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THE INVERSE OF A SYMMETRIC NONNEGATIVE MATRIX CAN BE COPOSITIVE*

ROBERT B. REAMS†

Abstract. Let A be an $n \times n$ symmetric matrix. We first show that if A and its pseudoinverse are strictly copositive, then A is positive semidefinite, which extends a similar result of Han and Mangasarian. Suppose A is invertible, as well as being symmetric. We showed in an earlier paper that if A^{-1} is nonnegative with n zero diagonal entries, then A can be copositive (for instance, this happens with the Horn matrix), and when A is copositive, it cannot be of form $P + N$, where P is positive semidefinite and N is nonnegative and symmetric. Here, we show that if A^{-1} is nonnegative with $n - 1$ zero diagonal entries and one positive diagonal entry, then A can be of the form $P + N$, and we show how to construct A. We also show that if A^{-1} is nonnegative with one zero diagonal entry and $n - 1$ positive diagonal entries, then A cannot be copositive.

Key words. Nonnegative matrix, Positive semidefinite matrix, Copositive matrix, Exceptional matrix, Pseudoinverse.

AMS subject classifications. 15A10, 15A63, 15B48.

1. Introduction. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *nonnegative*, if every entry of A is nonnegative. Similarly, a vector $x \in \mathbb{R}^n$ is *nonnegative* if all its components are nonnegative, and this is denoted $x \geq 0$. For $1 \leq i \leq n$, we will use e_i to denote the vector in $\mathbb{R}^n$ with a 1 in the ith position and all other entries zero. We will use e to denote the vector with all its n components equal to 1. A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is said to be *positive semidefinite* if $x^T Ax \geq 0$, for all $x \in \mathbb{R}^n$, while A is *positive definite* if $x^T Ax > 0$, for all $x \in \mathbb{R}^n$ such that $x \neq 0$. A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is said to be *copositive* if $x^T Ax \geq 0$, for all $x \in \mathbb{R}^n$ such that $x \geq 0$, while A is *strictly copositive* if $x^T Ax > 0$, for all $x \geq 0$, $x \neq 0$.

Clearly, for $A \in \mathbb{R}^{n\times n}$, if A is the sum of a positive semidefinite matrix P and a symmetric nonnegative matrix N, then A is copositive. However, it can be shown that not all copositive matrices are of the form $P + N$. Copositive matrices that are not of form $P + N$ are called *exceptional*. Diananda [6] proved there are no exceptional matrices when $n \leq 4$, i.e., every copositive matrix is of the form $P + N$, when $n \leq 4$. The most well-known example of an exceptional matrix is the Horn matrix, which is 5×5 [12, 17]. For a vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, we will denote the Euclidean norm by $||x||_2 = \sqrt{\sum_{i=1}^n x_i^2} = \sqrt{x^T x}$, and the one-norm by $||x||_1 = \sum_{i=1}^n |x_i|$, so that $||x||_1 = x^T e$, when $x \geq 0$. It can be shown ([10, 13, 21]) for a copositive matrix A that if we have $x^T Ax = 0$ for $x \geq 0$, then $Ax \geq 0$. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *interior* if $\min_{x\geq 0, ||x||_1=1} x^T Ax$ is achieved in the interior of $\{x \in \mathbb{R}^n | x \geq 0, ||x||_1 = 1\}$. In other words, there is a minimizing vector u with all positive components. (See [12], for instance, for some results about interior matrices.) $\mathcal{R}(A)$ will denote the *range* of a matrix, that is, to say $\mathcal{R}(A) = \{Ax | x \in \mathbb{R}^n\}$.

In Section 2, we extend a result of Han and Mangasarian. In Section 3, we present a copositive construction of the inverse of a nonnegative matrix with $n - 1$ zero diagonal entries and one positive diagonal entry. In Section 4, we show that it is not possible for the inverse of a nonnegative matrix, which has $n - 1$ positive diagonal entries and one zero diagonal entry, to be copositive. In Section 5, we consider some other special cases.

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†Department of Mathematics, SUNY Plattsburgh, Plattsburgh, New York 12901, USA (robert.reams@plattsburgh.edu).

