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DIAGONAL-SCHUR COMPLEMENTS OF NEKRASOV MATRICES*

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Abstract. The Schur and diagonal-Schur complements are important tools in many fields. It was revealed that the diagonal-Schur complements of Nekrasov matrices with respect to the index set $\{1\}$ are Nekrasov matrices by Cvetković and Nedović [*Appl. Math. Comput.,* 208:225–230, 2009]. In this paper, we prove that the diagonal-Schur complements of Nekrasov matrices with respect to any index set are Nekrasov matrices. Similar results hold for Σ-Nekrasov matrices. We also present some results on Nekrasov diagonally dominant degrees. Numerical examples are given to verify the correctness of the results.

Key words. Nekrasov matrix, Diagonal-Schur complement, Σ-Nekrasov matrix.

AMS subject classifications. 15A45, 15A48.

1. Introduction.

The diagonal-Schur complement is an important tool in numerical analysis, control theory, matrix theory, and statistics [1, 2, 3]. There is a very close connection between Schur complements and diagonal-Schur complements. In addition, the following structural perturbation of stationary linear large-scale systems is usually considered in control theory [1, 4]:

$$\frac{dx}{dt} = Ax, \tag{1.1}$$

where A is an $n \times n$ complex matrix and x is an n-dimensional vector. The matrix A is often written as: $A = \tilde{A} + \breve{A}$, where $\tilde{A}$ is a diagonal matrix and $\breve{A} = A - \tilde{A}$. Such a matrix $\breve{A}$ (can be written as a matrix minus a diagonal matrix) is related to some diagonal-Schur complement. The structures of $\breve{A}$ and $\tilde{A}$ are important to investigate the stability of (1.1). Hence, the closure property of diagonal-Schur complements for some special types of matrices, such as diagonally dominant matrices, H-matrices, and Nekrasov matrices, has always been one of the issues of concern.

It has been proved in [2] that the diagonal-Schur complement of an H-matrix is also an H-matrix. Similar results hold for some subclasses of H-matrices, such as strictly diagonally dominant (SDD) matrices [1], Σ-SDD matrices [5], strictly doubly diagonally dominant matrices and γ-SDD matrices [2], and Dashnic–Zumanovich matrices [6]. In 2020, Li, Huang, and Zhao [3] proved that the diagonal-Schur complements of Dashnic–Zumanovich-type matrices may be Dashnic–Zumanovich-type matrices under certain conditions. For more results about diagonal-Schur complements, one can refer to [7, 8, 9, 10] and the references therein.

Now we introduce some notations and symbols. Let $C^{n\times n}$ be the set of all $n \times n$ complex matrices (we always assume $n \geq 2$ in this paper) and $\langle n\rangle = \{1, 2, \cdots, n\}$. For any $A = (a_{ij}) \in C^{n\times n}$, denote $|A| = (|a_{ij}|)$. The determinant of A is denoted by $det(A)$. The comparison matrix of A is denoted by $\mu(A) = (u_{ij})_{n\times n}$ where

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$$u_{ij} = \begin{cases} |a_{ij}|, & i = j; \\ -|a_{ij}|, & i \neq j. \end{cases}$$

The symbol "$\circ$" stands for the Hadamard product of two matrices, i.e., for $A = (a_{ij}) \in C^{n\times n}$ and $B = (b_{ij}) \in C^{n\times n}$, $A \circ B$ is defined as $(a_{ij}b_{ij})$. Let α and β be given subsets of $\langle n\rangle$. $|\alpha|$ stands for the cardinal number of α. $A(\alpha, \beta)$ stands for the sub-matrix of A lying in the rows indexed by α and the columns indexed by β. $A(\alpha, \alpha)$ is called the principal sub-matrix of A, abbreviated to $A(\alpha)$, which stands for the sub-matrix of A lying in the rows and columns indexed by α. If $A(\alpha)$ is nonsingular, then the Schur complement of $A \in C^{n\times n}$ with respect to $A(\alpha)$ is denoted by A/α, i.e.,

$$A/\alpha = A(\bar{\alpha}) - A(\bar{\alpha}, \alpha)[A(\alpha)]^{-1}A(\alpha, \bar{\alpha}),$$

where $\bar{\alpha} = \langle n\rangle - \alpha$. The diagonal-Schur complement of $A \in C^{n\times n}$ with respect to $A(\alpha)$ is denoted by $A/_{\circ}\alpha$, i.e.,

$$A/_{\circ}\alpha = A(\bar{\alpha}) - \{A(\bar{\alpha}, \alpha)[A(\alpha)]^{-1}A(\alpha, \bar{\alpha})\} \circ I.$$

Remark that we adopt the convention that $A/_{\circ}\emptyset = A$. For Schur complements, if B is a nonsingular principal sub-matrix of A, and C is a nonsingular principal sub-matrix of B, then the following quotient formula $A/B = (A/C)/(B/C)$ holds [11]. However, the quotient formula does not hold in general for diagonal-Schur complements.

The class of Nekrasov matrices, an important subclass of H-matrices, has a wide range of applications in many fields, like computational mathematics, control, economics, and dynamic systems [12, 13, 14]. For convenience, let N_n denote the set of all $n \times n$ complex Nekrasov matrices. Many results about Nekrasov matrices have been obtained, such as infinity norm bounds for the inverse [15, 16], subdirect sums [17, 18], and Schur complements [14, 19]. Given $A \in N_n$. It is clear that $A/\{1\} \in N_{n-1}$. As an application of the quotient formula, one can easily get that $A/\{k\} \in N_{n-k}$. However, for a general $\alpha \subset \langle n\rangle$, A/α may not be a Nekrasov matrix [14]. In 2008, Cvetković and Nedović [6] proved that $A/_{\circ}\{1\}$ is also a Nekrasov matrix. For the lack of quotient formula, the problem of whether $A/_{\circ}\langle k\rangle$ $(1 < k < n)$ is also a Nekrasov matrix cannot be directly concluded. It is well known that for any $\alpha \subset \langle n\rangle$, $A(\alpha) \in N_{|\alpha|}$ and then $A(\alpha)$ is nonsingular. In this paper, we show that, for any $\alpha \subset \langle n\rangle$, $A/_{\circ}\alpha \in N_{n-|\alpha|}$. The similar results for Σ-Nekrasov matrices are obtained by using scaling matrices.

The dominant degree can measure the separations of the Geršgorin discs from the origin, and so it is an important tool in studying the location of the eigenvalues. The definition of diagonally dominant degree of SDD matrices was first proposed in [20], and the location of the eigenvalues for the Schur complements of SDD matrices, γ-SDD matrices, and DSDD matrices has been discussed in [20, 21, 22]. The Nekrasov diagonally dominant degree of Nekrasov matrices was proposed in [14] and it has been applied to estimate the bounds for the determinant of Nekrasov matrices [14]. We discuss the Nekrasov diagonally dominant degree of diagonal-Schur complements after we obtained the closure property of diagonal-Schur complements for Nekrasov matrices.

The rest of this paper is carried out as follows. In Section 2, we define some matrices, which are obtained by $\mu[A(\alpha \cup \{j_t\})]$ $(j_t \in \bar{\alpha})$. We prove that these matrices are Nekrasov matrices and the determinants are positive. These matrices and their properties play an important role in investigating the diagonal-Schur complements for Nekrasov matrices. In Section 3, we present that the diagonal-Schur complements of Nekrasov matrices are Nekrasov matrices. The Nekrasov diagonally dominant degrees are involved. Section 4 shows that the diagonal-Schur complements of Σ-Nekrasov matrices are also Σ-Nekrasov matrices.

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2. The preliminaries. In this section, we first introduce some notions and results related to this paper. Then we propose two classes of matrices C_t and D_t. The positiveness of their determinants plays an important role in the study of the diagonal-Schur complements for Nekrasov matrices.

