THE INVERSE OF A SYMMETRIC NONNEGATIVE MATRIX CAN BE COPOSITIVE*

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Abstract. Let A be an $n \times n$ symmetric matrix. We first show that if A and its pseudoinverse are strictly copositive, then A is positive semidefinite, which extends a similar result of Han and Mangasarian. Suppose A is invertible, as well as being symmetric. We showed in an earlier paper that if $A^{-1}$ is nonnegative with n zero diagonal entries, then A can be copositive (for instance, this happens with the Horn matrix), and when A is copositive, it cannot be of form $P + N$, where P is positive semidefinite and N is nonnegative and symmetric. Here, we show that if $A^{-1}$ is nonnegative with $n - 1$ zero diagonal entries and one positive diagonal entry, then A can be of the form $P + N$, and we show how to construct A. We also show that if $A^{-1}$ is nonnegative with one zero diagonal entry and $n - 1$ positive diagonal entries, then A cannot be copositive.

Key words. Nonnegative matrix, Positive semidefinite matrix, Copositive matrix, Exceptional matrix, Pseudoinverse.

AMS subject classifications. 15A10, 15A63, 15B48.

1. Introduction. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be nonnegative, if every entry of A is nonnegative. Similarly, a vector $x \in \mathbb{R}^n$ is nonnegative if all its components are nonnegative, and this is denoted $x \geq 0$. For $1 \leq i \leq n$, we will use $e_i$ to denote the vector in $\mathbb{R}^n$ with a 1 in the $i$th position and all other entries zero. We will use $e$ to denote the vector with all its n components equal to 1. A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is said to be positive semidefinite if $x^TAx \geq 0$, for all $x \in \mathbb{R}^n$, while A is positive definite if $x^TAx > 0$, for all $x \in \mathbb{R}^n$ such that $x \neq 0$. A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is said to be copositive if $x^TAx \geq 0$, for all $x \in \mathbb{R}^n$ such that $x \geq 0$, while A is strictly copositive if $x^TAx > 0$, for all $x \geq 0, x \neq 0$.

Clearly, for $A \in \mathbb{R}^{n \times n}$, if A is the sum of a positive semidefinite matrix $P$ and a symmetric nonnegative matrix $N$, then A is copositive. However, it can be shown that not all copositive matrices are of the form $P + N$. Copositive matrices that are not of form $P + N$ are called exceptional. Diananda [6] proved there are no exceptional matrices when $n \leq 4$, i.e., every copositive matrix is of the form $P + N$, when $n \leq 4$. The most well-known example of an exceptional matrix is the Horn matrix, which is $5 \times 5$ [12, 17]. For a vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, we will denote the Euclidean norm by $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2} = \sqrt{x^T x}$, and the one-norm by $\|x\|_1 = \sum_{i=1}^n |x_i|$, so that $\|x\|_1 = x^Te$, when $x \geq 0$. It can be shown ([10, 13, 21]) for a copositive matrix A that if we have $x^TAx = 0$ for $x \geq 0$, then $Ax \geq 0$. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be interior if $\min_{x \geq 0, \|x\|_1 = 1} x^TAx$ is achieved in the interior of $\{x \in \mathbb{R}^n | x \geq 0, \|x\|_1 = 1\}$. In other words, there is a minimizing vector $u$ with all positive components. (See [12], for instance, for some results about interior matrices.) $\mathcal{R}(A)$ will denote the range of a matrix, that is, to say $\mathcal{R}(A) = \{Ax | x \in \mathbb{R}^n\}$.

In Section 2, we extend a result of Han and Mangasarian. In Section 3, we present a copositive construction of the inverse of a nonnegative matrix with $n - 1$ zero diagonal entries and one positive diagonal entry. In Section 4, we show that it is not possible for the inverse of a nonnegative matrix, which has $n - 1$ positive diagonal entries and one zero diagonal entry, to be copositive. In Section 5, we consider some other special cases.

