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NEAR-BIPARTITE LEONARD PAIRS*

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Abstract. Let $\mathbb{F}$ denote a field, and let V denote a vector space over $\mathbb{F}$ with finite positive dimension. A Leonard pair on V is an ordered pair of diagonalizable $\mathbb{F}$-linear maps $A : V \to V$ and $A^* : V \to V$ that each act on an eigenbasis for the other in an irreducible tridiagonal fashion. Let A, A^* denote a Leonard pair on V. Let $\{v_i\}_{i=0}^d$ denote an eigenbasis for A^* on which A acts in an irreducible tridiagonal fashion. For $0 \le i \le d$, define an $\mathbb{F}$-linear map $E_i^* : V \to V$ such that $E_i^* v_i = v_i$ and $E_i^* v_j = 0$ if $j \ne i$ $(0 \le j \le d)$. The map $F = \sum_{i=0}^d E_i^* A E_i^*$ is called the flat part of A. The Leonard pair A, A^* is bipartite whenever $F = 0$. The Leonard pair A, A^* is said to be near-bipartite whenever the pair $A - F, A^*$ is a Leonard pair on V. In this case, the Leonard pair $A - F, A^*$ is bipartite and called the bipartite contraction of A, A^*. Let B, B^* denote a bipartite Leonard pair on V. By a near-bipartite expansion of B, B^*, we mean a near-bipartite Leonard pair on V with bipartite contraction B, B^*. In the present paper, we have three goals. Assuming $\mathbb{F}$ is algebraically closed, (i) we classify up to isomorphism the near-bipartite Leonard pairs over $\mathbb{F}$; (ii) for each near-bipartite Leonard pair over $\mathbb{F}$ we describe its bipartite contraction; (iii) for each bipartite Leonard pair over $\mathbb{F}$ we describe its near-bipartite expansions. Our classification (i) is summarized as follows. We identify two families of Leonard pairs, said to have Krawtchouk type and dual q-Krawtchouk type. A Leonard pair of dual q-Krawtchouk type is said to be reinforced whenever $q^{2i} \ne -1$ for $1 \le i \le d-1$. A Leonard pair A, A^* is said to be essentially bipartite whenever the flat part of A is a scalar multiple of the identity. Assuming $\mathbb{F}$ is algebraically closed, we show that a Leonard pair A, A^* over $\mathbb{F}$ with $d \ge 3$ is near-bipartite if and only if at least one of the following holds: (i) A, A^* is essentially bipartite; (ii) A, A^* has reinforced dual q-Krawtchouk type; and (iii) A, A^* has Krawtchouk type.

Key words. Bipartite, Leonard pair, Leonard system, Near-bipartite, Tridiagonal matrix.

AMS subject classifications. 05E30, 15A21, 15B10.

1. Introduction.

The notion of a Leonard pair was introduced by the second author [46]. We recall the definition of a Leonard pair. A square matrix is said to be tridiagonal whenever each nonzero entry lies on the diagonal, the subdiagonal, or the superdiagonal. A tridiagonal matrix is said to be irreducible whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero. Let $\mathbb{F}$ denote a field, and let V denote a vector space over $\mathbb{F}$ with finite positive dimension. A Leonard pair on V is an ordered pair of $\mathbb{F}$-linear maps $A : V \to V$ and $A^* : V \to V$ that satisfy (i) and (ii) below:

(i) there exists a basis of V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal;
(ii) there exists a basis of V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

In this case, we say that A, A^* is over $\mathbb{F}$. We call $d = \dim V - 1$ the diameter of A, A^*.

In the literature, there are two well-known families of Leonard pairs, said to be bipartite and almost-bipartite (see [16, Section 1]). In the present paper, we introduce a family of Leonard pairs called near-

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bipartite. We will describe this family shortly. In order to motivate things, we give some history and background.

The Leonard pairs arose from the study of Q-polynomial distance-regular graphs and orthogonal polynomials. Delsarte showed in [20] that a Q-polynomial distance-regular graph yields two orthogonal polynomial sequences that are related by what is now called Askey–Wilson duality. Motivated by Delsarte's result, Leonard showed in [32] that the q-Racah polynomials give the most general orthogonal polynomial systems that satisfy Askey–Wilson duality. Leonard's theorem was improved by Bannai and Ito [5, Theorem 5.1] by treating all the limiting cases. This version gives a complete classification of the orthogonal polynomial systems that satisfy Askey–Wilson duality. It shows that the orthogonal polynomial systems that satisfy Askey–Wilson duality all come from the terminating branch of the Askey scheme, consisting of the q-Racah, q-Hahn, dual q-Hahn, q-Krawtchouk, dual q-Krawtchouk, quantum q-Krawtchouk, affine q-Krawtchouk, Racah, Hahn, Krawtchouk, Bannai/Ito, and orphan polynomials. The Leonard theorem [5, Theorem 5.1] is rather complicated. The notion of a Leonard pair was introduced in [46] to simplify and clarify Leonard's theorem. A Leonard system [46, Definition 1.4] is a Leonard pair A, A^* together with appropriate orderings of the eigenvalues of A and A^*. The Leonard systems are classified up to isomorphism in [46, Theorem 1.9]. This result gives a linear algebraic version of Leonard's theorem.

We just mentioned how Leonard pairs are related to orthogonal polynomials. Leonard pairs have applications to many other areas of mathematics and physics, such as Lie theory [3, 21, 22, 25, 29, 37], quantum groups [1, 2, 13–15, 26–28], spin models [17–19, 41], double affine Hecke algebras [23, 24, 30, 31, 38], partially ordered sets [33, 34, 44, 52], and exactly solvable models in statistical mechanics [6–12]. For more information about Leonard pairs and related topics, see [4, 37, 39, 40, 42, 43, 45, 47, 51].

We now describe the near-bipartite Leonard pairs. Let A, A^* denote a Leonard pair on V. Let $\{v_i\}_{i=0}^d$ denote a basis of V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal. For $0 \leq i \leq d$ define an $\mathbb{F}$-linear map $E_i^* : V \to V$ such that $E_i^* v_i = v_i$ and $E_i^* v_j = 0$ for $j \neq i$ $(0 \leq j \leq d)$. Define $F = \sum_{i=0}^d E_i^* A E_i^*$. We call F the flat part of A. The Leonard pair A, A^* is bipartite if and only if $F = 0$. The Leonard pair A, A^* is said to be near-bipartite whenever the pair $A - F, A^*$ is a Leonard pair on V. In this case, the Leonard pair $A - F, A^*$ is bipartite and called the bipartite contraction of A, A^*. Let B, B^* denote a bipartite Leonard pair on V. By a near-bipartite expansion of B, B^*, we mean a near-bipartite Leonard pair on V with bipartite contraction B, B^*.

In the present paper, we have three main goals. Assuming $\mathbb{F}$ is algebraically closed,

(i) we classify up to isomorphism the near-bipartite Leonard pairs over $\mathbb{F}$;
(ii) for each near-bipartite Leonard pair over $\mathbb{F}$, we describe its bipartite contraction;
(iii) for each bipartite Leonard pair over $\mathbb{F}$, we describe its near-bipartite expansions.

We now summarize our classification for item (i). In this summary, we assume $d \geq 3$; the case $d \leq 2$ is a bit different and given in the main body of the paper. We will identify two families of Leonard pairs, said to have Krawtchouk type and dual q-Krawtchouk type. These are attached to the Krawtchouk polynomials and the dual q-Krawtchouk polynomials. A Leonard pair of dual q-Krawtchouk type is said to be reinforced whenever $q^{2i} \neq -1$ for $1 \leq i \leq d-1$. A Leonard pair A, A^* is said to be essentially bipartite whenever the flat part F of A is a scalar multiple of the identity. Assuming $\mathbb{F}$ is algebraically closed, we will show that a Leonard pair A, A^* over $\mathbb{F}$ is near-bipartite if and only if at least one of the following (i)–(iii) holds:

(i) A, A^* is essentially bipartite;

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(ii) A, A^* has reinforced dual q-Krawtchouk type;
(iii) A, A^* has Krawtchouk type.

The paper is organized as follows. In Section 2, we fix our notation and recall some basic materials from linear algebra. In Section 3, we recall some materials concerning Leonard pairs. In Section 4, we introduce the concept of a TD/D sequence for a Leonard pair. In Section 5, we introduce the normalized TD/D form of a Leonard pair. In Section 6, we recall the bipartite Leonard pairs and essentially bipartite Leonard pairs. In Section 7, we introduce the flat part F of a Leonard pair. In Section 8, we introduce the near-bipartite property for Leonard pairs. In Section 9, we classify the near-bipartite Leonard pairs with diameter $d = 1$. In Sections 10 and 11, we classify the near-bipartite Leonard pairs with diameter $d = 2$. Starting in Section 12 we assume $d \geq 3$. In Section 12, we recall the type of a Leonard pair. In Sections 13–16, we recall the primary data of a Leonard pair. In Section 17, we characterize the essentially bipartite Leonard pairs in terms of the primary data. In Section 18, we characterize the bipartite Leonard pairs in terms of the primary data. In Section 19, we describe the dual q-Krawtchouk Leonard pairs. In Section 20, we describe the Krawtchouk Leonard pairs. In Section 21, we describe the Leonard pairs that are bipartite and have dual q-Krawtchouk type. In Section 22, we describe the Leonard pairs that are bipartite and have Krawtchouk type. In Section 23, we determine the near-bipartite Leonard pairs of dual q-Krawtchouk type, and describe their bipartite contraction. In Section 24, we show that a Leonard pair of Krawtchouk type is near-bipartite, and we describe its bipartite contraction. In Section 25, we classify the near-bipartite Leonard pairs. In Sections 26 and 27, we describe the near-bipartite expansions of a bipartite Leonard pair.

2. Preliminaries. We now begin our formal argument. Throughout the paper, the following notation is in effect. Let $\mathbb{F}$ denote a field. Every vector space and algebra discussed in this paper is over $\mathbb{F}$. Fix an integer $d \geq 0$. The notation $\{x_i\}_{i=0}^d$ refers to the sequence $x_0, x_1, \ldots, x_d$. Let $\mathrm{Mat}_{d+1}(\mathbb{F})$ denote the algebra consisting of the $d+1$ by $d+1$ matrices that have all entries in $\mathbb{F}$. We index the rows and columns by $0, 1, \ldots, d$. The identity element of $\mathrm{Mat}_{d+1}(\mathbb{F})$ is denoted by I. Let $\mathbb{F}^{d+1}$ denote the vector space consisting of the column vectors with $d+1$ rows and all entries in $\mathbb{F}$. We index the rows by $0, 1, \ldots, d$. The algebra $\mathrm{Mat}_{d+1}(\mathbb{F})$ acts on $\mathbb{F}^{d+1}$ by left multiplication. Throughout the paper, V denotes a vector space with dimension $d+1$. Let $\mathrm{End}(V)$ denote the algebra consisting of the $\mathbb{F}$-linear maps $V \to V$. The identity element of $\mathrm{End}(V)$ is denoted by I. We recall how each basis $\{v_i\}_{i=0}^d$ of V gives an algebra isomorphism $\mathrm{End}(V) \to \mathrm{Mat}_{d+1}(\mathbb{F})$. For $A \in \mathrm{End}(V)$ and $M \in \mathrm{Mat}_{d+1}(\mathbb{F})$, we say that M *represents A with respect to* $\{v_i\}_{i=0}^d$ whenever $Av_j = \sum_{i=0}^d M_{i,j} v_i$ for $0 \leq j \leq d$. The isomorphism sends A to the unique matrix in $\mathrm{Mat}_{d+1}(\mathbb{F})$ that represents A with respect to $\{v_i\}_{i=0}^d$. Let $A \in \mathrm{End}(V)$. By an *eigenspace of* A, we mean a subspace $W \subseteq V$ such that $W \neq 0$ and there exists $\theta \in \mathbb{F}$ such that $W = \{v \in V \mid Av = \theta v\}$; in this case θ, is the *eigenvalue* of A associated with W. We say that A is *diagonalizable* whenever V is spanned by the eigenspaces of A. We say that A is *multiplicity-free* whenever A is diagonalizable and its eigenspaces all have dimension one. Assume that A is multiplicity-free. Let $\{\theta_i\}_{i=0}^d$ denote an ordering of the eigenvalues of A. For $0 \leq i \leq d$ let V_i denote the eigenspace of A associated with θ_i, and define $E_i \in \mathrm{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ for $j \neq i$ $(0 \leq j \leq d)$. We call E_i the *primitive idempotent of A associated with* θ_i. We have (i) $E_i E_j = \delta_{i,j} E_i$ $(0 \leq i, j \leq d)$; (ii) $I = \sum_{i=0}^d E_i$; (iii) $AE_i = \theta_i E_i = E_i A$ $(0 \leq i \leq d)$; (iv) $A = \sum_{i=0}^d \theta_i E_i$; (v) $V_i = E_i V$ $(0 \leq i \leq d)$; (vi) $\mathrm{rank}(E_i) = 1$; and (vii) $\mathrm{tr}(E_i) = 1$ $(0 \leq i \leq d)$, where tr means trace. Moreover,

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \qquad (0 \leq i \leq d).$$

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Let $\langle A\rangle$ denote the subalgebra of $\mathrm{End}(V)$ generated by A. The algebra $\langle A\rangle$ is commutative. The elements $\{A^i\}_{i=0}^d$ (resp. $\{E_i\}_{i=0}^d$) form a basis of $\langle A\rangle$.

Lemma 1. *Assume that $A \in \mathrm{End}(V)$ is multiplicity-free with primitive idempotents $\{E_i\}_{i=0}^d$. Then for $H \in \mathrm{End}(V)$, the following* (i)–(iii) *are equivalent:*

(i) *$H \in \langle A\rangle$;*
(ii) *H commutes with A;*
(iii) *H commutes with E_i for $0 \le i \le d$.*

Proof. This is a reformulation of the fact that a matrix $M \in \mathrm{Mat}_{d+1}(\mathbb{F})$ commutes with each diagonal matrix in $\mathrm{Mat}_{d+1}(\mathbb{F})$ if and only if M diagonal. $\square$

A square matrix is said to be *tridiagonal* whenever each nonzero entry lies on the diagonal, subdiagonal, or the superdiagonal. A tridiagonal matrix is said to be *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

Lemma 2. *Let $T \in \mathrm{Mat}_{d+1}(\mathbb{F})$ be irreducible tridiagonal. Then the following are equivalent:*

(i) *T is diagonalizable;*
(ii) *T is multiplicity-free.*

Proof. (i) $\Rightarrow$ (ii) Observe that $I, T, T^2, \ldots, T^d$ are linearly independent. Thus, the minimal polynomial of T has degree $d+1$. So T has $d+1$ mutually distinct eigenvalues.

(ii) $\Rightarrow$ (i) Clear. $\square$

For elements r, s in an algebra, the notation $[r, s]$ means the commutator $rs - sr$.

3. Leonard pairs. In this section, we recall the notion of a Leonard pair. Recall the vector space V with dimension $d+1$.

Definition 3. (See [46, Definition 1.1].) By a *Leonard pair on V* we mean an ordered pair A, A^* of elements in $\mathrm{End}(V)$ that satisfy (i) and (ii) below:

(i) there exists a basis of V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal;
(ii) there exists a basis of V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

We call d the *diameter* of A, A^*. We say that A, A^* is *over* $\mathbb{F}$. We call V the *underlying vector space.*

Note 4. *By a common notational convention, A^* denotes the conjugate-transpose of A. We are not using this convention. In a Leonard pair A, A^*, the elements A and A^* are arbitrary subject to (i) and (ii) above.*

We recall the notion of an isomorphism of Leonard pairs. Consider a Leonard pair A, A^* on V and a Leonard pair B, B^* on a vector space $\mathcal{V}$. By an *isomorphism of Leonard pairs from A, A^* to B, B^**, we mean an $\mathbb{F}$-linear bijection $S : V \to \mathcal{V}$ such that $SA = BS$ and $SA^* = B^*S$. We say that A, A^* and B, B^* are *isomorphic* whenever there exists an isomorphism of Leonard pairs from A, A^* to B, B^*.

Lemma 5. (See [36, Lemma 5.1].) *For a Leonard pair A, A^* on V and scalars ξ, ξ^*, ζ, ζ^* in $\mathbb{F}$ with $\xi\xi^* \neq 0$, the pair $\xi A + \zeta I$, $\xi^* A^* + \zeta^* I$ is a Leonard pair on V.*

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Lemma 6. (See [46, Lemma 3.1].) *For a Leonard pair A, A^* on V, each of A and A^* is multiplicity-free.*

Lemma 7. (See [46, Lemma 3.3].) *Let A, A^* denote a Leonard pair on V. Then there does not exist a subspace W of V such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.*

Let A, A^* denote a Leonard pair on V. An ordering $\{\theta_i\}_{i=0}^d$ of the eigenvalues of A is said to be *standard* whenever there exists a basis of V with respect to which the matrix representing A is $\operatorname{diag}(\theta_0, \theta_1, \ldots, \theta_d)$ and the matrix representing A^* is irreducible tridiagonal. For a standard ordering $\{\theta_i\}_{i=0}^d$ of the eigenvalues of A, the ordering $\{\theta_{d-i}\}_{i=0}^d$ is standard and no further ordering is standard. A standard ordering of the eigenvalues of A^* is similarly defined.

