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# MINIMIZING THE LEAST EIGENVALUE OF UNBALANCED SIGNED UNICYCLIC GRAPHS WITH GIVEN GIRTH OR PENDANT VERTICES*

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**Abstract.** A signed graph $\Gamma = (G, \sigma)$ consists of an underlying graph $G = (V, E)$ with a sign function $\sigma : E \to \{1, -1\}$. Let $A(\Gamma)$ be the adjacency matrix of $\Gamma$. Let $\lambda_1(A(\Gamma)) \geq \lambda_2(A(\Gamma)) \geq \cdots \geq \lambda_n(A(\Gamma))$ be the spectrum of the signed graph $\Gamma$, where $\lambda_n(A(\Gamma))$ is the least eigenvalue of $\Gamma$. Let $\mathcal{U}^-_{n,g,k}$ denote the set of all the unbalanced signed unicyclic graphs with order $n$, girth $g$ and $k$ pendant vertices, let $\mathcal{U}^-_n(k)$ denote the set of all the unbalanced signed unicyclic graphs with $n$ vertices and $k$ pendant vertices, and let $\mathcal{U}^-_{n,g}$ denote the set of all the unbalanced signed unicyclic graphs with order $n$ and girth $g$. Obviously, $\mathcal{U}^-_n(k) = \bigcup_{g=3}^{n-k} \mathcal{U}^-_{n,g,k}$ and $\mathcal{U}^-_{n,g} = \bigcup_{k=0}^{n-g} \mathcal{U}^-_{n,g,k}$. In this paper, we determine the signed unicyclic graphs whose least eigenvalues are minimal among all the graphs in $\mathcal{U}^-_{n,g,k}$, $\mathcal{U}^-_n(k)$ and $\mathcal{U}^-_{n,g}$, respectively.

**Key words.** Unbalanced signed unicyclic graph, Adjacency matrix, Least eigenvalue.

**AMS subject classifications.** 05C50, 05C35.

## 1. Introduction.

All the graphs we consider are simple and connected. For a simple graph $G$ with vertex set $\{v_1, v_2, \ldots, v_n\}$, the generic entry $a_{ij}$ of the adjacency matrix $A(G)$ is $a_{ij} = 1$ if $v_i \sim v_j$ and 0 otherwise. Denote by $\Phi(G, \lambda) = det(\lambda I - A(G))$ the characteristic polynomial of $G$. The spectrum of $G$ is the spectrum $Spec(A(G))$ of $A(G)$. The largest eigenvalue of $A(G)$ is called the spectral radius of $G$. A signed graph $\Gamma = (G, \sigma)$ consists of an underlying graph $G = (V, E)$ with a sign function $\sigma : E \to \{1, -1\}$. The (unsigned) graph $G$ is often called the underlying graph of $\Gamma$, while the function $\sigma$ is called the signature of $\Gamma$. In terms of signed graphs, the adjacency matrix of $\Gamma$ is defined as $A(\Gamma) = (a^\sigma_{i,j})$, where $a^\sigma_{i,j} = \sigma(v_i v_j)$ if $v_i \sim v_j$ and 0 otherwise. The characteristic polynomial $\Phi(\Gamma, \lambda) = det\,(\lambda I - A(\Gamma))$ of $A(\Gamma)$ is called the characteristic polynomial of $\Gamma$. As usual, we use $\lambda_1(A(\Gamma)) \geq \lambda_2(A(\Gamma)) \geq \cdots \geq \lambda_n(A(\Gamma))$ to denote the spectrum of $A(\Gamma)$. The spectrum of $A(\Gamma)$ is referred to as the spectrum of $\Gamma$ and denoted by $Spec(A(\Gamma))$. The largest eigenvalue $\lambda_1(A(\Gamma))$ is often called the index.

The notion of balance, introduced by Harary [11], plays a central role in matroid theory of signed graphs. A signed cycle is called negative (resp. positive) if it contains an odd (resp. even) number of negative edges. A signed graph is balanced if none of its cycles is negative, otherwise it is unbalanced. A unicyclic graph is a connected graph containing exactly one cycle. Many familiar notions related to unsigned graphs are directly extended to the signed graphs. For example, the degree of a vertex $v$ in $\Gamma = (G, \sigma)$ is its degree in $G$ and it is denoted by $d_\Gamma(v)$ or $d(v)$ which is the number of edges incident with $v$. A vertex of degree one is said to be a pendant vertex, and an edge of a graph is said to be pendant if one of its vertices is a pendant vertex. The girth of a signed graph $\Gamma$, denoted by $g$, is the length of a smallest cycle in its underlying graph $G$.

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Let $\Gamma = (G, \sigma)$ be a signed graph and $v \in V(\Gamma)$. A switching at vertex $v$ is the changing of the signs of all edges associated with $v$, and we call $v$ a switching vertex. A switching of a signed graph $\Gamma$ is a signed graph that can be obtained by the finite times of switching operations. We say that two signed graphs $\Gamma_1$ and $\Gamma_2$ are switching equivalent, and we write $\Gamma_1 \sim \Gamma_2$ if $\Gamma_2$ is a switching of $\Gamma_1$. Furthermore, two signed graphs $\Gamma_1$ and $\Gamma_2$ are said to be switching isomorphic if $\Gamma_1$ is isomorphic to a switching of $\Gamma_2$. Note that the adjacency matrices of switching equivalent signed graphs are similar, and hence, they have the same spectrum. It can be easily seen that the switching does not affect the signs of the cycles. Unsigned graphs are treated as (balanced) signed graphs where all edges get positive signs, that is, the all-positive signature. Obviously, a balanced signed graph is switching equivalent to its underlying graph.

In the past few decades, many scholars have paid more attention to the research of index of the signed graphs. Ghorbani and Majidi [9] characterized the signed graph with the maximum index of signed complete graphs with $n$ vertices and $t \leq \frac{n^2}{2}$ negative edges. Let $\mathcal{U}^-_{n,g,k}$ be the set of all the unbalanced signed unicyclic graphs with order $n$, girth $g$ and $k$ pendant vertices. Let $\mathcal{U}^-_{n,g}$ (resp. $\mathcal{U}_{n,g}$) denote the set of all the unbalanced (resp. balanced) signed unicyclic graphs with order $n$ and girth $g$. Denote by $\mathcal{U}^-_n(k)$ (resp. $\mathcal{U}_n(k)$) the set of all the unbalanced (resp. balanced) signed unicyclic graphs with $n$ vertices and $k$ pendant vertices. Up to switching equivalence, any signed graph in $\mathcal{U}^-_{n,g,k}$ (resp. $\mathcal{U}^-_{n,g}$ and $\mathcal{U}^-_n(k)$) can be obtained from the signed unicyclic graph containing exactly one negative edge in the unique cycle, and all the other edges being positively signed. Paths $P_{l_1}, P_{l_2}, \ldots, P_{l_k}$ are said to have almost equal lengths if $l_1, l_2, \ldots, l_k$ satisfy $|l_i - l_j| \leq 1$ for $1 \leq i, j \leq k$. Denote by $S^-_{n,g,k}$ (resp. $S_{n,g,k}$) the signed graph obtained from the negative cycle with girth $g$ (resp. $C_g$) by attaching $k$ paths of almost equal lengths at one vertex. Our motivation is from Guo [10] who characterized the graph $S_{n,3,k}$ with the maximal spectral radius among all the graphs in $\mathcal{U}_n(k)$. In this paper, we first focus on the signed unicyclic graph whose least eigenvalue is minimal among all the signed graphs in $\mathcal{U}^-_{n,g,k}$. The main result is posed as follows.

Theorem 1.1. *Let $U^- \in \mathcal{U}^-_{n,g,k}$ be the signed unicyclic graph with minimum least eigenvalue, where $1 \leq k \leq n - g$. Then, $U^-$ is switching isomorphic to $S^-_{n,g,k}$.*

Let $-\Gamma$ be obtained from $\Gamma$ by changing the sign of each edge. Note that $U^-$ is switching isomorphic to $-U^-$ when $g$ is even and $U^-$ is switching isomorphic to $-U$ when $g$ is odd, where $U^- = (U, \sigma) \in \mathcal{U}^-_{n,g,k}$. Then, the following corollary holds immediately.

Corollary 1.2. *Let $U^-$ (resp. $U$) be the signed (resp. unsigned) unicyclic graph with maximal index.*

*(i) If $g$ is odd, then $U$ is isomorphic to $S_{n,g,k}$.*
*(ii) If $g$ is even, then $U^-$ is switching isomorphic to $S^-_{n,g,k}$.*

Next, we characterize the signed unicyclic graph with minimum least eigenvalue in $\mathcal{U}^-_n(k)$. Note that $\mathcal{U}^-_n(k) = \bigcup_{g=3}^{n-k} \mathcal{U}^-_{n,g,k}$. Let $U^- \in \mathcal{U}^-_n(k)$ be the signed unicyclic graph with minimum least eigenvalue. By Theorem 1.1, we know that $U^-$ is switching isomorphic to one of the signed graphs in $\{S^-_{n,3,k}, S^-_{n,4,k}, \ldots, S^-_{n,n-k,k}\}$. We can derive the following result.