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*Received by the editors on May 6, 2023. Accepted for publication on September 8, 2023. Handling Editor: Michael Tsatsomeros. Corresponding Author: Robert B. Reams.
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2. A result of Han and Mangasarian extended. In [9], Han and Mangasarian proved that if $A$ and $A^{-1}$ are strictly copositive, then $A$ is positive definite. The pseudoinverse we shall use is the Moore–Penrose generalized inverse, denoted $A^\dagger$. The four defining properties of $A^\dagger$ are $A^\dagger AA^\dagger = A$, $AA^\dagger A = A$, $AA^\dagger$ and $A^\dagger A$ are symmetric, for which $A^\dagger$ is unique [15]. In fact, when $A$ is symmetric, it can be seen that $A^\dagger = U^T \text{diag}(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}, 0, \ldots, 0) U$, where $A = U^T \text{diag}(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0) U$, with $U$ an orthogonal matrix and $\lambda_1, \ldots, \lambda_k$, the nonzero eigenvalues of $A$. The pseudoinverse of a copositive matrix was studied in [20].

Theorem 2.1 will use a result by Marshall and Olkin [18], which states that any strictly copositive matrix $A$ is scalable. Scalable means that there is a diagonal matrix $D$, with positive diagonal entries, such that $DAD$ has row (and column) sums 1. Theorem 2.1 will also make use of the result that if a matrix $A \in \mathbb{R}^{n \times n}$ is copositive and interior, then $A$ is positive semidefinite. (See Corollary 2 of [12], or Lemma 1 of [6].) A useful observation quoted by Hiriart-Urruty and Seeger (Theorem 7.6 of [11]), as part of a theorem proved by Han and Mangasarian (although they do not state it this way), follows after making the assumption that $A$ and $A^{-1}$ are both copositive: one of them is strictly copositive if and only if the other is. To see this, write $0 = x^T Ax = x^T AA^{-1} Ax$, and remember that $x^T Ax = 0$ implies $Ax \geq 0$. Replacing $A^{-1}$ with $A^\dagger$ does not change the observation.

**Theorem 2.1.** Let $A \in \mathbb{R}^{n \times n}$. If $A$ and $A^\dagger$ are strictly copositive, then $A$ is positive semidefinite.

**Proof.** Since $A$ is strictly copositive, we know that there is a diagonal matrix $D$, with positive diagonal entries, such that $DAD e = e$. Write $DAD = V^T \Lambda V$, where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$ and $V$ is orthogonal. Then $(DAD)^\dagger = V^T \Lambda^\dagger V$, where $\Lambda^\dagger = \text{diag}(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_k}, 0, \ldots, 0)$, and $(DAD)^\dagger e = e$. In other words, without loss of generality when proving the theorem we may assume that $Ae = e$ and $A^\dagger e = e$.

Next, let $\min_{x \geq 0, \|x\|_1 = 1} x^T Ax = \lambda$. Then $\lambda > 0$, since $A$ is strictly copositive. Also, $(\frac{x}{x^T e})^T A(\frac{x}{x^T e}) \geq \lambda$, for all $x \geq 0$, i.e., $x^T Ax \geq \lambda(x^T e)^2$, which can be rewritten as $x^T (A - \lambda e e^T) x \geq 0$, and thus $A - \lambda e e^T$ is copositive. Now let $u$ be a minimizing vector, so that $u^T (A - \lambda e e^T) u = 0$, then $z = (A - \lambda e e^T) u \geq 0$.