We recall the notion of a parameter array.

Lemma 8. (See [46, Theorem 3.2].) *Let A, A^* denote a Leonard pair on V. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) denote a standard ordering of the eigenvalues of A (resp. A^*). Then there exists a unique sequence $\{\varphi_i\}_{i=1}^d$ of scalars in $\mathbb{F}$ with the following property: there exists a basis of V with respect to which the matrices representing A and A^* are*

$$A: \begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & 1 & \theta_d \end{pmatrix}, \qquad A^*: \begin{pmatrix} \theta^*_0 & \varphi_1 & & & \mathbf{0} \\ & \theta^*_1 & \varphi_2 & & \\ & & \theta^*_1 & \ddots & \\ & & & \ddots & \varphi_d \\ \mathbf{0} & & & & \theta^*_d \end{pmatrix}.$$

Definition 9. Referring to Lemma 8, the sequence $\{\varphi_i\}_{i=1}^d$ is called the *first split sequence* of A, A^* with respect to the standard orderings $\{\theta_i\}_{i=0}^d$ and $\{\theta^*_i\}_{i=0}^d$.

Lemma 10. *Let A, A^* denote a Leonard pair on V. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) denote a standard ordering of the eigenvalues of A (resp. A^*). Then there exists a unique sequence $\{\phi_i\}_{i=1}^d$ of scalars in $\mathbb{F}$ with the following property: there exists a basis of V with respect to which the matrices representing A and A^* are*

$$A: \begin{pmatrix} \theta_d & & & & \mathbf{0} \\ 1 & \theta_{d-1} & & & \\ & 1 & \theta_{d-2} & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & 1 & \theta_0 \end{pmatrix}, \qquad A^*: \begin{pmatrix} \theta^*_0 & \phi_1 & & & \mathbf{0} \\ & \theta^*_1 & \phi_2 & & \\ & & \theta^*_1 & \ddots & \\ & & & \ddots & \phi_d \\ \mathbf{0} & & & & \theta^*_d \end{pmatrix}.$$

Proof. Apply Lemma 8 to the eigenvalue sequence $\{\theta_{d-i}\}_{i=0}^d$ of A, A^*. □

Definition 11. Referring to Lemma 10, the sequence $\{\phi_i\}_{i=1}^d$ is called the *second split sequence* of A, A^* with respect to the standard orderings $\{\theta_i\}_{i=0}^d$ and $\{\theta^*_i\}_{i=0}^d$.

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Definition 12. (See [48, Definition 11.1].) Let A, A^* denote a Leonard pair on V. By a *parameter array of* A, A^*, we mean a sequence

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \tag{3.1}$$

where $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) is a standard ordering of the eigenvalues of A (resp. A^*), and $\{\varphi_i\}_{i=1}^d$ (resp. $\{\phi_i\}_{i=1}^d$) is the first split sequence (resp. second split sequence) with respect to the standard orderings $\{\theta_i\}_{i=0}^d$ and $\{\theta^*_i\}_{i=0}^d$.

We comment on the uniqueness of the parameter array.

Lemma 13. (See [46, Theorem 1.11].) *Let A, A^* denote a Leonard pair on V, and let*

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d),$$

denote a parameter array of A, A^*. Then each of the following is a parameter array of A, A^*:

$$\begin{aligned}
&(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d),\\
&(\{\theta_i\}_{i=0}^d; \{\theta^*_{d-i}\}_{i=0}^d; \{\phi_{d-i+1}\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d),\\
&(\{\theta_{d-i}\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d),\\
&(\{\theta_{d-i}\}_{i=0}^d; \{\theta^*_{d-i}\}_{i=0}^d; \{\varphi_{d-i+1}\}_{i=1}^d; \{\phi_{d-i+1}\}_{i=1}^d).
\end{aligned}$$

Moreover, A, A^* has no further parameter array.

We mention a significance of the parameter array.

Lemma 14. (See [46, Theorem 1.9].) *Two Leonard pairs over $\mathbb{F}$ are isomorphic if and only if they have a parameter array in common.*

Definition 15. By a *parameter array over* $\mathbb{F}$, we mean the parameter array of a Leonard pair over $\mathbb{F}$.

In the next result, we classify the parameter arrays over $\mathbb{F}$.

Lemma 16. (See [46, Theorem 1.9].) *Consider a sequence*

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \tag{3.2}$$

of scalars in $\mathbb{F}$. This sequence is a parameter array over $\mathbb{F}$ if and only if the following conditions (i)–(v) *hold:*

- (i) $\theta_i \neq \theta_j, \quad \theta^*_i \neq \theta^*_j \quad$ *if* $i \neq j \quad (0 \leq i, j \leq d)$;
- (ii) $\varphi_i \neq 0, \quad \phi_i \neq 0 \quad (1 \leq i \leq d)$;
- (iii) $\varphi_i = \phi_1 \sum_{\ell=0}^{i-1} \frac{\theta_\ell - \theta_{d-\ell}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{i-1} - \theta_d) \quad (1 \leq i \leq d)$;
- (iv) $\phi_i = \varphi_1 \sum_{\ell=0}^{i-1} \frac{\theta_\ell - \theta_{d-\ell}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0) \quad (1 \leq i \leq d)$;
- (v) *the expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}, \tag{3.3}$$

are equal and independent of i for $2 \leq i \leq d-1$.

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LEMMA 17. (See [36, Lemma 6.1].) *Let A, A^* denote a Leonard pair over $\mathbb{F}$ with parameter array:*

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d).$$

Then for scalars ξ, ξ^*, ζ, ζ^* in $\mathbb{F}$ with $\xi\xi^* \neq 0$, the sequence

$$(\{\xi\theta_i + \zeta\}_{i=0}^d; \{\xi^*\theta^*_i + \zeta^*\}_{i=0}^d; \{\xi\xi^*\varphi_i\}_{i=1}^d, \{\xi\xi^*\phi_i\}_{i=1}^d),$$

is a parameter array of the Leonard pair $\xi A + \zeta I$, $\xi^* A^* + \zeta^* I$.

DEFINITION 18. Let $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$ denote a parameter array over $\mathbb{F}$ with $d \geq 3$. Define $\beta \in \mathbb{F}$ such that $\beta + 1$ is equal to the common value of the two fractions in (3.3). We call β the *fundamental constant* of the parameter array.

DEFINITION 19. Let A, A^* denote a Leonard pair over $\mathbb{F}$ with diameter $d \geq 3$. The parameter arrays of A, A^* have the same fundamental constant β; we call β the *fundamental constant* of A, A^*.

4. The TD/D sequences of a Leonard pair. In this section, we introduce the notion of a TD/D sequence of a Leonard pair. To motivate this notion, we make some comments about Leonard pairs. An irreducible tridiagonal matrix is said to be *normalized* whenever the subdiagonal entires are all 1.

LEMMA 20. *For $A \in \mathrm{End}(V)$ and a basis $\{u_i\}_{i=0}^d$ of V, assume that with respect to $\{u_i\}_{i=0}^d$ the matrix B representing A is irreducible tridiagonal. Then for nonzero scalars $\{\alpha_i\}_{i=0}^d$ in $\mathbb{F}$, the following are equivalent:*

(i) *With respect to the basis $\{\alpha_i u_i\}_{i=0}^d$ of V, the matrix representing A is normalized irreducible tridiagonal;*
(ii) $\alpha_i/\alpha_{i-1} = B_{i,i-1}$ *for* $1 \leq i \leq d$.

Proof. By linear algebra, the matrix representing A with respect to the basis $\{\alpha_i u_i\}_{i=0}^d$ is equal to $D^{-1}BD$, where $D = \mathrm{diag}(\alpha_0, \alpha_1, \ldots, \alpha_d)$. The result is a routine consequence of this. □

PROPOSITION 21. *Let A, A^* denote a Leonard pair on V. Let $\{\theta^*_i\}_{i=0}^d$ denote a standard ordering of the eigenvalues of A^*. Then there exists a unique sequence $(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d)$ of scalars in $\mathbb{F}$ with the following property: there exists a basis of V with respect to which the matrices representing A and A^* are*

$$A: \begin{pmatrix} a_0 & x_1 & & & & \mathbf{0} \\ 1 & a_1 & x_2 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & x_d \\ \mathbf{0} & & & & 1 & a_d \end{pmatrix}, \qquad A^*: \mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d). \tag{4.4}$$

Moreover, $x_i \neq 0$ for $1 \leq i \leq d$.

Proof. We first show that the sequence exists. By Definition 3(ii) and the definition of a standard ordering, there exists a basis $\{u_i\}_{i=0}^d$ of V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is equal to $\mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d)$. After adjusting the basis $\{u_i\}_{i=0}^d$ using Lemma 20, the matrix representing A is normalized irreducible tridiagonal and the matrix representing A^* is $\mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d)$. We have shown that the sequence exists. Next, we show that the sequence is unique. Suppose we have another basis $\{v_i\}_{i=0}^d$ of V with respect to which the matrix representing A is normalized irreducible tridiagonal and the matrix representing A^* is $\mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d)$. Observe that for $0 \leq i \leq d$ each of u_i and v_i is an eigenvector for A^* with eigenvalue θ^*_i. Therefore, there exists a nonzero scalar $\alpha_i \in \mathbb{F}$ such

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that $v_i = \alpha_i u_i$. By Lemma 20 and the construction, $\alpha_i/\alpha_{i-1} = 1$ for $1 \le i \le d$. Therefore, α_i is independent of i for $0 \le i \le d$. Consequently, $v_i = \alpha_0 u_i$ for $0 \le i \le d$. This implies that the matrix representing A with respect to $\{u_i\}_{i=0}^d$ is equal to the matrix representing A with respect to $\{v_i\}_{i=0}^d$. It follows that the sequence is unique. We have $x_i \neq 0$ for $1 \le i \le d$ by Lemma 7. ∎

Definition 22. Let A, A^* denote a Leonard pair on V. By a *TD/D sequence of* A, A^* we mean a sequence:

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta_i^*\}_{i=0}^d), \tag{4.5}$$

where $\{\theta_i^*\}_{i=0}^d$ is a standard ordering of the eigenvalues of A^*, and $(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d)$ is the corresponding sequence of scalars from Proposition 21.

We comment on the uniqueness of the TD/D sequence.

Lemma 23. *Let A, A^* denote a Leonard pair on V. Let*

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta_i^*\}_{i=0}^d),$$

denote a TD/D sequence of A, A^*. Then

$$(\{a_{d-i}\}_{i=0}^d; \{x_{d-i+1}\}_{i=1}^d; \{\theta_{d-i}^*\}_{i=0}^d), \tag{4.6}$$

is a TD/D sequence of A, A^*, and A, A^* has no further TD/D sequence.

Proof. By Proposition 21, there exists a basis $\{u_i\}_{i=0}^d$ of V with respect to which the matrices representing A and A^* are as in (4.4). Define

$$v_i = x_d x_{d-1} \cdots x_{d-i+1} u_{d-i} \qquad (0 \le i \le d).$$

The sequence $\{v_i\}_{i=0}^d$ is a basis of V with respect to which the matrices representing A and A^* are

$$A : \begin{pmatrix} a_d & x_d & & & & \mathbf{0} \\ 1 & a_{d-1} & x_{d-1} & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & x_1 \\ \mathbf{0} & & & & 1 & a_0 \end{pmatrix}, \qquad A^* : \operatorname{diag}(\theta_d^*, \theta_{d-1}^*, \ldots, \theta_0^*).$$

By the paragraph below Lemma 6, the ordering $\{\theta_{d-i}^*\}_{i=0}^d$ is standard. By these comments and Definition 22, the sequence (4.6) is a TD/D sequence of A, A^*. The uniqueness follows from the paragraph below Lemma 6. ∎

We mention a significance of the TD/D sequence.

Proposition 24. *Two Leonard pairs over $\mathbb{F}$ are isomorphic if and only if they have a TD/D sequence in common.*

Proof. Let A, A^* denote a Leonard pair on V, and let B, B^* denote a Leonard pair on a vector space $\mathcal{V}$. First assume that A, A^* and B, B^* are isomorphic. Then clearly they have the same TD/D sequences. Next assume that A, A^* and B, B^* have a common TD/D sequence $(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta_i^*\}_{i=0}^d)$. By Definition 22, $\{\theta_i^*\}_{i=0}^d$ is a standard ordering of the eigenvalues of A^* and B^*. Moreover, there exists a basis $\{u_i\}_{i=0}^d$ of V

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(resp. basis $\{v_i\}_{i=0}^d$ of $\mathcal{V}$) with respect to which the matrix representing A (resp. B) is equal to the matrix on the left in (4.4), and the matrix representing A^* (resp. B^*) is equal to the matrix on the right in (4.4). By linear algebra, there exists an $\mathbb{F}$-linear bijection $V \to \mathcal{V}$ that sends $u_i \mapsto v_i$ for $0 \le i \le d$. By construction, this bijection is an isomorphism of Leonard pairs from A, A^* to B, B^*. □

Definition 25. By a *TD/D sequence over* $\mathbb{F}$ we mean a TD/D sequence of a Leonard pair over $\mathbb{F}$.

Definition 26. Consider a parameter array

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \tag{4.7}$$

over $\mathbb{F}$ and a TD/D sequence

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta_i^{*\prime}\}_{i=0}^d), \tag{4.8}$$

over $\mathbb{F}$. Then, (4.7) and (4.8) are said to *correspond* whenever $\theta_i^* = \theta_i^{*\prime}$ for $0 \le i \le d$, and there exists a Leonard pair A, A^* over $\mathbb{F}$ that has parameter array (4.7) and TD/D sequence (4.8).

Lemma 27. *Referring to Definition 25, assume that (4.7) and (4.8) correspond. Then* $\sum_{i=0}^d \theta_i = \sum_{i=0}^d a_i$.

Proof. Let the matrices A, A^* are as in (4.4). On the one hand, we have $\mathrm{tr}(A) = \sum_{i=0}^d a_i$. On the other hand, $\mathrm{tr}(A) = \sum_{i=0}^d \theta_i$, since the eigenvalues of A are $\{\theta_i\}_{i=0}^d$. The result follows. □

Lemma 28. *The following hold.*

(i) *Consider a parameter array over* $\mathbb{F}$*:*

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d). \tag{4.9}$$

Then there exists a unique TD/D sequence over $\mathbb{F}$ *that corresponds to (4.9).*

(ii) *Consider a TD/D sequence over* $\mathbb{F}$*:*

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta_i^*\}_{i=0}^d). \tag{4.10}$$

Then there exists at least one parameter array over $\mathbb{F}$ *that corresponds to (4.10). If*

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \tag{4.11}$$

is a parameter array over $\mathbb{F}$ *that corresponds to (4.10), then so is*

$$(\{\theta_{d-i}\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d),$$

and there is no other parameter array over $\mathbb{F}$ *that corresponds to (4.10).*

Proof. (i) By Lemma 14, up to isomorphism there exists a unique Leonard pair A, A^* over $\mathbb{F}$ that has parameter array (4.9). By Lemma 23 and Definition 26, there exists a unique TD/D sequence over $\mathbb{F}$ that corresponds to A, A^*. By these comments, we get the result.

(ii) By Proposition 24, up to isomorphism there exists a unique Leonard pair A, A^* over $\mathbb{F}$ that has TD/D sequence (4.10). In Lemma 13, the four parameter arrays of A, A^* are given. Among these four parameter arrays, only the first and third involves the eigenvalues of A^* in the same standard order as (4.10). By these comments, we get the result. □

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To prepare for the next results, we define some notation. For the moment, let $\{\theta^*_i\}_{i=0}^d$ denote any sequence of scalars in $\mathbb{F}$. For $0 \le i \le d$ define the following polynomials in an indeterminate λ:

$$\tau^*_i(\lambda) = (\lambda - \theta^*_0)(\lambda - \theta^*_1)\cdots(\lambda - \theta^*_{i-1}),$$
$$\eta^*_i(\lambda) = (\lambda - \theta^*_d)(\lambda - \theta^*_{d-1})\cdots(\lambda - \theta^*_{d-i+1}).$$

LEMMA 29. *Assume that $d \ge 1$. Consider a sequence of scalars in $\mathbb{F}$:*

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d). \tag{4.12}$$

The sequence (4.12) is a TD/D sequence over $\mathbb{F}$ if and only if there exists a parameter array:

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \tag{4.13}$$

over $\mathbb{F}$ such that

$$a_0 = \theta_0 + \frac{\varphi_1}{\theta^*_0 - \theta^*_1}, \tag{4.14}$$

$$a_i = \theta_i + \frac{\varphi_i}{\theta^*_i - \theta^*_{i-1}} + \frac{\varphi_{i+1}}{\theta^*_i - \theta^*_{i+1}} \qquad (1 \le i \le d-1), \tag{4.15}$$

$$a_d = \theta_d + \frac{\varphi_d}{\theta^*_d - \theta^*_{d-1}}, \tag{4.16}$$

$$x_i = \varphi_i \phi_i \frac{\tau^*_{i-1}(\theta^*_{i-1})\eta^*_{d-i}(\theta^*_i)}{\tau^*_i(\theta^*_i)\eta^*_{d-i+1}(\theta^*_{i-1})} \qquad (1 \le i \le d). \tag{4.17}$$

In this case, the TD/D sequence (4.12) and the parameter array (4.13) correspond.