Theorem 1.3. *Let $U^- = (U, \sigma) \in \mathcal{U}^-_n(k)$ be the signed unicyclic graph with minimum least eigenvalue, where $1 \leq k \leq n - 3$. Then, $U^-$ is switching isomorphic to $S^-_{n,3,k}$.*

Akbari, Belardo, Heydari, Maghasedi and Souri [1] proved that among all the unbalanced signed unicyclic graphs with order $n$, the signed graph achieving the maximal index is the unbalanced triangle with all

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remaining vertices being pendant at the same vertex of the triangle. Belardo, Li Marzi and Simić [3] studied that among all the graphs in $\mathcal{U}_{n,g}$, the graph achieving the maximal index is the graph $S_{n,g,n-g}$. However, there are few researches on the least eigenvalues of signed graphs. Fan, Wang and Guo [8] determined the unique graph with minimum least eigenvalue among all the graphs in $\mathcal{U}_{n,g}$. Note that $\mathcal{U}^-_{n,g} = \bigcup_{k=0}^{n-g} \mathcal{U}^-_{n,g,k}$. Next we will consider the similar result about the least eigenvalue in $\mathcal{U}^-_{n,g}$. Let $U^- \in \mathcal{U}^-_{n,g}$ be the signed unicyclic graph with minimum least eigenvalue, then we can note that $U^-$ is switching isomorphic to one of the signed graphs in $\{S^-_{n,g,0}, S^-_{n,g,1}, \ldots, S^-_{n,g,n-g}\}$ by Theorem 1.1. In subsequent proof of the Theorem 1.4, we observe that $\lambda_n(A(S^-_{n,g,1})) > \cdots > \lambda_n(A(S^-_{n,g,n-g}))$. Hence, the following result holds.

Theorem 1.4. *Let $U^- = (U, \sigma) \in \mathcal{U}^-_{n,g}$ be the signed unicyclic graph with minimum least eigenvalue. Then $U^-$ is switching isomorphic to $S^-_{n,g,n-g}$.*

Recall that $-\Gamma$ is obtained from $\Gamma$ by reversing the sign of each edge. Then, we can get the maximal index of $U$ and $U^-$ by Theorem 1.4, which has been confirmed by Belardo, Li Marzi, Simić [3] and Souri, Heydari, Maghasedi [15].

Corollary 1.5. *Let $U^-$ (resp. $U$) be the signed (resp. unsigned) unicyclic graph with maximal index.*

*(i) If $g$ is odd, then $U$ is isomorphic to $S_{n,g,n-g}$.*
*(ii) If $g$ is even, then $U^-$ is switching isomorphic to $S^-_{n,g,n-g}$.*

**2. Proof of Theorem 1.1.** Assume that $V_1 \subset V(\Gamma)$ and $V_1 \neq \emptyset$. Let $\Gamma[V_1]$ be the signed induced subgraph of $\Gamma$, whose vertex set is $V_1$ and edge set is the set of those edges that have both ends in $V_1$. Note that sign functions of signed induced subgraphs are the restrictions of the former ones to the corresponding edge subsets. An important tool works in a similar way for signed graphs, which is a consequence of ([7], Theorem 1.3.11).

Lemma 2.1. *(Interlacing Theorem for signed graphs) Let $\Gamma$ be a signed graph of order $n$ and the eigenvalues be $\lambda_1(A(\Gamma)) \geq \lambda_2(A(\Gamma)) \geq \cdots \geq \lambda_n(A(\Gamma))$, and let $\Gamma'$ be an induced subgraph of $\Gamma$ with $m$ vertices. If the eigenvalues of $\Gamma'$ are $\mu_1(A(\Gamma')) \geq \mu_2(A(\Gamma')) \geq \cdots \geq \mu_m(A(\Gamma'))$, then $\lambda_{n-m+i}(A(\Gamma)) \leq \mu_i(A(\Gamma')) \leq \lambda_i(A(\Gamma))$ for $i = 1, 2, \ldots, m$.*

For convenience, the least eigenvalue $\lambda_n(A(\Gamma))$ is denoted by $\lambda(A(\Gamma))$. Let $V(\Gamma) = \{v_1, \ldots, v_n\}$ and $X = (x_1, x_2, \ldots, x_n)^T \in R^n$, where $x_i$ corresponds to the vertex $v_i$ $(1 \leq i \leq n)$, and $X$ is a unit vector corresponding to $\lambda(A(\Gamma))$. Then by the Rayleigh quotient Theorem,

$$\lambda(A(\Gamma)) = \min_{Y \in R^n, ||Y||=1} Y^T A(\Gamma) Y = X^T A(\Gamma) X,$$

and the eigenvalue equation for $v$ is as follows:

$$\lambda(A(\Gamma)) x_v = \sum_{u \sim v} \sigma(uv) x_u.$$

We first give the following results about $\lambda(A(\Gamma))$, which are the most often used tools in the identifications of graphs with minimum least eigenvalue.

Lemma 2.2. *Let $r$, $s$ and $t$ be distinct vertices of a signed graph $\Gamma$ and let $X = (x_1, x_2, \ldots, x_n)^T$ be a unit eigenvector corresponding to $\lambda(A(\Gamma))$. Let $\Gamma'$ be obtained from either by rotating the positive edge $rs$ to*

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*non-edge position $rt$ or rotating the negative edge $rt$ to the non-edge position $rs$. If*

$$\begin{cases} x_r \geq 0, & x_s \geq x_t \\ x_r \leq 0, & x_s \leq x_t \end{cases},$$

*then $\lambda\left(A\left(\Gamma'\right)\right) \leq \lambda\left(A\left(\Gamma\right)\right)$. If $x_r \neq 0$ or $x_s \neq x_t$, then $\lambda(A(\Gamma')) < \lambda(A(\Gamma))$.*

*Proof.* Note that

$$\begin{aligned} \lambda\left(A\left(\Gamma'\right)\right) - \lambda\left(A\left(\Gamma\right)\right) &\leq X^T\left(A(\Gamma') - A(\Gamma)\right)X \\ &= 2x_r\left(x_t - x_s\right). \end{aligned}$$

If $x_r \geq 0$ and $x_s \geq x_t$ or $x_r \leq 0$ and $x_s \leq x_t$, then $\lambda(A(\Gamma')) \leq \lambda(A(\Gamma))$. Note that $X$ is also an eigenvector of $A(\Gamma')$ corresponding to $\lambda(A(\Gamma'))$ when $\lambda(A(\Gamma')) = \lambda(A(\Gamma))$. If $x_r \neq 0$ (resp. $x_s \neq x_t$), then the eigenvalue equation cannot hold for $s$ (resp. $r$) in $\Gamma$ and $\Gamma'$, and we are done. The second relocation is considered in the same way. ∎

LEMMA 2.3. *[13] Let $r$, $s$ and $t$ be distinct vertices of a signed graph $\Gamma$, let $\Gamma'$ be obtained from $\Gamma$ by reversing the sign of the positive edge $rs$ and negative edge $rt$ and let $X = (x_1, x_2, \ldots, x_n)^T$ be a unit eigenvector corresponding to $\lambda(A(\Gamma))$. If $x_r(x_s - x_t) \geq 0$, then $\lambda(A(\Gamma)) \geq \lambda(A(\Gamma'))$. If $x_r \neq 0$ or $x_s \neq x_t$, then $\lambda(A(\Gamma)) > \lambda(A(\Gamma'))$.*

LEMMA 2.4. *[14] Let $f_1(x) = x, f_i(x) = x - \frac{1}{f_{i-1}(x)}, i \geq 2$. For $x \leq -2$, we have*

*(i) $f_i(x) < -1, i.e., |f_i(x)| > 1$.*
*(ii) $|f_i(x)| > |f_{i+1}(x)|$.*

LEMMA 2.5. *[14] Let $v_0$ be a vertex of a connected graph $G$ with at least two vertices. Let $G_l (l \geq 1)$ be the graph obtained from $G$ by attaching a new path $P = v_0v_1 \cdots v_l$ of length $l$ at $v_0$, where $v_1, \ldots, v_l$ are distinct new vertices. Let $X$ be a unit eigenvector of $\lambda\left(A\left(G_l\right)\right)$. If $\lambda\left(A\left(G_l\right)\right) \leq -2$, then we have*

*(i) $x_{v_i} = f_{l-i}(\lambda)x_{v_{i+1}} (0 \leq i \leq l-1)$, where $f_i(\lambda)$ is a function on $\lambda$ defined in Lemma 2.4 and $\lambda = \lambda\left(A\left(G_l\right)\right)$;*
*(ii) For any fixed $i (i = 0, 1, \ldots, l-1)$, we have $|x_{v_{i+1}}| \leq |x_{v_i}|$ and $x_{v_i}x_{v_{i+1}} \leq 0$, with equalities if and only if $x_{v_0} = 0$.*

The consequences of Lemma 2.5 can be naturally extended to signed graphs.

COROLLARY 2.6. *Let $v_0$ be a vertex of a signed graph $\Gamma$ with at least two vertices. Let $\Gamma_l (l \geq 1)$ (Fig. 1) be the signed graph obtained from $\Gamma$ by attaching a new path $P = v_0v_1 \cdots v_l$ of length $l$ at $v_0$, where $v_1, \ldots, v_l$ are distinct new vertices and all the edges are positive. Let $X$ be a unit eigenvector of $\lambda\left(A\left(\Gamma_l\right)\right)$. If $\lambda\left(A\left(\Gamma_l\right)\right) \leq -2$, then we have*

*(i) $x_{v_i} = f_{l-i}(\lambda)x_{v_{i+1}} (0 \leq i \leq l-1)$, where $f_i(\lambda)$ is a function on $\lambda$ defined in Lemma 2.4 and $\lambda = \lambda\left(A\left(\Gamma_l\right)\right)$;*
*(ii) For any fixed $i (i = 0, 1, \ldots, l-1)$, we have $|x_{v_{i+1}}| \leq |x_{v_i}|$ and $x_{v_i}x_{v_{i+1}} \leq 0$, with equalities if and only if $x_{v_0} = 0$.*

Denoted by $W_n (n \geq 6)$ be the graph obtained from a path $v_1v_2 \cdots v_{n-4}$ by attaching two pendant vertices to $v_1$ and another two to $v_{n-4}$ (Fig. 1). Let $N_\Gamma(v)$ or $N(v)$ denote the neighbor set of vertex $v$ in $\Gamma$. The distance between vertices $u$ and $v$ of a signed graph $\Gamma$ is denoted by $d_\Gamma(u, v)$ or $d(u, v)$. By [6], we have $Spec(A(W_n)) = Spec(A(C_4)) \cup Spec(A(P_{n-4}))$. Note that $\lambda(A(W_n)) = -2$.