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2. A result of Han and Mangasarian extended. In [9], Han and Mangasarian proved that if A and A^{-1} are strictly copositive, then A is positive definite. The pseudoinverse we shall use is the Moore–Penrose generalized inverse, denoted $A^\dagger$. The four defining properties of $A^\dagger$ are $A^\dagger AA^\dagger = A$, $AA^\dagger A = A$, $AA^\dagger$ and $A^\dagger A$ are symmetric, for which $A^\dagger$ is unique [15]. In fact, when A is symmetric, it can be seen that $A^\dagger = U^T\text{diag}(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_k}, 0, \ldots, 0)U$, where $A = U^T\text{diag}(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)U$, with U an orthogonal matrix and $\lambda_1, \ldots, \lambda_k$, the nonzero eigenvalues of A. The pseudoinverse of a copositive matrix was studied in [20]. Theorem 2.1 will use a result by Marshall and Olkin [18], which states that any strictly copositive matrix A is scalable. *Scalable* means that there is a diagonal matrix D, with positive diagonal entries, such that DAD has row (and column) sums 1. Theorem 2.1 will also make use of the result that if a matrix $A \in \mathbb{R}^{n\times n}$ is copositive and interior, then A is positive semidefinite. (See Corollary 2 of [12], or Lemma 1 of [6].) A useful observation quoted by Hiriart-Urruty and Seeger (Theorem 7.6 of [11]), as part of a theorem proved by Han and Mangasarian (although they do not state it this way), follows after making the assumption that A and A^{-1} are both copositive: one of them is strictly copositive if and only if the other is. To see this, write $0 = x^TAx = x^TAA^{-1}Ax$, and remember that $x^TAx = 0$ implies $Ax \geq 0$. Replacing A^{-1} with $A^\dagger$ does not change the observation.

Theorem 2.1. *Let $A \in \mathbb{R}^{n\times n}$. If A and $A^\dagger$ are strictly copositive, then A is positive semidefinite.*

Proof. Since A is strictly copositive, we know that there is a diagonal matrix D, with positive diagonal entries, such that $DADe = e$. Write $DAD = V^T\Lambda V$, where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$ and V is orthogonal. Then $(DAD)^\dagger = V^T\Lambda^\dagger V$, where $\Lambda^\dagger = \text{diag}(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_k}, 0, \ldots, 0)$, and $(DAD)^\dagger e = e$. In other words, without loss of generality when proving the theorem we may assume that $Ae = e$ and $A^\dagger e = e$.

Next, let $\min_{x\geq 0, ||x||_1=1} x^TAx = \lambda$. Then $\lambda > 0$, since A is strictly copositive. Also, $(\frac{x}{x^Te})^TA(\frac{x}{x^Te}) \geq \lambda$, for all $x \geq 0$, i.e., $x^TAx \geq \lambda(x^Te)^2$, which can be rewritten as $x^T(A - \lambda ee^T)x \geq 0$, and thus $A - \lambda ee^T$ is copositive. Now let u be a minimizing vector, so that $u^T(A - \lambda ee^T)u = 0$, then $z = (A - \lambda ee^T)u \geq 0$.

Since $A^\dagger$ is copositive (actually strictly), we have

$$\begin{aligned} z^TA^\dagger z &= u^T(A - \lambda ee^T)A^\dagger(A - \lambda ee^T)u, \\ &= u^T(AA^\dagger A - \lambda ee^T - \lambda ee^T + \lambda^2 nee^T)u, \\ &= u^T(A - \lambda ee^T)u + \lambda(u^Te)^2(n\lambda - 1), \\ &= 0 + \lambda(n\lambda - 1) \geq 0. \end{aligned}$$

Then $n\lambda - 1 \geq 0$, and $\lambda \geq \frac{1}{n}$. But also $(\frac{1}{n}e)^TA(\frac{1}{n}e) \geq \lambda$, giving $\frac{1}{n} = \frac{e^Te}{n^2} \geq \lambda$, so that $\lambda = \frac{1}{n}$. We have just shown that $z = (A - \lambda ee^T)u = 0$, i.e., $Au = \frac{1}{n}e^Tue = \frac{1}{n}e$. The minimizing vector u may have zero components, but as $\frac{1}{n}e$ is (also) a minimizing vector, A is copositive and interior, and therefore positive semidefinite. ∎