Proof. First assume that the sequence (4.12) is a TD/D sequence over $\mathbb{F}$. By Definition 25, there exists a Leonard pair A, A^* over $\mathbb{F}$ that has TD/D sequence (4.12). Then, (4.14)–(4.17) hold by [49, Theorems 17.8, 17.9]. We have proved the result in one direction. Concerning the converse, assume that there exists a parameter array (4.13) over $\mathbb{F}$ that satisfies (4.14)–(4.17). Let A, A^* denote a Leonard pair over $\mathbb{F}$ that has parameter array (4.13). By (4.14)–(4.17) and [49, Theorems 17.8, 17.9], we find that the sequence (4.12) is a TD/D sequence of A, A^*. The last assertion follows from our above comments. $\square$

LEMMA 30. *Assume that $d \ge 1$. Consider a sequence of scalars in $\mathbb{F}$:*

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d). \tag{4.18}$$

The sequence (4.18) is a TD/D sequence over $\mathbb{F}$ if and only if there exists a parameter array:

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \tag{4.19}$$

over $\mathbb{F}$ such that

$$a_0 = \theta_d + \frac{\phi_1}{\theta^*_0 - \theta^*_1}, \tag{4.20}$$

$$a_i = \theta_{d-i} + \frac{\phi_i}{\theta^*_i - \theta^*_{i-1}} + \frac{\phi_{i+1}}{\theta^*_i - \theta^*_{i+1}} \qquad (1 \le i \le d-1), \tag{4.21}$$

$$a_d = \theta_0 + \frac{\phi_d}{\theta^*_d - \theta^*_{d-1}}, \tag{4.22}$$

$$x_i = \varphi_i \phi_i \frac{\tau^*_{i-1}(\theta^*_{i-1})\eta^*_{d-i}(\theta^*_i)}{\tau^*_i(\theta^*_i)\eta^*_{d-i+1}(\theta^*_{i-1})} \qquad (1 \le i \le d). \tag{4.23}$$

In this case, the TD/D sequence (4.18) and the parameter array (4.19) correspond.

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Proof. By Lemma 28, $(\{\theta_{d-i}\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d)$ is a parameter array over $\mathbb{F}$ corresponding to (4.18). Now apply Lemma 29 to this parameter array. ∎

LEMMA 31. *Let A, A^* denote a Leonard pair over $\mathbb{F}$ with TD/D sequence*

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d).$$

Then for scalars ξ, ζ, ξ^*, ζ^* in $\mathbb{F}$ with $\xi\xi^* \neq 0$, the sequence

$$(\{\xi a_i + \zeta\}_{i=0}^d; \{\xi^2 x_i\}_{i=1}^d; \{\xi^*\theta^*_i + \zeta^*\}_{i=0}^d),$$

is a TD/D sequence of the Leonard pair $\xi A + \zeta I, \xi^* A^* + \zeta^* I$.

Proof. By Lemmas 17 and 29. ∎

For the rest of this section, we use the following notation. Let A, A^* denote a Leonard pair on V with a TD/D sequence $(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d)$. For $0 \leq i \leq d$ let E^*_i denote the primitive idempotent of A^* for θ^*_i.

LEMMA 32. (See [49, Definition 7.1, Lemma 7.5].) *For $0 \leq i \leq d$ the following hold:*

(i) $E^*_i A E^*_i = a_i E^*_i$;
(ii) $a_i = \mathrm{tr}(AE^*_i)$.

LEMMA 33. (See [49, Definition 7.1, Lemma 7.5].) *For $1 \leq i \leq d$ the following hold:*

(i) $E^*_i A E^*_{i-1} A E^*_i = x_i E^*_i$;
(ii) $x_i = \mathrm{tr}(E^*_i A E^*_{i-1} A)$.

5. Leonard pairs in normalized TD/D form. Let A, A^* denote a Leonard pair over $\mathbb{F}$. In this section, we describe a normalization for A, A^*, called the normalized TD/D form. We show that the normalized TD/D forms of A, A^* are in bijection with the TD/D sequences of A, A^*.

DEFINITION 34. A Leonard pair A, A^* over $\mathbb{F}$ with diameter d is said to have *normalized TD/D form* whenever the following (i)–(iii) hold:

(i) the underlying vector space of A, A^* is $\mathbb{F}^{d+1}$;
(ii) A is a normalized irreducible tridiagonal matrix in $\mathrm{Mat}_{d+1}(\mathbb{F})$;
(iii) A^* is a diagonal matrix in $\mathrm{Mat}_{d+1}(\mathbb{F})$.

LEMMA 35. *Consider a Leonard pair A, A^* over $\mathbb{F}$ that is in normalized TD/D form:*

$$A = \begin{pmatrix} a_0 & x_1 & & & & \mathbf{0} \\ 1 & a_1 & x_2 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & x_d \\ \mathbf{0} & & & & 1 & a_d \end{pmatrix}, \qquad A^* = \mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d).$$

Then the following hold.

(i) *The sequence $(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d)$ is a TD/D sequence of A, A^*.*
(ii) *For $0 \leq i \leq d$ let E^*_i denote the primitive idempotent of A^* for θ^*_i. Then E^*_i has (i,i)-entry 1 and all other entries 0.*

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Proof. (i) Note by the paragraph below Lemma 6 that $\{\theta^*_i\}_{i=0}^d$ is a standard ordering of the eigenvalues of A^*. Now the result follows from Definition 22.

(ii) Clear. □

LEMMA 36. *Let A, A^* denote a Leonard pair on V with TD/D sequence*

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d).$$

Then the matrices

$$\begin{pmatrix} a_0 & x_1 & & & & \mathbf{0} \\ 1 & a_1 & x_2 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & x_d \\ \mathbf{0} & & & & 1 & a_d \end{pmatrix}, \qquad \operatorname{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d),$$

form a Leonard pair in normalized TD/D form that is isomorphic to A, A^*.

Proof. By Proposition 21 and Definition 22. □

Next, we give a variation on Lemma 36. Consider a sequence of scalars in $\mathbb{F}$:

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d).$$

Define matrices A, $A^* \in \operatorname{Mat}_{d+1}(\mathbb{F})$ by:

$$A = \begin{pmatrix} a_0 & x_1 & & & & \mathbf{0} \\ 1 & a_1 & x_2 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & x_d \\ \mathbf{0} & & & & 1 & a_d \end{pmatrix}, \qquad A^* = \operatorname{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d). \tag{5.24}$$

PROPOSITION 37. *With the above notation, the following are equivalent:*

(i) $(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d)$ *is a TD/D sequence over $\mathbb{F}$;*
(ii) *the matrices A, A^* form a Leonard pair over $\mathbb{F}$.*

Suppose (i) *and* (ii) *hold. Then the Leonard pair A, A^* is in normalized TD/D form, and $(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d)$ is a TD/D sequence of A, A^*.*

Proof. By Definitions 22 and 34. □

6. Bipartite and essentially bipartite Leonard pairs. In this section, we first recall the bipartite and essentially bipartite conditions on a Leonard pair. We then characterize these conditions in terms of the parameter array.

Throughout this section, let A, A^* denote a Leonard pair over $\mathbb{F}$ with parameter array:

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d),$$

and the corresponding TD/D sequence:

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d).$$

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Definition 38. (See [35, Section 1].)

(i) The Leonard pair A, A^* is said to be *bipartite* whenever $a_i = 0$ for $0 \leq i \leq d$.
(ii) The Leonard pair A, A^* is said to be *essentially bipartite* whenever a_i is independent of i for $0 \leq i \leq d$.

Note 39. *A bipartite Leonard pair is essentially bipartite.*

Note 40. *Assume that $d = 0$. Then any ordered pair A, A^* of elements in* $\mathrm{End}(V)$ *is an essentially bipartite Leonard pair. This Leonard pair is bipartite if and only if $A = 0$.*

Lemma 41. *The following hold.*

(i) *Assume that A, A^* is bipartite. Then for $\zeta \in \mathbb{F}$ the Leonard pair $A + \zeta I, A^*$ is essentially bipartite.*
(ii) *Assume that A, A^* is essentially bipartite. Then there exists a unique $\zeta \in \mathbb{F}$ such that $A - \zeta I, A^*$ is bipartite. The scalar ζ is equal to the common value of $\{a_i\}_{i=0}^d$.*

Proof. By Lemma 31 and Definition 38. □

Lemma 42. *Assume that A, A^* is bipartite. Then for scalars ξ, ξ^*, ζ^* in $\mathbb{F}$ with $\xi\xi^* \neq 0$, the Leonard pair $\xi A, \xi^* A^* + \zeta^* I$ is bipartite.*

Proof. By Lemma 31 and Definition 38. □

Lemma 43. *Assume that A, A^* is essentially bipartite. Then for scalars ξ, ζ, ξ^*, ζ^* in $\mathbb{F}$ with $\xi\xi^* \neq 0$, the Leonard pair $\xi A + \zeta I, \xi^* A^* + \zeta^* I$ is essentially bipartite.*

Proof. By Lemma 31 and Definition 38. □

Lemma 44. (See [35, Theorem 1.5].) *The following are equivalent:*

(i) *A, A^* is essentially bipartite;*
(ii) *$\theta_i + \theta_{d-i}$ is independent of i for $0 \leq i \leq d$, and $\varphi_i + \phi_i = 0$ for $1 \leq i \leq d$.*

Suppose (i), (ii) *hold. Then $\theta_i + \theta_{d-i} = 2\alpha$ for $0 \leq i \leq d$, where α is the common value of $\{a_i\}_{i=0}^d$.*

Lemma 45. *Assume that $d \geq 1$. Then the following are equivalent:*

(i) *A, A^* is bipartite;*
(ii) *$\theta_i + \theta_{d-i} = 0$ for $0 \leq i \leq d$, and $\varphi_i + \phi_i = 0$ for $1 \leq i \leq d$.*

Proof. (i) ⇒ (ii) By Lemma 44.

(ii) ⇒ (i) We first show that $\mathrm{Char}(\mathbb{F}) \neq 2$. By the assumption, $\theta_0 + \theta_d = 0$. By $d \geq 1$ and Lemma 16(i), $\theta_0 \neq \theta_d$. By these comments, $\theta_d \neq -\theta_d$. Thus, $\mathrm{Char}(\mathbb{F}) \neq 2$. By the assumption, $\theta_i + \theta_{d-i} = 0$ for $0 \leq i \leq d$. By this and the last assertion of Lemma 44, $2a_i = 0$ for $0 \leq i \leq d$. By this and $\mathrm{Char}(\mathbb{F}) \neq 2$, we get $a_i = 0$ for $0 \leq i \leq d$. □

Lemma 46. *Assume that $d \geq 1$ and A, A^* is essentially bipartite. Then* $\mathrm{Char}(\mathbb{F}) \neq 2$.

Proof. By way of contradiction, assume that $\mathrm{Char}(\mathbb{F}) = 2$. By Lemma 44, $\theta_0 + \theta_d = 0$. So $\theta_0 = \theta_d$, contradicting Lemma 16(i). □

Definition 47. A parameter array over $\mathbb{F}$ is said to be *bipartite* (resp. *essentially bipartite*) whenever the corresponding Leonard pair is bipartite (resp. essentially bipartite).

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7. The element F for a Leonard pair. Throughout this section, let A, A^* denote a Leonard pair over $\mathbb{F}$ with TD/D sequence:

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d). \tag{7.25}$$

We introduce the flat part F of A. We describe the bipartite condition and essentially bipartite condition in terms of F. For $0 \le i \le d$ let E^*_i denote the primitive idempotent of A^* associated with θ^*_i.

Definition 48. Define

$$F = \sum_{i=0}^d E^*_i A E^*_i.$$

We call F the *flat part of A*.

Note 49. *The element F is independent of the choice of the TD/D sequence (7.25) for A, A^*.*

Lemma 50. *Referring to Definition 48, $F = \sum_{i=0}^d a_i E^*_i$.*

Proof. By Lemma 32(i) and Definition 48. □

Lemma 51. *Assume that A, A^* is in normalized TD/D form:*

$$A = \begin{pmatrix} a_0 & x_1 & & & & \mathbf{0} \\ 1 & a_1 & x_2 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & x_d \\ \mathbf{0} & & & & 1 & a_d \end{pmatrix}, \qquad A^* = \mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d).$$

Then

$$F = \mathrm{diag}(a_0, a_1, \ldots, a_d), \qquad A - F = \begin{pmatrix} 0 & x_1 & & & & \mathbf{0} \\ 1 & 0 & x_2 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & x_d \\ \mathbf{0} & & & & 1 & 0 \end{pmatrix}. \tag{7.26}$$

Proof. By Lemmas 35(ii) and 50. □

Lemma 52. *Let F denote the flat part of A. Then F commutes with A^*. Moreover, $F \in \langle A^* \rangle$.*

Proof. By Lemma 1 and Definition 48. □

Lemma 53. *Let F denote the flat part of A. For scalars ξ, ζ, ξ^*, ζ^* in $\mathbb{F}$ with $\xi\xi^* \neq 0$, consider the Leonard pair $\xi A + \zeta I, \xi^* A^* + \zeta^* I$. For this Leonard pair, the flat part of $\xi A + \zeta I$ is $\xi F + \zeta I$.*

Proof. By Definition 48. □

Lemma 54. *Let F denote the flat part of A. Then the following hold:*

(i) *A, A^* is bipartite if and only if $F = 0$;*
(ii) *A, A^* is essentially bipartite if and only if F is a scalar multiple of I.*

Proof. By Definition 38 and Lemma 50. □

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8. Near-bipartite Leonard pairs. In this section, we introduce the near-bipartite condition on a Leonard pair. To motivate this condition, we make an observation. Let $\mathbb{O}$ denote the sequence $\{a_i\}_{i=0}^d$ of scalars in $\mathbb{F}$ such that $a_i = 0$ for $0 \le i \le d$.

LEMMA 55. *Let A, A^* denote a Leonard pair on V with TD/D sequence:*

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta_i^*\}_{i=0}^d).$$

Let F denote the flat part of A, and assume that $A - F, A^*$ is a Leonard pair on V. Then $A - F, A^*$ has a TD/D sequence:

$$(\mathbb{O}; \{x_i\}_{i=1}^d; \{\theta_i^*\}_{i=0}^d). \tag{8.27}$$

Moreover, the Leonard pair $A - F, A^*$ is bipartite.

Proof. By Lemma 36, we may assume that A, A^* is in normalized TD/D form, and A, A^* are as in (5.24). By Lemma 51, F and $A - F$ are as in (7.26). By this and Proposition 37, $A - F, A^*$ has a TD/D sequence (8.27). By this and Definition 38, the Leonard pair $A - F, A^*$ is bipartite. □

DEFINITION 56. Let A, A^* denote a Leonard pair on V, and let F denote the flat part of A. Then A, A^* is said to be *near-bipartite* whenever $A - F, A^*$ is a Leonard pair on V. In this case, we call the Leonard pair $A - F, A^*$ the *bipartite contraction of* A, A^*.

NOTE 57. *An essentially bipartite Leonard pair is near-bipartite.*

NOTE 58. *Assume that $d = 0$. Then any ordered pair of elements in* $\mathrm{End}(V)$ *is a near-bipartite Leonard pair.*

DEFINITION 59. Let B, A^* denote a bipartite Leonard pair on V. By a *near-bipartite expansion of* B, A^* we mean a near-bipartite Leonard pair A, A^* on V whose bipartite contraction is equal to B, A^*.

In the next result, we clarify Definitions 56 and 59.

LEMMA 60. *Let A, A^* denote a Leonard pair on V, and let F denote the flat part of A. Let B, A^* denote a bipartite Leonard pair on V. Then the following are equivalent:.*

(i) *B, A^* is the bipartite contraction of A, A^*;*
(ii) *A, A^* is a near-bipartite expansion of B, A^*;*
(iii) *$A - F = B$.*

Proof. (i) ⇔ (ii) By Definitions 56 and 59.

(i) ⇔ (iii) By Definition 56. □

In the next result, we give a variation on Lemma 60 that does not refer to F.

LEMMA 61. *Let A, A^* denote a Leonard pair on V, and let B, A^* denote a bipartite Leonard pair on V. Then the following are equivalent:*

(i) *B, A^* is the bipartite contraction of A, A^*;*
(ii) *A, A^* is a near-bipartite expansion of B, A^*;*
(iii) *$[A, A^*] = [B, A^*]$.*

Proof. (i) ⇔ (ii) By Lemma 60.

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(i), (ii) $\Rightarrow$ (iii) By Lemma 60, $A - F = B$. By Lemma 52, $[F, A^*] = 0$. By these comments, $[A, A^*] = [B, A^*]$.