Since $A^\dagger$ is copositive (actually strictly), we have

$$z^T A^\dagger z = u^T (A - \lambda e e^T) A^\dagger (A - \lambda e e^T) u,$$

$$= u^T (AA^\dagger A - \lambda e e^T - \lambda e e^T + \lambda^2 ee^T) u,$$

$$= u^T (A - \lambda e e^T) u + \lambda(u^T e)^2 (n\lambda - 1),$$

$$= 0 + \lambda(n\lambda - 1) \geq 0.$$ 

Then $n\lambda - 1 \geq 0$, and $\lambda \geq \frac{1}{n}$. But also $(\frac{1}{n} e)^T A(\frac{1}{n} e) \geq \lambda$, giving $\frac{1}{n} = \frac{\lambda}{n^2 \lambda} \geq \lambda$, so that $\lambda = \frac{1}{n}$. We have just shown that $z = (A - \lambda e e^T) u = 0$, i.e., $Au = \frac{1}{n} e^T ue = \frac{1}{n} e$. The minimizing vector $u$ may have zero components, but as $\frac{1}{n} e$ is (also) a minimizing vector, $A$ is copositive and interior, and therefore positive semidefinite.

Unlike for Han and Mangasarian’s result, the converse of Theorem 2.1 does not hold, as can be seen by considering $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. The matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, for which $A^\dagger =$

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 5 & 4 \\ 1 & 4 & 5 \end{pmatrix},$$

illustrates that it is possible to have both $A$ and $A^\dagger$ strictly copositive, and this situation is not covered by Han and Mangasarian’s result. Another way to state these results is to say that if we
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want to characterize those matrices for which both $A$ and $A^{-1}$ (or $A^T$) are strictly copositive, we need look no further than the positive (semi)definite matrices. The problem of characterizing the matrices for which both $A$ and $A^{-1}$ (or $A^T$) are copositive, but not strictly copositive, seems to be unsolved. (See Section 7.4 of [11].) If we only consider when $A$ and $A^{-1}$ are both nonnegative, then $A$ is the product of a permutation matrix and a diagonal matrix with positive diagonal entries (see [8], or Lemma 1.1 of Minc [16]). Such a matrix $A$ is called a generalized permutation matrix. If $A$ and $A^T$ are both nonnegative, then $A$ has been characterized in Theorem 5.2 of Berman and Plemmons [2]. It is well known that if $A$ is copositive with all zero diagonal entries then $A$ must be nonnegative. (This follows from $e_i^T A e_i = 0$, for each $i$, implies $A e_i \geq 0$.) In [14], we showed that if $A^{-1}$ is a nonnegative matrix with all zero diagonal entries, then $A$ may or may not be copositive, but when $A$ is copositive it cannot be of the form $P + N$, i.e., $A$ must be exceptional. Symmetric nonnegative matrices with all zeros on the diagonal are studied in [5], where such matrices are called hollow. It is easy to show that any nonnegative matrix, not necessarily invertible, with all positive diagonal entries must be strictly copositive. If $A^{-1}$ is nonnegative with all positive diagonal entries, then $A^{-1}$ is strictly copositive, and if in addition $A$ is copositive, then from Han and Mangasarian’s result, $A$ is positive definite. The consideration of what happens to $A$ as we make assumptions with some of the diagonal entries of nonnegative $A^{-1}$ being zero or positive leads us to the theorems of Sections 3 and 4.

3. When the inverse of a nonnegative matrix is of form $P+N$. For the proof of Theorem 3.1, we recall the well-known fact [15] for invertible $A \in \mathbb{R}^{n \times n}$, and $a, b \in \mathbb{R}^n$, such that $1 + b^T A^{-1} a \neq 0$, that when $A + ab^T$ is invertible, it can be written $(A + ab^T)^{-1} = A^{-1} - A^{-1} ab^T A^{-1} \frac{1}{1 + b^T A^{-1} a}$. Theorem 3.1 characterizes the invertible, nonnegative matrices $A^{-1}$, with $n - 1$ zero diagonal entries and one positive diagonal entry, such that $A$ is copositive of form $P + N$.