Unlike for Han and Mangasarian's result, the converse of Theorem 2.1 does not hold, as can be seen by considering $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. The matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, for which $A^\dagger = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 5 & 4 \\ 1 & 4 & 5 \end{pmatrix}$, illustrates that it is possible to have both A and $A^\dagger$ strictly copositive, and this situation is not covered by Han and Mangasarian's result. Another way to state these results is to say that if we

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want to characterize those matrices for which both A and A^{-1} (or $A^\dagger$) are strictly copositive, we need look no further than the positive (semi)definite matrices. The problem of characterizing the matrices for which both A and A^{-1} (or $A^\dagger$) are copositive, but not strictly copositive, seems to be unsolved. (See Section 7.4 of [11].) If we only consider when A and A^{-1} are both nonnegative, then A is the product of a permutation matrix and a diagonal matrix with positive diagonal entries (see [8], or Lemma 1.1 of Minc [16]). Such a matrix A is called a *generalized permutation matrix*. If A and $A^\dagger$ are both nonnegative, then A has been characterized in Theorem 5.2 of Berman and Plemmons [2]. It is well known that if A is copositive with all zero diagonal entries then A must be nonnegative. (This follows from $e_i^T Ae_i = 0$, for each i, implies $Ae_i \geq 0$.) In [14], we showed that if A^{-1} is a nonnegative matrix with all zero diagonal entries, then A may or may not be copositive, but when A is copositive it cannot be of the form $P + N$, i.e., A must be exceptional. Symmetric nonnegative matrices with all zeroes on the diagonal are studied in [5], where such matrices are called *hollow*. It is easy to show that any nonnegative matrix, not necessarily invertible, with all positive diagonal entries must be strictly copositive. If A^{-1} is nonnegative with all positive diagonal entries, then A^{-1} is strictly copositive, and if in addition A is copositive, then from Han and Mangasarian's result, A is positive definite. The consideration of what happens to A as we make assumptions with some of the diagonal entries of nonnegative A^{-1} being zero or positive leads us to the theorems of Sections 3 and 4.

3. When the inverse of a nonnegative matrix is of form $\boldsymbol{P+N}$. For the proof of Theorem 3.1, we recall the well-known fact [15] for invertible $A \in \mathbb{R}^{n\times n}$, and $a, b \in \mathbb{R}^n$, such that $1 + b^T A^{-1}a \neq 0$, that when $A + ab^T$ is invertible, it can be written $(A + ab^T)^{-1} = A^{-1} - \frac{A^{-1}ab^T A^{-1}}{1+b^T A^{-1}a}$. Theorem 3.1 characterizes the invertible, nonnegative matrices A^{-1}, with $n - 1$ zero diagonal entries and one positive diagonal entry, such that A is copositive of form $P + N$.

Theorem 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric and invertible. Suppose that A^{-1} is nonnegative with $n - 1$ zero diagonal entries and one positive diagonal entry, and A is of form $P + N$ (without loss of generality with N having all zeroes on the diagonal). Then, under permutation similarity, the positive semidefinite matrix P has form $P = \lambda uu^T$, where $\lambda > 0$, and u has its first $n-1$ components (when they are nonzero) having the same sign (as each other), and its nth component of opposite sign, while the symmetric nonnegative matrix N has its upper left $(n - 1) \times (n - 1)$ block as a generalized permutation matrix, with all zeros elsewhere in N. Also, n must be odd.*

Proof. Suppose that A^{-1} is nonnegative with exactly $n - 1$ zeroes on the diagonal, and without loss of generality (as this can be achieved with a permutation similarity) such that the lone positive diagonal entry lies in the bottom right corner of A^{-1}. If $N = 0$, then A would have to be positive definite, but then A^{-1} could not have any zeros on the diagonal. If $P = 0$, then N would have to be a generalized permutation matrix with a positive entry in the bottom right-hand corner, and the theorem holds, just that $u = e_n$. Suppose, for what follows, that we do not have the latter situation.