(iii) $\Rightarrow$ (i), (ii) For the Leonard pair A, A^* let $\{\theta^*_i\}_{i=0}^d$ denote a dual eigenvalue sequence of A, A^*. For $0 \le i \le d$ let E^*_i denote the primitive idempotent of A^* associated with θ^*_i. By Definition 48, the flat part of A is $F = \sum_{i=0}^d E^*_i A E^*_i$. By Lemma 54(i) and since the Leonard pair B, A^* is bipartite, we have $E^*_i B E^*_i = 0$ for $0 \le i \le d$. By assumption $[A, A^*] = [B, A^*]$ so $[A - B, A^*] = 0$. By this and Lemma 1, we have $A - B \in \langle A^* \rangle$. The elements $\{E^*_i\}_{i=0}^d$ form a basis of $\langle A^* \rangle$, so there exist scalars $\{\alpha_i\}_{i=0}^d$ in $\mathbb{F}$ such that $A - B = \sum_{i=0}^d \alpha_i E^*_i$. We may now argue

$$F = \sum_{i=0}^d E^*_i A E^*_i = \sum_{i=0}^d E^*_i (A - B) E^*_i = \sum_{i=0}^d \alpha_i E^*_i = A - B.$$

By these comments and Lemma 60, we obtain (i), (ii). □

Lemma 62. *Let A, A^* denote a near-bipartite Leonard pair on V, with bipartite contraction B, A^*. Let ξ, ζ, ξ^*, ζ^* denote scalars in $\mathbb{F}$ with $\xi\xi^* \neq 0$. Then the Leonard pair*

$$\xi A + \zeta I,\ \xi^* A^* + \zeta^* I, \tag{8.28}$$

is near-bipartite, with bipartite contraction $\xi B, \xi^* A^* + \zeta^* I$.

Proof. Let F denote the flat part of A. We have $B = A - F$ by Lemma 60. For the Leonard pair (8.28), the flat part of $\xi A + \zeta I$ is $\xi F + \zeta I$ by Lemma 53. We have

$$\xi A + \zeta I - (\xi F + \zeta I) = \xi(A - F) = \xi B.$$

Note that $\xi B, \xi^* A^* + \zeta^* I$ is a bipartite Leonard pair by Lemma 42 and since B, A^* is a bipartite Leonard pair. By these comments

$$\xi A + \zeta I - (\xi F + \zeta I),\ \xi^* A^* + \zeta^* I,$$

is a Leonard pair. By these comments and Definition 56, the pair (8.28) is near-bipartite, with bipartite contraction $\xi B, \xi^* A^* + \zeta^* I$. □

Lemma 63. *Let A, A^* denote a near-bipartite Leonard pair over $\mathbb{F}$ that is in normalized TD/D form. Then the bipartite contraction of A, A^* is in normalized TD/D form.*

Proof. By Definition 34 and Lemmas 51, 55. □

Lemma 64. *Let B, A^* denote a bipartite Leonard pair over $\mathbb{F}$ that is in normalized TD/D form. Then every near-bipartite expansion of B, A^* is in normalized TD/D form.*

Proof. By Definition 34 and Lemmas 51, 55. □

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LEMMA 65. *Referring to Lemma 61, assume that both A, A^* and B, A^* are in normalized TD/D form. Let*

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \qquad (\{\theta'_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi'_i\}_{i=1}^d; \{\phi'_i\}_{i=1}^d),$$

denote a parameter array of A, A^* and B, A^*, respectively, and let

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d), \qquad (\mathbb{O}; \{x'_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d),$$

denote the corresponding TD/D sequence of A, A^* and B, A^*, respectively. Then (i)–(iii) *hold if and only if*

$$x_i = x'_i \qquad (1 \le i \le d), \tag{8.29}$$

if and only if

$$\varphi_i \phi_i = \varphi'_i \phi'_i \qquad (1 \le i \le d). \tag{8.30}$$

Proof. By Lemma 51, $A^* = \mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d)$ and $F = \mathrm{diag}(a_0, a_1, \ldots, a_d)$. Also by Lemma 51,

$$A = \begin{pmatrix} a_0 & x_1 & & & & \mathbf{0} \\ 1 & a_1 & x_2 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & x_d \\ \mathbf{0} & & & & 1 & a_d \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & x'_1 & & & & \mathbf{0} \\ 1 & 0 & x'_2 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & x'_d \\ \mathbf{0} & & & & 1 & 0 \end{pmatrix}.$$

By this and Definition 56, the equivalent conditions (i)–(iii) in Lemma 61 hold if and only if $A - F = B$ if and only if (8.29) holds. By (4.17), the condition (8.29) is equivalent to the condition (8.30). ∎

LEMMA 66. *Let A, A^* denote a Leonard pair over $\mathbb{F}$ with parameter array*

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d). \tag{8.31}$$

Let

$$(\{\theta'_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi'_i\}_{i=1}^d; \{\phi'_i\}_{i=1}^d), \tag{8.32}$$

denote a bipartite parameter array over $\mathbb{F}$. Then the following are equivalent:

(i) *$\varphi_i \phi_i = \varphi'_i \phi'_i$ for $1 \le i \le d$;*
(ii) *A, A^* is near-bipartite, and its bipartite contraction has parameter array* (8.32).

Proof. By Lemma 36, we may assume that A, A^* is in normalized TD/D form. Let B, A^* denote a Leonard pair in normalized TD/D form that has parameter array (8.32). Note that B, A^* is bipartite. Now the result follows by Lemma 65. ∎

LEMMA 67. *Let B, A^* denote a bipartite Leonard pair over $\mathbb{F}$ with parameter array*

$$(\{\theta'_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi'_i\}_{i=1}^d; \{\phi'_i\}_{i=1}^d). \tag{8.33}$$

Let

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \tag{8.34}$$

denote a parameter array over $\mathbb{F}$. Then the following are equivalent:

(i) *$\varphi_i \phi_i = \varphi'_i \phi'_i$ for $1 \le i \le d$;*
(ii) *there exists a near-bipartite expansion of B, A^* that has parameter array* (8.34).

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Proof. By Lemma 36, we may assume that B, A^* is in normalized TD/D form.

(i) $\Rightarrow$ (ii) Define scalars $\{a_i\}_{i=0}^d$ by (4.14)–(4.16). Define scalars $\{x_i\}_{i=1}^d$ by (4.17). Define $A \in \mathrm{Mat}_{d+1}(\mathbb{F})$ as in (5.24). By Lemma 36, the pair A, A^* is a Leonard pair that has parameter array (8.34). By Lemma 65, A, A^* is a near-bipartite expansion of B, A^*.

(ii) $\Rightarrow$ (i) By Lemma 65. □

In the present paper, we have three main goals. Assuming $\mathbb{F}$ is algebraically closed,

(i) we classify up to isomorphism the near-bipartite Leonard pairs over $\mathbb{F}$;
(ii) for each near-bipartite Leonard pair over $\mathbb{F}$, we describe its bipartite contraction;
(iii) for each bipartite Leonard pair over $\mathbb{F}$, we describe its near-bipartite expansions.

Lemma 68. *Assume that $d \geq 1$ and $\mathrm{Char}(\mathbb{F}) = 2$. Then there does not exist a near-bipartite Leonard pair on V.*

Proof. By Definition 56 and Lemma 46. □

In view of Lemmas 46, 68, for the rest of this paper we assume

$$\mathrm{Char}(\mathbb{F}) \neq 2.$$

As a warmup, we will treat $d = 1$, and for this case we assume that $\mathbb{F}$ is arbitrary. Later, for $d \geq 2$ we will assume that $\mathbb{F}$ is algebraically closed.

9. Near-bipartite Leonard pairs with diameter one. In this section, we classify up to isomorphism the near-bipartite Leonard pairs with diameter one. Moreover, for each near-bipartite Leonard pair with diameter one, we describe its bipartite contraction. Also, for each bipartite Leonard pair with diameter one, we describe its near-bipartite expansions.

In view of Proposition 37, we consider the following matrices in $\mathrm{Mat}_2(\mathbb{F})$:

$$A = \begin{pmatrix} a_0 & x_1 \\ 1 & a_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & x_1 \\ 1 & 0 \end{pmatrix}, \qquad A^* = \begin{pmatrix} \theta_0^* & 0 \\ 0 & \theta_1^* \end{pmatrix}.$$

Let $f_A(\lambda)$ denote the characteristic polynomial of A. By linear algebra,

$$f_A(\lambda) = \lambda^2 - (a_0 + a_1)\lambda + a_0 a_1 - x_1.$$

Let θ_0 and θ_1 denote the roots of $f_A(\lambda)$. We remark that θ_0, θ_1 are contained in the algebraic closure of $\mathbb{F}$, and possibly not in $\mathbb{F}$. We have

$$(\theta_0 - \theta_1)^2 = (a_0 - a_1)^2 + 4x_1. \tag{9.35}$$

Lemma 69. *The pair A, A^* is a Leonard pair over $\mathbb{F}$ if and only if the following* (i) *and* (ii) *hold:*

(i) *θ_0, $\theta_1 \in \mathbb{F}$;*
(ii) *$x_1 \neq 0$, $\theta_0^* \neq \theta_1^*$, and*

$$(a_0 - a_1)^2 + 4x_1 \neq 0. \tag{9.36}$$

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Proof. First assume that A, A^* is a Leonard pair over $\mathbb{F}$. Clearly (i) holds. By Lemma 7, $x_1 \neq 0$. By Lemma 6, $\theta_0^* \neq \theta_1^*$. Also by Lemma 6, $\theta_0 \neq \theta_1$. By this and (9.35), we get (9.36). Thus, (ii) holds. Next assume that (i) and (ii) hold. We have $\theta_0 \neq \theta_1$ by (9.35) and (9.36), so A is diagonalizable. The matrices A and A^* do not have an eigenspace in common, since A is not upper or lower triangular. Since A is diagonalizable, there exists an invertible matrix $P \in \mathrm{Mat}_2(\mathbb{F})$ such that PAP^{-1} is diagonal. The matrix PA^*P^{-1} has off diagonal entries nonzero, because A and A^* have no eigenspace in common. Therefore, PA^*P^{-1} is irreducible tridiagonal. By these comments, A, A^* is a Leonard pair over $\mathbb{F}$. □

Let $f_B(\lambda)$ denote the characteristic polynomial of B. We have $f_B(\lambda) = \lambda^2 - x_1$. The roots of $f_B(\lambda)$ are $\pm\sqrt{x_1}$. We remark that $\sqrt{x_1}$ is contained in the algebraic closure of $\mathbb{F}$, and possibly not in $\mathbb{F}$. The following result classifies the bipartite Leonard pairs with diameter one.

Proposition 70. *The pair B, A^* is a Leonard pair over $\mathbb{F}$ if and only if the following* (i) *and* (ii) *hold:*

(i) $\sqrt{x_1} \in \mathbb{F}$;
(ii) $x_1 \neq 0$ *and* $\theta_0^* \neq \theta_1^*$.

Proof. By Lemma 69. □

The following result classifies the near-bipartite Leonard pairs with diameter one and describes its bipartite contraction.

Proposition 71. *Assume that the pair A, A^* is a Leonard pair over $\mathbb{F}$. Then the following* (i) *and* (ii) *are equivalent.*

(i) A, A^* *is near-bipartite;*
(ii) B, A^* *is a Leonard pair over* $\mathbb{F}$.

Assume that (i) *and* (ii) *hold. Then B, A^* is the bipartite contraction of A, A^*.*

Proof. Clear. □

The following result describes the near-bipartite expansions of a given bipartite Leonard pair over $\mathbb{F}$ with diameter one.

Proposition 72. *Assume that the pair B, A^* is a Leonard pair over $\mathbb{F}$. Then the following* (i)–(iii) *are equivalent:*

(i) A, A^* *is a near-bipartite expansion of* B, A^*;
(ii) A, A^* *is a Leonard pair over* $\mathbb{F}$;
(iii) θ_0, $\theta_1 \in \mathbb{F}$ *and* a_0, a_1, x_1 *satisfy* (9.36).

Proof. The equivalence of (i) and (ii) follows from Lemmas 51 and 60. The equivalence of (ii) and (iii) follows from Lemma 69 and Proposition 70. □

Next, we consider what the previous results in this section become if $\mathbb{F}$ is algebraically closed.

Lemma 73. *Assume that $\mathbb{F}$ is algebraically closed. Then the pair A, A^* is a Leonard pair over $\mathbb{F}$ if and only if $x_1 \neq 0$, $\theta_0^* \neq \theta_1^*$, and*

$$(a_0 - a_1)^2 + 4x_1 \neq 0.$$

Proposition 74. *Assume that $\mathbb{F}$ is algebraically closed. Then, the pair B, A^* is a Leonard pair over $\mathbb{F}$ if and only if $x_1 \neq 0$ and $\theta_0^* \neq \theta_1^*$.*

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Proposition 75. *Assume that $\mathbb{F}$ is algebraically closed, and the pair A, A^* is a Leonard pair over $\mathbb{F}$. Then A, A^* is near-bipartite if and only if B, A^* is a Leonard pair over $\mathbb{F}$. In this case, B, A^* is the bipartite contraction of A, A^*.*

Proposition 76. *Assume that $\mathbb{F}$ is algebraically closed, and the pair B, A^* is a Leonard pair over $\mathbb{F}$. Then A, A^* is a near-bipartite expansion of B, A^* if and only if A, A^* is a Leonard pair over $\mathbb{F}$ if and only if a_0, a_1, x_1 satisfy* (9.36).

For the rest of this paper, we assume that $\mathbb{F}$ is algebraically closed.

10. Leonard pairs with diameter two in TD/D form. In the previous section, we described the near-bipartite Leonard pairs with diameter one. In the present section and the next, we describe the near-bipartite Leonard pairs with diameter two. Consider the following matrices in $\mathrm{Mat}_3(\mathbb{F})$:

$$A = \begin{pmatrix} a_0 & x_1 & 0 \\ 1 & a_1 & x_2 \\ 0 & 1 & a_2 \end{pmatrix}, \qquad A^* = \begin{pmatrix} \theta_0^* & 0 & 0 \\ 0 & \theta_1^* & 0 \\ 0 & 0 & \theta_2^* \end{pmatrix}.$$

In this section, we prove the following result.

Proposition 77. *The pair A, A^* is Leonard pair over $\mathbb{F}$ if and only if*

$$\theta_i^* \neq \theta_j^* \quad (0 \leq i < j \leq 2), \qquad x_i \neq 0 \quad (1 \leq i \leq 2), \tag{10.37}$$

$$\frac{x_1}{\theta_2^* - \theta_1^*} + \frac{x_2}{\theta_0^* - \theta_1^*} = \frac{a_0 - a_2}{(\theta_0^* - \theta_2^*)^2}\Big(a_0(\theta_0^* - \theta_1^*) + a_1(\theta_2^* - \theta_0^*) + a_2(\theta_1^* - \theta_2^*)\Big), \tag{10.38}$$

$$0 \neq \frac{x_1}{(\theta_2^* - \theta_1^*)^2} + \frac{x_2}{(\theta_0^* - \theta_1^*)^2} + \frac{(a_0 - a_2)^2}{(\theta_0^* - \theta_2^*)^2}, \tag{10.39}$$

$$0 \neq \frac{x_1}{\theta_2^* - \theta_1^*} - \frac{x_2}{\theta_0^* - \theta_1^*} + \frac{(a_0 - a_2)^2}{2(\theta_2^* - \theta_0^*)} + \frac{\big(a_0(\theta_0^* - \theta_1^*) + a_1(\theta_2^* - \theta_0^*) + a_2(\theta_1^* - \theta_2^*)\big)^2}{2(\theta_2^* - \theta_0^*)^3}. \tag{10.40}$$

Lemma 78. *Assume that A, A^* is a Leonard pair over $\mathbb{F}$, and $\{\theta_i\}_{i=0}^2$ is a standard ordering of the eigenvalues of A. Then*

$$\theta_1 = \frac{a_0(\theta_0^* - \theta_1^*) + a_2(\theta_1^* - \theta_2^*)}{\theta_0^* - \theta_2^*}. \tag{10.41}$$

Proof. By Lemma 16(iii),

$$\varphi_2 = \phi_1 + (\theta_2^* - \theta_0^*)(\theta_1 - \theta_2).$$

In this equation, eliminate φ_2 and ϕ_1 using (4.16) and (4.20) and solve the result in θ_1. ∎

Lemma 79. *Assume that A, A^* is a Leonard pair over $\mathbb{F}$. Then* (10.38) *holds.*

Proof. By (4.17),

$$x_1 = -\varphi_1\phi_1 \frac{\theta_1^* - \theta_2^*}{(\theta_0^* - \theta_1^*)^2(\theta_0^* - \theta_2^*)}, \qquad x_2 = -\varphi_2\phi_2 \frac{\theta_0^* - \theta_1^*}{(\theta_0^* - \theta_2^*)(\theta_1^* - \theta_2^*)^2}.$$

In these equations, eliminate φ_1, φ_2, ϕ_1, ϕ_2 using (4.14), (4.16), (4.20), and (4.22). This gives

$$x_1 = -\frac{(a_0 - \theta_0)(a_0 - \theta_2)(\theta_1^* - \theta_2^*)}{\theta_0^* - \theta_2^*}, \qquad x_2 = -\frac{(a_2 - \theta_0)(a_2 - \theta_2)(\theta_0^* - \theta_1^*)}{\theta_0^* - \theta_2^*}.$$