Definition 2.1. Let $A \in C^{n\times n}$. The matrix A is called an M-matrix if it can be written in the form of $A = sI - P$, where I is the identity matrix, P is a nonnegative matrix, $s > \rho(P)$, and $\rho(P)$ is the spectral radius of P.

Definition 2.2. Let $A \in C^{n\times n}$. The matrix A is called an H-matrix if $\mu(A)$ is an M-matrix.

Definition 2.3. Let $A = (a_{ij}) \in C^{n\times n}$. We say that A is a Nekrasov matrix if

$$|a_{ii}| > R_i(A), \text{ for all } i \in \langle n\rangle,$$

and we call $|a_{ii}| - R_i(A)$ the Nekrasov diagonally dominant degree for the i-th row of A, where $R_i(A)$ is defined as follows:

$$R_1(A) = \sum_{j=2}^{n} |a_{ij}|, \quad R_i(A) = \sum_{j=1}^{i-1} \frac{R_j(A)}{|a_{jj}|}|a_{ij}| + \sum_{j=i+1}^{n} |a_{ij}|, \quad i \in \langle n\rangle - \{1\}.$$

It is well known that a Nekrasov matrix is a nonsingular H-matrix. For any given nonempty subset S of $\langle n\rangle$, denote

$$R_1^S(A) = \sum_{j\in S, j\neq 1} |a_{1j}|; \;\; R_i^S(A) = \sum_{j=1}^{i-1} \frac{R_j^S(A)}{|a_{jj}|}|a_{ij}| + \sum_{j=i+1, j\in S}^{n} |a_{ij}|, \;\; i \in \langle n\rangle - \{1\}.$$

Definition 2.4. Let $A = (a_{ij}) \in C^{n\times n}$ and S be a given nonempty subset of $\langle n\rangle$. A is called an S-Nekrasov matrix if

$$|a_{ii}| > R_i^S(A), \;\; |a_{jj}| > R_j^{\bar{S}}(A)$$

and

$$[|a_{ii}| - R_i^S(A)][|a_{jj}| - R_j^{\bar{S}}(A)] > R_i^{\bar{S}}(A) R_j^S(A),$$

for all $i \in S$ and $j \in \bar{S}$.

Definition 2.5. Let $A \in C^{n\times n}$. A is called a Σ-Nekrasov matrix if there exists a nonempty subset S of $\langle n\rangle$ such that A is an S-Nekrasov matrix.

Lemma 2.1. [23, p.131] *If A is an H-matrix, then* $[\mu(A)]^{-1} \geq |A^{-1}|$.

Lemma 2.2. [23, p.117] *If A is an M-matrix, then* $det(A) > 0$.

Lemma 2.3. [11, p.5] *Let $A \in C^{n\times n}$ and α be a nonempty proper subset of $\langle n\rangle$. If $A(\alpha)$ is nonsingular, then* $det(A) = det(A(\alpha))det(A/\alpha)$.

Lemma 2.4. *Let $b > c \geq 0$, $r > 0$ and $a \geq rb$. Then*

$$\frac{b-c}{a-rc} \leq \frac{b}{a}.$$

If $r = 1$, then

$$\frac{b-c}{a-c} \leq \frac{b}{a}.$$

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Proof. Since $r > 0$, $b > c$ and $a \geq rb$, we get that

$$b - c > 0, \quad a - rc \geq rb - rc = r(b - c) > 0.$$

Noticing that a, b, r are positive numbers and $c \geq 0$, it can be deduced that

$$\begin{aligned} a \geq rb &\Rightarrow -ac \leq -rbc \\ &\Rightarrow ab - ac \leq ab - rbc \\ &\Rightarrow a(b-c) \leq b(a - rc) \\ &\Rightarrow \frac{b-c}{a-rc} \leq \frac{b}{a}. \end{aligned}$$

Now we define some matrices, which can be used to observe the closure property of the diagonally Schur complements for Nekrasov matrices. For convenience, we denote the index set α ($\emptyset \neq \alpha \subset \langle n \rangle$) and $\bar{\alpha} = \langle n \rangle - \alpha$. The elements in both of α and $\bar{\alpha}$ are listed in increasing order. To be precise, let

$$\alpha = \{i_1, i_2, \cdots, i_k\} \subset \langle n \rangle, \quad i_1 < i_2 < \cdots < i_k, \tag{2.1}$$

and

$$\bar{\alpha} = \langle n \rangle - \alpha = \{j_1, j_2, \cdots, j_l\} \ (l = n - k), \quad j_1 < j_2 < \cdots < j_l. \tag{2.2}$$

Lemma 2.5. *Let $A \in N_n$ and let $\alpha = \{i_1, i_2, \cdots, i_k\}$ and $\bar{\alpha} = \{j_1, j_2, \cdots, j_l\}$ $(l = n - k)$ be defined in (2.1) and (2.2), respectively. Given $j_t \in \bar{\alpha}$ with $j_t < i_k$, let $\alpha \cup \{j_t\}$ be listed in increasing order, i.e., $\alpha \cup \{j_t\} = \{i_1, \cdots, i_{v_t}, j_t, i_{v_{t+1}}, \cdots, i_k\}$ where i_{v_t} is the biggest number less than j_t (if $j_t < i_1$, $\alpha \cup \{j_t\} = \{j_t, i_1, \cdots, i_k\}$). The matrix C_t is obtained from $\mu[A(\alpha \cup \{j_t\})]$ by replacing $|a_{j_t,j_t}|$ with γ_t^*, i.e.,*

$$C_t := \begin{bmatrix} |a_{i_1,i_1}| & \dots & -|a_{i_1,i_{v_t}}| & -|a_{i_1,j_t}| & -|a_{i_1,i_{v_t+1}}| & \dots & -|a_{i_1,i_k}| \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \\ -|a_{i_{v_t},i_1}| & \dots & |a_{i_{v_t},i_{v_t}}| & \vdots & \vdots & & \\ -|a_{j_t,i_1}| & \dots & \dots & \gamma_t^* & \vdots & & \vdots \\ -|a_{i_{v_t+1},i_1}| & \dots & \dots & \dots & |a_{i_{v_t+1},i_{v_t+1}}| & & \\ \vdots & & & & & \ddots & \\ -|a_{i_k,i_1}| & \dots & -|a_{i_k,i_{v_t}}| & -|a_{i_k,j_t}| & -|a_{i_k,i_{v_t+1}}| & \dots & |a_{i_k,i_k}| \end{bmatrix}, \tag{2.3}$$

where if

$$\sum_{m=1}^{v_t} \frac{R_{i_m}(A)}{|a_{i_m,i_m}|} |a_{j_t,i_m}| + \sum_{m=v_t+1}^{k} |a_{j_t,i_m}| \neq 0,$$

then

$$\gamma_t^* = \frac{|a_{j_t,j_t}|}{R_{j_t}(A)} \left(\sum_{m=1}^{v_t} \frac{R_{i_m}(A)}{|a_{i_m,i_m}|} |a_{j_t,i_m}| + \sum_{m=v_t+1}^{k} |a_{j_t,i_m}| \right);$$

Otherwise, $\gamma_t^* = \varepsilon$ where ε is an arbitrary positive number. Then $C_t \in N_{k+1}$ and $det(C_t) > 0$.

Proof. Denote $C_t = (c_{ij})$. Notice that the condition

$$\sum_{m=1}^{v_t} \frac{R_{i_m}(A)}{|a_{i_m,i_m}|} |a_{j_t,i_m}| + \sum_{m=v_t+1}^{k} |a_{j_t,i_m}| \neq 0,$$

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suggests $R_{j_t}(A) \neq 0$, then the matrix C_t is well-defined. We first show $C_t \in N_{k+1}$. It is sufficient to show that $R_s(C_t) < |c_{ss}|$ holds for all $s \in \langle k+1 \rangle$. We will prove it in the following three steps.