**Theorem 3.1.** Let $A \in \mathbb{R}^{n \times n}$ be symmetric and invertible. Suppose that $A^{-1}$ is nonnegative with $n - 1$ zero diagonal entries and one positive diagonal entry, and $A$ is of form $P + N$ (without loss of generality with $N$ having all zeroes on the diagonal). Then, under permutation similarity, the positive semidefinite matrix $P$ has form $P = \lambda uu^T$, where $\lambda > 0$, and $u$ has its first $n - 1$ components (when they are nonzero) having the same sign (as each other), and its $n$th component of opposite sign, while the symmetric nonnegative matrix $N$ has its upper left $(n-1) \times (n-1)$ block as a generalized permutation matrix, with all zeros elsewhere in $N$. Also, $n$ must be odd.

**Proof.** Suppose that $A^{-1}$ is nonnegative with exactly $n - 1$ zeroes on the diagonal, and without loss of generality (as this can be achieved with a permutation similarity) such that the lone positive diagonal entry lies in the bottom right corner of $A^{-1}$. If $N = 0$, then $A$ would have to be positive definite, but then $A^{-1}$ could not have any zeros on the diagonal. If $P = 0$, then $N$ would have to be a generalized permutation matrix with a positive entry in the bottom right-hand corner, and the theorem holds, just that $u = e_n$. Suppose, for what follows, that we do not have the latter situation.

Now, $0 = e_i^T A^{-1} e_i = e_i^T A^{-1} AA^{-1} e_i = e_i^T A^{-1} e_i (P + N) A^{-1} e_i$, implies $P x_i = 0$, for each $i = 1, \ldots, n - 1$, where $x_i = A^{-1} e_i$. It follows that $P$ has rank one, and $P = \lambda uu^T$, for some $\lambda > 0$ and $u \in \mathbb{R}^n$. Because $\lambda uu^T A^{-1} e_i = 0$, for each $i = 1, \ldots, n - 1$, this implies $u^T A^{-1} e_i = 0$, for each $i = 1, \ldots, n - 1$, and $u^T A^{-1} e_n > 0$ (actually $u^T A^{-1} e_n \neq 0$, but we can replace $u$ with $-u$ to achieve this). Next, $u^T A^{-1} = (0, \ldots, 0, u^T A^{-1} e_n)$,

or rewritten as a column vector $A^{-1} u = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ u^T A^{-1} e_n \end{pmatrix} = (u^T A^{-1} e_n) e_n$. Then
u = (N + \lambda uu^T)(u^TA^{-1}e_n)e_n, \\
= (u^TA^{-1}e_n)Ne_n + \lambda(u^Te_n)(u^TA^{-1}e_n)u, \\
and rearranging gives \[1 - \lambda(u^Te_n)(u^TA^{-1}e_n)\] u = (u^TA^{-1}e_n)Ne_n. If Ne_n \neq 0, then u \in \mathbb{R}(N), and setting 
\mu = \frac{\lambda^2 e_n^T e_n}{1 - \lambda(u^Te_n)^2}, we have u = \mu Ne_n, and A = N + \lambda uu^T = N + \lambda\mu Ne_n e_n^T N = N(I + \lambda\mu e_n e_n^T N).

If N were singular, so would A be singular, so we must have N nonsingular. But then (N + \lambda uu^T)^{-1} = N^{-1} - \frac{\lambda N^{-1} u u^T N^{-1}}{1 + \lambda u^T N^{-1} u}, as u^T N^{-1} u = \mu^2 e_n^T Ne_n = 0, which implies 1 + \lambda u^T N^{-1} u > 0. Then consider as e_i^T N A^{-1} Ne_i = e_i^T N \left(N^{-1} - \frac{\lambda N^{-1} u u^T N^{-1}}{1 + \lambda u^T N^{-1} u}\right) Ne_i = e_i^T N e_i - \frac{\lambda(u^Te_i)^2}{1 + \lambda u^T N^{-1} u} < 0, for some \(i \in \{1, \ldots, n - 1\}\), which contradicts \(A^{-1}\) being a nonnegative matrix. In other words, we must have Ne_n = 0.