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So

$$\frac{x_1}{\theta^*_1-\theta^*_2}-\frac{x_2}{\theta^*_0-\theta^*_1}=-\frac{(a_0-\theta_0)(a_0-\theta_2)-(a_2-\theta_0)(a_2-\theta_2)}{\theta^*_0-\theta^*_2}. \tag{10.42}$$

Considering the trace of A,

$$\theta_0+\theta_1+\theta_2=a_0+a_1+a_2.$$

Using this and (10.41), one finds that

$$\begin{aligned}&(a_0-\theta_0)(a_0-\theta_2)-(a_2-\theta_0)(a_2-\theta_2)\\&\qquad=\frac{(a_0-a_2)\big(a_0(\theta^*_0-\theta^*_1)+a_1(\theta^*_2-\theta^*_0)+a_2(\theta^*_1-\theta^*_2)\big)}{\theta^*_0-\theta^*_2}.\end{aligned}$$

By this and (10.42), we get (10.38). □

Lemma 80. *Assume that A, A^* is a Leonard pair over $\mathbb{F}$. Let $\{\theta_i\}_{i=0}^2$ denote a standard ordering of the eigenvalues of A. Then*

$$\theta_0+\theta_2=\frac{a_0(\theta^*_1-\theta^*_2)+a_1(\theta^*_0-\theta^*_2)+a_2(\theta^*_0-\theta^*_1)}{\theta^*_0-\theta^*_2}, \tag{10.43}$$

$$\theta_0\theta_2=\frac{a_1\big(a_0(\theta^*_1-\theta^*_2)+a_2(\theta^*_0-\theta^*_1)\big)}{\theta^*_0-\theta^*_2}+\frac{(a_0-a_2)^2(\theta^*_1-\theta^*_0)(\theta^*_1-\theta^*_2)}{(\theta^*_0-\theta^*_2)^2}-x_1-x_2. \tag{10.44}$$

Proof. Let $f(\lambda)$ denote the characteristic polynomial of A. Then

$$\begin{aligned}f(\lambda)=\lambda^3-(a_0+a_1+a_2)\lambda^2+(a_0a_1+a_0a_2+a_1a_2-x_1-x_2)\lambda&\\+a_0x_2+a_2x_1-a_0a_1a_2.&\end{aligned}$$

We have $f(\lambda)=(\lambda-\theta_0)(\lambda-\theta_1)(\lambda-\theta_2)$. Comparing the previous two equations, we obtain

$$\theta_0+\theta_2=a_0+a_1+a_2-\theta_1, \tag{10.45}$$

$$\theta_0\theta_2=a_0a_1+a_0a_2+a_1a_2-x_1-x_2-\theta_1(\theta_0+\theta_2). \tag{10.46}$$

In (10.45), eliminate θ_1 using (10.41) to get (10.43). In (10.46), eliminate θ_1 using (10.41) and simplify the result using (10.38) to get (10.44). □

Lemma 81. *Assume that A, A^* is a Leonard pair over $\mathbb{F}$. Let $\{\theta_i\}_{i=0}^2$ denote a standard ordering of the eigenvalues of A. Then*

$$\begin{aligned}\frac{(\theta_1-\theta_0)(\theta_1-\theta_2)}{(\theta^*_1-\theta^*_0)(\theta^*_1-\theta^*_2)}&=\frac{x_1}{(\theta^*_2-\theta^*_1)^2}+\frac{x_2}{(\theta^*_0-\theta^*_1)^2}+\frac{(a_0-a_2)^2}{(\theta^*_0-\theta^*_2)^2},\\\frac{(\theta_0-\theta_2)^2}{2(\theta^*_2-\theta^*_0)}&=\frac{x_1}{\theta^*_2-\theta^*_1}-\frac{x_2}{\theta^*_0-\theta^*_1}+\frac{(a_0-a_2)^2}{2(\theta^*_2-\theta^*_0)}+\frac{\big(a_0(\theta^*_0-\theta^*_1)+a_1(\theta^*_2-\theta^*_0)+a_2(\theta^*_1-\theta^*_2)\big)^2}{2(\theta^*_2-\theta^*_0)^3}.\end{aligned}$$

Proof. We have

$$\begin{aligned}(\theta_1-\theta_0)(\theta_1-\theta_2)&=\theta_1^2-(\theta_0+\theta_2)\theta_1+\theta_0\theta_2,\\(\theta_0-\theta_2)^2&=(\theta_0+\theta_2)^2-4\theta_0\theta_2.\end{aligned}$$

In the right-hand sides of these equations, eliminate θ_1, $\theta_0+\theta_2$, $\theta_0\theta_2$ using (10.41), (10.43), and (10.44) and simplify the result using (10.38). □

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Proof of Proposition 77. First assume that A, A^* is a Leonard pair over $\mathbb{F}$. Clearly (10.37) holds. By Lemma 79, (10.38) holds. The scalars $\{\theta_i\}_{i=0}^2$ are mutually distinct. By this and Lemma 81, we get (10.39) and (10.40).

Next assume that (10.37)–(10.40) hold. Define scalars $\{\theta_i\}_{i=0}^2$ that satisfy (10.41), (10.43), and (10.44). Define scalars

$$\varphi_1 = (a_0 - \theta_0)(\theta_0^* - \theta_1^*), \qquad \varphi_2 = (a_2 - \theta_2)(\theta_2^* - \theta_1^*), \tag{10.47}$$

$$\phi_1 = (a_0 - \theta_2)(\theta_0^* - \theta_1^*), \qquad \phi_2 = (a_2 - \theta_0)(\theta_2^* - \theta_1^*). \tag{10.48}$$

Our first goal is to show that the sequence

$$(\{\theta_i\}_{i=0}^2; \{\theta_i^*\}_{i=0}^2; \{\varphi_i\}_{i=1}^2; \{\phi_i\}_{i=1}^2), \tag{10.49}$$

is a parameter array over $\mathbb{F}$. We have

$$(a_0 - \theta_0)(a_0 - \theta_2) = a_0^2 - (\theta_0 + \theta_2)a_0 + \theta_0\theta_2.$$

In this equation, eliminate the factors $\theta_0 + \theta_2$ and $\theta_0\theta_2$ using (10.43) and (10.44). In the result, eliminate x_2 using (10.38) to get

$$(a_0 - \theta_0)(a_0 - \theta_2) = -\frac{\theta_0^* - \theta_2^*}{\theta_1^* - \theta_2^*} x_1. \tag{10.50}$$

In a similar way, we get

$$(a_2 - \theta_0)(a_2 - \theta_2) = -\frac{\theta_0^* - \theta_2^*}{\theta_0^* - \theta_1^*} x_2. \tag{10.51}$$

We show that

$$x_1 = -\varphi_1\phi_1 \frac{\theta_1^* - \theta_2^*}{(\theta_0^* - \theta_1^*)^2(\theta_0^* - \theta_2^*)}, \tag{10.52}$$

$$x_2 = -\varphi_2\phi_2 \frac{\theta_0^* - \theta_1^*}{(\theta_0^* - \theta_2^*)(\theta_1^* - \theta_2^*)^2}. \tag{10.53}$$

In the right-hand side of (10.52), eliminate φ_1, ϕ_1 using (10.47) and (10.48) and simplify the result using (10.50) to get (10.52). The line (10.53) is similarly obtained. We now verify conditions (i)–(v) in Lemma 16. We are assuming $d = 2$. We first verify that condition (i) holds. The scalars $\{\theta_i^*\}_{i=0}^2$ are mutually distinct by (10.37). As in the proof of Lemma 81, we get the two equations in Lemma 81. By this and (10.39) and (10.40), we find that the scalars $\{\theta_i\}_{i=0}^2$ are mutually distinct. By these comments, condition (i) holds. Condition (ii) holds by (10.52) and (10.53). Conditions (iii) and (iv) hold by (10.41), (10.47), and (10.48). Condition (v) vacuously holds. We have shown that (10.49) is a parameter array over $\mathbb{F}$. Our next goal is to show that the sequence

$$(\{a_i\}_{i=0}^2; \{x_i\}_{i=1}^2; \{\theta_i^*\}_{i=0}^2) \tag{10.54}$$

is a TD/D sequence. To do this, we verify (4.14)–(4.17). Lines (4.14) and (4.16) hold by (10.47) and (10.48). Using (10.47), we find that

$$\frac{\varphi_1}{\theta_1^* - \theta_0^*} + \frac{\varphi_2}{\theta_1^* - \theta_2^*} = \theta_0 + \theta_2 - a_0 - a_2.$$

In this line, use $\theta_0 + \theta_1 + \theta_2 = a_0 + a_1 + a_2$ to get (4.15). Line (4.17) holds by (10.52) and (10.53). We have shown that the sequence (10.54) is a TD/D sequence. Now by Proposition 37, A, A^* is a Leonard pair over $\mathbb{F}$.

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11. The classification of near-bipartite Leonard pairs with diameter two. In this section, we classify up to isomorphism the near-bipartite Leonard pairs with diameter two. Moreover, for each near-bipartite Leonard pair with diameter two, we describe its bipartite contraction. Also, for each bipartite Leonard pair with diameter two, we describe its near-bipartite expansions.

In view of Proposition 37, we consider the following matrices in $\mathrm{Mat}_3(\mathbb{F})$:

$$A = \begin{pmatrix} a_0 & x_1 & 0 \\ 1 & a_1 & x_2 \\ 0 & 1 & a_2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & x_1 & 0 \\ 1 & 0 & x_2 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A^* = \begin{pmatrix} \theta^*_0 & 0 & 0 \\ 0 & \theta^*_1 & 0 \\ 0 & 0 & \theta^*_2 \end{pmatrix}.$$

The following result classifies the bipartite Leonard pairs with diameter two.

Proposition 82. *The pair B, A^* is a Leonard pair over $\mathbb{F}$ if and only if*

$$\theta^*_i \neq \theta^*_j \quad (0 \leq i < j \leq 2), \qquad\qquad x_i \neq 0 \quad (1 \leq i \leq 2), \tag{11.55}$$

$$\frac{x_1}{\theta^*_1 - \theta^*_2} = \frac{x_2}{\theta^*_0 - \theta^*_1}. \tag{11.56}$$

Proof. Apply Proposition 77 with $a_i = 0$ $(0 \leq i \leq 2)$. ◻

The following result classifies the near-bipartite Leonard pairs with diameter two and describes its bipartite contraction.

Proposition 83. *Assume that A, A^* is a Leonard pair over $\mathbb{F}$. Then the pair A, A^* is near-bipartite if and only if* (11.56) *holds. In this case, B, A^* is the bipartite contraction of A, A^*.*

Proof. By Propositions 77 and 82. ◻

Our next goal is to determine all the near-bipartite expansions of a given bipartite Leonard pair. We will invoke Proposition 77. To do this, we consider what conditions (10.38)–(10.40) become under the assumption that B, A^* is a Leonard pair. We first consider the condition (10.38).

Lemma 84. *Assume that B, A^* is a Leonard pair over $\mathbb{F}$. Then* (10.38) *holds if and only if one of the following* (i), (ii) *holds:*

(i) $a_0 = a_2$;
(ii) $a_0 \neq a_2$ *and*

$$a_0(\theta^*_0 - \theta^*_1) + a_2(\theta^*_1 - \theta^*_2) + a_1(\theta^*_2 - \theta^*_0) = 0. \tag{11.57}$$

Proof. Note by Proposition 82 that (11.56) holds, and so in (10.38) the left-hand side is zero. First assume that (10.38) holds. In (10.38), the right-hand side is zero. Thus, one of (i) and (ii) holds. Next assume that one of (i) and (ii) holds. Then in (10.38), the right-hand side is zero. So (10.38) holds. ◻

Next, we consider conditions (10.39) and (10.40).

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LEMMA 85. *Assume that B, A^* is a Leonard pair over $\mathbb{F}$. Then (10.39) and (10.40) hold if and only if one of the following holds:*

(i) $a_0 = a_2$, *and*

$$(a_0 - a_1)^2 + 4(x_1 + x_2) \neq 0; \tag{11.58}$$

(ii) $a_0 \neq a_2$, *and*

$$(a_0 - a_2)^2 + 4(x_1 + x_2) \neq 0, \tag{11.59}$$

$$(a_0 - a_2)^2 + \frac{(x_1 + x_2)^3}{x_1 x_2} \neq 0. \tag{11.60}$$

Proof. Note by Proposition 82 that (11.56) holds. Using (11.56), one routinely finds that

$$(\theta_0^* - \theta_2^*) \left(\frac{x_1}{\theta_1^* - \theta_2^*} + \frac{x_2}{\theta_0^* - \theta_1^*} \right) = 2(x_1 + x_2), \tag{11.61}$$

$$(\theta_0^* - \theta_2^*)^2 \left(\frac{x_1}{(\theta_1^* - \theta_2^*)^2} + \frac{x_2}{(\theta_0^* - \theta_1^*)^2} \right) = \frac{(x_1 + x_2)^3}{x_1 x_2}. \tag{11.62}$$

For each of the cases $a_0 = a_2$ and $a_0 \neq a_2$, simplify (10.39) and (10.40) using (11.61), (11.62). For the case $a_0 \neq a_2$, also use (11.57). ∎

PROPOSITION 86. *Assume that B, A^* is a Leonard pair over $\mathbb{F}$. Then A, A^* is a near-bipartite expansion of B, A^* if and only if one of the following* (i) and (ii) *holds:*

(i) $a_0 = a_2$ *and* (11.58) *holds;*
(ii) $a_0 \neq a_2$ *and* (11.57), (11.59), (11.60) *hold.*

Proof. By Proposition 77 and Lemmas 84, 85. ∎

12. The type of a Leonard pair. In Sections 9 and 11, we classified the near-bipartite Leonard pairs of diameter one and two. To classify the near-bipartite Leonard pairs of diameter at least 3, we will divide the arguments into some cases. To describe these cases, we recall from [35] the type of a Leonard pair.

For the rest of this paper, we assume that $d \geq 3$.

DEFINITION 87. (See [35, Section 4].) Let A, A^* denote a Leonard pair over $\mathbb{F}$ with diameter d. Let β denote the fundamental constant of A, A^*. We define the *type* of A, A^* as follows:

Type	Description
I	$\beta \neq 2, \quad \beta \neq -2$
II	$\beta = 2$
III^+	$\beta = -2, \quad d$ even
III^-	$\beta = -2, \quad d$ odd

NOTE 88. *By Definition 18, the fundamental constant of a Leonard pair A, A^* is determined by A^*. By this and Definition 56, any near-bipartite Leonard pair has the same type as its bipartite contraction.*

DEFINITION 89. By the *type of a parameter array*, we mean the type of the corresponding Leonard pair.

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Lemma 90. *Let A, A^* denote a Leonard pair over $\mathbb{F}$. For scalars ξ, ζ, ξ^*, ζ^* in $\mathbb{F}$ with $\xi\xi^* \neq 0$, the Leonard pair $\xi A + \zeta I, \xi^* A^* + \zeta^* I$ has the same type as A, A^*.*

Proof. By Lemma 17 and Definition 18, the Leonard pairs A, A^* and $\xi A + \zeta I, \xi^* A^* + \zeta^* I$ have the same fundamental constant. ∎

By [35, Theorem 9.6] a Leonard pair of type III^- is not bipartite, so not near-bipartite by Note 88. In Sections 13–15, we will describe the parameter arrays of types I, II, III^+.

13. Parameter arrays of type I. In this section, we describe the parameter arrays of type I. Throughout this section, fix a nonzero $q \in \mathbb{F}$ such that $q^4 \neq 1$.

Lemma 91. (See [42, Lemma 16.1].) *For a sequence*

$$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau), \tag{13.63}$$

of scalars in $\mathbb{F}$, define

$$\theta_i = \delta + \mu q^{2i-d} + h q^{d-2i}, \tag{13.64}$$

$$\theta^*_i = \delta^* + \mu^* q^{2i-d} + h^* q^{d-2i}, \tag{13.65}$$

for $0 \leq i \leq d$ and

$$\varphi_i = (q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})(\tau - \mu\mu^* q^{2i-d-1} - hh^* q^{d-2i+1}), \tag{13.66}$$

$$\phi_i = (q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})(\tau - h\mu^* q^{2i-d-1} - \mu h^* q^{d-2i+1}), \tag{13.67}$$

for $1 \leq i \leq d$. Then the sequence

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \tag{13.68}$$

is a parameter array over $\mathbb{F}$ that has type I and fundamental constant $\beta = q^2 + q^{-2}$, provided that the inequalities in Lemma 16(i),(ii) *hold. Conversely, assume that the sequence* (13.68) *is a parameter array over $\mathbb{F}$ that has type I and fundamental constant $\beta = q^2 + q^{-2}$. Then there exists a unique sequence* (13.63) *of scalars in $\mathbb{F}$ that satisfies* (13.64)–(13.67).

