2 - \lambda(\lambda_2 + \lambda_3) + \lambda_2\lambda_3)$, or, equivalently, if and only if $\text{trace}\,(A) = 1 + \lambda_2 + \lambda_3$ and $\det(A) = \lambda_2\lambda_3$. These equations we shall view as two equations in the variables $\alpha, \beta, \gamma, \delta$, and ϕ. In fact, if we introduce $\Phi(\alpha, \beta, \gamma, \delta, \phi) = (\text{trace}\,(A), \det(A))$, then Φ, viewed as a map from $\mathbb{R}^5 \to \mathbb{R}^2$ is a polynomial

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expression in five variables, so certainly a C^1 map. For given λ_2 and λ_3, we consider the solutions of the set of two equations $\Phi(\alpha,\beta,\gamma,\delta,\phi) = (\text{trace}\,(A), \det(A))$. If we can find a nonnegative solution, that is, one where we allow one or more of the variables to be zero, then we can apply the implicit function theorem to show that there is a positive solution in a neighborhood.

Indeed, since

$$\det(A) = -\alpha\beta + \alpha\delta + \beta\phi - \delta\gamma, \tag{2}$$

$$\text{trace}\,(A) = 1 + \gamma - \delta - \phi \tag{3}$$

we have that

$$\Phi'(\alpha,\beta,\gamma,\delta,\phi) = \begin{bmatrix} 0 & 0 & 1 & -1 & -1 \\ \delta-\beta & \phi-\alpha & -\delta & \alpha-\gamma & \beta \end{bmatrix}.$$

It is easy to see that this has rank 2 unless $\beta = \delta$, $\alpha = \phi$, and $\gamma = \phi - \beta$. Hence in a neighborhood of any point where this is not the case, we may apply the implicit function theorem.

Case 1: $\det(A) < 0$. First, we consider the case where $\det(A) < 0$, then $0 < \lambda_2 < 1$ and $-1 < \lambda_3 < 0$. We make a specific choice: take $\alpha = 0, \beta = 0, \phi = 0$, and take $\gamma = \lambda_2, \delta = -\lambda_3$. So, we consider

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \lambda_2 & 1-\lambda_2 \\ -\lambda_3 & 0 & 1+\lambda_3 \end{bmatrix}.$$

One sees that A has Λ as its spectrum. We now show that γ and δ can be solved from the equations (2) and (3) as C^1 functions of α, β, and ϕ in a neighborhood of $(\alpha,\beta,\phi) = (0,1,0)$. Indeed, that is a consequence of the implicit function theorem if we show that the matrix formed by the third and fourth columns of Φ' (the columns corresponding to γ and δ) is an invertible matrix for the particular choice we made. This submatrix of Φ' is $\begin{bmatrix} 1 & -1 \\ -\delta & \alpha-\gamma \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ \lambda_3 & -\lambda_2 \end{bmatrix}$, which has determinant $-\lambda_2 + \lambda_3$. If this would be zero, then $\lambda_2 = \lambda_3$, which contradicts the assumption that $\det(A) < 0$. Applying the implicit function theorem gives us a positive solution for some point with $0 < \alpha < 1$, $0 < \beta < 1 - \gamma$, $0 < \phi < 1 - \delta$ and $\delta > 0$, $\gamma > 0$. So then we have a matrix A which has positive entries except for the $(1,1)$ entry with the desired eigenvalues $\{1, \lambda_2, \lambda_3\}$.

Case 2: $\det(A) > 0$. In this case, we shall construct in Subsection 4.2 a nonnegative matrix of the form $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \gamma & 1-\gamma \\ \delta & \phi & 1-\delta-\phi \end{bmatrix}$ for which $\sigma(A) = \Lambda$, both in case the two eigenvalues are real and in case the two eigenvalues are non-real. We can then apply a similar reasoning as above, using the implicit function theorem to show that there is also a nonnegative matrix of the form $A = \begin{bmatrix} 0 & \alpha & 1-\alpha \\ \beta & \gamma & 1-\beta-\gamma \\ \delta & \phi & 1-\delta-\phi \end{bmatrix}$ with only a zero in the $(1,1)$-entry that has spectrum equal to Λ. $\square$

Next, we consider the case where A has one zero, which is not on the diagonal. By using a permutation matrix as similarity if necessary, and taking into consideration the fact that the spectra of A and A^T coincide, we may assume that A has a zero in the $(2,1)$ entry.

PROPOSITION 9. *Let $\Lambda = \{1, \lambda_2, \lambda_3\}$ be a list which satisfies the conditions of Proposition 3. Then Λ is*

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the spectrum of a nonnegative row-stochastic matrix of the form $A = \begin{bmatrix} \alpha & \beta & 1-\alpha-\beta \\ 0 & \gamma & 1-\gamma \\ \delta & \phi & 1-\delta-\phi \end{bmatrix}$ *with* $\alpha, \beta, \gamma, \delta, \phi,$ $\alpha+\beta,\ \delta+\phi \in (0,1)$ *if and only if the following conditions are satisfied.*

i. *If* $\lambda_{2,3} = a \pm bi$ *are non-real, then* $-\frac{1}{2} < a < 1$ *and* $|b| < \frac{\sqrt{3}}{3}(1-a)$.
ii. *If* $\lambda_3 \leq \lambda_2$ *are real, then* $-1 < \lambda_3,\ \lambda_2 < 1$ *and* $\lambda_2 + \lambda_3 > -1$.

Proof. Part i: non-real eigenvalues. Consider $B = \begin{bmatrix} 0 & \alpha & 1-\alpha \\ 0 & 0 & 1 \\ \delta & 1-\delta & 0 \end{bmatrix}$. Note that B has trace zero and positive determinant equal to $\alpha\delta$, and it follows that the eigenvalues of B are 1, and $\lambda_{2,3} = -\frac{1}{2} \pm bi$ with $\alpha\delta = b^2 + \frac{1}{4}$. For given b with $0 < b < \frac{1}{2}\sqrt{3}$, we can find α and δ such that $\lambda_{2,3} = -\frac{1}{2} \pm bi$.

Let $0 < t < 1$ and take $A(t) = tI + (1-t)B$. If $Bx = \lambda x$, then $A(t)x = (t + (1-t)\lambda)x$. So the eigenvalues of $A(t)$, if we let t vary, trace out the line segment between 1 and λ. Note that $A(t)$ has only one zero entry, namely in the $(2,1)$ position.

It follows that for any list $\Lambda = \{1, \lambda_2, \lambda_3\}$ with $\lambda_{2,3} = a \pm bi$, where $-\frac{1}{2} < a < 1$ and $0 < b < \frac{\sqrt{3}}{3}(1-a)$, there is a nonnegative matrix A with one zero entry in the $(2,1)$ position such that $\sigma(A) = \Lambda$.

From results in Propositions 6 and 7, we have that the eigenvalues cannot be on the boundary of the Karpelevič region.

Part ii: real eigenvalues. Let A have one zero only, not on the diagonal, and assume that $\Lambda = \{1, \lambda_2, \lambda_3\}$ is a list of three real numbers with $-1 \leq \lambda_3 \leq \lambda_2 \leq 1$. We investigate what the conditions on Λ will be in order to have $\Lambda = \sigma(A)$. First note that $\operatorname{trace}(A) > 0$, so $\lambda_2 + \lambda_3 > -1$. Also note that A is a primitive matrix (A^2 is obviously positive). Hence, $\lambda_3 > -1$ and $\lambda_2 < 1$.

Now for $A = \begin{bmatrix} \alpha & \beta & 1-\alpha-\beta \\ 0 & \gamma & 1-\gamma \\ \delta & \phi & 1-\delta-\phi \end{bmatrix}$ consider $\Psi(\alpha, \beta, \gamma, \delta, \phi) = (\operatorname{trace}(A), \det(A))$, so

$$\Psi(\alpha, \beta, \gamma, \delta, \phi) = (\alpha + \gamma + 1 - \delta - \phi, \alpha\gamma - \alpha\phi + \delta\beta - \delta\gamma).$$

Recalling that $\operatorname{trace}(A) = 1 + \lambda_2 + \lambda_3$ and $\det(A) = \lambda_2\lambda_3$, we see that given the list Λ we are looking for a solution of the set of equations:

$$\Psi(\alpha, \beta, \gamma, \delta, \phi) = (1 + \lambda_2 + \lambda_3, \lambda_2\lambda_3),$$

with positive $\alpha, \beta, \gamma, \delta$, and ϕ. Now, if we can find a solution with nonnegative $\alpha, \beta, \gamma, \delta$, and ϕ, we may apply the implicit function theorem to show that there is also a solution with positive $\alpha, \beta, \gamma, \delta$, and ϕ.

Consider first the case where λ_2 and λ_3 are both nonnegative, with $\lambda_2 \neq \lambda_3$, then we can take $A = \begin{bmatrix} \lambda_2 & 0 & 1-\lambda_2 \\ 0 & \lambda_3 & 1-\lambda_3 \\ 0 & 0 & 1 \end{bmatrix}$, which has the desired spectrum. By the argument of the previous paragraph, there is then also a nonnegative row-stochastic matrix with only one zero in the (2,1) entry which has these eigenvalues. Here, we have to consider α and γ as functions of (β, δ, ϕ) in a neighborhood of $(0,0,0)$. If $\lambda_2 \neq \lambda_3$, then the implicit function theorem is applicable to this situation, and so for (β, δ, ϕ) in a neighborhood of $(0,0,0)$

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there is a solution depending continuously differentiable on these parameters. Hence, we can choose positive values for β, δ, and ϕ such that the corresponding α and γ are positive as well and such that also $1-\alpha-\beta$ and $1-\gamma$ are positive.

Now consider the case where $\lambda_2 \geq 0$ and $\lambda_3 \leq 0$. Take $\alpha=\beta=0$, $\gamma=\lambda_2$, $\delta=-\lambda_3$, and $\phi=0$, so $A=\begin{bmatrix} 0 & 0 & 1 \\ 0 & \lambda_2 & 1-\lambda_2 \\ -\lambda_3 & 0 & 1+\lambda_3 \end{bmatrix}$. Then A has the desired spectrum, however, again with four zero entries. Considering now γ and δ as functions of (α,β,ϕ) in a neighborhood of the origin, we apply again the implicit function theorem. Once again, this can be done provided $\lambda_2 \neq \lambda_3$.

Next, consider the case where $\lambda_2 \leq 0$ and $\lambda_3 < 0$ with $\lambda_2 \neq \lambda_3$. Take $\alpha=0$, $\beta=-\lambda_3$, $\gamma=0$, $\delta=-\lambda_2$, and $\phi=-\lambda_3$. So $A=\begin{bmatrix} 0 & -\lambda_3 & 1+\lambda_3 \\ 0 & 0 & 1 \\ -\lambda_2 & 0 & 1+\lambda_2+\lambda_3 \end{bmatrix}$. Now we consider β and δ as functions of (α,γ,ϕ) in a neighborhood of $(0,0,-\lambda_3)$. The implicit function theorem can be applied provided $\lambda_2 \neq 0$. That leaves the case $\lambda_2=0$, in which case, we can take $\gamma=\beta=\phi$ and α and δ such that $\lambda_3=\alpha-\delta$. Then, $A=\begin{bmatrix} \alpha & \beta & 1-\alpha-\beta \\ 0 & \beta & 1-\beta \\ \delta & \beta & 1-\delta-\beta \end{bmatrix}$ will have eigenvalues $\{1,0,\lambda_3\}$.