Step 1. Since $A \in N_n$, it is clear that

$$R_s(C_t) \leq R_{i_s}(A) < |c_{ss}|, \quad s \in \langle v_t \rangle. \tag{2.4}$$

Step 2. We show that

$$R_{v_t+1}(C_t) < \gamma_t^* = |c_{v_t+1,v_t+1}|, \tag{2.5}$$

and

$$\frac{R_{v_t+1}(C_t)}{|c_{v_t+1,v_t+1}|} \leq \frac{R_{j_t}(A)}{|a_{j_t,j_t}|}. \tag{2.6}$$

By (2.4), we know that

$$\begin{aligned} &R_{v_t+1}(C_t) \\ &= \sum_{m=1}^{v_t} \frac{R_m(C_t)}{|c_{mm}|}|c_{v_t+1,m}| + \sum_{m=v_t+2}^{k+1} |c_{v_t+1,m}| \\ &= \sum_{m=1}^{v_t} \frac{R_m(C_t)}{|a_{i_m,i_m}|}|a_{j_t,i_m}| + \sum_{m=v_t+1}^{k} |a_{j_t,i_m}| \\ &\leq \sum_{m=1}^{v_t} \frac{R_{i_m}(A)}{|a_{i_m,i_m}|}|a_{j_t,i_m}| + \sum_{m=v_t+1}^{k} |a_{j_t,i_m}|. \end{aligned} \tag{2.7}$$

If $\sum_{m=1}^{v_t} \frac{R_{i_m}(A)}{|a_{i_m,i_m}|}|a_{j_t,i_m}| + \sum_{m=v_t+1}^{k} |a_{j_t,i_m}| \neq 0$, by $A \in N_n$, we have

$$R_{v_t+1}(C_t) < \frac{|a_{j_t,j_t}|}{R_{j_t}(A)} \left(\sum_{m=1}^{v_t} \frac{R_{i_m}(A)}{|a_{i_m,i_m}|}|a_{j_t,i_m}| + \sum_{m=v_t+1}^{k} |a_{j_t,i_m}| \right) = \gamma_t^*.$$

Hence, the inequality (2.5) holds. Moreover, by (2.7) it holds that

$$\frac{R_{v_t+1}(C_t)}{|c_{v_t+1,v_t+1}|} = \frac{R_{v_t+1}(C_t)}{\gamma_t^*} \leq \frac{\sum_{m=1}^{v_t} \frac{R_{i_m}(A)}{|a_{i_m,i_m}|}|a_{j_t,i_m}| + \sum_{m=v_t+1}^{k} |a_{j_t,i_m}|}{\gamma_t^*} = \frac{R_{j_t(A)}}{|a_{j_t,j_t}|}.$$

Then the inequality (2.6) holds. If $\sum_{m=1}^{v_t} \frac{R_{i_m}(A)}{|a_{i_m,i_m}|}|a_{j_t,i_m}| + \sum_{m=v_t+1}^{k} |a_{j_t,i_m}| = 0$, then $\gamma_t^* = \varepsilon$ and $R_{v_t+1}(C_t) = 0$ and then inequalities in (2.5) and (2.6) hold trivially.

Step 3. We show that

$$R_s(C_t) < |c_{ss}|, \quad s = v_t+2, \ldots, k+1. \tag{2.8}$$

by mathematical induction. First, it follows from (2.4) and (2.6) that

$$R_{v_t+2}(C_t)$$

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$$
\begin{aligned}
&= \sum_{m=1}^{v_t+1} \frac{R_m(C_t)}{|c_{mm}|}|c_{v_t+2,m}| + \sum_{m=v_t+3}^{k+1} |c_{v_t+2,m}| \\
&= \sum_{m=1}^{v_t} \frac{R_m(C_t)}{|c_{mm}|}|c_{v_t+2,m}| + \frac{R_{v_t+1}(C_t)}{|c_{v_t+1,v_t+1}|}|c_{v_t+2,v_t+1}| + \sum_{m=v_t+3}^{k+1} |c_{v_t+2,m}| \\
&= \sum_{m=1}^{v_t} \frac{R_m(C_t)}{|a_{i_m,i_m}|}|a_{i_{v_t+1},i_m}| + \frac{R_{v_t+1}(C_t)}{\gamma_t^*}|a_{i_{v_t+1},j_t}| + \sum_{m=v_t+2}^{k} |a_{i_{v_t+1},i_m}| \\
&\le \sum_{m=1}^{v_t} \frac{R_{i_m}(A)}{|a_{i_m,i_m}|}|a_{i_{v_t+1},i_m}| + \frac{R_{j_t}(A)}{|a_{j_t,j_t}|}|a_{i_{v_t+1},j_t}| + \sum_{m=v_t+2}^{k} |a_{i_{v_t+1},i_m}|.
\end{aligned}
$$

Then, we have

$$R_{v_t+2}(C_t) \le R_{i_{v_t+1}}(A) < |a_{i_{v_t+1},i_{v_t+1}}| = |c_{v_t+2,v_t+2}|.$$

Suppose that $R_u(C_t) \le R_{i_{u-1}}(A)$ for $v_t+2 \le u \le s-1$, where s is a fixed integer with $v_t+2 \le s-1 \le k$. Then, by (2.4) and (2.6) we can obtain that

$$
\begin{aligned}
&R_s(C_t) \\
&= \sum_{m=1}^{s-1} \frac{R_m(C_t)}{|c_{mm}|}|c_{sm}| + \sum_{m=s+1}^{k+1} |c_{sm}| \\
&= \sum_{m=1}^{v_t} \frac{R_m(C_t)}{|c_{mm}|}|c_{sm}| + \frac{R_{v_t+1}(C_t)}{|c_{v_t+1,v_t+1}|}|c_{s,v_t+1}| + \sum_{m=v_t+2}^{s-1} \frac{R_m(C_t)}{|c_{mm}|}|c_{sm}| + \sum_{m=s+1}^{k+1} |c_{sm}| \\
&= \sum_{m=1}^{v_t} \frac{R_m(C_t)}{|a_{i_m,i_m}|}|a_{i_{s-1},i_m}| + \frac{R_{v_t+1}(C_t)}{\gamma_t^*}|a_{i_{s-1},j_t}| \\
&\quad + \sum_{m=v_t+2}^{s-1} \frac{R_m(C_t)}{|a_{i_{m-1},i_{m-1}}|}|a_{i_{s-1},i_{m-1}}| + \sum_{m=s+1}^{k+1} |a_{i_{s-1},i_{m-1}}| \\
&\le \sum_{m=1}^{v_t} \frac{R_{i_m}(A)}{|a_{i_m,i_m}|}|a_{i_{s-1},i_m}| + \frac{R_{j_t}(A)}{|a_{j_t,j_t}|}|a_{i_{s-1},j_t}| \\
&\quad + \sum_{m=v_t+2}^{s-1} \frac{R_{i_{m-1}}(A)}{|a_{i_{m-1},i_{m-1}}|}|a_{i_{s-1},i_{m-1}}| + \sum_{m=s+1}^{k+1} |a_{i_{s-1},i_{m-1}}| \\
&= \sum_{m=1}^{v_t} \frac{R_{i_m}(A)}{|a_{i_m,i_m}|}|a_{i_{s-1},i_m}| + \frac{R_{j_t}(A)}{|a_{j_t,j_t}|}|a_{i_{s-1},j_t}| \\
&\quad + \sum_{m=v_t+1}^{s-2} \frac{R_{i_m}(A)}{|a_{i_m,i_m}|}|a_{i_{s-1},i_m}| + \sum_{m=s}^{k} |a_{i_{s-1},i_m}|.
\end{aligned}
$$

It follows that $R_s(C_t) \le R_{i_{s-1}}(A) < |a_{i_{s-1},i_{s-1}}| = |c_{ss}|$. Thus, we conclude that inequalities in (2.8) hold.