Lemma 92. (See [42, Lemma 16.4].) *Referring to Lemma 91, the inequalities in Lemma 16*(i),(ii) *hold if and only if*

$$q^{2i} \neq 1 \qquad (1 \leq i \leq d), \tag{13.69}$$

$$\mu \neq h q^{2i} \qquad (1-d \leq i \leq d-1), \tag{13.70}$$

$$\mu^* \neq h^* q^{2i} \qquad (1-d \leq i \leq d-1), \tag{13.71}$$

$$\tau \neq \mu\mu^* q^{2i-d-1} + hh^* q^{d-2i+1} \qquad (1 \leq i \leq d), \tag{13.72}$$

$$\tau \neq h\mu^* q^{2i-d-1} + \mu h^* q^{d-2i+1} \qquad (1 \leq i \leq d). \tag{13.73}$$

Definition 93. A *primary q-data* is a sequence (13.63) of scalars in $\mathbb{F}$ that satisfy (13.69)–(13.73). The primary q-data (13.63) and the parameter array (13.68) are said to *correspond*.

Definition 94. Let A, A^* denote a Leonard pair of type I. By *a primary q-data of A, A^**, we mean the primary q-data that corresponds to a parameter array of A, A^*.

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LEMMA 95. *For the parameter arrays in Lemma 13, consider the corresponding primary q-data. These primary q-data are related as follows:*

Parameter array	Primary q-data
$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$	$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau)$
$(\{\theta_i\}_{i=0}^d; \{\theta^*_{d-i}\}_{i=0}^d; \{\phi_{d-i+1}\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d)$	$(\delta, \mu, h, \delta^*, h^*, \mu^*, \tau)$
$(\{\theta_{d-i}\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d)$	$(\delta, h, \mu, \delta^*, \mu^*, h^*, \tau)$
$(\{\theta_{d-i}\}_{i=0}^d; \{\theta^*_{d-i}\}_{i=0}^d; \{\varphi_{d-i+1}\}_{i=1}^d; \{\phi_{d-i+1}\}_{i=1}^d)$	$(\delta, h, \mu, \delta^*, h^*, \mu^*, \tau)$

Proof. Use Lemma 91 and Definition 93. ∎

14. Parameter arrays of type II. In this section, we describe the parameter arrays of type II.

LEMMA 96. (See [42, Lemma 19.1].) *For a sequence*

$$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau), \tag{14.74}$$

of scalars in $\mathbb{F}$, define

$$\theta_i = \delta + \mu(i - d/2) + hi(d-i), \tag{14.75}$$

$$\theta^*_i = \delta^* + \mu^*(i - d/2) + h^*i(d-i), \tag{14.76}$$

for $0 \le i \le d$ and

$$\varphi_i = i(d-i+1)\big(\tau - \mu\mu^*/2 + (h\mu^* + \mu h^*)(i-(d+1)/2) + hh^*(i-1)(d-i)\big), \tag{14.77}$$

$$\phi_i = i(d-i+1)\big(\tau + \mu\mu^*/2 + (h\mu^* - \mu h^*)(i-(d+1)/2) + hh^*(i-1)(d-i)\big), \tag{14.78}$$

for $1 \le i \le d$. Then the sequence

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \tag{14.79}$$

is a parameter array over $\mathbb{F}$ that has type II, provided that the inequalities in Lemma 16(i),(ii) *hold. Conversely, assume that the sequence* (14.79) *is a parameter array over $\mathbb{F}$ that has type II. Then there exists a unique sequence* (14.74) *of scalars in $\mathbb{F}$ that satisfies* (14.75)–(14.78).

LEMMA 97. (See [42, Lemma 19.4].) *Referring to Lemma 96, the inequalities in Lemma 16*(i),(ii) *hold if and only if*

$$\text{Char}(\mathbb{F}) \text{ is equal to 0 or greater than } d, \tag{14.80}$$

$$\mu \ne hi \qquad (1-d \le i \le d-1), \tag{14.81}$$

$$\mu^* \ne h^*i \qquad (1-d \le i \le d-1), \tag{14.82}$$

$$\tau \ne \mu\mu^*/2 - (h\mu^* + \mu h^*)(i-(d+1)/2) - hh^*(i-1)(d-i) \qquad (1 \le i \le d), \tag{14.83}$$

$$\tau \ne -\mu\mu^*/2 - (h\mu^* - \mu h^*)(i-(d+1)/2) - hh^*(i-1)(d-i) \qquad (1 \le i \le d). \tag{14.84}$$

DEFINITION 98. A *primary data of type II* is a sequence (14.74) of scalars in $\mathbb{F}$ that satisfy (14.80)–(14.84). The primary data (14.74) of type II and the parameter array (14.79) are said to *correspond*.

DEFINITION 99. Let A, A^* denote a Leonard pair of type II. By a *primary data of A, A^**, we mean the primary data of type II that corresponds to a parameter array of A, A^*.

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LEMMA 100. *For the parameter arrays in Lemma 13, consider the corresponding primary data of type II. These primary data are related as follows:*

Parameter array	Primary data
$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$	$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau)$
$(\{\theta_i\}_{i=0}^d; \{\theta^*_{d-i}\}_{i=0}^d; \{\phi_{d-i+1}\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d)$	$(\delta, \mu, h, \delta^*, -\mu^*, h^*, \tau)$
$(\{\theta_{d-i}\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d)$	$(\delta, -\mu, h, \delta^*, \mu^*, h^*, \tau)$
$(\{\theta_{d-i}\}_{i=0}^d; \{\theta^*_{d-i}\}_{i=0}^d; \{\varphi_{d-i+1}\}_{i=1}^d; \{\phi_{d-i+1}\}_{i=1}^d)$	$(\delta, -\mu, h, \delta^*, -\mu^*, h^*, \tau)$

Proof. Use Lemma 96 and Definition 98. □

15. Parameter arrays of type III$^+$. In this section, we describe the parameter arrays of type III$^+$.

LEMMA 101. (See [42, Lemma 22.1].) *Assume that d is even. For a sequence*

$$(\delta, s, h, \delta^*, s^*, h^*, \tau), \tag{15.85}$$

of scalars in $\mathbb{F}$, define

$$\theta_i = \begin{cases} \delta + s + h(i - d/2) & \text{if } i \text{ is even,} \\ \delta - s - h(i - d/2) & \text{if } i \text{ is odd,} \end{cases} \tag{15.86}$$

$$\theta^*_i = \begin{cases} \delta^* + s^* + h^*(i - d/2) & \text{if } i \text{ is even,} \\ \delta^* - s^* - h^*(i - d/2) & \text{if } i \text{ is odd} \end{cases} \tag{15.87}$$

for $0 \le i \le d$ and

$$\varphi_i = \begin{cases} i\big(\tau - sh^* - s^*h - hh^*(i - (d+1)/2)\big) & \text{if } i \text{ is even,} \\ (d - i + 1)\big(\tau + sh^* + s^*h + hh^*(i - (d+1)/2)\big) & \text{if } i \text{ is odd,} \end{cases} \tag{15.88}$$

$$\phi_i = \begin{cases} i\big(\tau - sh^* + s^*h + hh^*(i - (d+1)/2)\big) & \text{if } i \text{ is even,} \\ (d - i + 1)\big(\tau + sh^* - s^*h - hh^*(i - (d+1)/2)\big) & \text{if } i \text{ is odd} \end{cases} \tag{15.89}$$

for $1 \le i \le d$. Then the sequence

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \tag{15.90}$$

is a parameter array over $\mathbb{F}$ that has type III$^+$, provided that the inequalities in Lemma 16(i) and (ii) *hold. Conversely, assume that the sequence* (15.90) *is a parameter array over $\mathbb{F}$ that has type III$^+$. Then there exists a unique sequence* (15.85) *of scalars in $\mathbb{F}$ that satisfies* (15.86)–(15.89).

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Lemma 102. (See [42, Lemma 22.4].) *Referring to Lemma 101, the inequalities in Lemma 16*(i),(ii) *hold if and only if*

$$\text{Char}(\mathbb{F}) \text{ is equal to 0 or greater than } d/2, \tag{15.91}$$

$$h \neq 0, \qquad h^* \neq 0, \tag{15.92}$$

$$2s \neq ih \qquad \text{if } i \text{ is odd} \qquad (1-d \leq i \leq d-1), \tag{15.93}$$

$$2s^* \neq ih^* \qquad \text{if } i \text{ is odd} \qquad (1-d \leq i \leq d-1), \tag{15.94}$$

$$\tau \neq \begin{cases} sh^* + s^*h + hh^*\big(i-(d+1)/2\big) & \text{if } i \text{ is even,} \\ -sh^* - s^*h - hh^*\big(i-(d+1)/2\big) & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq d), \tag{15.95}$$

$$\tau \neq \begin{cases} sh^* - s^*h - hh^*\big(i-(d+1)/2\big) & \text{if } i \text{ is even,} \\ -sh^* + s^*h + hh^*\big(i-(d+1)/2\big) & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq d). \tag{15.96}$$

Definition 103. A *primary data of type III*$^+$ is a sequence (15.85) of scalars in $\mathbb{F}$ that satisfy (15.91)–(15.96). The primary data (15.85) of type III$^+$ and the parameter array (15.90) are said to *correspond*.

Definition 104. Let A, A^* denote a Leonard pair of type III$^+$. By a *primary data of* A, A^*, we mean the primary data of type III$^+$ that corresponds to a parameter array of A, A^*.

Lemma 105. *For the parameter arrays in Lemma 13, consider the corresponding primary data of type III*$^+$*. These primary data are related as follows:*

Parameter array	Primary data
$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$	$(\delta, s, h, \delta^*, s^*, h^*, \tau)$
$(\{\theta_i\}_{i=0}^d; \{\theta^*_{d-i}\}_{i=0}^d; \{\phi_{d-i+1}\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d)$	$(\delta, s, h, \delta^*, s^*, -h^*, \tau)$
$(\{\theta_{d-i}\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d)$	$(\delta, s, -h, \delta^*, s^*, h^*, \tau)$
$(\{\theta_{d-i}\}_{i=0}^d; \{\theta^*_{d-i}\}_{i=0}^d; \{\varphi_{d-i+1}\}_{i=1}^d; \{\phi_{d-i+1}\}_{i=1}^d)$	$(\delta, s, -h, \delta^*, s^*, -h^*, \tau)$

Proof. Use Lemma 101 and Definition 103. □

16. About the condition $\varphi_i\phi_i = \varphi'_i\phi'_i$.

In the proof of our main results, Lemma 65 will play an essential role. In that lemma, we see the condition (8.30). In this section, we recall from [42] the necessary and sufficient conditions for (8.30) in terms of the primary data.

Let A, A^* and B, A^* denote Leonard pairs over $\mathbb{F}$. Let

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \qquad (\{\theta'_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi'_i\}_{i=1}^d; \{\phi'_i\}_{i=1}^d), \tag{16.97}$$

denote a parameter array of A, A^* and B, A^*, respectively.

First consider the case of type I.

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LEMMA 106. (See [42, Theorem 17.1].) *Assume that A, A^* has type I. For the parameter arrays in (16.97), let*

$$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau), \qquad\qquad (\delta', \mu', h', \delta^*, \mu^*, h^*, \tau'),$$

denote the corresponding primary q-data. Then $\varphi_i \phi_i = \varphi'_i \phi'_i$ $(1 \le i \le d)$ if and only if

$$\mu h = \mu' h', \tag{16.98}$$

$$\tau(\mu + h) = \tau'(\mu' + h'), \tag{16.99}$$

$$\tau^2 + (\mu + h)^2 \mu^* h^* = \tau'^2 + (\mu' + h')^2 \mu^* h^*. \tag{16.100}$$

Next, consider the case of type II.

LEMMA 107. (See [42, Theorem 20.1].) *Assume that A, A^* has type II. For the parameter arrays in (16.97), let*

$$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau), \qquad\qquad (\delta', \mu', h', \delta^*, \mu^*, h^*, \tau'),$$

denote the corresponding primary data. Then $\varphi_i \phi_i = \varphi'_i \phi'_i$ $(1 \le i \le d)$ if and only if

$$h^2 = h'^2, \tag{16.101}$$

$$2h\tau + \mu^2 h^* = 2h'\tau' + \mu'^2 h^*, \tag{16.102}$$

$$4\tau^2 - \mu^2\big(\mu^{*2} + (d-1)^2 h^{*2}\big) = 4\tau'^2 - \mu'^2\big(\mu^{*2} + (d-1)^2 h^{*2}\big). \tag{16.103}$$

Next consider the case of type III$^+$.

LEMMA 108. (See [42, Theorem 23.1].) *Assume that A, A^* has type III$^+$. For the parameter arrays in (16.97), let*

$$(\delta, s, h, \delta^*, s^*, h^*, \tau), \qquad\qquad (\delta', s', h', \delta^*, s^*, h^*, \tau'),$$

denote the corresponding primary data. Then, $\varphi_i \phi_i = \varphi'_i \phi'_i$ $(1 \le i \le d)$ if and only if

$$h^2 = h'^2, \tag{16.104}$$

$$(\tau + s h^*)^2 = (\tau' + s' h^*)^2, \tag{16.105}$$

$$(\tau - s h^*)^2 = (\tau' - s' h^*)^2. \tag{16.106}$$

17. Some characterizations of the essentially bipartite Leonard pairs. In this section, we characterize the essentially bipartite Leonard pairs in terms of the primary data, the parameter array, and the TD/D sequence.

Throughout this section, let A, A^* denote a Leonard pair over $\mathbb{F}$, with parameter array:

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \tag{17.107}$$

and corresponding TD/D sequence:

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d).$$

We first consider the case of type I.

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Lemma 109. (See [35, Theorem 6.11].) *Fix a nonzero $q \in \mathbb{F}$, and assume that A, A^* has type I with fundamental constant $\beta = q^2 + q^{-2}$. Let*

$$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau),$$

denote the primary q-data corresponding to the parameter array (17.107). Then the following are equivalent:

(i) *A, A^* is essentially bipartite;*
(ii) *$\mu + h = 0$ and $\tau = 0$.*

Lemma 110. *Referring to Lemma 109, A, A^* is essentially bipartite if and only if*

$$\begin{aligned}
\theta_i &= \delta + \mu(q^{2i-d} - q^{d-2i}) && (0 \le i \le d),\\
\theta^*_i &= \delta^* + \mu^* q^{2i-d} + h^* q^{d-2i} && (0 \le i \le d),\\
\varphi_i &= -(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})\mu(\mu^* q^{2i-d-1} - h^* q^{d-2i+1}) && (1 \le i \le d),\\
\phi_i &= (q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})\mu(\mu^* q^{2i-d-1} - h^* q^{d-2i+1}) && (1 \le i \le d).
\end{aligned}$$

Proof. By Lemmas 91 and 109. ∎

Lemma 111. *Referring to Lemma 109, assume that A, A^* is essentially bipartite. Then*

$$\begin{aligned}
&q^{2i} \neq 1 && (1 \le i \le d),\\
&q^{2i} \neq -1 && (1 \le i \le d-1),\\
&\mu \neq 0,\\
&\mu^* q^i \neq h^* q^{-i} && (1-d \le i \le d-1).
\end{aligned}$$

Proof. Evaluate the inequalities in Lemma 92 using Lemma 109(ii). ∎

Lemma 112. *Referring to Lemma 109, A, A^* is essentially bipartite if and only if*

$$\begin{aligned}
a_i &= \delta && (0 \le i \le d),\\
x_i &= \frac{(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})\mu^2(\mu^* q^i - h^* q^{-i})(\mu^* q^{i-d-1} - h^* q^{d-i+1})}{(\mu^* q^{2i-d} - h^* q^{d-2i})(\mu^* q^{2i-d-2} - h^* q^{d-2i+2})} && (1 \le i \le d-1),\\
x_d &= \frac{(q^d - q^{-d})(q - q^{-1})\mu^2(\mu^* q^{-1} - h^* q)}{\mu^* q^{d-2} - h^* q^{2-d}}.
\end{aligned}$$

Proof. By Lemmas 29 and 110. ∎

Next consider the case of type II.

Lemma 113. (See [35, Theorem 7.11].) *Assume that A, A^* has type II. Let*

$$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau),$$

denote the primary data corresponding to the parameter array (17.107). Then the following are equivalent:

(i) *A, A^* is essentially bipartite;*
(ii) *$h = 0$ and $\tau = 0$.*

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LEMMA 114. *Referring to Lemma 113, A, A^* is essentially bipartite if and only if*

$$\begin{aligned}
\theta_i &= \delta + \mu(i - d/2) && (0 \le i \le d),\\
\theta^*_i &= \delta^* + \mu^*(i - d/2) + h^* i(d-i) && (0 \le i \le d),\\
\varphi_i &= -i(d-i+1)\mu\big(\mu^*/2 - h^*(i-(d+1)/2)\big) && (1 \le i \le d),\\
\phi_i &= i(d-i+1)\mu\big(\mu^*/2 - h^*(i-(d+1)/2)\big) && (1 \le i \le d).
\end{aligned}$$

Proof. By Lemmas 96 and 113. ∎

LEMMA 115. *Referring to Lemma 113, assume that A, A^* is essentially bipartite. Then*

$$\begin{aligned}
&\mathrm{Char}(\mathbb{F}) \text{ is equal to } 0 \text{ or greater than } d,\\
&\mu \neq 0,\\
&\mu^* \neq h^* i \qquad (1-d \le i \le d-1).
\end{aligned}$$

Proof. Evaluate the inequalities in Lemma 97 using Lemma 113(ii). ∎

LEMMA 116. *Referring to Lemma 113, A, A^* is essentially bipartite if and only if*

$$\begin{aligned}
a_i &= \delta && (0 \le i \le d),\\
x_i &= \frac{i(d-i+1)\mu^2(\mu^* - ih^*)\big(\mu^* + (d-i+1)h^*\big)}{4\big(\mu^* + (d-2i)h^*\big)\big(\mu^* + (d-2i+2)h^*\big)} && (1 \le i \le d-1),\\
x_d &= \frac{d\mu^2(\mu^* + h^*)}{4(\mu^* + (2-d)h^*)}.
\end{aligned}$$

Proof. By Lemmas 29 and 114. ∎

Next consider the case of type III^+.