Finally, we have to consider the cases where $\lambda_2=\lambda_3$. Notice that in that case $-\frac{1}{2}<\lambda_2<1$ since A is primitive and has nonzero trace. Consider the matrix $B=\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$. This matrix has $-\frac{1}{2}$ as a double eigenvalue and 1 as the third eigenvalue. Then $A(t)=tI+(1-t)B$ has eigenvalues $t-\frac{1}{2}(1-t)$ as a double eigenvalue and 1 as the third eigenvalue. $\square$

4. Zero patterns with three zeros. We now turn our attention to zero patterns with three zeros in the 3×3 matrix case. Observe that in principle, there are $\binom{9}{3}=84$ different zero patterns with three zeros in a 3×3 matrix. However, similarity with a permutation matrix keeps the number of zeros equal to three and does not change the eigenvalues. In addition, also taking the transpose keeps the number of zeros equal to 3 and does not change the eigenvalues. These observations reduce the possible number of cases considerably. This leads to a reduction to 12 essentially different cases, listed in the table below

Cases 1 to 6	$\begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$	$\begin{bmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{bmatrix}$	$\begin{bmatrix} * & 0 & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$	$\begin{bmatrix} * & 0 & * \\ * & * & 0 \\ 0 & * & * \end{bmatrix}$	$\begin{bmatrix} 0 & * & 0 \\ 0 & * & * \\ * & * & * \end{bmatrix}$
Cases 7 to 12	$\begin{bmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} 0 & * & * \\ * & 0 & * \\ 0 & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & 0 \\ * & 0 & * \\ 0 & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ 0 & 0 & * \\ 0 & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & 0 \\ 0 & 0 & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} 0 & * & * \\ 0 & 0 & * \\ * & * & * \end{bmatrix}$

Observe that case 3 has already been dealt with: it is the trace zero case. Notice also that there are four reducible cases: the cases 1, 2, 4, and 10. Apart from these four reducible cases, the matrices of the form in the table above are all primitive (either their square or their third power is apositive matrix). Hence, by

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the Perron–Frobenius theorem, for all but the reducible cases, the eigenvalue 1 is the only eigenvalue on the unit circle. We will not discuss the reducible cases, as these are trivial.

4.1. The circulant pattern. In case 5, the pattern of zeros is that of a circulant matrix. Consider the row-stochastic matrix

$$A = \begin{bmatrix} \alpha & 0 & 1-\alpha \\ 1-\beta & \beta & 0 \\ 0 & 1-\gamma & \gamma \end{bmatrix}, \tag{4}$$

with α, β, and γ in $(0,1)$. Observe that the determinant $\det(A) = \alpha\beta\gamma + (1-\alpha)(1-\beta)(1-\gamma)$ is positive, so zero cannot be an eigenvalue.

PROPOSITION 10. *If $\sigma(A) = \{1, \lambda_2, \lambda_3\}$ for a matrix of the type (4), then*

i. *if $\lambda_2 \geq \lambda_3$ are real, then λ_3 is positive,*
ii. *if $\lambda_{2,3} = a \pm bi$ are non-real, then we have $(a+1)^2 + b^2 > 1$ and $|b| \leq \frac{\sqrt{3}}{3}(1-a)$ (see Figure 4).*

Conversely, given a pair of real numbers $0 < \lambda_3 \leq \lambda_2 < 1$, or a pair of non-real numbers $\lambda_{2,3} = a \pm bi$ such that $(a+1)^2 + b^2 > 1$ and $|b| \leq \frac{\sqrt{3}}{3}(1-a)$, there is a matrix of the type (4) such that $\sigma(A) = \{1, \lambda_2, \lambda_3\}$.

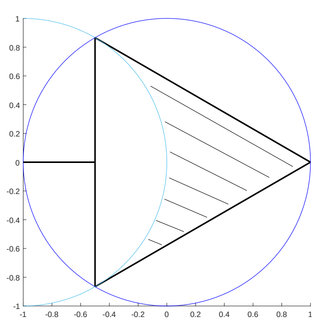

FIGURE 4. *The possible eigenvalue locations for the case of circulant zero pattern.*

Proof. Part 1. Necessity. That the conditions (i) and (ii) of the proposition are necessary can be shown easily. If $\{1, \lambda_2, \lambda_3\}$ is the list of eigenvalues of A, then $\lambda_2 + \lambda_3 + \lambda_2\lambda_3 = \alpha\beta + \alpha\gamma + \beta\gamma > 0$. Moreover, $\det(A) > 0$. Hence, if λ_2 and λ_3 are real, then they are either both positive or both negative, and the latter situation is excluded by the fact that $\lambda_2 + \lambda_3 + \lambda_2\lambda_3 > 0$. Hence, if λ_2 and λ_3 are real, then they are both positive. If λ_2 and λ_3 are non-real eigenvalues, say $\lambda_{2,3} = a \pm bi$, then $\lambda_2 + \lambda_3 + \lambda_2\lambda_3 = 2a + a^2 + b^2 > 0$, that is $(a+1)^2 + b^2 > 1$. The condition that $|b| \leq \frac{\sqrt{3}}{3}(1-a)$ holds for any nonnegative three-by-three matrix.

Part 2. Sufficiency. For the converse, we claim that for suitable choices of α, β, and γ in $(0,1)$, the non-real eigenvalues of matrices of the type (4) cover the set inside the unit circle bounded by the lines $|y| \leq \frac{\sqrt{3}}{3}(1-x)$ and outside the circle $(x+1)^2 + y^2 = 1$, and that any pair of positive numbers $0 < \lambda_3 \leq \lambda_2 < 1$ occurs as eigenvalues of a matrix of the type (4).

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Real eigenvalues. First consider the case of real eigenvalues. Let $0 < \lambda_3 \leq \lambda_2 < 1$ be a pair of real numbers. Consider the row-stochastic matrix

$$A = \begin{bmatrix} \alpha & 0 & 1-\alpha \\ 1-\beta & \beta & 0 \\ 0 & 1-\gamma & \gamma \end{bmatrix}.$$

Observe that the determinant $\det(A) = \alpha\beta\gamma + (1-\alpha)(1-\beta)(1-\gamma)$ is positive, so zero cannot be an eigenvalue. Then necessary and sufficient conditions for $\sigma(A) = \{1, \lambda_2, \lambda_3\}$ are

$$1 + \lambda_2 + \lambda_3 = \alpha + \beta + \gamma, \qquad \lambda_2 + \lambda_3 + \lambda_2\lambda_3 = \alpha\beta + \alpha\gamma + \beta\gamma.$$

One can check that the condition $\det(A) = \lambda_2\lambda_3 = \alpha\beta\gamma + (1-\alpha)(1-\beta)(1-\gamma)$ follows from the two conditions above.

Now we are viewing the two conditions as equations to be solved for (α, β, γ) with all three in the interval $(0,1)$. Note that there is an obvious solution: taking $\alpha = 1, \beta = \lambda_2$, and $\gamma = \lambda_3$ does solve the system of equations, but unfortunately, this solution violates the zero pattern, as $1 - \alpha = 0$. However, using the implicit function theorem, it is easy to see that there must be a solution such that $\alpha < 1$, at least as long as $0 < \lambda_3 < \lambda_2$. Indeed, let $\Phi : \mathbb{R}^3 \to \mathbb{R}^2$ be given by:

$$\Phi(\alpha, \beta, \gamma) = \begin{pmatrix} \alpha + \beta + \gamma - (1 + \lambda_2 + \lambda_3) \\ \alpha\beta + \alpha\gamma + \beta\gamma - (\lambda_2 + \lambda_3 + \lambda_2\lambda_3) \end{pmatrix}.$$

Then $\Phi(1, \lambda_2, \lambda_3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and

$$\Phi'(1, \lambda_2, \lambda_3) = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_2 + \lambda_3 & 1 + \lambda_3 & 1 + \lambda_2 \end{pmatrix}.$$

The last two columns of Φ' form an invertible matrix when $\lambda_2 \neq \lambda_3$, and hence we can apply the implicit function theorem in that case to see that there is an open interval U containing 1 and a C^1-function $g : U \to \mathbb{R}^2$ such that $\Phi(\alpha, g(\alpha)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all $\alpha \in U$ and $g(1) = (\lambda_2, \lambda_3)$. So in particular, taking $0 < \alpha < 1$ in U and $(\beta, \gamma) = g(\alpha)$, we obtain a matrix A such that $\sigma(A) = \{1, \lambda_2, \lambda_3\}$. Note that by the continuity of the map g, it is possible to get (α, β, γ) so that they are all bounded away from zero, for instance, we may assume that all of them are bigger than $\frac{1}{2}\lambda_3$.

It remains to consider the case $\lambda_2 = \lambda_3$. Take $0 < \varepsilon < \frac{1}{2}\lambda_2$ and let $\lambda_3 = \lambda_2 - \varepsilon$. Then find $\alpha(\varepsilon), \beta(\varepsilon)$, and $\gamma(\varepsilon)$ in $(0,1)$ such that with these values the spectrum of the matrix $A(\varepsilon)$ is $\{1, \lambda_2, \lambda_2 - \varepsilon\}$. Now let $\varepsilon \to 0$. As the eigenvalues are continuous functions of ε, we have that the matrix $A = \lim_{\varepsilon \to 0} A(\varepsilon)$ has eigenvalues 1 and λ_2, the latter with multiplicity 2. The only thing that remains to check is that A has the desired zero pattern and no extra zeroes. This can be seen by the fact that we can bound $\alpha(\varepsilon)$, $\beta(\varepsilon)$, and $\gamma(\varepsilon)$ from below, for instance by $\frac{1}{2}\lambda_2$, as observed in the previous paragraph.

Non-real eigenvalues. We will make use of Proposition 1. First, let $-\frac{1}{2} \leq a \leq 0$ be given, and let b be such that $a^2 + b^2 = -2a$. Consider the matrix

$$A(a) = \begin{bmatrix} 1+2a & 0 & -2a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

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Since $0 \leq 1+2a \leq 1$ and $-2a \geq 0$, $A(a)$ is a nonnegative row-stochastic matrix. It can easily be checked that the eigenvalues of $A(a)$ are $\lambda_{2,3} = a \pm ib$. Note that for $0 < t < 1$, the matrix $A(t,a) = (A(a) + tI)/(1+t)$ is nonnegative, row-stochastic and has the circulant zero pattern. By Proposition 1 for any pair of complex numbers $\lambda_{2,3} = a \pm bi$ satisfying $(a+1)^2 + b^2 > 1$ and $|b| \leq \frac{\sqrt{3}}{3}(1-a)$, there is a matrix of the form $A(t,a)$ with spectrum $\{1, \lambda_2, \lambda_3\}$. □

4.2. Case 6. In this case, we consider matrices of the form:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \alpha & 1-\alpha \\ \beta & \gamma & 1-\beta-\gamma \end{bmatrix}, \tag{5}$$

with $0 < \alpha < 1$ and $\beta > 0$, $\gamma > 0$ such that $\beta + \gamma < 1$. Note that $\det(A) = \beta(1-\alpha) > 0$. Further, the sum of the principal two times two submatrices is $\alpha(1-\beta-\gamma) - \gamma(1-\alpha) = \alpha - \gamma - \alpha\beta$. Finally, the trace of A is $1 + \alpha - \beta - \gamma$. Hence, the eigenvalues λ_2 and λ_3 satisfy

$$\alpha - \gamma - \alpha\beta = \lambda_2 + \lambda_3 + \lambda_2\lambda_3, \qquad \alpha - \beta - \gamma = \lambda_2 + \lambda_3, \qquad \beta(1-\alpha) = \lambda_2\lambda_3 > 0.$$

Note that A is always invertible, so 0 cannot be an eigenvalue.

Proposition 11. *Let A be of the form (5), and let $\sigma(A) = \{1, \lambda_2, \lambda_3\}$. Then the following hold*

i. *if $-1 < \lambda_3 \leq \lambda_2 < 1$, then either $\lambda_2 < 0$ and $\lambda_2 + \lambda_3 > -1$, or $\lambda_3 > 0$ and $\lambda_3 < 1 + \lambda_2 - 2\sqrt{\lambda_2}$,*
ii. *if $\lambda_{2,3} = a \pm bi$, then $a > -\frac{1}{2}$ and $b^2 < \frac{1}{4} - a$.*

See Figure 5.