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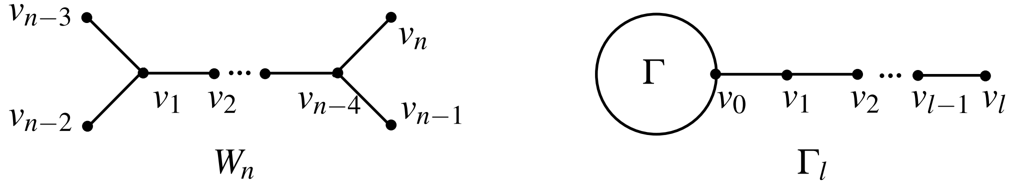


FIGURE 1. *The signed graphs $W_n$ and $\Gamma_l$.*

Now we begin to prove Theorem 1.1.

*Proof.* Suppose that $V(U^-) = \{v_1, \ldots, v_n\}$ and $(C_g, \sigma)$ is the unique negative cycle in $U^-$, where $C_g = v_1v_2v_3 \cdots v_gv_1$. Then, $U^-$ can be viewed as attaching some trees $T_i$ at the vertex $v_i(1 \leq i \leq g)$. Up to switching equivalence, let $v_1v_g$ be the unique negative edge of $U^-$. We apply induction on $n$. The result is clearly true for $k = 1$. Assume that $k \geq 2$. Let $X = (x_1, x_2, \ldots, x_n)^T$ be a unit eigenvector corresponding to $\lambda(A(U^-))$. Then the following claims can be obtained.

CLAIM 1. $x_u \neq 0$ *for any vertex* $u \in V(U^-) \setminus V((C_g, \sigma))$.

On the contrary, assume that there is a vertex $u_r \in V(T_p) \setminus \{v_p\}(1 \leq p \leq g)$ such that $x_{u_r} = 0$. Then, there must exist a unique path $P = v_pu_1 \cdots u_r$ between $v_p$ and $u_r$. If $r = 1$, then $x_{v_p} = 0$ and $x_{u_p} = 0$ for any vertex $u_p \in V(T_p)$ by Corollary 2.6. Let $T_q$ (if exists) be another tree of $U^-$, $z_l \in V(T_q)$ and $d(z_l) = 1$. Then, there exists a unique path $P = v_qz_1 \cdots z_l$ between $v_q$ and $z_l$. We assert that $x_{z_{l-1}} = 0$. Otherwise, we construct a new signed graph $U^*$ from $U^-$ by rotating the positive edge $u_rv_p$ to the non-edge position $u_rz_{l-1}$ such that $U^* \in \mathcal{U}^-_{n,g,k}$ and $\lambda(A(U^*)) < \lambda(A(U^-))$ by Lemma 2.2, a contradiction. Furthermore, we can see that $z_i = 0(i = 1, \ldots, l)$ by Corollary 2.6. By the same arguments, we can prove that $x_u = 0$ for any vertex $u \in V(U^-) \setminus V((C_g, \sigma))$. Now, if $r > 1$, then we assert that $x_{z_l} = 0$. Otherwise, a new signed graph $U^*$ can be obtained from $U^-$ by rotating the positive edge $u_ru_{r-1}$ to the non-edge position $u_rz_l$ such that $U^* \in \mathcal{U}^-_{n,g,k}$ and $\lambda(A(U^*)) < \lambda(A(U^-))$ by Lemma 2.2, a contradiction. We can obtain that $z_j = 0(j = 1, \ldots, l)$ by Corollary 2.6. Similarly, we can prove that $x_u = 0$ for any vertex $u \in V(U^-) \setminus V((C_g, \sigma))$. Recall that $x_{v_p} = 0$. Hence, $\lambda(A(U^-)) = \lambda(A((C_g, \sigma) - v_p)) = \lambda(A(P_{g-1})) = -2cos\frac{\pi}{g}$. However, since $(P_g, +)$ is an induced subgraph of $U^-$, $\lambda(A(U^-)) = -2cos\frac{\pi}{g} > -2cos\frac{\pi}{g+1} = \lambda(A(P_g)) \geq \lambda(A(U^-))$ by Lemma 2.1, a contradiction.

CLAIM 2. $(C_g, \sigma)$ *contains only one vertex with degree greater than* 2.

Otherwise, assume that $v_i$ and $v_j$ are two distinct vertices of the $(C_g, \sigma)$ such that $d(v_i) \geq 3$ and $d(v_j) \geq 3$. Let $\{v_iu_i, v_ju_j\} \in E(U^-)$, where $\{u_i, u_j\} \in V(U^-) \setminus V((C_g, \sigma))$. Note that $x_{u_i} \neq 0$ and $x_{u_j} \neq 0$ by Claim 1. We first assume that $x_{u_i}x_{u_j} > 0$. Without loss of generality, let $x_{u_i} < 0$ and $x_{u_j} < 0$. If $x_{v_i} \leq x_{v_j}$, then we can construct a new signed graph $U^*$ from $U^-$ by rotating the positive edge $u_iv_i$ to the non-edge position $u_iv_j$ such that $U^* \in \mathcal{U}^-_{n,g,k}$ and $\lambda(U^*) < \lambda(U^-)$ by Lemma 2.2, a contradiction. If $x_{v_j} < x_{v_i}$, then a new signed graph $U^*$ can be obtained from $U^-$ by deleting the positive edge $u_jv_j$ and adding the positive edge $u_jv_i$ such that $U^* \in \mathcal{U}^-_{n,g,k}$ and $\lambda(U^*) < \lambda(U^-)$ by Lemma 2.2, a contradiction. Now, we consider that $x_{u_i}x_{u_j} < 0$. Without loss of generality, let $x_{u_i} > 0$ and $x_{u_j} < 0$. Let $U_1^-$ be the unbalanced signed unicyclic graph obtained from $U^-$ by switching at the vertex $v_j$ and $X' = (x'_1, x'_2, \ldots, x'_n)^T$ be its eigenvector corresponding to $\lambda(U_1^-)$, where $x'_{v_j} = -x_{v_j}$ and $x'_s = x_s$ for any vertex $v_s \in V(U^-) \setminus \{v_j\}$. Thus, $x'_{u_i} = x_{u_i} > 0$ and $x'_{u_j} = x_{u_j} < 0$. If $x'_{v_i} \leq x'_{v_j}$, then a new signed graph $U^*$ can be obtained from $U_1^-$ by rotating the negative

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edge $u_jv_j$ to the non-edge position $u_jv_i$ such that $U^* \in \mathcal{U}^-_{n,g,k}$ and $\lambda(U^*) < \lambda(U_1^-) = \lambda(U^-)$ by Lemma 2.2, a contradiction. If $x'_{v_i} > x'_{v_j}$, then we construct a new signed graph $U^*$ from $U_1^-$ by rotating the positive edge $u_iv_i$ to the non-edge position $u_iv_j$ such that $U^* \in \mathcal{U}^-_{n,g,k}$ $\lambda(U^*) < \lambda\left(U_1^-\right) = \lambda\left(U^-\right)$ by Lemma 2.2, a contradiction.

Claim 3. *There exists an integer $i$ such that $x_{v_i} \neq 0$ for $1 \leq i \leq g$.*

On the contrary, we assume that $x_{v_1} = x_{v_2} = \cdots = x_{v_g} = 0$. Let $v_s$ and $v_t$ be two distinct vertices of $(C_g, \sigma)$, and assume that there exists a vertex $u_p (g < p \leq n)$ such that $v_su_p \in E(U^-)$. We assert that $x_{u_p} = 0$. Otherwise, we construct a new signed graph $U^*$ from $U^-$ by rotating the positive edge $u_pv_s$ to the non-edge position $u_pv_t$ such that $U^* \in \mathcal{U}^-_{n,g,k}$ and $\lambda(A(U^*)) < \lambda(A(U^-))$ by Lemma 2.2, a contradiction. If $d(u_p) = 2$, then $\lambda(A(U^-))x_{u_p} = x_{v_s} + x_{u_q}$ by the eigenvalue equation for $u_p$, i.e., $x_{u_q} = 0$, where $u_q \in N(u_p) \setminus \{v_s\}$. If $d(u_p) > 2$, then we assert that $x_u = 0$, where $u \in N(u_p) \setminus \{v_s\}$. Otherwise, we construct a new signed graph $U^*$ from $U^-$ by rotating the positive edge $uu_p$ to the non-edge positive $uv_s$ such that $U^* \in \mathcal{U}^-_{n,g,k}$ and $\lambda(A(U^*)) < \lambda(A(U^-))$ by Lemma 2.2, a contradiction. By the same arguments, $x_{v_1} = x_{v_2} = \cdots = x_{v_n} = 0$, which means that $X = 0$, a contradiction.