LEMMA 117. (See [35, Theorem 8.11].) *Assume that A, A^* has type III^+, and let*

$$(\delta, s, h, \delta^*, s^*, h^*, \tau),$$

denote the primary data corresponding to the parameter array (17.107). Then the following are equivalent:

(i) *A, A^* is essentially bipartite;*
(ii) *$s = 0$ and $\tau = 0$.*

LEMMA 118. *Referring to Lemma 117, A, A^* is essentially bipartite if and only if*

$$\begin{aligned}
\theta_i &= \begin{cases} \delta + h(i-d/2) & \text{if } i \text{ is even},\\ \delta - h(i-d/2) & \text{if } i \text{ is odd}\end{cases} && (0 \le i \le d),\\
\theta^*_i &= \begin{cases} \delta^* + s^* + h^*(i-d/2) & \text{if } i \text{ is even},\\ \delta^* - s^* - h^*(i-d/2) & \text{if } i \text{ is odd}\end{cases} && (0 \le i \le d),\\
\varphi_i &= \begin{cases} -ih\big(s^* + h^*(i-(d+1)/2)\big) & \text{if } i \text{ is even},\\ (d-i+1)h\big(s^* + h^*(i-(d+1)/2)\big) & \text{if } i \text{ is odd}\end{cases} && (1 \le i \le d),\\
\phi_i &= \begin{cases} ih\big(s^* + h^*(i-(d+1)/2)\big) & \text{if } i \text{ is even},\\ -(d-i+1)h\big(s^* + h^*(i-(d+1)/2)\big) & \text{if } i \text{ is odd}\end{cases} && (1 \le i \le d).
\end{aligned}$$

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Proof. By Lemmas 101 and 117. ∎

Lemma 119. *Referring to Lemma 117, assume that A, A^* is essentially bipartite. Then*

$$\begin{aligned}
&\text{Char}(\mathbb{F}) \text{ is equal to 0 or greater than } d,\\
&h \neq 0, \qquad h^* \neq 0,\\
&2s^* \neq h^* i \qquad \text{if } i \text{ is odd} \qquad (1-d \leq i \leq d-1).
\end{aligned}$$

Proof. Evaluate the inequalities in Lemma 102 using Lemma 117(ii). ∎

Lemma 120. *Referring to Lemma 117, A, A^* is essentially bipartite if and only if*

$$a_i = \delta \qquad (0 \leq i \leq d),$$

$$x_i = \begin{cases} ih^2\big((d-i+1)h^* - 2s^*\big)/(4h^*) & \text{if } i \text{ is even},\\ (d-i+1)h^2(ih^* + 2s^*)/(4h^*) & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq d).$$

Proof. By Lemmas 29 and 118. ∎

Lemma 121. (See [35, Theorem 9.6].) *A Leonard pair of type III$^-$ is not essentially bipartite.*

18. Some characterizations of the bipartite Leonard pairs. In this section, we characterize the bipartite Leonard pairs in terms of the primary data, the parameter array, and the TD/D sequence.

Throughout this section, let A, A^* denote a Leonard pair over $\mathbb{F}$, with parameter array

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \tag{18.108}$$

and corresponding TD/D sequence

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d).$$

We first consider the case of type I.

Lemma 122. *Fix a nonzero $q \in \mathbb{F}$, and assume that A, A^* has type I with fundamental constant $\beta = q^2 + q^{-2}$. Let*

$$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau),$$

denote the primary q-data corresponding to the parameter array (18.108). Then the following are equivalent:

(i) *A, A^* is bipartite;*
(ii) *$\delta = 0$ and $\mu + h = 0$ and $\tau = 0$.*

Proof. By Lemmas 109 and 112. ∎

Lemma 123. *Referring to Lemma 122, A, A^* is bipartite if and only if*

$$\begin{aligned}
\theta_i &= \mu(q^{2i-d} - q^{d-2i}) && (0 \leq i \leq d),\\
\theta^*_i &= \delta^* + \mu^* q^{2i-d} + h^* q^{d-2i} && (0 \leq i \leq d),\\
\varphi_i &= -(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})\mu(\mu^* q^{2i-d-1} - h^* q^{d-2i+1}) && (1 \leq i \leq d),\\
\phi_i &= (q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})\mu(\mu^* q^{2i-d-1} - h^* q^{d-2i+1}) && (1 \leq i \leq d).
\end{aligned}$$

Proof. By Lemmas 110 and 122. ∎

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Lemma 124. *Referring to Lemma 122, assume that A, A^* is bipartite. Then*

$$\begin{aligned} &q^{2i} \neq 1 && (1 \leq i \leq d),\\ &q^{2i} \neq -1 && (1 \leq i \leq d-1),\\ &\mu \neq 0, &&\\ &\mu^* q^i \neq h^* q^{-i} && (1-d \leq i \leq d-1). \end{aligned}$$

Proof. Evaluate the inequalities in Lemma 92 using Lemma 122(ii). □

Lemma 125. *Referring to Lemma 122, A, A^* is bipartite if and only if*

$$\begin{aligned} a_i &= 0 && (0 \leq i \leq d),\\ x_i &= \frac{(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})\mu^2(\mu^* q^i - h^* q^{-i})(\mu^* q^{i-d-1} - h^* q^{d-i+1})}{(\mu^* q^{2i-d} - h^* q^{d-2i})(\mu^* q^{2i-d-2} - h^* q^{d-2i+2})} && (1 \leq i \leq d-1),\\ x_d &= \frac{(q^d - q^{-d})(q - q^{-1})\mu^2(\mu^* q^{-1} - h^* q)}{\mu^* q^{d-2} - h^* q^{2-d}}. && \end{aligned}$$

Proof. By Lemma 112. □

Next consider the case of type II.

Lemma 126. *Assume that A, A^* has type II, and let*

$$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau),$$

denote the primary data corresponding to the parameter array (18.108). Then the following are equivalent:

(i) *A, A^* is bipartite;*
(ii) *$\delta = 0$ and $h = 0$ and $\tau = 0$.*

Proof. By Lemmas 113 and 116. □

Lemma 127. *Referring to Lemma 126, A, A^* is bipartite if and only if*

$$\begin{aligned} \theta_i &= \mu(i - d/2) && (0 \leq i \leq d),\\ \theta^*_i &= \delta^* + \mu^*(i - d/2) + h^* i(d-i) && (0 \leq i \leq d),\\ \varphi_i &= -i(d-i+1)\mu\big(\mu^*/2 - h^*(i-(d+1)/2)\big) && (1 \leq i \leq d),\\ \phi_i &= i(d-i+1)\mu\big(\mu^*/2 - h^*(i-(d+1)/2)\big) && (1 \leq i \leq d). \end{aligned}$$

Proof. By Lemmas 114 and 126. □

Lemma 128. *Referring to Lemma 126, assume that A, A^* is bipartite. Then*

$$\begin{aligned} &\text{Char}(\mathbb{F}) \text{ is equal to 0 or greater than } d, &&\\ &\mu \neq 0, &&\\ &\mu^* \neq h^* i && (1-d \leq i \leq d-1). \end{aligned}$$

Proof. Evaluate the inequalities in Lemma 97 using Lemma 126(ii). □

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Lemma 129. *Referring to Lemma 126, A, A^* is bipartite if and only if*

$$\begin{aligned} a_i &= 0 && (0 \le i \le d),\\ x_i &= \frac{i(d-i+1)\mu^2(\mu^*-ih^*)\big(\mu^*+(d-i+1)h^*\big)}{4\big(\mu^*+(d-2i)h^*\big)\big(\mu^*+(d-2i+2)h^*\big)} && (1 \le i \le d-1),\\ x_d &= \frac{d\mu^2(\mu^*+h^*)}{4(\mu^*+(2-d)h^*)}. \end{aligned}$$

Proof. By Lemma 116. ☐

Next consider the case of type III$^+$.

Lemma 130. *Assume that A, A^* has type III$^+$, and let*

$$(\delta, s, h, \delta^*, s^*, h^*, \tau),$$

denote the primary data corresponding to the parameter array (18.108). Then the following are equivalent:

(i) *A, A^* is bipartite;*
(ii) *$\delta = 0$ and $s = 0$ and $\tau = 0$.*

Proof. By Lemmas 117 and 120. ☐

Lemma 131. *Referring to Lemma 130, A, A^* is bipartite if and only if*

$$\theta_i = \begin{cases} h(i-d/2) & \text{if } i \text{ is even},\\ -h(i-d/2) & \text{if } i \text{ is odd} \end{cases} \qquad (0 \le i \le d),$$

$$\theta^*_i = \begin{cases} \delta^*+s^*+h^*(i-d/2) & \text{if } i \text{ is even},\\ \delta^*-s^*-h^*(i-d/2) & \text{if } i \text{ is odd} \end{cases} \qquad (0 \le i \le d),$$

$$\varphi_i = \begin{cases} -ih\big(s^*+h^*(i-(d+1)/2)\big) & \text{if } i \text{ is even},\\ (d-i+1)h\big(s^*+h^*(i-(d+1)/2)\big) & \text{if } i \text{ is odd} \end{cases} \qquad (1 \le i \le d),$$

$$\phi_i = \begin{cases} ih\big(s^*+h^*(i-(d+1)/2)\big) & \text{if } i \text{ is even},\\ -(d-i+1)h\big(s^*+h^*(i-(d+1)/2)\big) & \text{if } i \text{ is odd} \end{cases} \qquad (1 \le i \le d).$$

Proof. By Lemmas 118 and 130. ☐

Lemma 132. *Referring to Lemma 130, assume that A, A^* is bipartite. Then*

$$\begin{aligned} &\mathrm{Char}(\mathbb{F}) \text{ is equal to 0 or greater than } d,\\ &h \ne 0, \qquad h^* \ne 0,\\ &2s^* \ne h^* i \qquad \text{if } i \text{ is odd} \qquad (1-d \le i \le d-1). \end{aligned}$$

Proof. Evaluate the inequalities in Lemma 102 using Lemma 130(ii). ☐

Lemma 133. *Referring to Lemma 130, A, A^* is bipartite if and only if*

$$a_i = 0 \qquad (0 \le i \le d),$$

$$x_i = \begin{cases} ih^2\big((d-i+1)h^*-2s^*\big)/(4h^*) & \text{if } i \text{ is even},\\ (d-i+1)h^2(ih^*+2s^*)/(4h^*) & \text{if } i \text{ is odd} \end{cases} \qquad (1 \le i \le d).$$

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Proof. By Lemma 120. ∎

LEMMA 134. *A Leonard pair of type III^- is not bipartite.*

Proof. By Lemma 121. ∎

19. Leonard pairs of dual q-Krawtchouk type. In this section, we describe a family of Leonard pairs, said to have dual q-Krawtchouk type.

Throughout this section, let A, A^* denote a Leonard pair over $\mathbb{F}$ with parameter array

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d). \tag{19.109}$$

DEFINITION 135. The Leonard pair A, A^* is said to have *dual q-Krawtchouk type* whenever the following (i)–(iii) hold:

(i) A, A^* has type I;
(ii) the expression $(\theta^*_{i-1} - \theta^*_i)/(\theta^*_i - \theta^*_{i+1})$ is independent of i for $1 \leq i \leq d-1$;
(iii) the expression φ_i/ϕ_i is independent of i for $1 \leq i \leq d$.

NOTE 136. *The Leonard pairs of dual q-Krawtchouk type are attached to the dual q-Krawtchouk polynomials [50, Example 35.8].*

For the rest of this section, assume that A, A^* has type I, and let

$$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau), \tag{19.110}$$

denote the primary q-data corresponding to the parameter array (19.109).

LEMMA 137. *We have*

$$\frac{\theta^*_{i-1} - \theta^*_i}{\theta^*_i - \theta^*_{i+1}} = \frac{\mu^* q^{4i} - h^* q^{2d+2}}{\mu^* q^{4i+2} - h^* q^{2d}} \qquad (1 \leq i \leq d-1), \tag{19.111}$$

$$\frac{\varphi_i}{\phi_i} = \frac{\tau - \mu\mu^* q^{2i-d-1} - hh^* q^{d-2i+1}}{\tau - h\mu^* q^{2i-d-1} - \mu h^* q^{d-2i+1}} \qquad (1 \leq i \leq d). \tag{19.112}$$

Proof. Use (13.65)–(13.67). ∎

Note by (13.71) that μ^* and h^* are not both zero.

LEMMA 138. *The following are equivalent:*

(i) *A, A^* has dual q-Krawtchouk type;*
(ii) *$\mu^* h^* = 0$ and $\tau = 0$.*

Suppose the equivalent conditions (i) *and* (ii) *hold. Then*

$$(\theta^*_{i-1} - \theta^*_i)/(\theta^*_i - \theta^*_{i+1}) = \begin{cases} q^2 & \text{if } \mu^* = 0, \\ q^{-2} & \text{if } h^* = 0 \end{cases} \qquad (1 \leq i \leq d-1), \tag{19.113}$$

$$\varphi_i/\phi_i = \begin{cases} h/\mu & \text{if } \mu^* = 0, \\ \mu/h & \text{if } h^* = 0 \end{cases} \qquad (1 \leq i \leq d). \tag{19.114}$$

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Proof. (i) ⇒ (ii) Using Definition 135(ii), compare the values of (19.111) for $i = 1$ and $i = 2$ to find that

$$\mu^* h^* q^{2d+4}(q^4 - 1)^2 = 0.$$

By this and (13.69), we get $\mu^* h^* = 0$. First assume that $\mu^* = 0$. By (19.112),

$$\frac{\varphi_i}{\phi_i} = \frac{\tau - hh^* q^{d-2i+1}}{\tau - \mu h^* q^{d-2i+1}} \qquad (1 \le i \le d). \tag{19.115}$$

Using Definition 135(iii), compare the values of (19.115) for $i = 1$ and $i = 2$ to find that

$$\tau h^*(\mu - h)(q^2 - 1) = 0.$$

By this and (13.69) and (13.70), we get $\tau = 0$. Next assume that $h^* = 0$. We get $\tau = 0$ in a similar way.

(ii) ⇒ (i) By (19.111) and (19.112) we find that (19.113) and (19.114) hold. Thus, in Definition 135 the conditions (ii) and (iii) hold. □

Lemma 139. *The following hold:*

(i) *A, A^* has dual q-Krawtchouk type with $\mu^* = 0$ if and only if*

$$\begin{aligned}
\theta_i &= \delta + \mu q^{2i-d} + h q^{d-2i} && (0 \le i \le d),\\
\theta^*_i &= \delta^* + h^* q^{d-2i} && (0 \le i \le d),\\
\varphi_i &= -hh^* q^{d-2i+1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}) && (1 \le i \le d),\\
\phi_i &= -\mu h^* q^{d-2i+1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}) && (1 \le i \le d);
\end{aligned}$$

(ii) *A, A^* has dual q-Krawtchouk type with $h^* = 0$ if and only if*

$$\begin{aligned}
\theta_i &= \delta + \mu q^{2i-d} + h q^{d-2i} && (0 \le i \le d),\\
\theta^*_i &= \delta^* + \mu^* q^{2i-d} && (0 \le i \le d),\\
\varphi_i &= -\mu\mu^* q^{2i-d-1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}) && (1 \le i \le d),\\
\phi_i &= -h\mu^* q^{2i-d-1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}) && (1 \le i \le d).
\end{aligned}$$

Proof. By Lemmas 91 and 138. □

Lemma 140. *Assume that A, A^* has dual q-Krawtchouk type. Then*

$$\begin{aligned}
&q^{2i} \ne 1 \qquad (1 \le i \le d),\\
&\mu \ne h q^{2i} \qquad (1 - d \le i \le d - 1),\\
&\mu \ne 0, \qquad h \ne 0,\\
&\mu^*,\ h^* \text{ not both zero.}
\end{aligned}$$

Proof. Evaluate the inequalities in Lemma 92 using Lemma 138(ii). □

Let $(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d)$ denote the TD/D sequence of A, A^* corresponding to the parameter array (19.109).