Conversely, if a pair (λ_2, λ_3) satisfies the conditions above, then there is a matrix of the form (5) with spectrum $\{1, \lambda_2, \lambda_3\}$.

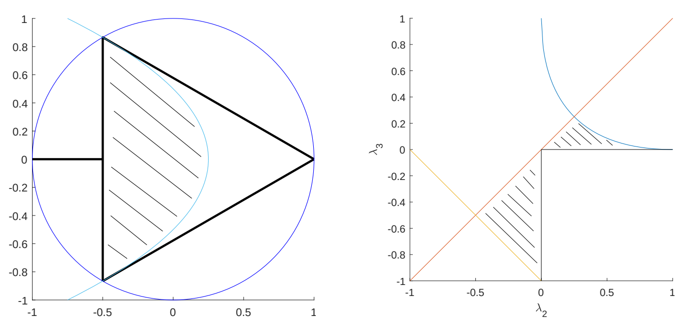

Figure 5. *The possible eigenvalue locations in case 6. Left side: The possibilities for a pair of complex conjugate eigenvalues. Right side: The possibilities for a pair of real eigenvalues with $\lambda_2 \geq \lambda_3$.*

Proof. Necessity. We recall an argument from the proof of Lemma 4. As a first step notice that $-\gamma(1-\alpha) < 0$. On the other hand, we can express this in terms of the sum of the two-by-two principal minors and the eigenvalues:

$$-\gamma(1-\alpha) = \lambda_2 + \lambda_3 + \lambda_2\lambda_3 - \alpha(1-\beta-\gamma).$$

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Using the expression of the trace, this can be expressed in terms of α and the eigenvalues as follows:

$$-\gamma(1-\alpha) = \lambda_2 + \lambda_3 + \lambda_2\lambda_3 - \alpha(1 + \lambda_2 + \lambda_3 - \alpha).$$

So the right-hand side must be negative for some α between 0 and 1. Rewriting this: there is a $0 < \alpha < 1$ such that

$$h(\alpha) := \alpha^2 - \alpha(1 + \lambda_2 + \lambda_3) + (\lambda_2 + \lambda_3 + \lambda_2\lambda_3) < 0.$$

Consider $h(\alpha)$ as a quadratic expression in α, and note that $h(1) = \lambda_2\lambda_3 = \det(A) > 0$. The fact that $h(\alpha) < 0$ for some value of α in $(0, 1)$, then implies that $h(\alpha)$ must have at least one real simple zero in $(0, 1)$. For that to happen, the discriminant must be positive:

$$(1 + \lambda_2 + \lambda_3)^2 - 4(\lambda_2 + \lambda_3 + \lambda_2\lambda_3) > 0. \tag{6}$$

Now we distinguish between the case of complex conjugate eigenvalues and the case of real distinct eigenvalues. If λ_2 and λ_3 are real, then because of the fact that $\det(A) > 0$ either both are positive or both are negative. Moreover, in case they are negative, then the trace of A is $1 + \lambda_2 + \lambda_3 > 0$, so the condition for negative eigenvalues is necessary. In case λ_2 and λ_3 are positive, then (6) can be rewritten as:

$$(\lambda_3 - (1 + \lambda_2))^2 - 4\lambda_2 > 0.$$

Since $\lambda_3 < 1 + \lambda_2$, this can be rewritten as $\lambda_3 < 1 + \lambda_2 - 2\sqrt{\lambda_2}$.

If $\lambda_{2,3} = a \pm bi$, then rewrting (6) in terms of a and b gives

$$0 < (1 + 2a)^2 - 4(2a + a^2 + b^2) = 1 - 4a - 4b^2,$$

which is equivalent to one of the conditions in the proposition. The condition that $a \geq -\frac{1}{2}$ holds for any nonnegative 3×3 matrix, as was already observed before. However, in this case the trace is strictly positive, and hence $1 + \lambda_2 + \lambda_3 = 1 + 2a > 0$, so that $a > -\frac{1}{2}$.

Sufficiency. First consider the case of complex eigenvalues $\lambda_{2,3} = a \pm bi$. In that case take

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} + a & \frac{1}{2} - a \\ \beta & \gamma & \frac{1}{2} + a \end{bmatrix},$$

with

$$\beta = \frac{a^2 + b^2}{\frac{1}{2} - a}, \quad \gamma = \frac{1}{2} - a - \frac{a^2 + b^2}{\frac{1}{2} - a} = \frac{\frac{1}{4} - a - b^2}{\frac{1}{2} - a}.$$

As $-\frac{1}{2} < a < \frac{1}{4}$ and $\frac{1}{4} - a - b^2 > 0$, the matrix A is row-stochastic and has the right zero pattern. Moreover, one easily checks that this matrix has the right characteristic polynomial, and hence the right eigenvalues.

Next, consider the case of real eigenvalues λ_2 and λ_3. First assume that λ_2 and λ_3 are both positive and that the conditions in the proposition are met. Then (6) holds, and therefore there is an $\alpha \in (0, 1)$ such that the quadratic expression $h(\alpha)$ is negative. In (5), take

$$\beta = \frac{\lambda_2\lambda_3}{1 - \alpha},$$
$$\gamma = \frac{-(1-\alpha)(\lambda_2 + \lambda_3) + \alpha - \alpha^2 - \lambda_2\lambda_3}{1 - \alpha} = \frac{-h(\alpha)}{1 - \alpha}.$$

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Then $\beta > 0$ and $\gamma > 0$. First, we check that $1-\beta-\gamma > 0$:

$$\begin{aligned} 1-\beta-\gamma &= 1 - \frac{\lambda_2\lambda_3}{1-\alpha} + \frac{h(\alpha)}{1-\alpha} \\ &= \frac{1-\alpha-\lambda_2\lambda_3+h(\alpha)}{1-\alpha}, \end{aligned}$$

which, after some computation, becomes $1-\beta-\gamma = 1-\alpha+\lambda_2+\lambda_3$, which is positive. Hence, A is row-stochastic with the desired zero pattern, and as the determinant is equal to $\beta(1-\alpha) = \lambda_2\lambda_3$ and the trace is $1+\alpha-\beta-\gamma = 1+\lambda_2+\lambda_3$, the spectrum of A is $\{1, \lambda_2, \lambda_3\}$ as desired.

Note that we can give an explicit choice for α such that $h(\alpha) < 0$, namely $\alpha = \frac{1}{2}(1+\lambda_2+\lambda_3)$, because the minimum value of $h(\alpha)$ is attained for this value of α.

Now consider the case where λ_2 and λ_3 are both negative and $\lambda_2+\lambda_3+1 > 0$. In (5), we take $\alpha = \frac{1}{2}(1+\lambda_2+\lambda_3)$ so that also $1-\beta-\gamma = \frac{1}{2}(1+\lambda_2+\lambda_3)$. Then $1-\alpha = \frac{1}{2}(1-\lambda_2-\lambda_3)$ which is also positive. Take $\beta = \frac{\lambda_2\lambda_3}{\frac{1}{2}(1-\lambda_2-\lambda_3)}$ and finally, $\gamma = \frac{1}{\frac{1}{2}(1-\lambda_2-\lambda_3)}\left(\frac{1}{4}((1+\lambda_2+\lambda_3)^2 - 4(\lambda_2+\lambda_3+\lambda_2\lambda_3)\right)$. Note that $\lambda_2+\lambda_3+\lambda_2\lambda_3$ is negative, so γ is positive. Then A is nonnegative, row-stochastic, has the desired zero pattern, and has spectrum $\{1, \lambda_2, \lambda_3\}$ since the trace, determinant, and the sum of the principal two-by-two minors are all as dictated by the spectrum. ☐

4.3. The lower anti-triangular pattern. In this section, we consider case 7, where A is lower anti-triangular:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \alpha & 1-\alpha \\ \beta & \gamma & 1-\beta-\gamma \end{bmatrix}, \tag{7}$$

with α, β, γ, and $\beta+\gamma$ in $(0,1)$.

Proposition 12. *Let A be a row-stochastic matrix of the form (7). Then the spectrum of A is of the form $\{1, \lambda_2, \lambda_3\}$ with $-1 < \lambda_3 < 0 < \lambda_2 < 1$. Conversely, for any such pair (λ_2, λ_3), there is a row-stochastic matrix of the form (7) with spectrum $\{1, \lambda_2, \lambda_3\}$.*

Proof. Necessity: Note that the determinant is always negative in this case: $\det(A) = -\beta\alpha$. Hence, A cannot have a pair of non-real eigenvalues, and if we order (λ_2, λ_3) as usual $\lambda_3 \leq \lambda_2$, then we must have $-1 < \lambda_3 < 0 < \lambda_2 < 1$.

Sufficiency: Now take a list $\Lambda = \{1, \lambda_2, \lambda_3\}$ such that $-1 < \lambda_3 < 0 < \lambda_2 < 1$. Then for $0 < \alpha < 1$, consider the matrix

$$A(\alpha) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \alpha & 1-\alpha \\ \frac{\lambda_2\lambda_3}{-\alpha} & \alpha + \frac{\lambda_2\lambda_3}{\alpha} - (\lambda_2+\lambda_3) & 1+\lambda_2+\lambda_3-\alpha \end{bmatrix}.$$

Note that 1 is an eigenvalue of $A(\alpha)$ as the row sums are all 1, that the trace of $A(\alpha)$ is equal to $1+\lambda_2+\lambda_3$ and that the determinant of $A(\alpha)$ is equal to $\lambda_2\lambda_3$. Hence, $A(\alpha)$ has spectrum Λ.

It remains to show that $A(\alpha)$ is nonnegative and has the right zero pattern for some choice of $\alpha \in (0,1)$. For this, we need to choose α so that the $(3,2)$ and $(3,3)$ entries are positive. One easily checks that the $(3,2)$ entry is positive for $\alpha > \lambda_2$. So taking $\lambda_2 < \alpha < \min(1, 1+\lambda_2+\lambda_3)$ will have the desired result. Note that this is possible as $1+\lambda_3 > 0$. ☐

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4.4. Case 8. Now we deal with case 8, where A has the following pattern:

$$A = \begin{bmatrix} 0 & \alpha & 1-\alpha \\ \beta & 0 & 1-\beta \\ 0 & \gamma & 1-\gamma \end{bmatrix}. \tag{8}$$

where $0 < \alpha, \beta, \gamma < 1$. Observe that $\operatorname{trace} A = 1-\gamma$, and that $\det A = -\beta(\alpha-\gamma)$. Moreover, the sum of the principal 2×2 submatrices is

$$\lambda_2 + \lambda_3 + \lambda_2\lambda_3 = -\alpha\beta - \gamma(1-\beta) < 0. \tag{9}$$

PROPOSITION 13. *Let A be a row-stochastic matrix of the form (8). Then the spectrum of A is of the form $\{1, \lambda_2, \lambda_3\}$ with*

i. *If $\lambda_{2,3} = a \pm bi$ with $b \neq 0$, then $a \in (-\frac{1}{2}, 0)$ and $(a+1)^2 + b^2 < 1$;*
ii. *If $\lambda_2 \geq \lambda_3$ are real, then $0 > \lambda_2 + \lambda_3 > -1$.*

Conversely, for a pair (λ_2, λ_3) satisfying the conditions above, there is a row-stochastic matrix of the form (8) with spectrum $\{1, \lambda_2, \lambda_3\}$.

Proof. Necessity. The conditions of the eigenvalues follow from the necessity part of Lemma 5.

Sufficiency. First, suppose $\lambda_{2,3} = a \pm bi$. Then we can construct the following matrix with the desired zero pattern that has 1 and $a \pm bi$ as its eigenvalues:

$$A_1 = \begin{bmatrix} 0 & \alpha & 1-\alpha \\ -\frac{a^2+b^2}{\alpha+2a} & 0 & 1+\frac{a^2+b^2}{\alpha+2a} \\ 0 & -2a & 1+2a \end{bmatrix},$$

where $\alpha \in (0, -(a^2+2a+b^2))$.