Claim 4. *$x_{v_1} \neq 0$ or $x_{v_g} \neq 0$.*

On the contrary, let $x_{v_1} = x_{v_g} = 0$. We can divide into the two cases. Firstly, assume that $g > 3$. We assert that $x_{v_2} = 0$. Otherwise, we can construct a new signed graph $U^*$ from $U^-$ by reversing the sign of the positive edge $v_1v_2$ and the negative edge $v_1v_g$ such that $U^* \in \mathcal{U}^-_{n,g,k}$ and $\lambda(A(U^*)) < \lambda(A(U^-))$ by Lemma 2.3. However, $U^*$ is switching isomorphic to $U^-$ and $\lambda(A(U^*)) = \lambda(A(U^-))$, a contradiction. If $N(v_2) \setminus \{v_1, v_3\} \neq \emptyset$, then we assert that $x_z = 0$ for any vertex $z \in N(v_2) \setminus \{v_1, v_3\}$. Otherwise, there exists a vertex $z \in N(v_2) \setminus \{v_1, v_3\}$ such that $x_z \neq 0$, we can construct a new signed graph $U^*$ from $U^-$ by rotating the positive edge $zv_2$ to the non-edge position $zv_1$ such that $U^* \in \mathcal{U}^-_{n,g,k}$ and $\lambda(A(U^*)) < \lambda(A(U^-))$ by Lemma 2.2, a contradiction. Note that $\lambda(A(U^-))x_{v_2} = x_{v_3}$ by the eigenvalue equation for $v_2$, i.e., $x_{v_3} = 0$. Similarly, we obtain that $x_{v_1} = x_{v_2} = \cdots = x_{v_g} = 0$, which contradicts with Claim 3. Now, we consider the case that $g = 3$. If $N(v_1) \setminus \{v_3\} = N(v_3) \setminus \{v_1\} = \{v_2\}$, then $\lambda(A(U^-))x_{v_1} = x_{v_2}$ by the eigenvalue equation for $v_1$, i.e., $x_{v_2} = 0$. This contradicts with Claim 3. Assume that $N(v_1) \setminus \{v_2, v_3\} \neq \emptyset$ or $N(v_3) \setminus \{v_1, v_2\} \neq \emptyset$. Without loss of generality, we just consider that $N(v_1) \setminus \{v_2, v_3\} \neq \emptyset$. Let $v_p \in N(v_1) \setminus \{v_2, v_3\}$. Then we assert that $x_{v_p} = 0$. Otherwise, we can construct a new signed graph $U^*$ from $U^-$ by rotating the positive edge $v_pv_1$ to the non-edge position $v_pv_3$ such that $U^* \in \mathcal{U}^-_{n,g,k}$ and $\lambda(A(U^*)) < \lambda(A(U^-))$ by Lemma 2.2, a contradiction. Note that $\lambda(A(U^-))x_{v_1} = x_{v_2}$ by the eigenvalue equation for $v_1$, i.e., $x_{v_2} = 0$, contradicting Claim 3. Hence, $x_{v_1} \neq 0$ or $x_{v_g} \neq 0$.

Note that $-X$ is also an eigenvector corresponding to $\lambda(A(U^-))$ if $X$ is an eigenvector corresponding to $\lambda(A(U^-))$. Without loss of generality, we always assume that $x_{v_1} > 0$.

Claim 5. $\max\limits_{v_i \in V((C_g,\sigma))} d(v_i) = k + 2$.

Otherwise, there must be a vertex $w \in V(U^-) \setminus V((C_g, \sigma))$ such that $d(w) > 2$. Up to switching equivalence, let $d(v_1) > 2$. Then, there exists a unique path $P$ between $v_1$ and $w$. Let $N(w) = \{w_1, w_2, \ldots, w_t\}$ $(t \geq 3)$ and $w_1 \in V(P)$. Since $(C_3, \sigma)$ or $(W_m, \sigma)(m \geq 6)$ with the negative edge $v_1v_g$ is an induced subgraph of $U^-$, $\lambda(A(U^-)) \leq -2$ by Lemma 2.1. It follows that $x_{w_i}x_w < 0$ for any $i \in \{2, \ldots, t\}$ by Corollary 2.6. If $d(v_1, w)$ is even, then $x_wx_{v_1} > 0 > x_{w_i}x_{v_1}$ and $|x_w| < |x_{v_1}|$ by Corollary 2.6. Combining this with $x_{v_1} > 0$, thus, $x_{v_1} > x_w > 0$ and $x_{w_i} < 0$ for any $i \in \{3, \ldots, t\}$. Then, a new signed graph $U^*$ can be obtained from $U^-$ by rotating the positive edge $w_iw$ to the non-edge position $w_iv_1$ such that $U^* \in \mathcal{U}^-_{n,g,k}$ and $\lambda(A(U^*)) < \lambda(A(U^-))$

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by Lemma 2.2, a contradiction. If $d(v_1,w)$ is odd, then $x_w x_{v_1} < 0 < x_{w_i} x_{v_1}$ and $|x_w| < |x_{v_1}|$ by Corollary 2.6. Hence, $x_{v_1} > 0 > x_w$ and $x_{w_i} > 0$ for any $i \in \{3,\dots,t\}$. Let $U_2^-$ be the unbalanced signed unicyclic graph obtained from $U^-$ by switching at the vertex $w$ and $X' = (x'_1, x'_2, \dots, x'_n)^T$ be its eigenvector corresponding to $\lambda\left(A\left(U_2^-\right)\right)$, where $x'_w = -x_w$ and $x'_{v_s} = x_{v_s}$ for any vertex $v_s \in V(U^-) \setminus \{w\}$. So $x'_{w_i} = x_{w_i} > 0$ and $x'_{v_1} > x'_w > 0$. Then, we can construct a new signed graph $U^*$ from $U_2^-$ by rotating the negative edge $w_i w$ to the non-edge position $w_i v_1$ such that $U^* \in \mathcal{U}^-_{n,g,k}$ and $\lambda\left(A\left(U^*\right)\right) < \lambda\left(A\left(U_2^-\right)\right) = \lambda\left(A\left(U^-\right)\right)$ by Lemma 2.2, a contradiction.

Hence, by Claim 5, $U^-$ is an unbalanced signed unicyclic graph which is obtained from $(C_g,\sigma)$ by attaching $k$ paths to $v_1$. Denote by $P_{l_1}, P_{l_2}, \dots, P_{l_k}$ the $k$ paths.

Claim 6. $|l_i - l_j| \le 1$ *for* $1 \le i, j \le k$.

Otherwise, there exist two paths, say $P_{l_1} = v_1 u_1 \cdots u_{l_1}$ and $P_{l_2} = v_1 z_1 \cdots z_{l_2}$, such that $l_1 = l_2 + t$ $(t \ge 2)$. Since $(C_3,\sigma)$ or $(K_{1,4},\sigma)$ with the negative edge $v_1 v_g$ is an induced subgraph of $U^-$, $\lambda(A(U^-)) \le -2$ by Lemma 2.1. Firstly, let $l_2$ be even and $t$ be odd. Then, $l_1$ is odd, and hence $x_{z_{l_2}} x_{v_1} > 0$, $x_{u_{l_1}} x_{v_1} < 0$ and $x_{u_{l_1-1}} x_{v_1} > 0$ by Corollary 2.6. Combining this with $x_{v_1} > 0$, it follows that $x_{u_{l_1}} < 0 < x_{z_{l_2}}, x_{u_{l_1-1}}$. We assert that $|x_{u_{l_1-1}}| > |x_{z_{l_2}}|$. Otherwise, we can construct a new signed graph $U^*$ from $U^-$ by rotating the positive edge $u_{l_1} u_{l_1-1}$ to the non-edge position $u_{l_1} z_{l_2}$ such that $U^* \in \mathcal{U}^-_{n,g,k}$ and $\lambda(A(U^*)) < \lambda(A(U^-))$ by Lemma 2.2, a contradiction. Note that $x_{u_{l_1-2}} x_{v_1} < 0$ and $x_{z_{l_2-1}} x_{v_1} < 0$ by Corollary 2.6. Recall that $x_{v_1} > 0$, then $x_{u_{l_1-2}} < 0$ and $x_{z_{l_2-1}} < 0$. If $|x_{u_{l_1-2}}| \le |x_{z_{l_2-1}}|$, then $x_{z_{l_2-1}} \le x_{u_{l_1-2}} < 0$. Let $U^* = U^- - u_{l_1-1}u_{l_1-2} - z_{l_2}z_{l_2-1} + u_{l_1-2}z_{l_2} + z_{l_2-1}u_{l_1-1}$, then $U^* \in \mathcal{U}^-_{n,g,k}$, clearly. By Rayleigh quotient,

$$\begin{aligned}
\lambda(A(U^*)) - \lambda(A(U^-)) &\le X^T \left(A(U^*) - A(U^-)\right) X \\
&= 2\left(x_{u_{l_1-1}} - x_{z_{l_2}}\right)\left(x_{z_{l_2-1}} - x_{u_{l_1-2}}\right) \\
&\le 0.
\end{aligned}$$

If $\lambda\left(A\left(U^*\right)\right) = \lambda\left(A\left(U^-\right)\right)$, then $X$ is also an eigenvector of $A(U^*)$ corresponding to $\lambda(A(U^*))$. By the eigenvalue equation for vertex $u_{l_1-2}$,

$$\begin{cases}
\lambda(A(U^-)) x_{u_{l_1-2}} = x_{u_{l_1-1}} + x_{u_{l_1-3}}, \\
\lambda(A(U^*)) x_{u_{l_1-2}} = x_{z_{l_2}} + x_{u_{l_1-3}}.
\end{cases}$$

By the above equations, $x_{u_{l_1-1}} = x_{z_{l_2}}$, which contradicts with $|x_{u_{l_1-1}}| > |x_{z_{l_2}}|$. Thus, $\lambda(A(U^*)) < \lambda(A(U^-))$, a contradiction. We can obtain that $|x_{u_{l_1-2}}| > |x_{z_{l_2-1}}|$. Similarly, we can assert that (if exists) $|x_{u_{l_1-3}}| > |x_{l_2-2}|$. For convenience, let $v_1 = z_0$. Therefore, using repeatedly the same arguments, we have $|x_{u_{t-1}}| > |x_{z_0}| = |x_{v_1}|$. However, $|x_{v_1}| > |x_{u_i}|$ for $1 \le i \le l_2$ by Corollary 2.6, a contradiction.