By (2.4), (2.5), and (2.8), we have $C_t \in N_{k+1}$, which implies that C_t is an H-matrix. We have $\mu(C_t) = C_t$ is an M-matrix. Then $det(C_t) > 0$ by Lemma 2.2. □

Lemma 2.6. *Let $A \in N_n$ and let $\alpha = \{i_1, i_2, \cdots, i_k\}$ and $\bar{\alpha} = \{j_1, j_2, \cdots, j_l\}$ $(l = n-k)$ be defined in (2.1) and (2.2), respectively. Given $j_t \in \bar{\alpha}$ with $j_t > i_k$. Define*

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(2.9)
$$D_t := \begin{bmatrix} & & & -|a_{i_1,j_t}| \\ & \mu[A(\alpha)] & & \vdots \\ & & & -|a_{i_k,j_t}| \\ -|a_{j_t,i_1}| & \dots & -|a_{j_t,i_k}| & \sum_{m=1}^{k} \frac{|a_{j_t,i_m}|}{|a_{i_m,i_m}|} R_m(D_t) + \varepsilon \end{bmatrix},$$

where ε is an arbitrary positive number. Then $D_t \in N_{k+1}$, $det(D_t) > 0$ and

(2.10)
$$R_s(D_t) \le \begin{cases} R_{i_s}(A), & s \in \langle k \rangle; \\ R_{j_t}(A), & s = k+1. \end{cases}$$

Proof. Since $A \in N_n$, we can easily testify that

$$|a_{i_s,i_s}| > R_{i_s}(A) \ge R_s(D_t), \quad s \in \langle k \rangle.$$

Moreover, it holds trivially that

$$\sum_{m=1}^{k} \frac{R_m(D_t)}{|a_{i_m,i_m}|}|a_{j_t,i_m}| + \varepsilon > \sum_{m=1}^{k} \frac{R_m(D_t)}{|a_{i_m,i_m}|}|a_{j_t,i_m}| = R_{k+1}(D_t),$$

and

$$R_{k+1}(D_t) \le R_{j_t}(A).$$

Then $D_t \in N_{k+1}$. Since $D_t = \mu(D_t)$, we have $det(D_t) > 0$ by Lemma 2.2 and (2.10) holds. ☐

3. Diagonal-Schur complements of Nekrasov matrices. Given $A = (a_{ij}) \in N_n$, let $\alpha = \{i_1, i_2, \cdots, i_k\}$ and $\bar{\alpha} = \{j_1, j_2, \cdots, j_l\}$ $(l = n-k)$ be defined in (2.1) and (2.2), respectively. This section will prove $A/_\circ\alpha \in N_{n-k}$. To begin with, we study the upper bound for

$$(|a_{j_t,i_1}|, \dots, |a_{j_t,i_k}|)[\mu(A(\alpha))]^{-1} \begin{pmatrix} |a_{i_1,j_t}| \\ \vdots \\ |a_{i_k,j_t}| \end{pmatrix}.$$

Lemma 3.1. *Let $A \in N_n$ and let $\alpha = \{i_1, i_2, \cdots, i_k\}$ and $\bar{\alpha} = \{j_1, j_2, \cdots, j_l\}$ $(l = n-k)$ be defined in (2.1) and (2.2), respectively.*

(i) For $j_t < i_k$, if $\sum_{i_v<j_t} \frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_t,i_v}| + \sum_{i_v>j_t} |a_{j_t,i_v}| \ne 0$, it holds that

(3.1)
$$(|a_{j_t,i_1}|, \dots, |a_{j_t,i_k}|)[\mu(A(\alpha))]^{-1} \begin{pmatrix} |a_{i_1,j_t}| \\ \vdots \\ |a_{i_k,j_t}| \end{pmatrix} < \frac{|a_{j_t,j_t}|}{R_{j_t}(A)} \left(\sum_{i_v<j_t} \frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_t,i_v}| + \sum_{i_v>j_t} |a_{j_t,i_v}| \right).$$

Otherwise, it holds that

(3.2)
$$(|a_{j_t,i_1}|, \dots, |a_{j_t,i_k}|)[\mu(A(\alpha))]^{-1} \begin{pmatrix} |a_{i_1,j_t}| \\ \vdots \\ |a_{i_k,j_t}| \end{pmatrix} = 0.$$

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(ii) For $j_t > i_k$, we have

$$(|a_{j_t,i_1}|,\dots,|a_{j_t,i_k}|)[\mu(A(\alpha))]^{-1}\begin{pmatrix}|a_{i_1,j_t}|\\ \vdots \\ |a_{i_k,j_t}|\end{pmatrix} \le \sum_{v=1}^{k}\frac{R_v(D_t)}{|a_{i_v,i_v}|}|a_{j_t,i_v}|. \tag{3.3}$$

Proof. For convenience, for $t \in \langle l\rangle$, we denote

$$x_t = (a_{j_t,i_1},\dots,a_{j_t,i_k})^T, \tag{3.4}$$

and

$$y_t = (a_{i_1,j_t},\dots,a_{i_k,j_t})^T. \tag{3.5}$$

We first prove (i). Let C_t be the same as in (2.3) and $\beta = \langle k+1\rangle - \{v_t+1\}$ (if $j_t < i_1$, $\beta = \langle k+1\rangle - \{1\}$). By Lemma 2.3, we have

$$det(C_t) = det(C_t(\beta))det(C_t/\beta),$$

where

$$C_t(\beta) = \mu(A(\alpha)),\ \ C_t/\beta = \gamma_t^* - |x_t^T|[\mu(A(\alpha))]^{-1}|y_t|.$$

Since $A \in N_n$, then $A(\alpha) \in N_k$ and $\mu(A(\alpha))$ is an M-matrix. By Lemma 2.2 and Lemma 2.5, we have $det(C_t(\beta)) > 0$ and $det(C_t) > 0$. Then it holds that

$$\gamma_t^* > |x_t^T|[\mu(A(\alpha))]^{-1}|y_t|. \tag{3.6}$$

Recalling the definition of γ_t^*, equalities in (3.1) and (3.2) hold.

Now we prove (ii). Let D_t be the same as in (2.9) and $\beta = \langle k\rangle$. By Lemma 2.3, we have

$$det(D_t) = det(D_t(\beta))det(D_t/\beta),$$

where

$$D_t(\beta) = \mu(A(\alpha)),\ \ D_t/\beta = \sum_{v=1}^{k}\frac{R_v(D_t)}{|a_{i_v,i_v}|}|a_{j_t,i_v}| + \varepsilon - |x_t^T|[\mu(A(\alpha))]^{-1}|y_t|.$$

By Lemma 2.2 and Lemma 2.6, we have $det(D_t(\beta)) > 0$ and $det(D_t) > 0$. Taking $\varepsilon \to 0^+$, then equalities in (3.3) hold immediately. ∎

Theorem 3.1. *Let $A \in N_n$ and let $\alpha = \{i_1, i_2, \cdots, i_k\}$ and $\bar{\alpha} = \{j_1, j_2, \cdots, j_l\}$ $(l = n-k)$ be defined in (2.1) and (2.2), respectively. Denote $A/_\circ\alpha = (a'_{tu})$. Then*

$$R_t(A/_\circ\alpha) \le R_{j_t}(A) - \left(\sum_{i_v<j_t}\frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_t,i_v}| + \sum_{i_v>j_t}|a_{j_t,i_v}|\right), \quad t \in \langle n-k\rangle. \tag{3.7}$$