Now, $0 = e_i^T A^{-1}e_i = e_i^T A^{-1}AA^{-1}e_i = e_i^T A^{-1}(P + N)A^{-1}e_i$, implies $Px_i = 0$, for each $i = 1, \ldots, n - 1$, where $x_i = A^{-1}e_i$. It follows that P has rank one, and $P = \lambda uu^T$, for some $\lambda > 0$ and $u \in \mathbb{R}^n$. Because $\lambda uu^T A^{-1}e_i = 0$, for each $i = 1, \ldots, n-1$, this implies $u^T A^{-1}e_i = 0$, for each $i = 1, \ldots, n-1$, and $u^T A^{-1}e_n > 0$ (actually $u^T A^{-1}e_n \neq 0$, but we can replace u with $-u$ to achieve this). Next, $u^T A^{-1} = (0, \ldots, 0, u^T A^{-1}e_n)$, or rewritten as a column vector $A^{-1}u = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ u^T A^{-1}e_n \end{pmatrix} = (u^T A^{-1}e_n)e_n$. Then

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$$\begin{aligned} u &= (N+\lambda uu^T)(u^TA^{-1}e_n)e_n, \\ &= (u^TA^{-1}e_n)Ne_n + \lambda(u^Te_n)(u^TA^{-1}e_n)u, \end{aligned}$$

and rearranging gives $[1-\lambda(u^Te_n)(u^TA^{-1}e_n)]u = (u^TA^{-1}e_n)Ne_n$. If $Ne_n \neq 0$, then $u \in \mathcal{R}(N)$, and setting $\mu = \frac{u^TA^{-1}e_n}{1-\lambda(u^Te_n)(u^TA^{-1}e_n)}$, we have $u = \mu Ne_n$, and $A = N+\lambda uu^T = N+\lambda\mu^2Ne_ne_n^TN = N(I+\lambda\mu^2e_ne_n^TN)$. If N were singular, so would A be singular, so we must have N nonsingular. But then $(N+\lambda uu^T)^{-1} = N^{-1} - \frac{\lambda N^{-1}uu^TN^{-1}}{1+\lambda u^TN^{-1}u}$, as $u^TN^{-1}u = \mu^2e_n^TNe_n = 0$, which implies $1+\lambda u^TN^{-1}u > 0$. Then consider $e_i^TNA^{-1}Ne_i = e_i^TN(N+\lambda uu^T)^{-1}Ne_i = e_i^TN[N^{-1} - \frac{\lambda N^{-1}uu^TN^{-1}}{1+\lambda u^TN^{-1}u}]Ne_i = e_i^TNe_i - \frac{\lambda(u^Te_i)^2}{1+\lambda u^TN^{-1}u} = -\frac{\lambda(u^Te_i)^2}{1+\lambda u^TN^{-1}u} < 0$, for some $i \in \{1,\ldots,n-1\}$, which contradicts A^{-1} being a nonnegative matrix. In other words, we must have $Ne_n = 0$.

Finally, write $u = \begin{pmatrix} \hat{u} \\ u_n \end{pmatrix}$, where $\hat{u} \in \mathbb{R}^{n-1}$, $u_n \in \mathbb{R}$, and $N = \begin{pmatrix} \hat{N} & 0 \\ 0 & 0 \end{pmatrix}$, where $\hat{N} \in \mathbb{R}^{(n-1)\times(n-1)}$. Then $A = \begin{pmatrix} \hat{N}+\lambda\hat{u}\hat{u}^T & \lambda u_n\hat{u} \\ \lambda u_n\hat{u}^T & \lambda u_n^2 \end{pmatrix}$, and because A is invertible we must have $u_n \neq 0$. If $\hat{N}$ were singular, with zero eigenvector $\hat{v}$ (say), then $\begin{pmatrix} \hat{v} \\ \frac{-\hat{u}^T\hat{v}}{u_n} \end{pmatrix}$, would be a zero eigenvector for A, which is not possible. Thus, $\hat{N}$ is nonsingular. Since $\begin{pmatrix} \hat{N}+\lambda\hat{u}\hat{u}^T & \lambda u_n\hat{u} \\ \lambda u_n\hat{u}^T & \lambda u_n^2 \end{pmatrix}^{-1} = \begin{pmatrix} \hat{N}^{-1} & -\frac{1}{u_n}\hat{N}^{-1}\hat{u} \\ -\frac{1}{u_n}\hat{u}^T\hat{N}^{-1} & \frac{1+\lambda\hat{u}^T\hat{N}^{-1}\hat{u}}{\lambda u_n^2} \end{pmatrix}$ is nonnegative, then $\hat{N}$ must be a generalized permutation matrix. Also, the nonzero components of $\hat{u}$ must all have the same sign, with u_n of opposite sign. Since $\hat{N}^{-1}$ has all zeros on its diagonal, n is odd. $\square$

Determining whether there are exceptional matrices A, for which A^{-1} is nonnegative with exactly $n-1$ zero entries on the diagonal, even when $n = 5$ or 6 (see [1, 7]), is beyond the scope of this paper.