Finally, write \(u = \left(\begin{array}{c} \hat{u} \\ u_n \end{array}\right)\), where \(\hat{u} \in \mathbb{R}^{n-1}\), \(u_n \in \mathbb{R}\), and \(N = \left(\begin{array}{cc} \hat{N} & 0 \\ 0 & 0 \end{array}\right)\), where \(\hat{N} \in \mathbb{R}^{(n-1) \times (n-1)}\).

Then \(A = \left(\begin{array}{cc} \hat{N} + \lambda \hat{u} \hat{u}^T & \lambda u_n \hat{u} \\ \lambda u_n \hat{u}^T & \lambda u_n^2 \end{array}\right)\), and because A is invertible we must have \(u_n \neq 0\). If \(\hat{N}\) were singular, with zero eigenvector \(\hat{v}\) (say), then \(\left(\begin{array}{c} \hat{v} \\ \frac{-\hat{u}^T \hat{v}}{u_n} \end{array}\right)\), would be a zero eigenvector for A, which is not possible.

Thus, \(\hat{N}\) is nonsingular. Since \(\left(\begin{array}{cc} \hat{N} + \lambda \hat{u} \hat{u}^T & \lambda u_n \hat{u} \\ \lambda u_n \hat{u}^T & \lambda u_n^2 \end{array}\right)^{-1} = \left(\begin{array}{cc} \hat{N}^{-1} & -\frac{1}{u_n} \hat{N}^{-1} \hat{u} \\ -\frac{1}{u_n} \hat{u}^T \hat{N}^{-1} & \frac{1}{u_n^2} \hat{N}^{-1} \hat{u}^T \hat{N}^{-1} \hat{u} \end{array}\right)\) is nonnegative, then \(\hat{N}\) must be a generalized permutation matrix. Also, the nonzero components of \(\hat{u}\) must all have the same sign, with \(u_n\) of opposite sign. Since \(\hat{N}^{-1}\) has all zeros on its diagonal, \(n\) is odd.

Determining whether there are exceptional matrices A, for which \(A^{-1}\) is nonnegative with exactly \(n - 1\) zero entries on the diagonal, even when \(n = 5\) or \(6\) (see [1, 7]), is beyond the scope of this paper.

4. When the inverse of a nonnegative matrix is not copositive. If \(A = (a_{ij}) \in \mathbb{R}^{n \times n}\) is a symmetric, nonnegative matrix with all positive diagonal entries (and therefore strictly copositive), and \(D = \text{diag}(\frac{1}{\sqrt{a_{11}}}, \ldots, \frac{1}{\sqrt{a_{nn}}})\), then \(DAD\) has all diagonal entries equal to 1. Replacing \(DAD\) with A, we have

\[
Ax = \begin{pmatrix} 
1 & a_{12} & a_{13} & \cdots & a_{1n} \\
12 & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{13} & a_{23} & 1 & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{1n} & a_{2n} & a_{3n} & \cdots & 1 
\end{pmatrix} \begin{pmatrix} x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n 
\end{pmatrix} \geq \begin{pmatrix} x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n 
\end{pmatrix}
\]

In other words, for such an A, we have \(Ax \geq x\), for all \(x \geq 0\). We will use this below. Theorem 4.1 was proved in [13]. (See also [3, 4, 19])

THEOREM 4.1. Let \(A \in \mathbb{R}^{n \times n}\) be symmetric, and \(A = \begin{pmatrix} A_1 & b \\
b^T & c \end{pmatrix}\), where \(A_1 \in \mathbb{R}^{(n-1) \times (n-1)}, b \in \mathbb{R}^{n-1}\), and \(c \in \mathbb{R}\). Then A is copositive if and only if \(c \geq 0\); \(A_1\) is copositive; if \(c > 0\) then \(x^T(A_1 - \frac{b b^T}{c})x \geq 0\), for all \(x \geq 0\), with \(x \in \mathbb{R}^{n-1}\), such that \(x^Tb \leq 0\); if \(c = 0\) then \(b \geq 0\).