Moreover, it holds that

$$\frac{R_t(A/_\circ\alpha)}{|a'_{tt}|} \le \frac{R_{j_t}(A)}{|a_{j_t,j_t}|}, \quad t \in \langle n-k\rangle. \tag{3.8}$$

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Proof. Let x_t and y_t be defined in (3.4) and (3.5), respectively. Then $A/_{\circ}\alpha$ can be obtained from $A(\bar{\alpha})$ by replacing a_{j_t,j_t} with $a_{j_t,j_t} - x_t^T[A(\alpha)]^{-1}y_t$ for each $t \in \langle l \rangle$, i.e.,

$$A/_{\circ}\alpha = \begin{bmatrix} a_{j_1,j_1} - x_1^T[A(\alpha)]^{-1}y_1 & \dots & a_{j_1,j_l} \\ \vdots & \ddots & \vdots \\ a_{j_l,j_1} & \dots & a_{j_l,j_l} - x_l^T[A(\alpha)]^{-1}y_l \end{bmatrix}.$$

We show that (3.7) and (3.8) hold by mathematical induction. Recall that j_1 is the smallest number in $\bar{\alpha}$, then $\langle j_1 - 1 \rangle \subseteq \alpha$. Then

$$\begin{aligned} R_{j_1}(A) &= \sum_{i=1}^{j_1-1} \frac{R_i(A)}{|a_{ii}|}|a_{j_1,i}| + \sum_{i=j_1+1}^{n} |a_{j_1,i}| \\ &= \sum_{i_v<j_1} \frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_1,i_v}| + \left(\sum_{i=j_1+1, i\in\alpha}^{n} |a_{j_1,i}| + \sum_{i=j_1+1, i\in\bar{\alpha}}^{n} |a_{j_1,i}| \right) \\ &= \sum_{i_v<j_1} \frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_1,i_v}| + \sum_{i_v>j_1} |a_{j_1,i_v}| + \sum_{u=2}^{l} |a_{j_1,j_u}|. \end{aligned}$$

Since $R_1(A/_{\circ}\alpha) = \sum_{u=2}^{l} |a_{j_1,j_u}|$, we have

$$R_1(A/_{\circ}\alpha) = R_{j_1}(A) - \left(\sum_{i_v<j_1} \frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_1,i_v}| + \sum_{i_v>j_1} |a_{j_1,i_v}| \right), \tag{3.9}$$

which implies that (3.7) holds for $t = 1$.

By Lemma 2.1, we have

$$|a'_{11}| = |a_{j_1,j_1} - x_1^T[A(\alpha)]^{-1}y_1| \geq |a_{j_1,j_1}| - |x_1^T|[\mu(A(\alpha))]^{-1}|y_1|. \tag{3.10}$$

We show (3.8) holds for $t = 1$ under two cases: $j_1 < i_k$ and $j_1 > i_k$ where i_k is the biggest number in α.

Case 1: $j_1 < i_k$. If

$$\sum_{i_v<j_1} \frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_1,i_v}| + \sum_{i_v>j_1} |a_{j_t,i_v}| = 0,$$

together with (3.2), (3.9), and (3.10), we have

$$|x_1^T|[\mu(A(\alpha))]^{-1}|y_1| = 0, \quad R_1(A/_{\circ}\alpha) = R_{j_1}(A), \quad |a'_{11}| = |a_{j_1,j_1}|.$$

Then (3.8) holds for $t = 1$ trivially.

Otherwise, by (3.1), we have

$$|x_1^T|[\mu(A(\alpha))]^{-1}|y_1| < \frac{|a_{j_1,j_1}|}{R_{j_1}(A)} \left(\sum_{i_v<j_1} \frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_1,i_v}| + \sum_{i_v>j_1} |a_{j_1,i_v}| \right).$$

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By (3.10), we have

$$|a'_{11}| \geq |a_{j_1,j_1}| - \frac{|a_{j_1,j_1}|}{R_{j_1}(A)} \left(\sum_{i_v<j_1} \frac{R_{i_v}(A)}{|a_{i_v,i_v}|} |a_{j_1,i_v}| + \sum_{i_v>j_1} |a_{j_1,i_v}| \right).$$

Then it can be deduced from (3.9) that

$$\frac{R_1(A/_\circ\alpha)}{|a'_{11}|} \leq \frac{R_{j_1}(A) - \left(\sum\limits_{i_v<j_1} \frac{R_{i_v}(A)}{|a_{i_v,i_v}|} |a_{j_1,i_v}| + \sum\limits_{i_v>j_1} |a_{j_1,i_v}| \right)}{|a_{j_1,j_1}| - \frac{|a_{j_1,j_1}|}{R_{j_1}(A)} \left(\sum\limits_{i_v<j_1} \frac{R_{i_v}(A)}{|a_{i_v,i_v}|} |a_{j_1,i_v}| + \sum\limits_{i_v>j_1} |a_{j_1,i_v}| \right)}.$$

Recalling Lemma 2.4, we get inequality (3.8) holds for $t = 1$ under the case $j_1 < i_k$.

Case 2: $j_1 > i_k$, i.e., $j_1 > i_v$ for all $v \in \langle k \rangle$. It follows from (3.3) that

$$|x_1^T|[\mu(A(\alpha))]^{-1}|y_1| \leq \sum_{v=1}^{k} \frac{R_v(D_1)}{|a_{i_v,i_v}|} |a_{j_1,i_v}|.$$

By (3.10) it holds that

$$|a'_{11}| \geq |a_{j_1,j_1}| - \sum_{v=1}^{k} \frac{R_v(D_1)}{|a_{i_v,i_v}|} |a_{j_1,i_v}|.$$

Then by (2.10), (3.9), and Lemma 2.4, we have

$$\frac{R_1(A/_\circ\alpha)}{|a'_{11}|} \leq \frac{R_{j_1}(A) - \sum\limits_{v=1}^{k} \frac{R_{i_v}(A)}{|a_{i_v,i_v}|} |a_{j_1,i_v}|}{|a_{j_1,j_1}| - \sum\limits_{v=1}^{k} \frac{R_v(D_1)}{|a_{i_v,i_v}|} |a_{j_1,i_v}|} \leq \frac{R_{j_1}(A) - \sum\limits_{v=1}^{k} \frac{R_{i_v}(A)}{|a_{i_v,i_v}|} |a_{j_1,i_v}|}{|a_{j_1,j_1}| - \sum\limits_{v=1}^{k} \frac{R_{i_v}(A)}{|a_{i_v,i_v}|} |a_{j_1,i_v}|} \leq \frac{R_{j_1}(A)}{|a_{j_1,j_1}|}.$$

Then the inequality (3.8) holds for $t = 1$ under the case $j_1 > i_k$. Now we have already proved that (3.7) and (3.8) always hold for $t = 1$.