Notice that $a^2+b^2+2a = \lambda_2\lambda_3 + \lambda_2 + \lambda_3 < 0$ based on equation (9); hence, $-(a^2+2a+b^2) > 0$. It is also true that $-(a^2+2a+b^2) < 1$; otherwise, we would get $(a+1)^2 + b^2 < 0$ which is a contradiction. So, $0 < \alpha < 1$ is surely satisfied, and we can apply the characteristic polynomial to verify that A_1 has eigenvalues $\{1, \lambda_2, \lambda_3\}$. Since $-\frac{1}{2} < a < 0$, it is quite obvious that $\gamma = -2a \in (0, 1)$. In addition, since $\alpha + 2a < -a^2 - b^2$, we have that $\beta > 0$. To check whether β is smaller than 1 or not, assume that $\beta \geq 1$. We then obtain that $a^2 + b^2 \geq -(\alpha+2a)$, or equivalently, $\alpha \geq -(a^2+b^2+2a)$, which is impossible.

Next, suppose λ_2 and λ_3 are both real such that $\lambda_2 \geq \lambda_3$ and $\lambda_2 + \lambda_3 \in (-1, 0)$. In this case, a nonnegative row-stochastic matrix with eigenvalues $\{1, \lambda_2, \lambda_3\}$ can be constructed as follows:

$$A_2 = \begin{bmatrix} 0 & -\lambda_3 & 1+\lambda_3 \\ -\lambda_3 & 0 & 1+\lambda_3 \\ 0 & -(\lambda_2+\lambda_3) & 1+\lambda_2+\lambda_3 \end{bmatrix}.$$

Here, we must have $\lambda_3 < 0$, since $\lambda_3 \leq \lambda_2$ and $\lambda_2 + \lambda_3 < 0$. Further, A_2 is clearly a nonnegative row-stochastic matrix. Also, one can verify that A_2 has $\{1, \lambda_2, \lambda_3\}$ as its spectrum using the characteristic equation. $\square$

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4.5. The anti-diagonal pattern. In this section, we deal with nonnegative matrices with the anti-diagonal zero pattern, which is case 9. So, let

$$A = \begin{bmatrix} \alpha & 1-\alpha & 0 \\ \beta & 0 & 1-\beta \\ 0 & \gamma & 1-\gamma \end{bmatrix}, \tag{10}$$

with α, β, and γ in $(0,1)$.

PROPOSITION 14. *Let A be a row-stochastic matrix of the form (10). Then the spectrum of A is of the form $\{1, \lambda_2, \lambda_3\}$ with $-1 < \lambda_3 < 0 < \lambda_2 < 1$. On the other hand, for a pair (λ_2, λ_3) such that $1 > \lambda_2 > 0 > \lambda_3 > -1$, there is a row-stochastic matrix of the form (10) with spectrum $\{1, \lambda_2, \lambda_3\}$.*

Proof. Necessity: Observe that the determinant here equals $\det(A) = \lambda_2\lambda_3 = -\alpha\gamma(1-\beta)-\beta(1-\alpha)(1-\gamma)$. Since the entries satisfy that $0 < \alpha, \beta, \gamma < 1$, it is always true that $\det(A) < 0$. Thus, A can only have real eigenvalues. Additionally, if we set as usual $\lambda_3 \le \lambda_2$, then we obtain $-1 < \lambda_3 < 0 < \lambda_2 < 1$.

Sufficiency: Suppose that there is a list $\Lambda = \{1, \lambda_2, \lambda_3\}$ such that $-1 < \lambda_3 < 0 < \lambda_2 < 1$. Then for $\alpha \in (\max\{0, \lambda_2 + \lambda_3\}, \min\{\lambda_2, 1 + \lambda_3\})$, consider the matrix

$$A = \begin{bmatrix} \alpha & 1-\alpha & 0 \\ \frac{(\alpha-\lambda_2)(\alpha-\lambda_3)}{2\alpha-(1+\lambda_2+\lambda_3)} & 0 & \frac{-\alpha^2+(2+\lambda_2+\lambda_3)\alpha-(\lambda_2+1)(\lambda_3+1)}{2\alpha-(1+\lambda_2+\lambda_3)} \\ 0 & \alpha-(\lambda_2+\lambda_3) & 1+\lambda_2+\lambda_3-\alpha \end{bmatrix}.$$

We can see that A indeed has the right zero pattern and it is row-stochastic. Besides, we can again use the characteristic polynomial to check that the spectrum of A is Λ as expected.

It remains to show that the entries which are not on the anti-diagonal are positive. Note that the condition $\alpha \in (\max\{0, \lambda_2 + \lambda_3\}, \min\{\lambda_2, 1 + \lambda_3\})$ indicates that $\lambda_2 + \lambda_3 < \alpha < 1 + \lambda_2 + \lambda_3$, that is, $0 < \alpha - (\lambda_2 + \lambda_3) < 1$. Thus, in terms of (10), we obtain that $\gamma = \alpha - (\lambda_2 + \lambda_3) \in (0, 1)$.

Meanwhile, we get that $\alpha - \lambda_2 < 0$ and hence $(\alpha - \lambda_2)(\alpha - \lambda_3) < 0$. To prove $\beta = \frac{(\alpha-\lambda_2)(\alpha-\lambda_3)}{2\alpha-(1+\lambda_2+\lambda_3)} > 0$, we need to make sure $2\alpha - (1 + \lambda_2 + \lambda_3) < 0$. Note that $\frac{1+\lambda_2+\lambda_3}{2}$ is the average value of λ_2 and $1 + \lambda_3$, so we indeed have $\alpha < \frac{1+\lambda_2+\lambda_3}{2}$, that is, $2\alpha < 1 + \lambda_2 + \lambda_3$. Therefore, $\beta > 0$ is satisfied. In addition, assume that $\beta \ge 1$, which implies that $(\alpha - \lambda_2)(\alpha - \lambda_3) \le 2\alpha - (1 + \lambda_2 + \lambda_3)$, which is equivalent to $\alpha^2 - \alpha(2 + \lambda_2 + \lambda_3) + (1 + \lambda_2)(1 + \lambda_3) \le 0$.

This produces that

$$\frac{2+\lambda_2+\lambda_3-\sqrt{\Delta}}{2} \le \alpha \le \frac{2+\lambda_2+\lambda_3+\sqrt{\Delta}}{2}, \tag{11}$$

where $\Delta = (2 + \lambda_2 + \lambda_3)^2 - 4(1 + \lambda_2)(1 + \lambda_3) = (\lambda_2 - \lambda_3)^2$. Simplifying the expression, we get that $\alpha \in (1 + \lambda_3, 1 + \lambda_2)$, which is impossible. Hence, we know that $\beta < 1$. ☐

4.6. Case 11. In this case, we consider the following type of nonnegative matrices:

$$A = \begin{bmatrix} \alpha & 1-\alpha & 0 \\ 0 & 0 & 1 \\ \beta & \gamma & 1-\beta-\gamma \end{bmatrix}, \tag{12}$$

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in which $0 < \alpha < 1$ and $\beta, \gamma > 0$ with $\beta + \gamma < 1$. Observe that the trace of A equals $1 + \alpha - (\beta + \gamma)$, and that $\det(A) = -\alpha(\beta + \gamma) + \beta$. Moreover, the sum of the principal two times two submatrices produces

$$-\gamma + \alpha(1 - \beta - \gamma) = \lambda_2 + \lambda_3 + \lambda_2\lambda_3. \tag{13}$$

PROPOSITION 15. *The spectrum of a row-stochastic matrix A of the form (12) is of the form $\{1, \lambda_2, \lambda_3\}$ with*

i. *If $\lambda_{2,3} = a \pm bi$ with $b \neq 0$, then and $a + b^2 < \frac{1}{4}$;*
ii. *If $\lambda_3 \leq \lambda_2$ are real, then $-1 < \lambda_2 + \lambda_3$, and the point (λ_2, λ_3) lies in the region in the real plane bounded by the lines $\lambda_2 = \lambda_3$, $\lambda_2 + \lambda_3 = -1$, $\lambda_3 = -1$, $\lambda_2 = 1$ and the parabola $1 - 2\lambda_2 - 2\lambda_3 - 2\lambda_2\lambda_3 + \lambda_2^2 + \lambda_3^2 = 0$.*

See Figure 6.

On the other hand, for a pair (λ_2, λ_3) satisfying the conditions above, there is a row-stochastic matrix of the form (12) with spectrum $\{1, \lambda_2, \lambda_3\}$.

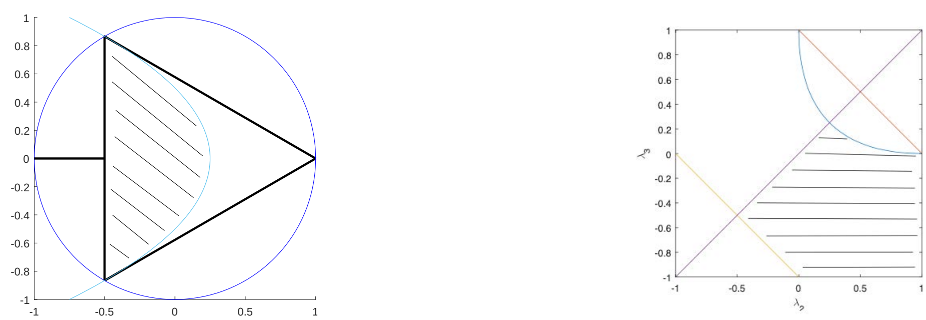

FIGURE 6. *The possible eigenvalues in case 11, left: Complex eigenvalues, right: Real eigenvalues.*

Proof. **Necessity:** The necessity in this case follows from Lemma 4. Observe that the parabola given in item ii here is the same as the parabola given in item ii of Lemma 4.

Sufficiency: First consider the case $\lambda_{2,3} = a \pm bi$. To construct a desired matrix that has 1 and $a \pm bi$ as eigenvalues, we take

$$A_1 = \begin{bmatrix} \frac{1}{2} + a & \frac{1}{2} - a & 0 \\ 0 & 0 & 1 \\ \frac{1}{4} + b^2 & \frac{1}{4} - a - b^2 & \frac{1}{2} + a \end{bmatrix}.$$

Since we have the conditions that $-\frac{1}{2} < a < \frac{1}{4}$ and $a + b^2 < \frac{1}{4}$, A_1 is row-stochastic with the right required zero pattern. Besides, we can apply the characteristic polynomial to make sure that A has the right spectrum.