Theorem 4.1 is needed to prove Theorem 4.2.

THEOREM 4.2. Let \(A \in \mathbb{R}^{n \times n}\) be symmetric and invertible. Suppose that \(A^{-1}\) is nonnegative with one zero diagonal entry and \(n - 1\) positive diagonal entries. Then A cannot be copositive.
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Proof. Suppose, for the sake of obtaining a contradiction, that $A$ is copositive. With $A^{-1}$ nonnegative and having only one zero on the diagonal, under permutation similarity and positive diagonal congruence we may assume $A^{-1}$ has the form $\begin{pmatrix} A_1 & b \\ b^T & 0 \end{pmatrix}$, where $A_1 \in \mathbb{R}^{(n-1) \times (n-1)}$, $b \in \mathbb{R}^{n-1}$, and nonnegative $A_1$ has all ones on the diagonal (so $A_1$ is strictly copositive).

$A_1$ must be nonsingular, since suppose $A_1$ is singular, then $z^T A_1 = 0$, for nonzero $z \in \mathbb{R}^{n-1}$. Next, since there is $\begin{pmatrix} B_1 & b_2 \\ b_2^T & c \end{pmatrix}$, with $B_1 \in \mathbb{R}^{(n-1) \times (n-1)}$, $b_2 \in \mathbb{R}^n$, $c \geq 0$, such that $\begin{pmatrix} A_1 & b \\ b^T & 0 \end{pmatrix} \begin{pmatrix} B_1 & b_2 \\ b_2^T & c \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix}$, this implies $A_1 b_2 + cb = 0$. But $z^T A_1 b_2 + c z^T b = 0$, means $c z^T b = 0$, so $z^T b = 0$ or $c = 0$. We also have that $A_1 B_1 + b b_2^T = I$, then $z^T A_1 B_2 + z^T b b_2 = z$. If $z^T b = 0$, the preceding equation implies $z = 0$, so we are left with $c = 0$. But $c = c_n A e_n = 0$ implies $A e_n \geq 0$, which implies $b \geq 0$. Then $A_1 b_2 = 0$ implies $b_2^T A_1 b_2 = 0$, which is not possible since $A_1$ is strictly copositive, unless $b = 0$. But then $c = 0$ would make $A$ singular. So we must have that $A_1$ is nonsingular. Moreover, $b_2 = -c A^{-1} b$ and $1 = b^T b_2 = -c b^T A^{-1} b$, implies that $b^T A_1^{-1} b < 0$.

We also have $A = \begin{pmatrix} A_1^{-1} - \frac{A_1^{-1} b b^T A_1^{-1}}{b^T A_1^{-1} b} & A_1^{-1} b \\ b^T A_1^{-1} & -\frac{1}{b^T A_1^{-1} b} \end{pmatrix}$. From Theorem 4.1, since $A$ is copositive, we know that $A_1^{-1} - \frac{A_1^{-1} b b^T A_1^{-1}}{b^T A_1^{-1} b}$ is copositive, and $x^T A_1^{-1} x \geq 0$, for all $x \geq 0$ such that $x^T A_1^{-1} b \geq 0$.