Assume that the following inequality holds for any u with $1 \leq u < t$, where t is a given positive integer with $1 < t \leq l$:

$$\frac{R_u(A/_\circ\alpha)}{|a'_{uu}|} \leq \frac{R_{j_u}(A)}{|a_{j_u,j_u}|}, \ \ 1 \leq u < t. \tag{3.11}$$

Note that

$$\begin{aligned} R_t(A/_\circ\alpha) &= \sum_{u=1}^{t-1} \frac{R_u(A/_\circ\alpha)}{|a'_{uu}|} |a'_{tu}| + \sum_{u=t+1}^{l} |a'_{tu}| \\ &= \sum_{u=1}^{t-1} \frac{R_u(A/_\circ\alpha)}{|a'_{uu}|} |a_{j_t,j_u}| + \sum_{u=t+1}^{l} |a_{j_t,j_u}|, \end{aligned}$$

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and

$$\begin{aligned}
&R_{j_t}(A)\\
&=\sum_{m=1}^{j_t-1}\frac{R_m(A)}{|a_{mm}|}|a_{j_t,m}|+\sum_{m=j_t+1}^{n}|a_{j_t,m}|\\
&=\left(\sum_{m=1,m\in\alpha}^{j_t-1}\frac{R_m(A)}{|a_{mm}|}|a_{j_t,m}|+\sum_{m=1,m\in\bar{\alpha}}^{j_t-1}\frac{R_m(A)}{|a_{mm}|}|a_{j_t,m}|\right)\\
&\quad+\left(\sum_{m=j_t+1,m\in\alpha}^{n}|a_{j_t,m}|+\sum_{m=j_t+1,m\in\bar{\alpha}}^{n}|a_{j_t,m}|\right)\\
&=\sum_{i_v<j_t}\frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_t,i_v}|+\sum_{u=1}^{t-1}\frac{R_{j_u}(A)}{|a_{j_u,j_u}|}|a_{j_t,j_u}|+\sum_{i_v>j_t}|a_{j_t,i_v}|+\sum_{u=t+1}^{l}|a_{j_t,j_u}|.
\end{aligned}$$

By (3.11), we have

$$\begin{aligned}
&R_{j_t}(A)-R_t(A/_\circ\alpha)\\
&=\sum_{u=1}^{t-1}\left(\frac{R_{j_u}(A)}{|a_{j_u,j_u}|}-\frac{R_u(A/_\circ\alpha)}{|a'_{uu}|}\right)|a_{j_t,j_u}|+\sum_{i_v<j_t}\frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_t,i_v}|+\sum_{i_v>j_t}|a_{j_t,i_v}|\\
&\geq\sum_{i_v<j_t}\frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_t,i_v}|+\sum_{i_v>j_t}|a_{j_t,i_v}|,
\end{aligned}$$

i.e.,

$$R_t(A/_\circ\alpha)\leq R_{j_t}(A)-\left(\sum_{i_v<j_t}\frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_t,i_v}|+\sum_{i_v>j_t}|a_{j_t,i_v}|\right),$$

which implies that (3.7) holds.

By Lemma 2.1, we have

$$|a'_{tt}|=|a_{j_t,j_t}-x_t^T[A(\alpha)]^{-1}y_t|\geq|a_{j_t,j_t}|-|x_t^T|[\mu(A(\alpha))]^{-1}|y_t|. \tag{3.12}$$

We consider two cases: $j_t<i_k$ and $j_t>i_k$.

Case 1: $j_t<i_k$. If $\sum\limits_{i_v<j_t}\frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_t,i_v}|+\sum\limits_{i_v>j_t}|a_{j_t,i_v}|=0$, by (3.2), (3.7), and (3.12), we know (3.8) holds trivially. Otherwise, by (3.1) and (3.12), we have

$$|a'_{tt}|\geq|a_{j_t,j_t}|-\frac{|a_{j_t,j_t}|}{R_{j_t}(A)}\left(\sum_{i_v<j_t}\frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_t,i_v}|+\sum_{i_v>j_t}|a_{j_t,i_v}|\right).$$

Then it can be deduced from (3.7) that

$$\frac{R_t(A/_\circ\alpha_k)}{|a'_{tt}|}\leq\frac{R_{j_t}(A)-\left(\sum\limits_{i_v<j_t}\frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_t,i_v}|+\sum\limits_{i_v>j_t}|a_{j_t,i_v}|\right)}{|a_{j_t,j_t}|-\frac{|a_{j_t,j_t}|}{R_{j_t}(A)}\left(\sum\limits_{i_v<j_t}\frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_t,i_v}|+\sum\limits_{i_v>j_t}|a_{j_t,i_v}|\right)}.$$

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By Lemma 2.4, we get (3.8) holds under the case $j_t < i_k$.

Case 2: $j_t > i_k$, i.e., $j_t > i_v$ for all $v \in \langle k\rangle$. It follows from (3.3) and (3.12) that

$$|a'_{tt}| \geq |a_{j_t,j_t}| - \sum_{v=1}^{k} \frac{R_v(D_t)}{|a_{i_v,i_v}|}|a_{j_t,i_v}|,$$

which, together with (2.10) and (3.7), implies that

$$\frac{R_t(A/_\circ\alpha)}{|a'_{tt}|} \leq \frac{R_{j_t}(A) - \sum\limits_{v=1}^{k} \frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_t,i_v}|}{|a_{j_t,j_t}| - \sum\limits_{v=1}^{k} \frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_t,i_v}|} \leq \frac{R_{j_t}(A)}{|a_{j_t,j_t}|}.$$

We get (3.8) holds under the case $j_t > i_k$.

Now we have already proved that inequalities in (3.7) and (3.8) always hold for any $t \in \langle l\rangle$. The proof is completed. ∎

Corollary 3.1. *Let $A \in N_n$ and $\alpha \subset \langle n\rangle$. Then $A/_\circ\alpha \in N_{n-|\alpha|}$.*

Now we study Nekrasov diagonally dominant degrees of $A/_\circ\alpha$.

Theorem 3.2. *Let $A \in N_n$ and let $\alpha = \{i_1, i_2, \cdots, i_k\}$ and $\bar{\alpha} = \{j_1, j_2, \cdots, j_l\}$ $(l = n-k)$ be defined in (2.1) and (2.2), respectively. Denote $A/_\circ\alpha = (a'_{tu})$. Then*

$$|a'_{tt}| - R_t(A/_\circ\alpha) > 0, \quad t \in \langle n-k\rangle.$$

Moreover, given $j_t \in \bar{\alpha}$ with $j_t > i_k$, then

$$|a'_{tt}| - R_t(A/_\circ\alpha) \geq |a_{j_t,j_t}| - R_{j_t}(A) + g(t) \geq |a_{j_t,j_t}| - R_{j_t}(A),$$

where D_t is given by (2.9) and

$$g(t) = \sum_{v=1}^{k} \frac{|a_{j_t,i_v}|}{|a_{i_v,i_v}|}(R_{i_v}(A) - R_v(D_t)) \geq 0.$$

Proof. By Theorem 3.1, we have $A/_\circ\alpha \in N_{n-k}$, then we get $|a'_{tt}| - R_t(A/_\circ\alpha) > 0$ for all $t \in \langle n-k\rangle$. Suppose $j_t > i_k$. We know that $j_t > i_v$ for all $v \in \langle k\rangle$. Combining Lemma 2.1, (3.3) with (3.7), we have

$$\begin{aligned}
&|a'_{tt}| - R_t(A/_\circ\alpha)\\
&= |a_{j_t,j_t} - x_t^T[A(\alpha)]^{-1}y_t| - R_t(A/_\circ\alpha)\\
&\geq |a_{j_t,j_t}| - |x_t^T|[\mu(A(\alpha))]^{-1}|y_t| - R_t(A/_\circ\alpha)\\
&\geq |a_{j_t,j_t}| - \sum_{v=1}^{k} \frac{R_v(D_t)}{|a_{i_v,i_v}|}|a_{j_t,i_v}| - R_{j_t}(A) + \sum_{v=1}^{k} \frac{R_{i_v}(A)}{|a_{i_v,i_v}|}|a_{j_t,i_v}|\\
&= |a_{j_t,j_t}| - R_{j_t}(A) + g(t)\\
&\geq |a_{j_t,j_t}| - R_{j_t}(A).
\end{aligned}$$

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Remark 3.1. Theorem 3.2 suggests that $|a'_{tt}| - R_t(A/_\circ\alpha) > 0$ always holds for all $t \in \langle n-k \rangle$. If $j_t < i_k$, both $|a'_{tt}| - R_t(A/_\circ\alpha) \leq |a_{j_t,j_t}| - R_{j_t}(A)$ and $|a'_{tt}| - R_t(A/_\circ\alpha) \geq |a_{j_t,j_t}| - R_{j_t}(A)$ may occur. If $j_t > i_k$, it always holds that $|a'_{tt}| - R_t(A/_\circ\alpha) \geq |a_{j_t,j_t}| - R_{j_t}(A)$.