4. When the inverse of a nonnegative matrix is not copositive. If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a symmetric, nonnegative matrix with all positive diagonal entries (and therefore strictly copositive), and $D = \mathrm{diag}(\frac{1}{\sqrt{a_{11}}},\ldots,\frac{1}{\sqrt{a_{nn}}})$, then DAD has all diagonal entries equal to 1. Replacing DAD with A, we have

$$Ax = \begin{pmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{12} & 1 & a_{23} & \cdots & a_{2n} \\ a_{13} & a_{23} & 1 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \geq \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

In other words, for such an A, we have $Ax \geq x$, for all $x \geq 0$. We will use this below. Theorem 4.1 was proved in [13]. (See also [3, 4, 19])

Theorem 4.1. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric, and $A = \begin{pmatrix} A_1 & b \\ b^T & c \end{pmatrix}$, where $A_1 \in \mathbb{R}^{(n-1)\times(n-1)}$, $b \in \mathbb{R}^{n-1}$, and $c \in \mathbb{R}$. Then A is copositive if and only if $c \geq 0$; A_1 is copositive; if $c > 0$ then $x^T(A_1 - \frac{bb^T}{c})x \geq 0$, for all $x \geq 0$, with $x \in \mathbb{R}^{n-1}$, such that $x^Tb \leq 0$; if $c = 0$ then $b \geq 0$.*

Theorem 4.1 is needed to prove Theorem 4.2.

Theorem 4.2. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric and invertible. Suppose that A^{-1} is nonnegative with one zero diagonal entry and $n-1$ positive diagonal entries. Then A cannot be copositive.*

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Proof. Suppose, for the sake of obtaining a contradiction, that A is copositive. With A^{-1} nonnegative and having only one zero on the diagonal, under permutation similarity and positive diagonal congruence we may assume A^{-1} has the form $\begin{pmatrix} A_1 & b \\ b^T & 0 \end{pmatrix}$, where $A_1 \in \mathbb{R}^{(n-1)\times(n-1)}$, $b \in \mathbb{R}^{n-1}$, and nonnegative A_1 has all ones on the diagonal (so A_1 is strictly copositive).

A_1 must be nonsingular, since suppose A_1 is singular, then $z^T A_1 = 0$, for nonzero $z \in \mathbb{R}^{n-1}$. Next, since there is $\begin{pmatrix} B_1 & b_2 \\ b_2^T & c \end{pmatrix}$, with $B_1 \in \mathbb{R}^{(n-1)\times(n-1)}$, $b_2 \in \mathbb{R}^n$, $c \geq 0$, such that $\begin{pmatrix} A_1 & b \\ b^T & 0 \end{pmatrix}\begin{pmatrix} B_1 & b_2 \\ b_2^T & c \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix}$, this implies $A_1 b_2 + cb = 0$. But $z^T A_1 b_2 + cz^T b = 0$, means $cz^T b = 0$, so $z^T b = 0$ or $c = 0$. We also have that $A_1 B_1 + bb_2^T = I$, then $z^T A_1 B_2 + z^T bb_2 = z$. If $z^T b = 0$, the preceding equation implies $z = 0$, so we are left with $c = 0$. But $c = e_n^T A e_n = 0$ implies $Ae_n \geq 0$, which implies $b_2 \geq 0$. Then $A_1 b_2 = 0$ implies $b_2^T A_1 b_2 = 0$, which is not possible since A_1 is strictly copositive, unless $b_2 = 0$. But then $c = 0$ would make A singular. So we must have that A_1 is nonsingular. Moreover, $b_2 = -cA^{-1}b$ and $1 = b^T b_2 = -cb^T A^{-1} b$, implies that $b^T A_1^{-1} b < 0$.