Next consider the real case with $\lambda_2 + \lambda_3 > -1$ and (λ_2, λ_3) lies in the region indicated in item ii. Note that the condition $\lambda_2 + \lambda_3 < 1$ is also satisfied. Then consider the following matrix:

$$A_2 = \begin{bmatrix} \alpha & 1 - \alpha & 0 \\ 0 & 0 & 1 \\ (\alpha - \lambda_2)(\alpha - \lambda_3) & -h(\alpha) & 1 + \lambda_2 + \lambda_3 - \alpha \end{bmatrix},$$

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where

$$h(\alpha) = \alpha^2 - (1+\lambda_2+\lambda_3)\alpha + (\lambda_2+\lambda_3+\lambda_2\lambda_3),$$

and where

$$\alpha \in \begin{cases} \left(\frac{1+\lambda_2+\lambda_3-\sqrt{\Delta}}{2},\, \frac{1+\lambda_2+\lambda_3}{2}\right) & \lambda_2,\lambda_3>0 \\ \left(\lambda_2,\, \frac{1+\lambda_2+\lambda_3}{2}\right) & \lambda_2>0>\lambda_3 \\ (0,\, 1+\lambda_2+\lambda_3) & \lambda_2,\lambda_3<0 \end{cases}.$$

Let $\Delta = 1-2(\lambda_2+\lambda_3)+(\lambda_2-\lambda_3)^2$ be the discriminant of $h(\alpha)$. Note that $\Delta = 1-2\lambda_2-2\lambda_3-2\lambda_2\lambda_3+\lambda_2^2+\lambda_3^2 > 0$ as the point (λ_2,λ_3) is in the region indicated in item ii. Observe that $\frac{1+\lambda_2+\lambda_3-\sqrt{\Delta}}{2} > \lambda_2+\lambda_3$ when $\lambda_2,\lambda_3>0$. Indeed, otherwise $\frac{1}{2}(1+\lambda_2+\lambda_3-\sqrt{\Delta)} \le \lambda_2+\lambda_3$, which is equivalent to $1-(\lambda_2+\lambda_3) \le \sqrt{\Delta}$, which in turn is equivalent to $(\lambda_2+\lambda_3)^2 \le (\lambda_2-\lambda_3)^2$, which is impossible when $\lambda_3>0$. So according to the restrictions on α, we immediately have that $\lambda_2+\lambda_3 < \alpha < 1+\lambda_2+\lambda_3$, that is, $0 < \alpha-(\lambda_2+\lambda_3) < 1$. Thus together with the expression for the trace of A_2, we get that $\beta+\gamma = \alpha-(\lambda_2+\lambda_3) \in (0,1)$. Furthermore, it also implies that $\beta = (\alpha-\lambda_2)(\alpha-\lambda_3) \in (0,1)$.

Now we check that $0<\gamma<1$. First, observe that $\gamma = \alpha-(\lambda_2+\lambda_3)-\beta < 1$. Next, note that γ is a quadratic polynomial in α, which we denote as $-h(\alpha)$. Assume that $\gamma \le 0$. Then $h(\alpha) \ge 0$, which gives that $\alpha \ge \frac{1+\lambda_2+\lambda_3+\sqrt{\Delta}}{2}$ or $\alpha \le \frac{1+\lambda_2+\lambda_3-\sqrt{\Delta}}{2}$. The former is impossible as $\frac{1+\lambda_2+\lambda_3+\sqrt{\Delta}}{2} \ge \frac{1+\lambda_2+\lambda_3}{2}$ when λ_2 or λ_3 is positive, while $\frac{1+\lambda_2+\lambda_3+\sqrt{\Delta}}{2} \ge 1+\lambda_2+\lambda_3$ when both λ_2 and λ_3 are both negative.

To show that the latter is not possible, we show that $\frac{1+\lambda_2+\lambda_3-\sqrt{\Delta}}{2} \le \lambda_2$ if $\lambda_2>0>\lambda_3$ and $\frac{1+\lambda_2+\lambda_3-\sqrt{\Delta}}{2} \le 0$ if $\lambda_2,\lambda_3<0$. Indeed, if $\lambda_3<0<\lambda_2$ we have $\Delta = 1-2\lambda_2-2\lambda_3-2\lambda_2\lambda_3+\lambda_2^2+\lambda_3^2 > 1-2\lambda_2+\lambda_2^2 = (1-\lambda_2)^2$. So, $\sqrt{\Delta} > 1-\lambda_2$, and hence $\frac{1+\lambda_2+\lambda_3-\sqrt{\Delta}}{2} < \frac{2\lambda_2+\lambda_3}{2} < \lambda_2$. In case $\lambda_2,\lambda_3 \le 0$, we have $\Delta 1-2\lambda_2-2\lambda_3+(\lambda_2-\lambda_3)^2 > 1$ and so $\frac{1+\lambda_2+\lambda_3-\sqrt{\Delta}}{2} < \frac{1+\lambda_2+\lambda_3-1}{2} = \frac{\lambda_2+\lambda_3}{2} < 0$.

Therefore, we obtain that $\gamma \in (0,1)$.

It is additionally quite clear that the matrix A_2 has eigenvalues $\{1,\lambda_2,\lambda_3\}$, while it is row-stochastic with the right zero pattern. □

4.7. Case 12. Now we are left with the type of nonnegative matrices below:

$$A = \begin{bmatrix} 0 & \alpha & 1-\alpha \\ 0 & 0 & 1 \\ \beta & \gamma & 1-\beta-\gamma \end{bmatrix}, \tag{14}$$

where $0<\alpha<1$ and $\beta,\gamma>0$ with $\beta+\gamma<1$. Note that here we have that trace $A = 1-(\beta+\gamma)$, and that $\det A = \alpha\beta$ which is always positive. Moreover, the sum of the principal two times two submatrices gives

$$\lambda_2+\lambda_3+\lambda_2\lambda_3 = -\gamma+\beta(\alpha-1) < 0. \tag{15}$$

Also, zero cannot be an eigenvalue as A is invertible.

PROPOSITION 16. *Given a row-stochastic matrix A of the form (14), the spectrum of A is of the form $\{1,\lambda_2,\lambda_3\}$ with*

i. *If $\lambda_{2,3} = a \pm bi$ with $b \neq 0$, then $(a+1)^2+b^2<1$;*
ii. *If $\lambda_2 \ge \lambda_3$ are real, then $0>\lambda_2 \ge \lambda_3 > -1$ and $\lambda_2+\lambda_3>-1$.*

See Figure 7.

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On the other hand, for a pair (λ_2, λ_3) *satisfying the conditions above, there is a row-stochastic matrix of the form* (14) *with spectrum* $\{1, \lambda_2, \lambda_3\}$.

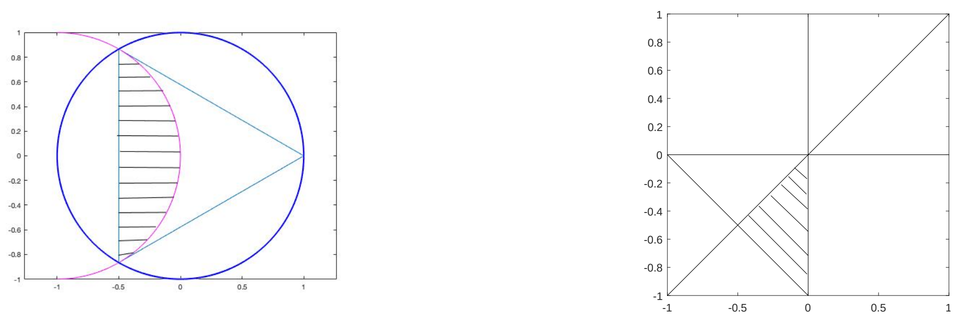

FIGURE 7. *The eigenvalue locations in case 12. Left: Complex eigenvalues, right: Real conjugate eigenvalues.*

Proof. **Necessity:** First consider the case when A has complex eigenvalues $\lambda_{2,3} = a \pm ib$. The necessity of the condition then follows from Lemma 5.

Next we check the case when A has only real eigenvalues. From the trace, we have $\lambda_2 + \lambda_3 = -(\beta + \gamma)$, and $-1 < \lambda_2 + \lambda_3 < 0$ because of the condition $\beta + \gamma < 1$. Meanwhile, observe that $\det A = \lambda_2\lambda_3$ is always positive. This indicates that λ_2 and λ_3 must be both negative. Hence, we obtain that $0 > \lambda_2, \lambda_3 > -1$.

Sufficiency: Suppose $\lambda_{2,3} = a \pm bi$. Then for $\gamma \in (0, -(a^2 + 2a + b^2))$, the matrix

$$A_1 = \begin{bmatrix} 0 & -\frac{a^2+b^2}{2a+\gamma} & 1 + \frac{a^2+b^2}{2a+\gamma} \\ 0 & 0 & 1 \\ -(2a+\gamma) & \gamma & 1+2a \end{bmatrix},$$

has the desired zero pattern and has 1 and $a \pm bi$ as its eigenvalues.

Notice that $a^2 + 2a + b^2 < 0$ based on equation (15); hence, $-(a^2 + 2a + b^2) > 0$. It is also true that $-(a^2 + 2a + b^2) < 1$; otherwise, we would get $(a + 1)^2 + b^2 < 0$ which is impossible. So γ is between 0 and 1, and we can use the characteristic polynomial to verify that it indeed has $\{1, \lambda_2, \lambda_3\}$ as its spectrum.

It remains to check that A_1 is nonnegative. It is easy to see that $\beta + \gamma = -2a \in (0, 1)$. Additionally, we obtain $\beta = -2a - \gamma > 0$, for $\gamma < -(a^2 + b^2) - 2a < -2a$. To make sure β is smaller than 1, assume that $\beta \geq 1$. It then follows that $-2a - \gamma \geq 1$, equivalently, $\gamma \leq -(1 + 2a) < 0$, which is a contradiction. Further, it is clear that $\alpha = \frac{a^2+b^2}{-(2a+\gamma)} = \frac{a^2+b^2}{\beta} > 0$. Since $\gamma < -(a^2 + b^2) - 2a$, we can rewrite the expression to get that $-(2a + \gamma) > a^2 + b^2$, implying that $\frac{a^2+b^2}{-(2a+\gamma)} < 1$, that is, $\alpha < 1$ is satisfied.

Next, suppose λ_2 and λ_3 are both real such that $-1 < \lambda_2, \lambda_3 < 0$ and $\lambda_2 + \lambda_3 > -1$. In this case, a row-stochastic matrix that has eigenvalues $\{1, \lambda_2, \lambda_3\}$ can be constructed as follows:

$$A_2 = \begin{bmatrix} 0 & -\lambda_3 & 1+\lambda_3 \\ 0 & 0 & 1 \\ -\lambda_2 & -\lambda_3 & 1+\lambda_2+\lambda_3 \end{bmatrix}.$$

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Obviously, A_2 is indeed a nonnegative row-stochastic matrix, and one can check that A_2 has $\{1, \lambda_2, \lambda_3\}$ as its spectrum using the characteristic polynomial. □

This completes the discussion of all possible zero patterns with three zeros.

5. Zero patterns with two zeros. There are $\binom{9}{2} = 36$ possible ways to distribute two zeros over nine entries; however, using similarity by permutations and the fact that the transpose pattern yields the same eigenvalues, this is reduced to a much smaller number of cases, namely the following six:

$$\begin{bmatrix} 0 & * & * \\ * & 0 & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, \begin{bmatrix} * & 0 & * \\ 0 & * & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} * & 0 & * \\ * & * & 0 \\ * & * & * \end{bmatrix}, \begin{bmatrix} 0 & * & * \\ * & * & 0 \\ * & * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & * \\ * & * & * \\ * & * & * \end{bmatrix}.$$

We will refer to these cases as cases 1 to 6 in this order.

In case 1, Lemma 5 gives a necessary condition. This can be shown to be sufficient as well. Indeed, from Proposition 13, there is a matrix with two zeros on the diagonal and an extra zero in the left lower entry which has the desired spectrum. Then we can use the implicit function theorem in a similar way as in the proof of Proposition 10 to show the desired sufficiency.

Case 2 is a reducible case, we shall omit further discussion of that case. In cases 5 and 6, Lemma 4 can be applied to give a necessary condition, which can be shown to be sufficient as well; see [11]. We shall discuss the cases 3 and 4 in the following two subsections.

5.1. Case 3.

PROPOSITION 17. *Let* $A = \begin{bmatrix} \alpha & 0 & 1-\alpha \\ 0 & \beta & 1-\beta \\ \gamma & \delta & 1-\delta-\gamma \end{bmatrix}$ *be a nonnegative primitive matrix, and let* $\{1, \lambda_2, \lambda_3\}$ *be the spectrum. Then* λ_2 *and* λ_3 *are both real and nonzero,* $\lambda_3 < \lambda_2$ *and* $\lambda_2 > 0$.

Conversely, for any pair of real numbers $\lambda_2 > \lambda_3$ *both non-zero and with* $\lambda_2 > 0$*, there is a nonnegative matrix* A *of this form with spectrum* $\{1, \lambda_2, \lambda_3\}$.