Corollary 3.2. *Let $A \in N_n$ and $\alpha = \langle k \rangle$ where k is a positive integer with $k < n$. Denote $A/_\circ\alpha = (a'_{tu})$. Then*

$$|a'_{tt}| - R_t(A/_\circ\alpha) \geq |a_{k+t,k+t}| - R_{k+t}(A) \text{ for all } t \in \langle n-k \rangle.$$

Example 3.1. *Consider the matrix*

$$A = \begin{bmatrix} 20 & 0 & 10 & 0 & 0 & 0 \\ 4 & 10 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 & 10 \\ 0 & 0 & 0 & -10 & 20 & 0 \\ 0 & 0 & 0 & 9 & 1 & 1 \end{bmatrix}.$$

It is easy to verify that A is a Nekrasov matrix. For any $\alpha \subset \langle 6 \rangle$, by computation, we know that $A/_\circ\alpha \in N_{6-|\alpha|}$ always holds but A/α is not necessarily a Nekrasov matrix (see Table 1).

TABLE 1
The closure properties of Schur and diagonal-Schur complements of the matrix A.

α	$A/_\circ\alpha$	A/α	α	$A/_\circ\alpha$	A/α
$\{3\}$	Yes	No	$\{1,3,6\}$	Yes	No
$\{6\}$	Yes	No	$\{2,3,6\}$	Yes	No
$\{1,6\}$	Yes	No	$\{3,4,5\}$	Yes	No
$\{2,6\}$	Yes	No	$\{3,4,6\}$	Yes	No
$\{3,4\}$	Yes	No	$\{3,5,6\}$	Yes	No
$\{3,5\}$	Yes	No	$\{1,2,3,6\}$	Yes	No
$\{3,6\}$	Yes	No	$\{3,4,5,6\}$	Yes	No
$\{1,2,6\}$	Yes	No	others	Yes	Yes

Example 3.2. *Consider the matrix*

$$A = \begin{bmatrix} A_1 & O & O & O \\ O & A_1 & O & O \\ O & O & A_1 & O \\ I & O & O & A_2 \end{bmatrix},$$

where"O" denotes the 4×4 zero matrix, "I" denotes the 4×4 identity matrix and

$$A_1 = \begin{bmatrix} 8 & 1 & 1 & 0 \\ 12 & 8 & 0 & 1 \\ 5 & 7 & 8 & 0 \\ 16 & 7 & 0 & 8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 8 & 1 & 1 & 0 \\ 12 & 8 & 0 & 1 \\ 5 & 7 & 8 & 0 \\ 16 & 7 & 0 & 10 \end{bmatrix}.$$

It is easy to testify that $A \in N_{16}$. Let $\alpha = \{3,6,9,12\} = \{i_1, i_2, i_3, i_4\}$. Then

$$\bar{\alpha} = \{1,2,4,5,7,8,10,11,13,14,15,16\} = \{j_1, j_2, \ldots, j_{12}\}.$$

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Denote $A = (a_{ij})_{16\times 16}$ *and* $A/_\circ\alpha = (a'_{tu})_{12\times 12}$. *For each* $1 \le t \le 12$, *let*

$$V_t = \frac{R_{j_t}(A)}{|a_{j_t,j_t}|};\ \ v_t = \frac{R_t(A/_\circ\alpha)}{|a'_{tt}|}.$$

In Fig.1, the red "○" denotes the value of V_t *and the blue "○" denotes the value of* v_t. *It can be seen from Fig.1 that*

$$0 \le \frac{R_t(A/_\circ\alpha)}{|a'_{tt}|} = v_t \le V_t = \frac{R_{j_t}(A)}{|a_{j_t,j_t}|} < 1,\ \ 1 \le t \le 12,$$

which is consistent with (3.8) *in Theorem* 3.1.

For each $1 \le t \le 12$, *we know that* $j_t < i_4 = 12$ *if and only if* $1 \le t \le 8$. *Let*

$$U_t = |a_{j_t,j_t}| - R_{j_t}(A);\ \ u_t = |a'_{tt}| - R_t(A/_\circ\alpha).$$

In Fig.2, the red "∗" denotes the value of U_t *and the blue "∗" denotes the value of* u_t. *It can be seen from Fig.2 that both* $U_t \ge u_t$ *and* $U_t \le u_t$ *may occur for* $1 \le t \le 8$ *and only* $U_t \le u_t$ *occurs for* $9 \le t \le 12$. *In fact, we can see that* $U_4 > u_4$ *and* $U_t \le u_t$ *for* $t \ne 4$. *This is consistent with Theorem* 3.2.

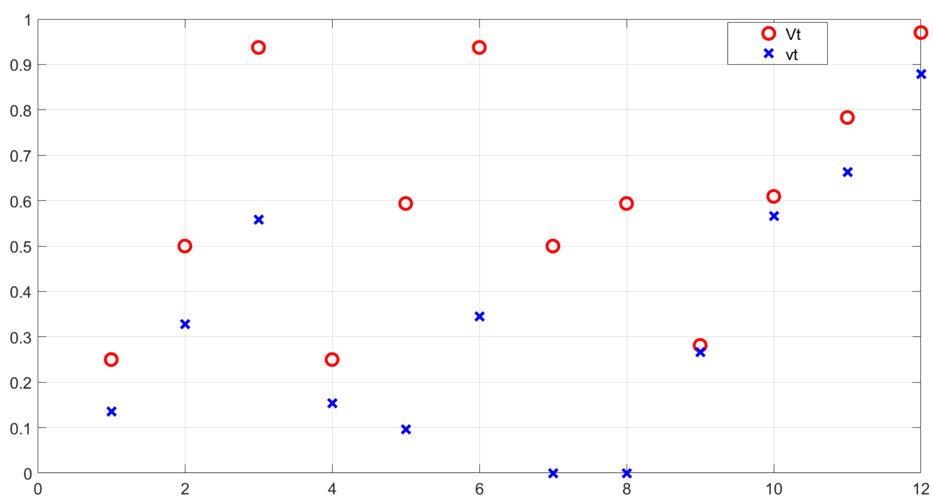

FIG. 1. *The comparison of* V_t *and* v_t.

4. Diagonal-Schur complements of Σ-Nekrasov matrices. This section discusses the diagonal-Schur complements for Σ-Nekrasov matrices by using scaling matrices.

Lemma 4.1. [6] *Let* $A \in C^{n\times n}$. *Then* A *is a* Σ*-Nekrasov matrix if and only if there exists a diagonal matrix* $W = diag\{w_1, \dots, w_n\}$ *such that* AW *is a Nekrasov matrix where* $w_i = \gamma > 0$ *for* $i \in S$ *and* $w_i = 1$ *for* $i \in \bar{S}$.

Theorem 4.1. *Let* S *be a nonempty proper subset of* $\langle n\rangle$ *and let* $\alpha = \{i_1, i_2, \cdots, i_k\}$ *and* $\bar{\alpha} = \{j_1, j_2, \cdots, j_l\}$ *be defined in* (2.1) *and* (2.2), *respectively. If* $A = (a_{ij}) \in C^{n\times n}$ *is an* S*-Nekrasov matrix, then* $A/_\circ\alpha$ *is an* $\{s_1, \dots, s_t\}$*-Nekrasov matrix where* $S \cap \bar{\alpha} = \{j_{s_1}, \dots, j_{s_t}\}$.