We also have $A = \begin{pmatrix} A_1^{-1} - \frac{A_1^{-1} bb^T A_1^{-1}}{b^T A^{-1} b} & \frac{A_1^{-1} b}{b^T A_1^{-1} b} \\ \frac{b^T A_1^{-1}}{b^T A_1^{-1} b} & \frac{-1}{b^T A_1^{-1} b} \end{pmatrix}$. From Theorem 4.1, since A is copositive, we know that $A_1^{-1} - \frac{A_1^{-1} bb^T A_1^{-1}}{b^T A^{-1} b}$ is copositive, and $x^T A_1^{-1} x \geq 0$, for all $x \geq 0$ such that $x^T A_1^{-1} b \geq 0$.

Finally, consider the nonnegative vector $z = A_1 b - b$, for which $z^T A_1^{-1} b = b^T b - b^T A^{-1} b \geq 0$. Then we know $z^T A_1^{-1} z = b^T (A_1 - I) A_1^{-1} (A_1 - I) b = b^T (A_1 + A_1^{-1} - 2I) b \geq 0$, which can be rearranged to say $b^T A_1 b - 2b^T b \geq -b^T A_1^{-1} b > 0$, i.e., $b^T A_1 b > 2b^T b$, or $\frac{b^T A_1 b}{b^T b} > 2$. Let $x = \max_{b_i > 0} \frac{(A_1 b)_i}{b_i} = \max_{b_i > 0} \frac{b_i (A_1 b)_i}{b_i^2}$, so that $(A_1 b)_i \geq x b_i$, for all $i \in \{1, \ldots, n-1\}$. We also have that $b^T A_1 b = \sum_{i=1}^n b_i (A_1 b)_i = \sum_{j=i_1}^{i_k} b_j^2 \frac{b_j (A_1 b)_j}{b_j^2}$, where we only include the positive b_j's in the sum (k of them, say). Then $b^T A_1 b \leq (b_{i_1}^2 + \cdots + b_{i_k}^2) \max_{b_i > 0} \frac{b_i (A_1 b)_i}{b_i^2} = b^T b x$, i.e., $x \geq \frac{b^T A_1 b}{b^T b} > 2$. Now we notice that $Ab \geq xb \geq 2b$ and redefine the nonnegative vector z as $z = A_1 b - 2b$, for which we have $z^T A_1^{-1} b = b^T b - 2b^T A_1^{-1} b \geq 0$. Then (again) we know that $z^T A_1^{-1} z = b^T (A_1 - 2I) A_1^{-1} (A_1 - 2I) b = b^T (A_1 + 4A_1^{-1} - 4I) b \geq 0$, which (again) can be rearranged to say $b^T A_1 b - 4b^T b \geq -4b^T A_1^{-1} b > 0$, i.e., $b^T A_1 b > 4b^T b$, or $\frac{b^T A_1 b}{b^T b} > 4$, and continuing like this we arrive at a contradiction. So, A must not be copositive. ∎

5. Concluding remarks. The author does not know how to improve on Theorem 3.1 to the case where there are exactly k zeroes on the diagonal, and $1 < k < n-1$, without additional assumptions on the nonnegative A^{-1}. If we could make assumptions on the entries of A^{-1} to conclude $A = \begin{pmatrix} \hat{N} + UU^T & UV^T \\ VU^T & VV^T \end{pmatrix}$, where V is invertible, then $A^{-1} = \begin{pmatrix} \hat{N}^{-1} & -\hat{N}^{-1} UV^{-1} \\ -(V^{-1})^T U^T \hat{N}^{-1} & (VV^T)^{-1} + (UV^{-1})^T \hat{N}^{-1} (UV^{-1}) \end{pmatrix}$, would enable us to deduce an analogous description of the form of A^{-1}. Or, Theorem 5 of [14] would be a different way to make assumptions on A^{-1}.