Proof. Necessity. The characteristic polynomial of A is of the form:

$$p_A(\lambda) = \det(\lambda I_3 - A) = (\lambda - 1)(\lambda^2 - (\operatorname{trace} A - 1)\lambda + \det A) = (\lambda - 1)(\lambda - \lambda_2)(\lambda - \lambda_3).$$

So the eigenvalues will be real and different when $(\operatorname{trace} A - 1)^2 - 4\det A = (\lambda_2 - \lambda_3)^2 > 0$.

Computing $\det A$ gives $\det A = \alpha\beta - \alpha\delta - \gamma\beta$. Also, $\operatorname{trace} A - 1 = \alpha + \beta - \delta - \gamma$. Hence,

$$\begin{aligned}
&(\operatorname{trace} A - 1)^2 - 4\det A = (\alpha + \beta - \delta - \gamma)^2 - 4(\alpha\beta - \alpha\delta - \gamma\beta) \\
=&\alpha^2 + \beta^2 + \delta^2 + \gamma^2 + 2(\alpha\beta - \alpha\gamma - \alpha\delta - \beta\gamma - \beta\delta + \gamma\delta) \\
&- 4\alpha\beta + 4\alpha\delta + 4\beta\gamma \\
=&\alpha^2 + \beta^2 + \delta^2 + \gamma^2 - 2\alpha\beta + 2\alpha\delta - 2\alpha\gamma + 2\beta\gamma - 2\beta\delta + 2\gamma\delta \\
=&(\alpha - \beta + \delta - \gamma)^2 + 4\gamma\delta > 0.
\end{aligned}$$

So both λ_2 and λ_3 are real and $\lambda_2 > \lambda_3$.

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It remains to show that λ_2 is positive. First observe that the columns of A are clearly independent, so $\det A \neq 0$. Further, if the determinant of A is negative, then $\lambda_2\lambda_3 < 0$, and so, since $\lambda_3 \leq \lambda_2$ it then follows that $\lambda_2 > 0$. So we may assume that $\det A > 0$.

Now we claim that if $\operatorname{trace} A - 1 < 0$, then $\det A < 0$. To see this introduce some notation first. Introduce

$$f(\alpha,\beta,\gamma,\delta) = \det A = \alpha\beta - \alpha\delta - \gamma\beta,$$
$$g(\alpha,\beta,\gamma,\delta) = \operatorname{trace} A - 1 = \alpha + \beta - \gamma - \delta,$$

and finally, let S be the set in $\mathbb{R}^4$ given by $\operatorname{trace} A - 1 \leq 0$ and all conditions on α, β, γ, and δ which make A a nonnegative row-stochastic matrix:

$$S = \{(\alpha,\beta,\gamma,\delta) \mid 0 \leq \alpha \leq 1, 0 \leq \beta \leq 1, 0 \leq \gamma, 0 \leq \delta, \gamma + \delta \leq 1, \alpha + \beta - \gamma - \delta \leq 0\}.$$

Observe that S is a compact set, so f has a maximum on S, either in the interior or on the boundary of the set. We shall show that the maximum of f on S is 0, and it is attained in the origin $\alpha = \beta = \delta = \gamma = 0$. First we show that f has no extremal value in the interior of S. For this compute $f'(\alpha,\beta,\gamma,\delta) = (\beta - \delta, \alpha - \gamma, -\beta, -\alpha)$, which is zero only in the origin. So we consider the boundary, piece by piece. If either $\alpha = 0$ or $\beta = 0$, then the values of f are less than or equal to 0, and 0 only in boundary points. Likewise when $\beta = 1$ and $\gamma + \delta = 1$. If $\alpha = 1$, then, because on S we have $\alpha + \beta \leq \gamma + \delta \leq 1$, it follows that $\beta = 0$. So again, in that case f has values less than or equal to 0. If $\gamma = 0$, then $f(\alpha,\beta,0,\delta) = \alpha(\beta - \delta)$. Since on S we have $\alpha + \beta \leq \gamma + \delta = \delta$, it follows that $\beta \leq \delta$, and hence in these points f has values less than or equal to 0. Likewise when $\delta = 0$. So it remains to consider the part of the boundary of S where $\alpha + \beta - \gamma - \delta = 0$. We use Euler–Lagrange on that set. Put $H = f - \mu g$, where μ is the Lagrange parameter. Then

$$H'(\alpha,\beta,\gamma,\delta,\mu) = ((\beta - \delta) - \mu, (\alpha - \gamma) - \mu, -\beta + \mu, -\alpha + \mu, -g(\alpha,\beta,\gamma,\delta)).$$

This has to be zero. It follows that $\mu = \beta - \delta = \alpha - \gamma = \beta = \alpha$. It is straightforward to check that together with $g(\alpha,\beta,\gamma,\delta) = 0$ this implies that $\alpha = \beta = \gamma = \delta = 0$. So the maximum of f on S is zero.

Hence, if $\det A \geq 0$ also $\operatorname{trace} A - 1 \geq 0$. Now

$$\lambda_2 = \frac{1}{2}(\operatorname{trace} A - 1) + \frac{1}{2}\sqrt{(\operatorname{trace} A - 1)^2 - 4\det A},$$

and we know that this is real by what we showed earlier. Then it follows that if $\det A \geq 0$ then $\lambda_2 \geq 0$, while if $\det A < 0$ the fact that $\lambda_2 > 0$ is automatic.

Finally, we show that $\lambda_2 > 0$. Indeed, suppose that $\lambda_2 = 0$, then $\det A = 0$, and so $\alpha\beta = \alpha\delta + \gamma\beta$, so dividing by $\alpha\beta$, we obtain $\frac{\delta}{\beta} + \frac{\gamma}{\alpha} = 1$. Since α, β, γ, and δ are all positive, this implies that $\gamma < \alpha$ and $\delta < \beta$. So $\operatorname{trace} A - 1 = \alpha + \beta - \gamma - \delta > 0$. Now if $\lambda_2 = 0$, then $\lambda_3 \leq 0$ and so $\operatorname{trace} A - 1 = \lambda_2 + \lambda_3 \leq 0$. Hence, we must have that $\det A = 0$ can only occur when $\lambda_3 = 0$.

This completes the proof of the necessity part.

Sufficiency. Let $0 < \lambda_2 < 1$ and $-1 < \lambda_3 < \lambda_2$ be given. Consider the following matrix with three zeros:

$$A_0 = \begin{bmatrix} \lambda_2 & 0 & 1 - \lambda_2 \\ 0 & \beta & 1 - \beta \\ 0 & \beta - \lambda_3 & 1 - \beta + \lambda_3 \end{bmatrix},$$

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with the conditions that $\max(0, \lambda_3) < \beta < \min(1, 1 + \lambda_3)$. Then A_0 is nonnegative and has the desired eigenvalues as one easily sees (eigenvalues 1 and λ_2 are obvious and from the trace one obtains the third eigenvalue equal to λ_3). For $A = \begin{bmatrix} \alpha & 0 & 1-\alpha \\ 0 & \beta & 1-\beta \\ \gamma & \delta & 1-\delta-\gamma \end{bmatrix}$ define

$$\Phi(\alpha, \beta, \gamma, \delta) = \begin{bmatrix} f(\alpha, \beta, \gamma, \delta) \\ g(\alpha, \beta, \gamma, \delta) \end{bmatrix} = \begin{bmatrix} \det A \\ \operatorname{trace} A - 1 \end{bmatrix} = \begin{bmatrix} \alpha\beta - \alpha\delta - \gamma\beta \\ \alpha + \beta - \gamma - \delta \end{bmatrix}.$$

Take a fixed $\beta_0 \in (\max(0, \lambda_3), \min(1, 1 + \lambda_3))$. Then

$$\Phi(\lambda_2, \beta_0, 0, \beta_0 - \lambda_3) = \begin{bmatrix} \lambda_2\lambda_3 \\ \lambda_2 + \lambda_3 \end{bmatrix}.$$

View (α, β) as functions of (γ, δ) determined by the equation $\Phi(\alpha, \beta, \gamma, \delta) = \begin{bmatrix} \lambda_2\lambda_3 \\ \lambda_2 + \lambda_3 \end{bmatrix}$ in a neighborhood of the point $(\lambda_2, \beta_0, 0, \beta_0 - \lambda_3)$. From the implicit function theorem, this is possible when the matrix $\begin{bmatrix} \frac{\partial f}{\partial \alpha} & \frac{\partial f}{\partial \beta} \\ \frac{\partial g}{\partial \alpha} & \frac{\partial g}{\partial \beta} \end{bmatrix}$ is invertible in the point $(\lambda_2, \beta_0, 0, \beta_0 - \lambda_3)$. We have in this point

$$\begin{bmatrix} \frac{\partial f}{\partial \alpha} & \frac{\partial f}{\partial \beta} \\ \frac{\partial g}{\partial \alpha} & \frac{\partial g}{\partial \beta} \end{bmatrix} = \begin{bmatrix} \beta - \delta & \alpha - \gamma \\ 1 & 1 \end{bmatrix} |_{(\alpha,\beta,\gamma,\delta)=(\lambda_2,\beta_0,0,\beta_0-\lambda_3)} = \begin{bmatrix} \lambda_3 & \lambda_2 \\ 1 & 1 \end{bmatrix}.$$

Since $\lambda_2 \neq \lambda_3$, this matrix is indeed invertible. Hence, by the implicit function theorem, for small positive values of γ and for δ in a (positive) neighborhood of $\beta_0 - \lambda_3$, there are solutions α and β, close to $\alpha = \lambda_2$ and $\beta = \beta_0$. Hence, there is a nonnegative matrix A with the desired zero pattern and spectrum $\{1, \lambda_2, \lambda_3\}$. This finishes the proof. □

5.2. Case 4. In this case, we have

$$A = \begin{bmatrix} \alpha & 0 & 1-\alpha \\ 1-\beta & \beta & 0 \\ \delta & 1-\gamma-\delta & \gamma \end{bmatrix}, \tag{16}$$

where $\alpha, \beta, \gamma, \delta \in (0, 1)$ and $0 < \delta + \gamma < 1$. Notice that we have $\operatorname{trace} A = \alpha + \beta + \gamma$, and $\det A = (1 - \alpha)(1 - \beta - \gamma - \delta) + \beta\gamma$. Furthermore, the sum of the principal two times two submatrices produces

$$\alpha\beta + \alpha\gamma + \beta\gamma - \delta(1 - \alpha) = \lambda_2 + \lambda_3 + \lambda_2\lambda_3.$$

Proposition 18. *Let A be a row-stochastic matrix of the form (16). Then the spectrum of A is of the form $\{1, \lambda_2, \lambda_3\}$ with*

i. *If $\lambda_{2,3} = a \pm bi$ with $b \neq 0$, then $a > -\frac{1}{2}$ and $|b| < \frac{\sqrt{3}}{3}(1 - a)$;*
ii. *If λ_2, λ_3 are real, then $-1 < \lambda_3 \leq \lambda_2 < 1$ and $\lambda_2 + \lambda_3 > -1$.*

Conversely, for any such pair (λ_2, λ_3), there is a row-stochastic matrix of the form (16) with spectrum $\{1, \lambda_2, \lambda_3\}$.

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Note that this means that the only restriction is that the eigenvalues cannot be on the boundary of the Karpelevich region. This case is, up to scaling and permutational similarity, the same type as the matrix occuring in Theorem 6 in [13].

Proof. Necessity. Note that the boundary points of the Karpelevič region cannot be eigenvalues of a matrix of the form (16) by Propositions 6 and 7.