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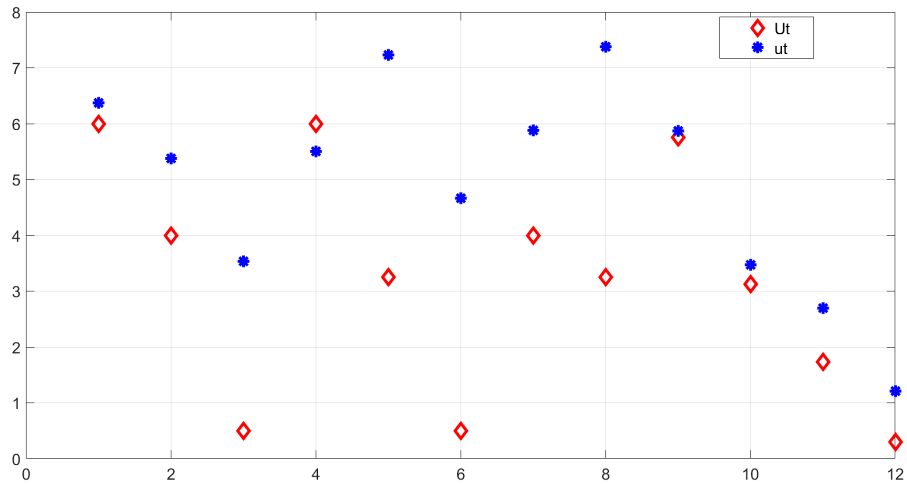

FIG. 2. *The comparison of U_t and u_t.*

Proof. Since $A = (a_{ij})$ is an S-Nekrasov matrix, by Lemma 4.1, there exists a diagonal matrix $W = diag\{w_1, \ldots, w_n\}$ such that AW is a Nekrasov matrix where

$$w_i = \begin{cases} \gamma > 0, & i \in S, \\ 1, & i \in \bar{S}. \end{cases} \tag{4.1}$$

It follows from Theorem 3.1 that the matrix $(AW)/_\circ\alpha$ is a Nekrasov matrix. By [2], we have

$$(AW)/_\circ\alpha = (A/_\circ\alpha)W(\bar{\alpha}). \tag{4.2}$$

Recalling that $\bar{\alpha}$ is the same as in (2.2), we know $W(\bar{\alpha})$ can be written as $W(\bar{\alpha}) = diag\{w_{j_1}, \cdots, w_{j_l}\}$ where w_{j_u} is determined by (4.1), i.e.,

$$w_{j_u} = \begin{cases} \gamma > 0, & j_u \in S \cap \bar{\alpha} = \{j_{s_1}, \ldots, j_{s_t}\}, \\ 1, & j_u \in \bar{S} \cap \bar{\alpha} = \bar{\alpha} - \{j_{s_1}, \ldots, j_{s_t}\}. \end{cases} \tag{4.3}$$

Combining (4.2) and Lemma 4.1, we have $A/_\circ\alpha$ is an $\{s_1, \ldots, s_t\}$-Nekrasov matrix. ∎

Remark 4.1. Generally, the set $\{j_{s_1}, \ldots, j_{s_t}\}$ may not be equal to $\{s_1, \ldots, s_t\}$.

Corollary 4.1. *Let S and α be nonempty proper subsets of $\langle n\rangle$. Let $A = (a_{ij}) \in C^{n\times n}$ be an S-Nekrasov matrix. If $S \subseteq \alpha$ or $\bar{S} \subseteq \alpha$, $A/_\circ\alpha$ is a Nekrasov matrix.*

Proof. It is clear that $S \cap \bar{\alpha} = \emptyset$ if $S \subseteq \alpha$. Then by (4.3), $W(\bar{\alpha}) = I$ and hence $A/_\circ\alpha = (AW)/_\circ\alpha$ is a Nekrasov matrix. It is clear that $S \cap \bar{\alpha} = \bar{\alpha}$ if $\bar{S} \subseteq \alpha$. Analogously, we can obtain that $A/_\circ\alpha = \gamma^{-1}(AW)/_\circ\alpha$ is a Nekrasov matrix. ∎

Example 4.1. *Consider the following matrix:*

$$A = \begin{bmatrix} 5 & 4 & 2 & 0 & 0 \\ 0 & 5 & 0 & 0 & 2 \\ 5 & 0 & 10 & 0 & 0 \\ 0 & 15 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

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It is clear that $A = (a_{ij})_{5\times 5}$ *is not a Nekrasov matrix, but it is an* S*-Nekrasov matrix for* $S = \{2,4,5\}$ *because it can be computed that*

$$R_2^{\bar{S}}(A) = R_4^{\bar{S}}(A) = R_5^{\bar{S}}(A) = R_5^{S}(A) = 0, \quad R_1^{S}(A) = R_3^{S}(A) = 4,$$

$$R_1^{\bar{S}}(A) = R_2^{S}(A) = R_3^{\bar{S}}(A) = 2, \quad R_4^{S}(A) = 6,$$

and for all $i \in S = \{2,4,5\}$ *and* $j \in \bar{S} = \{1,3\}$*, it holds that*

$$|a_{ii}| > R_i^{S}(A),\ |a_{jj}| > R_j^{\bar{S}}(A),$$

$$[|a_{ii}| - R_i^{S}(A)][|a_{jj}| - R_j^{\bar{S}}(A)] > R_i^{\bar{S}}(A)R_j^{S}(A).$$

Take $\alpha = \{3,5\}$*, then* $\bar{\alpha} = \{j_1, j_2, j_3\} = \{1,2,4\}$*. By computation, we get*

$$A/_{\circ}\alpha = \begin{bmatrix} 4 & 4 & 0 \\ 0 & 5 & 0 \\ 0 & 15 & 10 \end{bmatrix}.$$

It is clear that $A/_{\circ}\alpha = (a'_{tu})_{3\times 3}$ *is not a Nekrasov matrix and* $S \cap \bar{\alpha} = \{2,4\} = \{j_2, j_3\}$*. Denote* $S^* = \{2,3\}$*. By computations, we have*

$$R_1^{\bar{S^*}}(A/_{\circ}\alpha) = R_2^{\bar{S^*}}(A/_{\circ}\alpha) = R_2^{S^*}(A/_{\circ}\alpha) = R_3^{\bar{S^*}}(A/_{\circ}\alpha) = R_3^{S^*}(A/_{\circ}\alpha) = 0$$

and

$$R_1^{S^*}(A/_{\circ}\alpha) = 4.$$

Then it holds that

$$|a'_{22}| > R_2^{S^*}(A/_{\circ}\alpha), |a'_{33}| > R_3^{S^*}(A/_{\circ}\alpha), |a'_{11}| > R_1^{\bar{S^*}}(A/_{\circ}\alpha),$$

and

$$\left[|a'_{22}| - R_2^{S^*}(A/_{\circ}\alpha)\right]\left[|a'_{11}| - R_1^{\bar{S^*}}(A/_{\circ}\alpha)\right] > R_2^{\bar{S^*}}(A/_{\circ}\alpha)R_1^{S^*}(A/_{\circ}\alpha),$$

$$\left[|a'_{33}| - R_3^{S^*}(A/_{\circ}\alpha)\right]\left[|a'_{11}| - R_1^{\bar{S^*}}(A/_{\circ}\alpha)\right] > R_3^{\bar{S^*}}(A/_{\circ}\alpha)R_1^{S^*}(A/_{\circ}\alpha).$$

Hence, $A/_{\circ}\alpha$ *is a* $\{2,3\}$*-Nekrasov matrix, which is consistent with Theorem 4.1. Remark that* $A/_{\circ}\alpha$ *is not a* $\{j_2, j_3\}$*-Nekrasov matrix obviously.*

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