Now, let $l_2$ be even and $t$ be even. Then, $l_1$ is even, and hence, $x_{u_{l_1}} x_{v_1} > 0$ and $x_{u_{l_1-1}} x_{v_1} < 0$ by Corollary 2.6. Combining this with $x_{v_1} > 0$, then $x_{u_{l_1}}, x_{z_{l_2}} > 0 > x_{u_{l_1-1}}$. Let $U_3^-$ be the unbalanced signed unicyclic graph obtained from $U^-$ by switching at the vertex $u_{l_1-1}$ and $X' = (x'_1, x'_2, \dots, x'_n)^T$ be its eigenvector corresponding to $\lambda\left(A\left(U_3^-\right)\right)$, where $x'_{u_{l_1-1}} = -x_{u_{l_1-1}}$ and $x'_{v_s} = x_{v_s}$ for any vertex $v_s \in V(U^-) \setminus \{u_{l_1-1}\}$. We assert that $|x'_{u_{l_1-1}}| > |x'_{z_{l_2}}|$. Otherwise, a new signed graph $U^*$ can be obtained from $U_3^-$ by rotating the negative edge $u_{l_1} u_{l_1-1}$ to the non-edge position $u_{l_1} z_{l_2}$ such that $U^* \in \mathcal{U}^-_{n,g,k}$ and $\lambda\left(A\left(U^*\right)\right) < \lambda\left(A\left(U_3^-\right)\right) = \lambda\left(A\left(U^-\right)\right)$ by Lemma 2.2, a contradiction. Note that $x'_{u_{l_1-2}} x'_{v_1} > 0 > x'_{z_{l_2-1}} x'_{v_1}$ by Corollary 2.6, and hence $x'_{u_{l_1-2}} > 0 > x'_{z_{l_2-1}}$. We assume that $|x'_{u_{l_1-2}}| \le |x'_{z_{l_2-1}}|$. Let $U^* = U_3^- - u_{l_1-1}u_{l_1-2} - z_{l_2}z_{l_2-1} + u_{l_1-2}z_{l_2} + z_{l_2-1}u_{l_1-1}$ $(u_{l_1-1}u_{l_1-2}, u_{l_1-2}z_{l_2}$ are negative edges). By

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Rayleigh quotient, If $\lambda\left(A\left(U^{*}\right)\right)=\lambda\left(A\left(U_{3}^{-}\right)\right)$, then $X$ is also an eigenvector of $A(U^{*})$ corresponding to $\lambda(A(U^{*}))$. Since $x'_{u_{l_1-1}} \neq x'_{z_{l_2}}$, the eigenvalue equation cannot hold for $u_{l_1-2}$ in $U^{*}$ and $U_3^{-}$. Thus, $\lambda\left(A\left(U^{*}\right)\right)<\lambda\left(A\left(U_{3}^{-}\right)\right)=\lambda\left(A\left(U^{-}\right)\right)$, a contradiction. We can derive that $|x'_{u_{l_1-2}}|>|x'_{z_{l_2-1}}|$. Similarly, $|x_{u_{t-1}}|>|x_{z_0}|=|x_{v_1}|$, a contradiction.

Similarly, we can get a contradiction if $l_2$ is odd. This completes the proof. ∎

**3. Proof of Theorem 1.3.** Subdividing an edge $uv$ of a graph means replacing edge $uv$ by two edges $uw$ and $wv$, with $w$ being a new vertex. Let $G_{uv}$ denote a new graph can obtained from $G$ by subdividing the edge $uv$, where $G$ is a connected graph and $uv \in E(G)$. A walk $w_1w_2\cdots w_s(s\geq 2)$ in a graph $G$ is called an internal path, if these $k$ vertices are distinct (except possibly $w_1=w_s$), $d_G(w_1)>2$, $d_G(w_s)>2$ and $d_G(w_2)=\cdots=d_G(w_{s-1})=2$ (unless $s=2$).

Lemma 3.1. *[12] Let $G$ be a connected graph with $uv\in E(G)$. If $uv$ belongs to an internal path of $G$ and $G$ is not isomorphic to $W_n$, then $\lambda_1(G_{uv})<\lambda_1(G)$.*

Denote by $G_1\cup G_2$ the disjoint union of two graphs $G_1$ and $G_2$. Let $k$ and $n_1,\ldots,n_k$ be some positive integers. Let $S(n_1,\ldots,n_k)$ be the tree $T$ with a unique vertex $v$ of degree greater than 2, such that $T\setminus v\cong P_{n_1}\cup\cdots\cup P_{n_k}$. The tree $S(n_1,\ldots,n_k)$ $(k\geq 3)$ is often called starlike tree.

The proof of Theorem 1.3 as follows.

*Proof.* Suppose that $V(U^{-})=\{v_1,\ldots,v_n\}$ and $(C_g,\sigma)$ is the unique negative cycle in $U^{-}$, where $C_g=v_1v_2v_3\cdots v_gv_1$. Up to switching equivalence, let $v_1v_g$ be the unique negative edge of $U^{-}$. Note that $\mathcal{U}_n^{-}(k)=\bigcup_{g=3}^{n-k}\mathcal{U}_{n,g,k}^{-}$, then we can obtain that $U^{-}$ is switching isomorphic to one of the signed graphs in $\{S_{n,3,k}^{-},S_{n,4,k}^{-},\ldots,S_{n,n-k,k}^{-}\}$ by Theorem 1.1. Let $P_{l_1},P_{l_2},\ldots,P_{l_k}$ be the $k$ paths, $i=1,\ldots,k$. We can assume that $l_1\leq l_2\leq\cdots\leq l_k$, where $P_{l_1}=v_1z_1\cdots z_{l_1}$ and $P_{l_2}=v_1w_1\cdots w_{l_2}$. The result is trivial for $g=3$. Then assume that $g>3$. Let $X=(x_1,x_2,\ldots,x_n)^T$ be a unit eigenvector corresponding to $\lambda(A(U^{-}))$. It is known that $x_{v_1}>0$ by the proof of Theorem 1.1. Then, we divide the proof into the following two cases.

Case 1. *$g$ is odd.*

Since $g\geq 5$, there must exist $v_{i-1}v_i,v_iv_j,v_jv_{j+1}\in E((C_g,\sigma))$ for $3\leq i\leq g-2$. Let $U^{*}=U^{-}-v_{i-1}v_i-v_iv_j-v_jv_{j+1}+v_{i-1}v_{j+1}+v_iz_{l_1}+v_jw_{l_2}$, then $U^{*}=S_{n,g-2,k}^{-}$, clearly. Assume that $(U_1,\sigma)=U_1^{-}=U^{-}-v_{i-1}v_i-v_iv_j-v_jv_{j+1}+v_{i-1}v_{j+1}$. It is easy to notice that $U$ is the graph from $U_1$ by subdividing the edge $v_{i-1}v_{j+1}$. By Lemma 3.1, $\lambda_1(A(U))<\lambda_1(A(U_1))$, thus,

$$\lambda(A(U^{-}))=-\lambda_1(A(U))>-\lambda_1(A(U_1))=\lambda\left(A\left(U_1^{-}\right)\right).$$

Note that $U_1^{-}$ is an induced subgraph of $U^{*}$, then $\lambda\left(A\left(S_{n,g-2,k}^{-}\right)\right)=\lambda(A(U^{*}))\leq\lambda\left(A\left(U_1^{-}\right)\right)$ by Lemma 2.1. Hence, $\lambda\left(A\left(S_{n,g-2,k}^{-}\right)\right)<\lambda(A(U^{-}))$. Therefore, from a repeated use of the same arguments, we can obtain $U^{-}\cong S_{n,3,k}^{-}$.

Case 2. *$g$ is even.*

By the eigenvalue equations for vertices $v_1,v_2,\ldots,v_g$, we can obtain

$$\lambda(A(U^{-}))x_{v_g}=-x_{v_1}+x_{v_{g-1}}, \tag{3.1}$$

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and

$$\lambda(A(U^-))x_{v_i} = x_{v_{i-1}} + x_{v_{i+1}} \text{ for } i = 2, 3, \dots, g-1. \tag{3.2}$$

Let $f_i(\lambda(A(U^-)))$ $(i = 2, \dots, \frac{g}{2} - 1)$ be a function on $\lambda(A(U^-))$ defined in Lemma 2.4. By (3.2) for $i = 2$, we notice that

$$\lambda(A(U^-))x_{v_2} = x_{v_1} + x_{v_3}. \tag{3.3}$$

Add equations (3.1) and (3.3), then

$$\lambda(A(U^-))(x_{v_2} + x_{v_g}) = x_{v_3} + x_{v_{g-1}}. \tag{3.4}$$

By (3.2) for $i = 3$ and $i = g - 1$, we have

$$\lambda(A(U^-))x_{v_3} = x_{v_2} + x_{v_4}, \tag{3.5}$$

and

$$\lambda(A(U^-))x_{v_{g-1}} = x_{v_{g-2}} + x_{v_g}. \tag{3.6}$$

Thus, add equations (3.5) and (3.6),

$$\lambda(A(U^-))(x_{v_3} + x_{v_{g-1}}) = x_{v_2} + x_{v_4} + x_{v_{g-2}} + x_{v_g}. \tag{3.7}$$

According to (3.4) and (3.7),

$$f_2(\lambda(A(U^-)))(x_{v_3} + x_{v_{g-1}}) = x_{v_4} + x_{v_{g-2}}. \tag{3.8}$$

Similarly,

$$\lambda(A(U^-))(x_{v_4} + x_{v_{g-2}}) = x_{v_3} + x_{v_5} + x_{v_{g-3}} + x_{v_{g-1}}. \tag{3.9}$$

Then by (3.8) and (3.9),

$$f_3(\lambda(A(U^-)))(x_{v_4} + x_{v_{g-2}}) = x_{v_5} + x_{v_{g-3}}. \tag{3.10}$$