Sufficiency. Now we prove the converse, so we show that for each pair of complex conjugates λ_2 and λ_3 inside the triangle bounded by $x > -\frac{1}{2}$ and $|y| < \frac{\sqrt{3}}{3}(1-x)$ or for any pair of real numbers $\lambda_{2,3} \in (0, 1)$ such that $\lambda_2 + \lambda_3 > -1$, there is a nonnegative matrix of form (16) with $\sigma(A) = \{1, \lambda_2, \lambda_3\}$.
Notice that the necessary and sufficient conditions for $\sigma(A) = \{1, \lambda_2, \lambda_3\}$, with A in the form of (16), are

$$1 + \lambda_2 + \lambda_3 = \alpha + \beta + \gamma, \qquad \lambda_2 + \lambda_3 + \lambda_2\lambda_3 = \alpha\beta + \alpha\gamma + \beta\gamma - \delta(1-\alpha). \tag{17}$$

Based on the above two equations, we need to find solutions of $(\alpha, \beta, \gamma, \delta)$ such that $\alpha, \beta, \gamma, \delta \in (0, 1)$ and $0 < \delta + \gamma < 1$.

Real eigenvalues. In the real eigenvalue case, first let both $\lambda_2 \geq \lambda_3$ be positive. If we take $\beta = \alpha$, then a possible solution is $(\alpha, \beta, \gamma, \delta) = (\frac{1+\lambda_2}{2}, \frac{1+\lambda_2}{2}, \lambda_3, \frac{1-\lambda_2}{2})$, and we can easily verify that $\gamma + \delta = \frac{1-\lambda_2+2\lambda_3}{2} < 1$.

Then we are left with the case when both λ_2 and λ_3 are negative, or when $\lambda_2 \geq 0 \geq \lambda_3$ and $\lambda_2 + \lambda_3 > -1$. Recall that from Case 11 of the three-zero pattern, for any such pair of λ_2 and λ_3 we can construct a nonnegative matrix $B = \begin{bmatrix} \hat{\alpha} & 1-\hat{\alpha} & 0 \\ 0 & 0 & 1 \\ \hat{\delta} & \hat{\gamma} & 1-\hat{\gamma}-\hat{\delta} \end{bmatrix}$ that has eigenvalues $\{1, \lambda_2, \lambda_3\}$. As one can see, performing a similarity with $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ on B results in the matrix A with $\alpha_0 = 0$, $\beta_0 = \hat{\alpha}$, $\delta_0 = \hat{\gamma}$, and $\gamma_0 = 1-\hat{\gamma}-\hat{\delta}$ in this case. Hence, we can find a matrix $A(\alpha = 0)$ such that $\sigma(A(\alpha = 0)) = \{1, \lambda_2, \lambda_3\}$ for each pair of (λ_2, λ_3). Thus, using the implicit function theorem, we obtain an open interval U of (α, δ) containing $(0, \delta_0)$ and a continuous function $g : U \longrightarrow \mathbb{R}^2$ such that $f(\alpha, g(\alpha, \delta), \delta) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thereby, we can indeed find a pair of entries $(\alpha, \delta) \in U$ with $(\beta, \gamma) = g(\alpha, \delta)$ to construct the matrix A of the form (16) which has spectrum $\{1, \lambda_2, \lambda_3\}$.

Now consider a specific solution $(\alpha, 1, \gamma, \delta)$ to equation (17). Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ be defined as:

$$f(\alpha, \beta, \gamma, \delta) = \begin{pmatrix} \alpha + \beta + \gamma - (1 + \lambda_2 + \lambda_3) \\ \alpha\beta + \alpha\gamma + \beta\gamma - \delta(1-\alpha) - (\lambda_2 + \lambda_3 + \lambda_2\lambda_3) \end{pmatrix}.$$

We get that

$$f'(\alpha, \beta, \gamma, \delta) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ \beta + \gamma + \delta & \alpha + \gamma & \alpha + \beta & \alpha - 1 \end{pmatrix}.$$

The columns 2 and 3 clearly form an invertible matrix when $\alpha = 0$ and $\beta \neq \gamma$, and so in that case we can apply the implicit function theorem to show that there is a solution such that $\alpha > 0$ and small. Notice that the freedom we have in the construction in Case 11 of the three-zero pattern allows to make a choice such that $\beta \neq \gamma$.

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Complex eigenvalues. Next if $\lambda_{2,3} = a \pm bi$ ($b \neq 0$) by the proof given for Case 5 (the circulant pattern) of the three-zero pattern, we can find a nonnegative matrix as in the following form:

$$\tilde{A}_2 = \begin{bmatrix} \alpha & 0 & 1-\alpha \\ 1-\beta & \beta & 0 \\ 0 & 1-\beta & \beta \end{bmatrix},$$

such that $a \pm bi$ are eigenvalues of $\tilde{A}_2$. .

However, compared with A in (16), $\tilde{A}_2$ is obtained by taking $\gamma = \beta$ and $\delta = 0$. Consider $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ that is given by:

$$f(\alpha, \beta, \gamma, \delta) = \begin{pmatrix} \alpha + \beta + \gamma - (1 + 2a) \\ \alpha\beta + \alpha\gamma + \beta\gamma - \delta(1-\alpha) - (2a + a^2 + b^2) \end{pmatrix},$$

where we view (α, γ) as a function of (β, δ). For a specific solution $(\alpha, \beta, \gamma, \delta) = (\hat{\alpha}, \hat{\beta}, \hat{\beta}, 0)$, we have that $f(\hat{\alpha}, \hat{\beta}, \hat{\beta}, 0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and that

$$f'(\hat{\alpha}, \hat{\beta}, \hat{\beta}, 0) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2\hat{\beta} & \hat{\alpha} + \hat{\beta} & \hat{\alpha} + \hat{\beta} & \hat{\alpha} - 1 \end{pmatrix}.$$

Obviously, the matrix formed by the first and third columns of $\tilde{A}_2$ is invertible. Thus, we can apply the implicit function theorem to conclude that there must be a solution with $\delta > 0$. $\square$

6. Zero patterns with four zeros.

Next, we consider 3×3 matrices with four zero entries. Note that we only are interested in the irreducible cases. Performing a similarity permutation or taking a transpose keeps the number of zeros equal to 4 and does not change the eigenvalues. This results in a reduction to 6 essentially different cases which are listed in the table below, together with the conditions on the spectrum $\Lambda = \{1, \lambda_2, \lambda_3\}$ for each of the cases. The proofs use the same techniques as before and may be found in [11]. In each case, for any pair $\lambda_{2,3}$ satisfying the given conditions, there is a nonnegative, irreducible, row-stochastic matrix with the given zero pattern and the list Λ as its spectrum.

$\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{bmatrix}$	$\lambda_2 = 0, -1 < \lambda_3 < 0.$
$\begin{bmatrix} 0 & * & * \\ 0 & 0 & * \\ * & 0 & * \end{bmatrix}$	either $\lambda_{2,3} = a \pm bi$ with $b \neq 0$ and $-\frac{1}{2} < a < 0$, $(a+1)^2 + b^2 < 1$, or $\lambda_{2,3} < 0$ and $-1 < \lambda_2 + \lambda_3 < 0$.
$\begin{bmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & 0 \end{bmatrix}$	$-1 < \lambda_3 < 0 < \lambda_2 < 1$ and $-1 < \lambda_2 + \lambda_3 < 0$.

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$\begin{bmatrix} 0 & * & * \\ 0 & * & * \\ * & 0 & 0 \end{bmatrix}$	$\lambda_{2,3} = a \pm bi$ with $b \neq 0$ and $-\frac{1}{2} < a < 0$, $(a+1)^2 + b^2 < 1$, or $\lambda_{2,3} < 0$ and $-1 < \lambda_2 + \lambda_3 < 0$.
$\begin{bmatrix} 0 & * & * \\ 0 & 0 & * \\ * & * & 0 \end{bmatrix}$	$\lambda_{2,3} = -\frac{1}{2} \pm ib$ with $0 < b < \frac{\sqrt{3}}{2}$ or $\lambda_{2,3} = -\frac{1}{2} \pm 1$ with $0 \leq a < \frac{1}{2}$.
$\begin{bmatrix} 0 & * & 0 \\ 0 & * & * \\ * & 0 & * \end{bmatrix}$	either $\lambda_{2,3} = a \pm bi$ with $b \neq 0$ and $(a+1)^2 + b^2 > 1$ and $a + b^2 < \frac{1}{4}$ or $0 < \lambda_3 \leq \lambda_2$ and $1 + \lambda_2^2 + \lambda_3^2 - 2(\lambda_2 + \lambda_3 + \lambda_2\lambda_3) > 0$. See Figure 8.

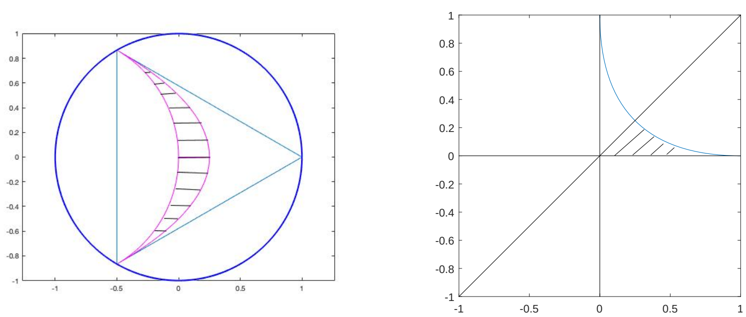

FIGURE 8. *Possible eigenvalue locations for the final case.*

7. Five zero patterns. When there are five zero entries in a 3×3 matrix, there are at least two columns with two zeroes in it. That results in a small number of irreducible forms. By permutation, we may assume that there are two zeroes in columns 1 and 2. If the two zeroes in the first column are in positions 2 and 3, then the matrix is reducible. Likewise, if the two zeroes in the second column are in positions 1 and 3, then the matrix is reducible.

So there remain the following three cases.

Case 1. Consider the following nonnegative matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \alpha & 1-\alpha & 0 \end{bmatrix}$. The characteristic polynomial of A is given by $\lambda(\lambda^2 - 1)$ independently of α, so the spectrum is $\{1, 0, -1\}$.

Case 2. $A = \begin{bmatrix} 0 & \alpha & 1-\alpha \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Observe that $\operatorname{trace}(A) = 1 + \lambda_2 + \lambda_3 = 0$ and that $\det(A) = \alpha > 0$. The spectrum of A is of the form $\{1, \lambda_2, \lambda_3\}$ such that either $\lambda_{2,3} = -\frac{1}{2} \pm bi$ with $b \in (0, \frac{\sqrt{3}}{2})$, or $\lambda_{2,3} = -\frac{1}{2} \pm a$ with $a \in [0, \frac{1}{2})$. Conversely, for any such pair (λ_2, λ_3), there is a row-stochastic matrix of this form with spectrum $\{1, \lambda_2, \lambda_3\}$.

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Case 3. Next suppose we have a nonnegative matrix with the following pattern: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1-\alpha & 0 & \alpha \end{bmatrix}$. In this case ,trace $A = 1 + \lambda_2 + \lambda_3 = \alpha \in (0,\, 1)$ and det $A = 1 - \alpha > 0$. Besides, the sum of the two times two subdeterminants is $\lambda_2 + \lambda_3 + \lambda_2\lambda_3 = 0$. The spectrum of A is of the form $\{1, \lambda_2, \lambda_3\}$ with $\lambda_{2,3} = a \pm bi$ such that $b \neq 0$ and $(a+1)^2 + b^2 = 1$. Conversely, for any such pair (λ_2, λ_3), there is a row-stochastic matrix of this form with spectrum $\{1, \lambda_2, \lambda_3\}$.

8. Six zero patterns. Now we take a further step to analyze the matrices with six zero entries. There is little choice since we are only interested in the irreducible cases, that is, those without a whole column or row of zeros. Up to permutation and transpose, this leaves the case $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Obviously, the spectrum is now $\{1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\}$.

9. Some remarks concerning dimension four. Next we consider the 4×4 case, with three zeroes on the diagonal. Let A be a row-stochastic irreducible matrix with zeroes on the $(1,1), (2,2)$, and $(3,3)$ entries, and let $\sigma(A) = \{1, \lambda_2, \lambda_3, \lambda_4\}$. So A has the form:

$$A = \begin{bmatrix} 0 & * & * & * \\ * & 0 & * & * \\ * & * & 0 & * \\ * & * & * & * \end{bmatrix}.$$

Note that the trace of A is less than 1, and that all principal 2×2 submatrices are negative.