Finally, consider the nonnegative vector $z = A_1 b - b$, for which $z^T A_1^{-1} b = b^T b - b^T A^{-1} b \geq 0$. Then we know $z^T A_1^{-1} z = b^T (A_1 - I) A_1^{-1} (A_1 - I) b = b^T (A_1 + A_1^{-1} - 2I) b \geq 0$, which can be rearranged to say $b^T A_1 b - 2 b^T b \geq -b^T A_1^{-1} b > 0$, i.e., $b^T A_1 b > 2 b^T b$, or $\frac{b^T A_1 b}{b^T b} > 2$. Let $x = \max_{b_i > 0} \frac{(A_1) b_i}{b_i} = \max_{b_i > 0} \frac{b_i (A_1) b_i}{b_i^2}$, so that $(A_1) b_i \geq x b_i$, for all $i \in \{1, \ldots, n-1\}$. We also have that $b^T A_1 b = \sum_{i=1}^{n} b_i (A_1) b_i = \sum_{j=1}^{n} \sum_{k=j}^{n} b_j b_k (A_1) b_k b_j$, where we only include the positive $b_j$'s in the sum ($k$ of them, say). Then $b^T A_1 b \leq (b_1^2 + \cdots + b_{n-1}^2) \max_{b_i > 0} \frac{b_i (A_1) b_i}{b_i^2} = b^T b x$, i.e., $x \geq \frac{b^T A_1 b}{b^T b} > 2$. Now we notice that $A b \geq x b \geq 2 b$ and redefine the nonnegative vector $z$ as $z = A_1 b - 2b$, for which we have $z^T A_1^{-1} b = b^T b - 2 b^T A_1^{-1} b \geq 0$. Then (again) we know that $z^T A_1^{-1} z = b^T (A_1 - 2I) A_1^{-1} (A_1 - 2I) b = b^T (A_1 + 4 A_1^{-1} - 4I) b \geq 0$, which (again) can be rearranged to say $b^T A_1 b - 4 b^T b \geq -4 b^T A_1^{-1} b > 0$, i.e., $b^T A_1 b > 4 b^T b$, or $\frac{b^T A_1 b}{b^T b} > 4$, and continuing like this we arrive at a contradiction. So, $A$ must not be copositive.

5. Concluding remarks. The author does not know how to improve on Theorem 3.1 to the case where there are exactly $k$ zeroes on the diagonal, and $1 < k < n - 1$, without additional assumptions on the nonnegative $A^{-1}$. If we could make assumptions on the entries of $A^{-1}$ to conclude $A = \begin{pmatrix} \hat{N} + U U^T & U V^T \\ U V^T & V V^T \end{pmatrix}$, where $V$ is invertible, then $A^{-1} = \begin{pmatrix} \hat{N}^{-1} & -\hat{N}^{-1} U V^{-1} \\ -(V^{-1})^T U U^T \hat{N}^{-1} & (V V^T)^{-1} + (V^{-1})^T \hat{N}^{-1} (U U^T)^{-1} \end{pmatrix}$, would enable us to deduce an analogous description of the form of $A^{-1}$. Or, Theorem 5 of [14] would be a different way to make assumptions on $A^{-1}$.

It does not appear that Theorem 4.2 can be improved on to suppose $A^{-1}$ has two zero diagonal entries and $n-2$ positive diagonal entries, without additional assumptions on the nonnegative $A^{-1}$, because although...
and choosing \( \hat{e} \) is not copositive. These are not just examples for \( n = 4 \), because we can easily make these matrices be the upper left \( 4 \times 4 \) blocks of \( n \times n \) matrices which have lower right blocks as the identity matrix \( I_{n-4} \), and all zeroes elsewhere. Or, making different additional assumptions on nonnegative \( A^{-1} \), suppose \( n = 2k \) and \( A^{-1} = \begin{pmatrix} 0 & B \\ B^T & C \end{pmatrix} \), where \( B \) is \( k \times k \) and invertible, and \( C \) is nonzero. Then \( A = \begin{pmatrix} -(B^T)^{-1}CB^{-1} & (B^T)^{-1} \\ B^{-1} & 0 \end{pmatrix} \), and choosing \( \hat{y} = Bc \), where \( e \in \mathbb{R}^k \), we have \( y = \begin{pmatrix} \hat{y} \\ 0 \end{pmatrix} \in \mathbb{R}^n \) nonnegative and \( \hat{y}^T Ay = \hat{y}^T[-(B^T)^{-1}CB^{-1}]\hat{y} = -e^T Ce < 0 \), so that \( A \) cannot be copositive.

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