By parity of reasoning, we can obtain

$$f_{\frac{g}{2}-1}(\lambda(A(U^-)))(x_{v_{\frac{g}{2}}} + x_{v_{\frac{g}{2}+2}}) = x_{v_{\frac{g}{2}+1}} + x_{v_{\frac{g}{2}+1}}. \tag{3.11}$$

Hence, $\lambda(A(U^-))f_{\frac{g}{2}-1}(\lambda(A(U^-)))x_{v_{\frac{g}{2}+1}} = 2x_{v_{\frac{g}{2}+1}}$. So $\lambda(A(U^-))f_{\frac{g}{2}-1}(\lambda(A(U^-))) = 2$ or $x_{v_{\frac{g}{2}+1}} = 0$. We claim that $\lambda(A(U^-))f_{\frac{g}{2}-1}(\lambda(A(U^-))) \neq 2$. Firstly, assume that $2 \leq k \leq n-3$. Since $(K_{1,4}, \sigma)$ with the negative edge $v_1v_g$ is an induced subgraph of $U^-$, $\lambda(A(U^-)) \leq -2$ by Lemma 2.1. Note that $f_{\frac{g}{2}-1}(\lambda(A(U^-))) < -1$ by Lemma 2.4, and then $\lambda(A(U^-))f_{\frac{g}{2}-1}(\lambda(A(U^-))) > 2$. Now we need to consider the case $k = 1$. If $g = 4$, then $\lambda(A(U^-))f_1(A(U^-)) = \lambda^2(A(U^-))$. Since $(K_{1,3}, \sigma)$ with the negative edge $v_1v_4$ is an induced subgraph of $U^-$, $\lambda(A(U^-)) \leq -\sqrt{3}$ by Lemma 2.1. Then, $\lambda(A(U^-))f_1(A(U^-)) = \lambda^2(A(U^-)) \geq 3$. If $g = 6$, then $\lambda(A(U^-))f_2(A(U^-)) = \lambda^2(A(U^-)) - 1$. Since $(S^-_{7,6,1}, \sigma)$ with the negative edge $v_1v_6$ is an induced subgraph of $U^-$, $\lambda(A(U^-)) < -1.9$ by Lemma 2.1. Then, $\lambda(A(U^-))f_2(A(U^-)) = \lambda^2(A(U^-)) - 1 > 2$. Next, assume that $g \geq 8$. Since $(S(1,3,3), \sigma)$ with a negative edge is an induced subgraph of $U^-$, $\lambda(A(U^-)) \leq -2$ by Lemma 2.1. Note that $f_{\frac{g}{2}-1}(A(U^-)) < -1$ by Lemma 2.4 and then $\lambda(A(U^-))f_{\frac{g}{2}-1}(\lambda(A(U^-))) > 2$. Hence, $x_{v_{\frac{g}{2}+1}} =$

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0, and then $x_{v_{\frac{g}{2}}}+x_{v_{\frac{g}{2}+2}}=0$. If $x_{v_{\frac{g}{2}}}=x_{v_{\frac{g}{2}+2}}=0$, then $x_{v_1}=\cdots=x_{v_g}=0$ by the above equation, and hence $X=0$, a contradiction. Therefore, $x_{v_{\frac{g}{2}+2}}=-x_{v_{\frac{g}{2}}}$. Let $U^*=U^- - v_{\frac{g}{2}}v_{\frac{g}{2}+1} - v_{\frac{g}{2}+1}v_{\frac{g}{2}+2} + v_{\frac{g}{2}}v_{\frac{g}{2}+2} + v_{\frac{g}{2}+1}z_{l_1}$, then $U^*=S^-_{n,g-1,k}$, clearly. By Rayleigh quotient,

$$\begin{aligned}\lambda(A(U^*))-\lambda(A(U^-)) &\leq X^T\left(A(U^*)-A(U^-)\right)X\\ &= x_{v_{\frac{g}{2}}}x_{v_{\frac{g}{2}+2}} + x_{v_{\frac{g}{2}+1}}x_{z_{l_1}} - x_{v_{\frac{g}{2}}}x_{v_{\frac{g}{2}+1}} - x_{v_{\frac{g}{2}+1}}x_{v_{\frac{g}{2}+2}}\\ &= -x^2_{v_{\frac{g}{2}}}\\ &< 0.\end{aligned}$$

Note that the girth of $U^*$ is odd. Refer to the Case 1, $U^-\cong S^-_{n,3,k}$. The proof is complete. ∎

**4. Proof of Theorem 1.4.** Let $(C_n,\sigma)$ be the unbalanced cycle with order $n$, by [4],

$$Spec\left(A\left(C_n,\sigma\right)\right)=\{2cos\frac{(2k+1)\pi}{n}, k=0,1,\ldots,n-1\}.$$

Then,

$$\lambda\left(A\left(C_n,\sigma\right)\right)=2cos\frac{\left(2\left\lfloor\frac{n}{2}\right\rfloor+1\right)\pi}{n}=\begin{cases}-2cos\frac{\pi}{n}, & \text{if } n \text{ is even.}\\ -2, & \text{if } n \text{ is odd.}\end{cases}$$

Next we give the proof of Theorem 1.4.

*Proof.* Suppose that $V(U^-)=\{v_1,\ldots,v_n\}$ and $(C_g,\sigma)$ is the unique negative cycle in $U^-$, where $C_g=v_1v_2v_3\cdots v_gv_1$. Note that $\mathcal{U}^-_{n,g}=\bigcup\limits_{k=0}^{n-g}\mathcal{U}^-_{n,g,k}$, then $U^-$ is switching isomorphic to one of the signed graphs in $\{S^-_{n,g,0},S^-_{n,g,1},\ldots,S^-_{n,g,n-g}\}$ by Theorem 1.1. The result is trivial for $k=0$ and $k=n-g$. Thus, assume that $k\geq 1$. Let $X=(x_1,x_2,\ldots,x_n)^T$ be a unit eigenvector corresponding to $\lambda(A(U^-))$. It is known that $x_{v_1}>0$ by the proof of Theorem 1.1.

Case 1. $2\leq k<n-g$.

Let $P_{l_1},P_{l_2},\ldots,P_{l_k}$ be the $k$ paths of $U^-$, where $P_{l_i}=v_1u_{i1}u_{i2}\cdots u_{il_i}$ for $i=1,2,\ldots,k$. Since $(C_3,\sigma)$ or $(K_{1,4},\sigma)$ with a negative edge is an induced subgraph of $S^-_{n,g,k}$, $\lambda\left(A\left(S^-_{n,g,k}\right)\right)\leq -2$ by Lemma 2.1. If $l_i=1$ for $1\leq i\leq k$, then the result is trivial. Next, we consider that there exists $l_i\geq 2$ $(1\leq i\leq k)$. We might as well assume that $l_1\geq 2$. If $l_1$ is odd, then $x_{u_{1l_1}}x_{v_1}<0<x_{u_{1l_1-1}}x_{v_1}$ and $|x_{u_{1l_1-1}}|<|x_{v_1}|$ by Corollary 2.6. Combining this with $x_{v_1}>0$, thus $x_{v_1}>x_{u_{1l_1-1}}>0$ and $x_{u_{1l_1}}<0$. Then, we can construct a new signed graph $U^*$ from $S^-_{n,g,k}$ by rotating the positive edge $u_{1l_1}u_{1l_1-1}$ to the non-edge position $u_{1l_1}v_1$ such that $U^*\in\mathcal{U}^-_{n,g,k+1}$ and $\lambda\left(A\left(U^*\right)\right)<\lambda\left(A\left(S^-_{n,g,k}\right)\right)$ by Lemma 2.2. Note that $\lambda\left(A\left(S^-_{n,g,k+1}\right)\right)\leq\lambda(A(U^*))$ by Theorem 1.1. If $l_1$ is even, then $x_{u_{1l_1}}x_{v_1}>0>x_{u_{1l_1-1}}x_{v_1}$ and $|x_{u_{1l_1-1}}|<|x_{v_1}|$ by Corollary 2.6. Recall that $x_{v_1}>0$, it follows $x_{u_{1l_1}}>0>x_{u_{1l_1-1}}$. Let $U_1^-$ be the unbalanced signed unicyclic graph obtained from $S^-_{n,g,k}$ by switching at the vertex $u_{1l_1-1}$ and $X'=(x'_1,x'_2,\ldots,x'_n)^T$ be its eigenvector corresponding to $\lambda\left(A\left(U_1^-\right)\right)$, where $x'_{u_{1l_1-1}}=-x_{u_{1l_1-1}}$ and $x'_{v_i}=x_{v_i}$ for any vertex $v_i\in V(U^-)\setminus\{u_{1l_1-1}\}$. Thus, $x'_{u_{1l_1}}>0$ and $x'_{v_1}>x'_{u_{1l_1-1}}$. Then, a new signed graph $U^*$ can be obtained from $U_1^-$ by rotating the negative edge $u_{1l_1}u_{1l_1-1}$ to the non-edge position $u_{1l_1}v_1$ such that $U^*\in\mathcal{U}^-_{n,g,k+1}$ and $\lambda\left(A\left(U^*\right)\right)<\lambda\left(A\left(U_1^-\right)\right)=\lambda\left(A\left(S^-_{n,g,k}\right)\right)$ by Lemma 2.2. Note that $\lambda\left(A\left(S^-_{n,g,k+1}\right)\right)\leq\lambda(A(U^*))$ by Theorem 1.1.

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Therefore, $\lambda\left(A\left(S^-_{n,g,k+1}\right)\right) < \lambda\left(A\left(S^-_{n,g,k}\right)\right)$. Hence, one use repeatedly the same arguments, we can obtain $\lambda\left(A\left(S^-_{n,g,n-g}\right)\right) \leq \lambda\left(A\left(S^-_{n,g,k}\right)\right)$.