PROPOSITION 19. *Let A be a row-stochastic matrix with three zeros on the diagonal, and let $\sigma(A) = \{1, \lambda_2, \lambda_3, \lambda_4\}$. Then the following hold*

i. *if $\sigma(A)$ consists of four real numbers, then λ_2, λ_3, and λ_4 satisfy the following two conditions:*

$$-1 < \lambda_2 + \lambda_3 + \lambda_4 < 0, \tag{18}$$

$$\lambda_2 + \lambda_3 + \lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 < 0, \tag{19}$$

ii. *if $\sigma(A)$ contains a pair of non-real eigenvalues, which we denote by $\lambda_{3,4} = a \pm bi$, then λ_2 and $a \pm bi$ satisfy the following two conditions besides the condition that $a \pm bi$ are contained in the Karpelevič region:*

$$-1 < \lambda_2 + 2a < 0, \tag{20}$$

$$(a + \lambda_2 + 1)^2 + b^2 < (\lambda_2 + \tfrac{1}{2})^2 + \tfrac{3}{4}. \tag{21}$$

Proof. Conditions (18) and (20) are consequences of the fact that $0 <$ trace $(A) < 1$ so that $0 < 1 + \lambda_2 + \lambda_3 + \lambda_4 < 1$.

The fact that the two-by-two principal minors are all negative means that the coefficient of λ^2 in the characteristic polynomial of det $(\lambda I - A)$ is negative, so the sum over all products of two different eigenvalues is negative. In other words,

$$\lambda_2 + \lambda_3 + \lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 < 0,$$

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as is stated by condition (19). For the case where there is a pair of non-real eigenvalues, insert $\lambda_{3,4} = a \pm bi$, then condition (21) is a straightforward rewriting of (19). □

Using the formula for the coefficients of the characteristic polynomial in terms of traces of powers of A, the conditions can be rewritten as:

$$0 < \operatorname{trace}(A) < 1, \qquad (\operatorname{trace}(A))^2 < \operatorname{trace}(A^2).$$

Note that the conditions do indeed exclude possibilities. Some obvious conclusions can be drawn from them: for instance, if A is of the form as stated in the proposition and has a pair of non-real eigevalues $a \pm bi$, then $a < \frac{1}{2}$ as follows from (20) taking into account that $-\lambda_2 < 1$. Also, if A has all eigenvalues real, and we order them as $\lambda_4 \le \lambda_3 \le \lambda_2$, then from (18) we see that $\lambda_4 < 0$ and $\lambda_2 > -\frac{1}{3}$.

Let us analyze a bit more the location of possible non-real eigenvalues. Denote the region bordered by the Karpelevič curves, that is, the possible locations of the eigenvalues, by K. For a fixed value of λ_2, the region described by condition (21) is a disc, and combined with $-\frac{1+\lambda_2}{2} < a < -\frac{\lambda_2}{2}$, which follows from (20) the two conditions describe a portion of a disc between the two vertical lines through $-\frac{1+\lambda_2}{2}$ and $-\frac{\lambda_2}{2}$. The disc is bounded by a circumference, and all these circumferences pass through the points $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. So for a fixed λ_2, the possible location of non-real eigenvalues is in the region:

$$V_{\lambda_2} = \{a + bi \in K \mid (a, b) \text{ satisfies (20) and (21)}\}.$$

One checks that the union of all these regions is the set of all elements of K with real part less than $\frac{1}{2}$. For $\lambda_2 = 1, 1/2, 0, -1/2, -1$, the regions V_{λ_2} are depicted in Figure 9.

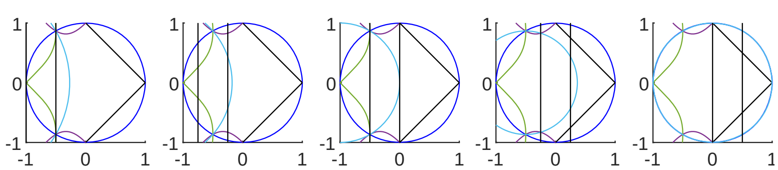

FIGURE 9. *The regions V_{λ_2} with (left to right) $\lambda_2 = 1, 1/2, 0, -1/2, -1$. The regions between the two vertical lines, inside the light colored circumference and inside the Karpelevič region.*

Let us also consider another pattern that has interesting behavior, namely

$$A = \begin{bmatrix} \alpha & 1-\alpha & 0 & 0 \\ 0 & \beta & 1-\beta & 0 \\ 0 & 0 & \gamma & 1-\gamma \\ 1-\delta & 0 & 0 & \delta \end{bmatrix}, \tag{22}$$

with α, β, γ, and δ in $(0,1)$. Note that for $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, we have that PAP^{-1} is a matrix of the same pattern as A, with α replaced by β, β replaced by γ, γ replaced by δ, and finally, δ replaced by α. So under this permutation of the parameters, the eigenvalues remain the same. Introduce the region

$$R = \{z = a + bi \mid z \in K, a > 0, (b^2 + a^2 + a)^2 + 2a^2 - b^2 > 0\}.$$

We conjecture the following.

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Conjecture 20. *For any irreducible nonnegative matrix A of the form (22), the non-real eigenvalues of A are in the region R.*

We have substantial evidence for this conjecture. In Figure 10, the eigenvalues of 10^4 matrices of this type are plotted in red. In addition, we consider the matrices:

$$A(\alpha) = \begin{bmatrix} \alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{23}$$

The eigenvalues of $A(\alpha)$ are the roots of the polynomial:

$$\begin{aligned} p_{A(\alpha)}(\lambda) &= \lambda^4 - \alpha\lambda^3 - 1 + \alpha = (\lambda^4 - 1) - \alpha(\lambda^3 - 1) \\ &= (\lambda - 1)\left((\lambda^3 + \lambda^2 + \lambda + 1) - \alpha(\lambda^2 + \lambda + 1)\right) \\ &= (\lambda - 1)(\lambda^3 + (1-\alpha)(\lambda^2 + \lambda + 1)). \end{aligned}$$

We consider the curve which is implicitly given in the complex plane by $\lambda^3 + (1-\alpha)(\lambda^2 + \lambda + 1) = 0$. Note that besides 1 the matrix $A(\alpha)$ must have at least one other real eigenvalue, and it is easy to see that for $\lambda > 0$ and $0 < \alpha < 1$ we have $\lambda^3 + (1-\alpha)(\lambda^2 + \lambda + 1) > 0$, so any real eigenvalue besides 1 must be negative. Suppose $\lambda = a + bi$ is a non-real eigenvalue of $A(\alpha)$. Splitting the equation $\lambda^3 + (1-\alpha)(\lambda^2 + \lambda + 1) = 0$ into real and imaginary parts, we find

$$a^3 - 3ab^2 + (1-\alpha)(a^2 - b^2 + a + 1) = 0, \qquad 3a^2 b - b^3 + (1-\alpha)(2a+1)b = 0.$$

Since we are interested in non-real eigenvalues, we have $b \neq 0$. The two equations above give two expressions for $1 - \alpha$, and equating those we arrive after some computation at

$$(b^2 - 3a^2)(a^2 - b^2 + a + 1) = (2a+1)(3ab^2 - a^3).$$

Working out both sides, and taking terms together, this is equivalent to

$$b^4 + a^4 + 2a^2b^2 + 2ab^2 + 2a^3 + 3a^2 - b^2 = 0,$$

which can be rewritten as:

$$(b^2 + a^2 + a)^2 + 2a^2 - b^2 = 0.$$

This curve in the plane is the magenta curve in Figure 10 together with the dark blue curves that are not on the negative real axis. Non-real eigenvalues of matrices of type $A(\alpha)$ for $0 \le \alpha \le 1$ are on the curve, they are shown in dark blue. By continuity of the eigenvalues, it is easy to see that this part of the curve is exactly traced out by the non-real eigenvalues of $A(\alpha)$ for $0 \le \alpha \le 1$. Note that this part of the curve is on the boundary of the region R, the remaining part of the boundary of R is on the boundary of K.

Now we use Proposition 1. For $0 < t < 1$ let

$$A(t) = tA(\alpha) + (1-t)I = \begin{bmatrix} t\alpha + (1-t) & t(1-\alpha) & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 0 & 1-t & t \\ t & 0 & 0 & 1-t \end{bmatrix}.$$

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Note that if we set $\hat{\alpha} = t\alpha + 1 - t$, then $1 - \hat{\alpha} = t(1 - \alpha)$. So $A(t)$ is of the form (22). Redefining $\alpha = \hat{\alpha}$ and $\beta = 1 - t$, we have from Proposition 1 that for any matrix of the form (22) with $\gamma = \delta = \beta$ the non-real eigenvalues are in the region R. Because of the permutation invariance of the list of eigenvalues with respect to the permutation induced by P, we have that the same holds whenever three of the four parameters are equal.

As another example, consider the matrix

$$B(\alpha) = \begin{bmatrix} \alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1-\alpha & 0 & 0 & \alpha \end{bmatrix},$$

with $0 < \alpha < 1$. The eigenvalues are the roots of

$$p_{B(\alpha)}(\lambda) = (\alpha - \lambda)^2\lambda^2 - (1-\alpha)^2$$
$$(\lambda - 1)(\lambda - (\alpha - 1))(\lambda^2 - \alpha\lambda - \alpha + 1).$$

So the eigenvalues are 1, $\alpha - 1$ and the two roots of $\lambda^2 - \alpha\lambda - \alpha + 1$. The latter are non-real for $0 < \alpha < 2\sqrt{2} - 2$. Denoting them by $\lambda = a \pm bi$, we get two equations expressing a and b in terms of α:

$$a^2 - b^2 - \alpha a - \alpha + 1 = 0$$
$$2ab - \alpha b = 0.$$

Since we are interested in non-real eigenvalues, we assume $b \neq 0$, then $\alpha = 2a$. Inserting that in the first of the two equations, we obtain that (a, b) must satisfy $a^2 + b^2 + 2a - 1 = 0$, that is, $(a + 1)^2 + b^2 = 2$. So the eigenvalues are on a part of the circumference centered at $(-1, 0)$ and radius $\sqrt{2}$ that is in K. Note that this part of the circumference certainly lies in R.

Again we use Proposition 1 and consider $A = tB(\alpha) + (1 - t)I$, which is a matrix of the form (22). Replacing again $t\alpha + 1 - t$ by α and t by β, we see from Proposition 1, using also the permutation invariance under P, that any matrix of the form (22) which has two of $\alpha, \beta, \gamma, \delta$ equal and the other two also equal must have its non-real eigenvalues (when it has non-real eigenvalues) inside R.

All this does not prove the conjecture, but it is a strong supporter.

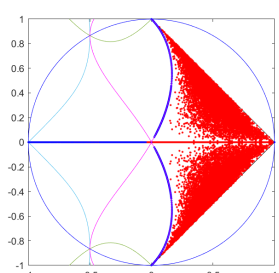

FIGURE 10. *The regions R together with the eigenvalues of 10^4 matrices of the form (22). The blue lines trace the eigenvalues of the matrices (23).*

As a third pattern in the four-dimensional case, consider a nonnegative row-stochastic irreducible matrix of the form:

$$A = \begin{bmatrix} \alpha & 1-\alpha & 0 & 0 \\ \beta & \gamma & 1-\beta-\gamma & 0 \\ 0 & \delta & \phi & 1-\delta-\phi \\ 0 & 0 & 1-\kappa & \kappa \end{bmatrix}.$$

Matrices of this form always have four real eigenvalues, as it is well known that a tri-diagonal matrix is always similar to a symmetric tri-diagonal matrix, and the similarity can be taken to be diagonal. We conjecture that the second eigenvalue is always positive.

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