It does not appear that Theorem 4.2 can be improved on to suppose A^{-1} has two zero diagonal entries and $n-2$ positive diagonal entries, without additional assumptions on the nonnegative A^{-1}, because although

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$\begin{pmatrix} 0 & \frac{1}{4} & 1 & 0 \\ \frac{1}{4} & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -16 & -16 & 12 & 4 \\ -16 & -16 & 4 & 12 \\ 12 & 4 & -1 & -3 \\ 4 & 12 & -3 & -1 \end{pmatrix}$ is not copositive, $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ is copositive. These are not just examples for $n = 4$, because we can easily make these matrices be the upper left 4×4 blocks of $n \times n$ matrices which have lower right blocks as the identity matrix I_{n-4}, and all zeroes elsewhere. Or, making different additional assumptions on nonnegative A^{-1}, suppose $n = 2k$ and $A^{-1} = \begin{pmatrix} 0 & B \\ B^T & C \end{pmatrix}$, where B is $k \times k$ and invertible, and C is nonzero. Then $A = \begin{pmatrix} -(B^T)^{-1}CB^{-1} & (B^T)^{-1} \\ B^{-1} & 0 \end{pmatrix}$, and choosing $\hat{y} = Be$, where $e \in \mathbb{R}^k$, we have $y = \begin{pmatrix} \hat{y} \\ 0 \end{pmatrix} \in \mathbb{R}^n$ nonnegative and $y^T Ay = \hat{y}^T[-(B^T)^{-1}CB^{-1}]\hat{y} = -e^T Ce < 0$, so that A cannot be copositive.

REFERENCES

[1] A. Afonin, R. Hildebrand, and P.J.C. Dickinson. The extreme rays of the 6×6 copositive cone. *J. Global Optim.*, 79:153–190, 2021.

[2] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences.* Classics in Applied Math, vol. 9. SIAM, Philadelphia, 1994.

[3] I. Bomze. Remarks on the recursive structure of copositivity. *J. Inform. Optim. Sci.,* 8(3):243–260, 1987.

[4] I. Bomze. Perron-Frobenius property of copositive matrices, and a block copositivity criterion. *Linear Algebra Appl.,* 429(1):68–71, 2008.

[5] Z.B. Charles, M. Farber, C.R. Johnson, and L. Kennedy-Shaffer. Nonpositive eigenvalues of hollow, symmetric, nonnegative matrices. *SIAM J. Matrix Anal. Appl.,* 34(3):1384–1400, 2013.

[6] P.H. Diananda. On non-negative forms in real variables some or all of which are non-negative. *Proc. Camb. Philos. Soc.,* 58:17–25, 1962.

[7] P.J.C. Dickinson, M. Dür, L. Gijben, and R. Hildebrand. Irreducible elements of the copositive cone. *Linear Algebra Appl.,* 439:1605–1626, 2013.

[8] R. DeMarr. Nonnegative matrices with nonnegative inverses. *Proc. Amer. Math. Soc.*, 35(1):307–308, 1972.

[9] S.-P. Han and O.L. Mangasarian. Conjugate cone characterization of positive definite and semidefinite matrices. *Linear Algebra Appl.*, 56:89–103, 1984.

[10] A.J. Hoffman and F. Pereira. On copositive matrices with −1, 0, 1 entries. *J. Comb. Theory (A)*, 14:302–309, 1973.

[11] J.-B. Hiriart-Urruty and A. Seegar. A variational approach to copositive matrices. *SIAM Rev.*, 52(4):593–629, 2010.

[12] C.R. Johnson and R. Reams. Constructing copositive matrices from interior matrices. *Electron. J. Linear Algebra*, 17:9–20, 2008.

[13] C.R. Johnson and R. Reams. Spectral theory of copositive matrices. *Linear Algebra Appl.*, 395:275–281, 2005.

[14] C.R. Johnson and R. Reams. Sufficient Conditions to be Exceptional. *Spec. Matrices*, 4:67–72, 2016.

[15] P. Lancaster and M. Tismenetsky. *The Theory of Matrices.* Academic Press, New York, 1985.

[16] H. Minc. *Nonnegative Matrices.* Wiley, New York, 1988.

[17] A.W. Marshall, I. Olkin, and B.C. Arnold. *Inequalities: Theory of Majorization and Its Applications.* Second edition. Springer, New York, 2011.

[18] A. W. Marshall and I. Olkin. Scaling of matrices to achieve specified row and column sums. *Numer. Math.*, 12:83–90, 1968.

[19] P. Li and F.Y. Yu. Criteria for copositive matrices of order four. *Linear Algebra Appl.*, 194:109–124, 1993.

[20] N. Shaked-Monderer. On the generalized inverse of a copositive matrix. *Linear Multilinear Algebra*, 68:1384–1394, 2020.

[21] H. Väliaho. Criteria for copositive matrices. *Linear Algebra Appl.*, 81:19–34, 1986.