Case 2. $k = 1$.

If $n-g=1$, then the assertion is true. Hence, we just need to consider $n-g \geq 2$. Firstly, assume that $g$ is odd. Since $(C_g, \sigma)$ with a negative edge is an induced subgraph of $S^-_{n,g,1}$, then $\lambda\left(A\left(S^-_{n,g,1}\right)\right) \leq -2$ by Lemma 2.1. Let $P_l = v_1u_1\cdots u_l$ be the unique path in $S^-_{n,g,1}$, where $l \geq 2$. If $l$ is odd, then $x_{u_l}x_{v_1} < 0 < x_{u_{l-1}}x_{v_1}$ and $|x_{u_{l-1}}| < |x_{v_1}|$ by Corollary 2.6. Combining this with $x_{v_1} > 0$, thus $x_{v_1} > x_{u_{l-1}} > 0$ and $x_{u_l} < 0$. Hence, we can construct a new signed graph $U^*$ from $S^-_{n,g,1}$ by rotating the positive edge $u_lu_{l-1}$ to the non-edge position $u_lv_1$ such that $U^* \in \mathcal{U}^-_{n,g,2}$ and $\lambda\left(A\left(U^*\right)\right) < \lambda\left(A\left(S^-_{n,g,1}\right)\right)$ by Lemma 2.2. Note that $\lambda\left(A\left(S^-_{n,g,2}\right)\right) \leq \lambda(A(U^*))$ by Theorem 1.1. If $l$ is even, then $x_{u_l}x_{v_1} > 0 > x_{u_{l-1}}x_{v_1}$ and $|x_{u_{l-1}}| < |x_{v_1}|$ by Corollary 2.6. Recall that $x_{v_1} > 0$, thus $x_{u_l} > 0 > x_{u_{l-1}}$. Let $U^-_2$ be the unbalanced signed unicyclic graph obtained from $S^-_{n,g,1}$ by switching at the vertex $u_{l-1}$ and $X' = (x'_1, x'_2, \ldots, x'_n)^T$ be its eigenvector corresponding to $\lambda\left(A\left(U^-_2\right)\right)$, where $x'_{u_{l-1}} = -x_{u_{l-1}}$ and $x'_{v_i} = x_{v_i}$ for any vertex $v_i \in V(U^-) \setminus \{u_{l-1}\}$. Note that $x'_{u_l} > 0$ and $x'_{v_1} > x'_{u_{l-1}}$. Hence, the new signed graph $U^*$ can be obtained from $U^-_2$ by rotating the negative edge $u_lu_{l-1}$ to the non-edge position $u_lv_1$ such that $U^* \in \mathcal{U}^-_{n,g,2}$ and $\lambda\left(A\left(U^*\right)\right) < \lambda\left(A\left(U^-_2\right)\right) = \lambda\left(A\left(S^-_{n,g,1}\right)\right)$ by Lemma 2.2. Note that $\lambda\left(A\left(S^-_{n,g,2}\right)\right) \leq \lambda(A(U^*))$ by Theorem 1.1.

Next, assume that $g$ is even. If $n-g=2$, then there exists a unique path $P_3 = v_1u_1u_2$ in $U^- = S^-_{g+2,g,1}$. We have $f_2(\lambda)x_{u_1} = x_{v_1}$ by the eigenvalue equations for vertices $x_{u_1}$ and $x_{u_2}$, where $f_2(\lambda)$ is defined in Lemma 2.4 and $\lambda = \lambda\left(A\left(S^-_{g+2,g,1}\right)\right)$. Since $(K_{1,3}, \sigma)$ with a negative edge is an induced subgraph of $S^-_{g+2,g,1}$, $\lambda\left(A\left(S^-_{g+2,g,1}\right)\right) \leq -\sqrt{3}$ by Lemma 2.1. Hence, $f_2(\lambda) = \lambda - \frac{1}{\lambda} < -1$, and then, $|x_{u_1}| < |x_{v_1}|$. Combining this with $x_{v_1} > 0$, then $x_{u_1} < 0 < x_{u_2}$. Let $U^-_3$ be the unbalanced signed unicyclic graph obtained from $S^-_{g+2,g,1}$ by switching at the vertex $u_1$ and $X' = (x'_1, x'_2, \ldots, x'_n)^T$ be its eigenvector corresponding to $\lambda\left(A\left(U^-_3\right)\right)$, where $x'_{u_1} = -x_{u_1}$ and $x'_{v_i} = x_{v_i}$ for any vertex $v_i \in V(U^-) \setminus \{u_1\}$. Then, $x'_{u_2} > 0$ and $x'_{v_1} > x'_{u_1}$. Hence, we can construct a new signed graph $U^*$ from $U^-_3$ by rotating the negative edge $u_2u_1$ to the non-edge position $u_2v_1$ such that $U^* = S^-_{g+2,g,2}$ and $\lambda\left(A\left(S^-_{g+2,g,2}\right)\right) < \lambda\left(A\left(U^-_3\right)\right) = \lambda\left(A\left(S^-_{g+2,g,1}\right)\right)$ by Lemma 2.2, as desired. If $n-g \geq 3$, then there is a unique path $P_l = v_1u_1\cdots u_l$ in $U^- \in S^-_{n,g,1}$, where $l \geq 3$. We first assume that $x_{u_3} \geq 0$. If $x_{u_2} > x_{u_1}$, then we can construct a new signed graph $U^*$ from $U^-$ by rotating the positive edge $u_3u_2$ to the non-edge position $u_3u_1$ such that $U^* \in \mathcal{U}^-_{n,g,2}$ and $\lambda(U^*) < \lambda\left(A\left(S^-_{n,g,1}\right)\right)$ by Lemma 2.2. Obviously, $\lambda\left(A\left(S^-_{n,g,2}\right)\right) < \lambda\left(A(U^*)\right)$ by Theorem 1.1. If $x_{u_1} \geq x_{u_2}$, then a new signed graph $U^*$ can be obtained from $U^-$ by deleting the positive edge $v_1u_1$ and adding the positive edge $v_1u_2$ such that $U^* \in \mathcal{U}^-_{n,g,2}$ and $\lambda(U^*) < \lambda\left(A\left(S^-_{n,g,1}\right)\right)$ by Lemma 2.2. Note that $\lambda\left(A\left(S^-_{n,g,2}\right)\right) < \lambda\left(A(U^*)\right)$ by Theorem 1.1. Next assume that $x_{u_3} < 0$. Let $U^-_4$ be the unbalanced signed unicyclic graph obtained from $S^-_{n,g,1}$ by switching at the vertex $u_2$ and $X' = (x'_1, x'_2, \ldots, x'_n)^T$ be its eigenvector corresponding to $\lambda\left(A\left(U^-_4\right)\right)$, where $x'_{u_2} = -x_{u_2}$ and $x'_{v_i} = x_{v_i}$ for any vertex $v_i \in V(U^-) \setminus \{u_2\}$. If $x'_{u_1} \leq x'_{u_2}$, then a new signed graph $U^*$ can be obtained from $U^-_4$ by deleting the negative edge $u_3u_2$ and adding the negative edge $u_3u_1$ such that $U^* \in \mathcal{U}^-_{n,g,2}$ and $\lambda(U^*) < \lambda\left(U^-_4\right) = \lambda\left(A\left(S^-_{n,g,1}\right)\right)$ by Lemma 2.2. Obviously, $\lambda\left(A\left(S^-_{n,g,2}\right)\right) < \lambda\left(A(U^*)\right)$ by Theorem 1.1. If $x'_{u_1} > x'_{u_2}$, then a new signed graph $U^*$ can be obtained from $U^-_4$ by rotating the positive edge $v_1u_1$ to the non-edge position $v_1u_2$ such that $U^* \in \mathcal{U}^-_{n,g,2}$ and $\lambda(U^*) < \lambda\left(U^-_4\right) = \lambda\left(A\left(S^-_{n,g,1}\right)\right)$ by Lemma 2.2. Note that $\lambda\left(A\left(S^-_{n,g,2}\right)\right) \leq \lambda(A(U^*))$ by Theorem 1.1. Then, $\lambda\left(A\left(S^-_{n,g,2}\right)\right) < \lambda\left(A\left(S^-_{n,g,1}\right)\right)$. Hence, according to the previous Case 1 for $2 \leq k < n-g$, from a repeated use of the same arguments, we can draw the conclusion. ∎

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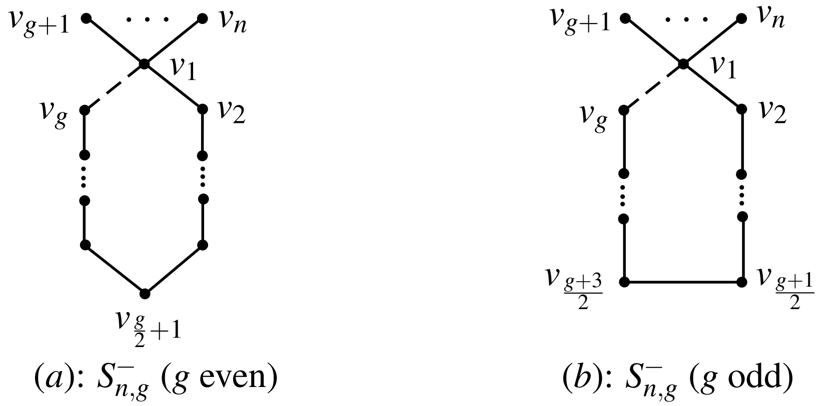


FIGURE 2. *The signed graphs* $S^-_{n,g,n-g}$.

LEMMA 4.1. *[2] Let* $\Gamma$ *be a signed graph and* $v$ *be one of its vertices. Then*

$$\Phi(\Gamma,\lambda)=\lambda\Phi(\Gamma-v,\lambda)-\sum_{u\sim v}\Phi(\Gamma-\{u,v\},\lambda)-2\sum_{C\in\mathcal{C}_v}\sigma(C)\Phi(\Gamma-C,\lambda),$$

*where* $\mathcal{C}_v$ *denotes the set of signed cycles passing through* $v$ *(we assume that* $\Phi(\emptyset,\lambda)=1$*).*

REMARK 4.2. *Let* $S^-_{n,g,n-g}$ *be the signed graph as shown in Fig. 2. By the proof of the Theorem 1.4, if* $g$ *is even, then we will get* $x_{v_2}+x_{v_g}=0$ *and* $0\neq x_{v_2}=-x_{v_g}$ *in* $S^-_{n,g,n-g}$*. If* $g$ *is odd, then by similar method of the Theorem 1.4, we can gain* $f_{\frac{g-1}{2}}\left(A\left(S^-_{n,g,n-g}\right)\right)\left(x_{v_{\frac{g+1}{2}}}+x_{v_{\frac{g+3}{2}}}\right)=x_{v_{\frac{g+1}{2}}}+x_{v_{\frac{g+3}{2}}}$*. Thus,* $f_{\frac{g-1}{2}}(A(S^-_{n,g,n-g}))=1$ *or* $x_{v_{\frac{g+1}{2}}}+x_{v_{\frac{g+3}{2}}}=0$*. Since* $(C_g,\sigma)$ *with a negative edge is an induced subgraph of* $S^-_{n,g,n-g}$*,* $\lambda(A(S^-_{n,g,n-g}))\leq -2$ *by Lemma 2.1. Note that* $f_{\frac{g-1}{2}}(A(S^-_{n,g,n-g}))<-1$ *by Lemma 2.4. Then,* $x_{v_{\frac{g+1}{2}}}+x_{v_{\frac{g+3}{2}}}=0$*. If* $x_{v_{\frac{g+1}{2}}}=x_{v_{\frac{g+3}{2}}}=0$*, then* $x_{v_1}=\cdots=x_{v_g}=0$ *and* $X=0$*, a contradiction. By parity of reasoning,* $x_{v_2}+x_{v_g}=0$*. Therefore,* $0\neq x_{v_2}=-x_{v_g}$*. By the eigenvalue equations of* $v_{g+i}$*,* $\lambda\left(A(S^-_{n,g,n-g})\right)x_{v_{g+i}}=x_{v_1}$ *for* $i=1,\ldots,n-g$*. Hence, we can derive that* $x_{v_{g+1}}=\cdots=x_{v_n}\neq 0$ *by the proof of Theorem 1.4.*

By [5],

$$\Phi(P_n,\lambda)=\prod_{j=1}^{n}\left(\lambda-2cos\frac{j\pi}{n+1}\right)=\frac{sin\left((n+1)\,arccos\frac{\lambda}{2}\right)}{sin\left(arccos\frac{\lambda}{2}\right)}.$$

Then,

$$\Phi(P_n,-2)=\frac{sin\left((n+1)\,arccos(-1)\right)}{sin\left(arccos(-1)\right)}=\frac{sin\left((n+1)\,\pi\right)}{sin\pi}=\begin{cases}n+1, & \text{if } n \text{ is even.}\\ -(n+1), & \text{if } n \text{ is odd.}\end{cases}$$

Let $Z_n$ denote the tree of order $n+2$ consisting of three paths $P_2$, $P_2$ and $P_n$ sharing one end vertex. By [6], we have

$$Spec(A(Z_n))=\{2cos\frac{(2k+1)\pi}{2n+2},k=0,1,\ldots,n\}\cup\{0\}.$$

Then, the following corollary holds.

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Corollary 4.3. For $n > g$, we have the following statements.

*(i)* *If* $n = g+1$, $g$ *is even and* $g > 8$, *then* $\lambda\left(A\left(S^-_{n,g+1,n-g-1}\right)\right) > \lambda\left(A\left(S^-_{n,g,n-g}\right)\right)$.
*(ii)* *If* $n = g+1$ *and* $g$ *is odd, then* $\lambda\left(A\left(S^-_{n,g+1,n-g-1}\right)\right) > \lambda\left(A\left(S^-_{n,g,n-g}\right)\right)$.
*(iii)* *If* $n > g+1$, *then* $\lambda\left(A\left(S^-_{n,g+1,n-g-1}\right)\right) > \lambda\left(A\left(S^-_{n,g,n-g}\right)\right)$.

*Proof.* (i) By Lemma 4.1,

$$\Phi\left(S^-_{g+1,g,1},\lambda\right) = \left(\lambda^2-1\right)\Phi\left(P_{g-1},\lambda\right) + 2\lambda\left(1-\Phi\left(P_{g-2},\lambda\right)\right).$$

Then,

$$\Phi\left(S^-_{g+1,g,1},-2\right) = 3\Phi\left(P_{g-1},-2\right) - 4\left(1-\Phi\left(P_{g-2},-2\right)\right) = -3g-4\left(1-(g-1)\right) = g-8.$$

Note that $\Phi\left(S^-_{g+1,g,1},-2\right) > 0$ when $g > 8$. Thus, $\lambda\left(A\left(C_{g+1},\sigma\right)\right) = -2 > \lambda\left(A\left(S^-_{g+1,g,1}\right)\right)$.

(ii) It is true for $g = 3$. Hence, assume that $g > 3$. Note that $(Z_{g-2},\sigma)$ with a negative edge is an induced subgraph of $S^-_{g+1,g,1}$. Then, $\lambda\left(A\left(S^-_{g+1,g,1}\right)\right) \le \lambda\left(A\left(Z_{g-2}\right)\right) = -2cos\frac{\pi}{2g-2} < -2cos\frac{\pi}{g+1} = \lambda\left(A\left(C_{g+1},\sigma\right)\right)$ by Lemma 2.1.

(iii) We consider a pendant edge $vt$ and two positive edges $ts$ and $sr$ of the cycle of $S^-_{n,g+1,n-g-1}$, where $d(v) = 1$, $d(t) > 2$ and $d(s) = d(r) = 2$. Let $X = (x_1,x_2,\ldots,x_n)^T$ be a unit eigenvector corresponding to $\lambda\left(A\left(S^-_{n,g+1,n-g-1}\right)\right)$. By Remark 4.2, $x_v \neq 0$. Without loss of generality, assume that $x_v < 0$. Note that $S^-_{n,g+1,n-g-1}$ has the minimum least eigenvalue in $\mathcal{U}^-_{n,g+1}$ by Theorem 1.4. Then we assert that $x_s < x_t$. Otherwise, we can construct a new signed graph $S^*$ from $S^-_{n,g+1,n-g-1}$ by rotating the positive edge $vt$ to the non-edge position $vs$ such that $S^* \in \mathcal{U}^-_{n,g+1}$ and $\lambda\left(A\left(S^*\right)\right) < \lambda\left(A\left(S^-_{n,g+1}\right)\right)$ by Lemma 2.2, a contradiction. By Remark 4.2 and the eigenvalue equations for $v$ and $t$, $\lambda\left(A\left(S^-_{n,g+1,n-g-1}\right)\right)x_v = x_t$, and then $(n-g-1)x_v + 2x_s = \lambda\left(A\left(S^-_{n,g+1,n-g-1}\right)\right)x_t = \lambda^2\left(A\left(S^-_{n,g+1,n-g-1}\right)\right)x_v$. This implies that $2x_s = \left(\lambda^2\left(A\left(S^-_{n,g+1,n-g-1}\right)\right) - n + g + 1\right)x_v$. Since $(K_{1,n-g+1},+)$ is an induced subgraph of $S^-_{n,g+1,n-g-1}$, $\lambda\left(A\left(S^-_{n,g+1,n-g-1}\right)\right) \le \lambda\left(A\left(K_{1,n-g+1}\right)\right) = -\sqrt{n-g+1}$ by Lemma 2.1. Thus, $x_s \ge x_v$. Next, we assert that $x_r \le 0$. Otherwise, we construct a new signed graph $S^*$ from $S^-_{n,g+1,n-g-1}$ by rotating the positive edge $rs$ to the non-edge position $rv$ such that $S^* \in \mathcal{U}^-_{n,g+1}$ and $\lambda\left(A\left(S^*\right)\right) < \lambda\left(A\left(S^-_{n,g+1,n-g-1}\right)\right)$ by Lemma 2.2. However, $S^*$ is isomorphic to $S^-_{n,g+1,n-g-1}$ and $\lambda\left(A\left(S^*\right)\right) = \lambda\left(A\left(S^-_{n,g+1,n-g-1}\right)\right)$, a contradiction. Recall that $x_s < x_t$. Hence, we can construct the signed graph $S^-_{n,g,n-g}$ from $S^-_{n,g+1,n-g-1}$ by rotating the positive edge $rs$ to the non-edge position $rt$. Thus, $\lambda\left(A\left(S^-_{n,g,n-g}\right)\right) < \lambda\left(A\left(S^-_{n,g+1,n-g-1}\right)\right)$ by Lemma 2.2, as required. ∎

Denote by $\mathcal{U}^-_{n,g^+}$ the set of all unbalanced unicyclic graphs with order $n$ and girth at least $g$. Then, the following Corollary can be obtained by Theorem 1.4 and Corollary 4.3 immediately.

Corollary 4.4. For any graph $U^- \in \mathcal{U}^-_{n,g^+}$ where $n > g+1$, we have

$$\lambda\left(A\left(U^-\right)\right) \ge \lambda\left(A\left(S^-_{n,g,n-g}\right)\right),$$

the equality holds if and only if $U^- = S^-_{n,g,n-g}$.

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