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Lemma 141. *The following hold:*

(i) A, A^* *has dual* q*-Krawtchouk type with* $\mu^* = 0$ *if and only if*

$$a_i = \delta + (\mu + h)q^{2i-d} \qquad (0 \le i \le d), \tag{19.116}$$

$$x_i = -\mu h q^{2i-d-1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}) \qquad (1 \le i \le d), \tag{19.117}$$

$$\theta^*_i = \delta^* + h^* q^{d-2i} \qquad (0 \le i \le d); \tag{19.118}$$

(ii) A, A^* *has dual* q*-Krawtchouk type with* $h^* = 0$ *if and only if*

$$a_i = \delta + (\mu + h)q^{d-2i} \qquad (0 \le i \le d),$$
$$x_i = -\mu h q^{d-2i+1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}) \qquad (1 \le i \le d),$$
$$\theta^*_i = \delta^* + \mu^* q^{2i-d} \qquad (0 \le i \le d).$$

Proof. (i) First assume that A, A^* has dual q-Krawtchouk type with $\mu^* = 0$. Using Lemmas 29 and 139, we get (19.116)–(19.118). Next assume that (19.116)–(19.118) hold. Comparing (19.118) with (13.65), we find that $\mu^* = 0$. Now by (4.14) and (13.64)–(13.66),

$$a_0 = \delta + (\mu + h)q^{-d} + \frac{\tau}{h^*} q(1 - q^{-2d}).$$

Comparing this with (19.116), we find that $\tau q(1 - q^{-2d}) = 0$. By this and (13.69), we get $\tau = 0$. So by Lemma 138, we find that A, A^* has dual q-Krawtchouk type.

(ii) Similar. □

Definition 142. A parameter array over $\mathbb{F}$ is said to have *dual* q*-Krawtchouk type* whenever the corresponding Leonard pair has dual q-Krawtchouk type.

20. Leonard pairs of Krawtchouk type.

In this section, we describe a family of Leonard pairs, said to have Krawtchouk type.

Throughout this section, let A, A^* denote a Leonard pair over $\mathbb{F}$ with parameter array:

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d). \tag{20.119}$$

Definition 143. The Leonard pair A, A^* is said to have *Krawtchouk type* whenever the following (i)–(iii) hold:

(i) A, A^* has type II;
(ii) $\theta_i - \theta_{i-1}$ is independent of i for $1 \le i \le d$;
(iii) $\theta^*_i - \theta^*_{i-1}$ is independent of i for $1 \le i \le d$.

Note 144. *The Leonard pairs of Krawtchouk type are attached to the Krawtchouk polynomials [50, Example 35.12].*

For the rest of this section, assume that A, A^* has type II, and let

$$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau), \tag{20.120}$$

denote the primary data corresponding to the parameter array (20.119).

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Lemma 145. *For $1 \le i \le d$ we have*

$$\theta_i - \theta_{i-1} = \mu + (d+1)h - 2hi, \tag{20.121}$$

$$\theta^*_i - \theta^*_{i-1} = \mu^* + (d+1)h^* - 2h^*i. \tag{20.122}$$

Proof. Use (14.75) and (14.76). ◻

Lemma 146. *The following are equivalent:*

(i) *A, A^* has Krawtchouk type;*
(ii) *$h = 0$ and $h^* = 0$.*

Suppose the equivalent conditions (i) *and* (ii) *hold. Then*

$$\theta_i - \theta_{i-1} = \mu, \qquad \theta^*_i - \theta^*_{i-1} = \mu^* \qquad (1 \le i \le d).$$

Proof. Use (20.121) and (20.122). ◻

Note 147. *Definition 143 is not the type II version of Definition 135. For the case of type II, the conditions (ii) and (iii) in Definition 135 hold if and only if $\mu h = 0$ and $h^* = 0$.*

Lemma 148. *The Leonard pair A, A^* has Krawtchouk type if and only if*

$$\begin{aligned}
\theta_i &= \delta + \mu(i - d/2) && (0 \le i \le d),\\
\theta^*_i &= \delta^* + \mu^*(i - d/2) && (0 \le i \le d),\\
\varphi_i &= i(d-i+1)(\tau - \mu\mu^*/2) && (1 \le i \le d),\\
\phi_i &= i(d-i+1)(\tau + \mu\mu^*/2) && (1 \le i \le d).
\end{aligned}$$

Proof. By Lemmas 96 and 146. ◻

Lemma 149. *Assume that the Leonard pair A, A^* has Krawtchouk type. Then*

$$\begin{aligned}
&\mathrm{Char}(\mathbb{F}) \text{ is equal to 0 or greater than } d,\\
&\mu \neq 0, \qquad \mu^* \neq 0,\\
&2\tau \neq \mu\mu^*, \qquad 2\tau \neq -\mu\mu^*.
\end{aligned}$$

Proof. Evaluate the inequalities in Lemma 97 using Lemma 146(ii). ◻

Let $(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d)$ denote the TD/D sequence of A, A^* corresponding to the parameter array (20.119).

Lemma 150. *The Leonard pair A, A^* has Krawtchouk type if and only if*

$$a_i = \delta + (2i-d)\tau/\mu^* \qquad (0 \le i \le d), \tag{20.123}$$

$$x_i = i(d-i+1)\left(\frac{\mu^2}{4} - \frac{\tau^2}{\mu^{*2}}\right) \qquad (1 \le i \le d), \tag{20.124}$$

$$\theta^*_i = \delta^* + \mu^*(i - d/2) \qquad (0 \le i \le d). \tag{20.125}$$

Proof. First assume that A, A^* has Krawtchouk type. By Lemma 146, we have $h = 0$ and $h^* = 0$. Now using Lemmas 29 and 148, we get (20.123)–(20.125). Next assume that (20.123)–(20.125) hold. Comparing (14.76) with (20.125), we get $h^* = 0$. By (4.15) and (14.75)–(14.77) with $h^* = 0$,

$$a_0 = \delta - \frac{d\tau}{\mu^*} + \frac{d(d-1)h}{2}.$$

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By (20.123),

$$a_0 = \delta - \frac{d\tau}{\mu^*}.$$

Comparing the above two equations, we find that $h = 0$. Now A, A^* has Krawtchouk type by Lemma 146. □

DEFINITION 151. A parameter array over $\mathbb{F}$ is said to have *Krawtchouk type* whenever the corresponding Leonard pair has Krawtchouk type.

21. Leonard pairs that are bipartite and have dual q-Krawtchouk type. In Section 18, we described the bipartite Leonard pairs. In Section 19, we described the Leonard pairs of dual q-Krawtchouk type. In this section, we describe the Leonard pairs that are bipartite and have dual q-Krawtchouk type.

Throughout this section, let A, A^* denote a Leonard pair over $\mathbb{F}$ that has type I with parameter array

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d). \tag{21.126}$$

Denote the corresponding primary q-data by

$$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau). \tag{21.127}$$

LEMMA 152. *The Leonard pair A, A^* is bipartite and has dual q-Krawtchouk type if and only if*

$$\delta = 0, \qquad \mu + h = 0, \qquad \mu^* h^* = 0, \qquad \tau = 0.$$

Proof. By Lemmas 122 and 138. □

LEMMA 153. *The following hold:*

(i) *A, A^* is bipartite and has dual q-Krawtchouk type with $\mu^* = 0$ if and only if*

$$\begin{aligned}
\theta_i &= \mu(q^{2i-d} - q^{d-2i}) && (0 \le i \le d),\\
\theta^*_i &= \delta^* + h^* q^{d-2i} && (0 \le i \le d),\\
\varphi_i &= \mu h^* q^{d-2i+1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}) && (1 \le i \le d),\\
\phi_i &= -\mu h^* q^{d-2i+1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}) && (1 \le i \le d);
\end{aligned}$$

(ii) *A, A^* is bipartite and has dual q-Krawtchouk type with $h^* = 0$ if and only if*

$$\begin{aligned}
\theta_i &= \mu(q^{2i-d} - q^{d-2i}) && (0 \le i \le d),\\
\theta^*_i &= \delta^* + \mu^* q^{2i-d} && (0 \le i \le d),\\
\varphi_i &= -\mu\mu^* q^{2i-d-1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}) && (1 \le i \le d),\\
\phi_i &= \mu\mu^* q^{2i-d-1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}) && (1 \le i \le d).
\end{aligned}$$

Proof. By Lemmas 122 and 139. □

LEMMA 154. *Assume that A, A^* is bipartite and has dual q-Krawtchouk type. Then*

$$\begin{aligned}
&q^{2i} \ne 1 \qquad (1 \le i \le d),\\
&q^{2i} \ne -1 \qquad (1 \le i \le d-1),\\
&\mu \ne 0,\\
&\mu^*,\ h^* \text{ not both zero.}
\end{aligned}$$

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Proof. By Lemmas 124 and 152. ∎

Let $(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta_i^*\}_{i=0}^d)$ denote the TD/D sequence of A, A^* corresponding to the parameter array (21.126).

LEMMA 155. *The following hold:*

(i) *A, A^* is bipartite and has dual q-Krawtchouk type with $\mu^* = 0$ if and only if*

$$\begin{aligned} a_i &= 0 && (0 \le i \le d), \\ x_i &= \mu^2 q^{2i-d-1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}) && (1 \le i \le d), \\ \theta_i^* &= \delta^* + h^* q^{d-2i} && (0 \le i \le d); \end{aligned}$$

(ii) *A, A^* is bipartite and has dual q-Krawtchouk type with $h^* = 0$ if and only if*

$$\begin{aligned} a_i &= 0 && (0 \le i \le d), \\ x_i &= \mu^2 q^{d-2i+1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}) && (1 \le i \le d), \\ \theta_i^* &= \delta^* + \mu^* q^{2i-d} && (0 \le i \le d). \end{aligned}$$

Proof. By Lemmas 141 and 152. ∎

22. Leonard pairs that are bipartite and have Krawtchouk type. In Section 18, we described the bipartite Leonard pairs. In Section 20, we described the Leonard pairs of Krawtchouk type. In this section, we describe the Leonard pairs that are bipartite and have Krawtchouk type.

Throughout this section, let A, A^* denote a Leonard pair over $\mathbb{F}$ that has type II with parameter array

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d). \tag{22.128}$$

Denote the corresponding primary data by:

$$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau). \tag{22.129}$$

LEMMA 156. *The Leonard pair A, A^* is bipartite and has Krawtchouk type if and only if*

$$\delta = 0, \qquad h = 0, \qquad h^* = 0, \qquad \tau = 0.$$

Proof. By Lemmas 126 and 146. ∎

LEMMA 157. *The Leonard pair A, A^* is bipartite and has Krawtchouk type if and only if*

$$\begin{aligned} \theta_i &= \mu(i - d/2) && (0 \le i \le d), \\ \theta_i^* &= \delta^* + \mu^*(i - d/2) && (0 \le i \le d), \\ \varphi_i &= -\mu\mu^* i(d-i+1)/2 && (1 \le i \le d), \\ \phi_i &= \mu\mu^* i(d-i+1)/2 && (1 \le i \le d). \end{aligned}$$

Proof. By Lemmas 126 and 148. ∎

LEMMA 158. *Assume that the Leonard pair A, A^* is bipartite and has Krawtchouk type. Then*

$$\begin{gathered} \mathrm{Char}(\mathbb{F}) \text{ is equal to 0 or greater than } d, \\ \mu \neq 0, \qquad \mu^* \neq 0. \end{gathered}$$

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Proof. By Lemmas 149 and 156 □

Let $(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d)$ denote the TD/D sequence of A, A^* corresponding to the parameter array (22.128).

LEMMA 159. *The Leonard pair A, A^* is bipartite and has Krawtchouk type if and only if*

$$\begin{aligned} a_i &= 0 && (0 \le i \le d), \\ x_i &= \mu^2 i(d-i+1)/4 && (1 \le i \le d), \\ \theta^*_i &= \delta^* + \mu^*(i - d/2) && (0 \le i \le d). \end{aligned}$$

Proof. Use Lemmas 126 and 150. □

23. Near-bipartite Leonard pairs of dual q-Krawtchouk type. In this section, we determine the near-bipartite Leonard pairs of dual q-Krawtchouk type and describe their bipartite contraction.

Throughout this section, let A, A^* denote a Leonard pair over $\mathbb{F}$ that has dual q-Krawtchouk type. Let

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \tag{23.130}$$

denote a parameter array of A, A^* and let

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d), \tag{23.131}$$

denote the corresponding TD/D sequence. Let

$$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau), \tag{23.132}$$

denote the primary q-data corresponding to the parameter array (23.130). Note by Lemma 138 that $\mu^* h^* = 0$ and $\tau = 0$. In view of Lemma 122 and (16.98), we make a definition.

DEFINITION 160. Define scalars δ', μ', h', τ' in $\mathbb{F}$ such that

$$\delta' = 0, \qquad \mu' + h' = 0, \qquad \mu' h' = \mu h, \qquad \tau' = 0. \tag{23.133}$$

DEFINITION 161. Define scalars $\{\theta'_i\}_{i=0}^d$, $\{\varphi'_i\}_{i=1}^d$, $\{\phi'_i\}_{i=1}^d$ in $\mathbb{F}$ as follows.

(i) If $\mu^* = 0$, then

$$\begin{aligned} \theta'_i &= \mu' q^{2i-d} + h' q^{d-2i} && (0 \le i \le d), \\ \varphi'_i &= \mu' h^* q^{d-2i+1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}) && (1 \le i \le d), \\ \phi'_i &= h' h^* q^{d-2i+1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}) && (1 \le i \le d). \end{aligned}$$

(ii) If $h^* = 0$, then

$$\begin{aligned} \theta'_i &= \mu' q^{2i-d} + h' q^{d-2i} && (0 \le i \le d), \\ \varphi'_i &= h' \mu^* q^{2i-d-1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}) && (1 \le i \le d), \\ \phi'_i &= \mu' \mu^* q^{2i-d-1}(q^i - q^{-i})(q^{d-i+1} - q^{i-d-1}) && (1 \le i \le d). \end{aligned}$$

LEMMA 162. *Let F denote the flat part of A. Then the eigenvalues of $A - F$ are $\{\theta'_i\}_{i=0}^d$.*

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Proof. We may assume that A, A^* is in normalized TD/D form. The matrix $A - F$ is shown in (7.26). Define a matrix $B \in \mathrm{Mat}_{d+1}(\mathbb{F})$ that has (i,i)-entry θ'_i for $0 \le i \le d$ and $(i, i-1)$-entry 1 for $1 \le i \le d$. All other entries of B are zero. Define the upper triangular matrix $P \in \mathrm{Mat}_{d+1}(\mathbb{F})$ that has (i,j)-entry

$$P_{i,j} = \frac{\varphi'_1 \varphi'_2 \cdots \varphi'_j}{\varphi'_1 \varphi'_2 \cdots \varphi'_i} \frac{\eta^*_{d-j}(\theta^*_i)}{\eta^*_{d-i}(\theta^*_i)},$$

for $0 \le i \le j \le d$. Observe that $P_{i,i} = 1$ for $0 \le i \le d$. Therefore, P is invertible. By matrix multiplication, we routinely find

$$(A - F)P = PB.$$

The matrices $A - F$ and B are similar, so they have the same eigenvalues. The eigenvalues of B are $\{\theta'_i\}_{i=0}^d$. The result follows. ∎

Lemma 163. *With reference to Definition 161, the following are equivalent:*

(i) $\{\theta'_i\}_{i=0}^d$ *are mutually distinct;*
(ii) $q^{2i} \neq -1$ *for* $1 \le i \le d-1$.

Proof. For $0 \le i, j \le d$,

$$\theta'_i - \theta'_j = \mu'(q^{i-j} - q^{j-i})(q^{d-i-j} + q^{i+j-d}).$$

Thus for $0 \le i < j \le d$, $\theta'_i \neq \theta'_j$ if and only if $q^{2(j-i)} \neq 1$ and $q^{2(d-i-j)} \neq -1$. By (13.69), $q^{2(i-j)} \neq 1$ for $0 \le i < j \le d$. So $\theta'_i \neq \theta'_j$ for $0 \le i < j \le d$ if and only if $q^{2(d-i-j)} \neq -1$ for $0 \le i < j \le d$ if and only if $q^{2i} \neq -1$ for $1 \le i \le d-1$. ∎

Lemma 164. *With reference to Lemma 163, assume that the equivalent conditions* (i) *and* (ii) *hold. Then the following hold:*

(i) *the sequence*

$$(\{\theta'_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi'_i\}_{i=1}^d; \{\phi'_i\}_{i=1}^d), \tag{23.134}$$

is a parameter array over $\mathbb{F}$;

(ii) *the sequence*

$$(\delta', \mu', h', \delta^*, \mu^*, h^*, \tau'), \tag{23.135}$$

is the primary q-data corresponding to the parameter array (23.134);

(iii) *the parameter array* (23.134) *is bipartite and has dual q-Krawtchouk type.*

Proof. (i), (ii) Use Lemmas 91 and 92.

(iii) By Lemma 153. ∎

Lemma 165. *We have* $\varphi_i \phi_i = \varphi'_i \phi'_i$ *for* $1 \le i \le d$.

Proof. Use Lemma 106. ∎

Proposition 166. *Let F denote the flat part of A. Then the following are equivalent:*

(i) *the Leonard pair A, A^* is near-bipartite;*
(ii) $A - F$ *is diagonalizable;*
(iii) $A - F$ *is multiplicity-free;*
(iv) $q^{2i} \neq -1$ *for* $1 \le i \le d-1$.

Suppose (i)–(iv) *hold. Then the bipartite contraction of A, A^* has parameter array* (23.134).

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Proof. (i) $\Rightarrow$ (ii) By Definition 56, $A-F, A^*$ is a Leonard pair. By this and Definition 3, $A-F$ is diagonalizable.

(ii) $\Rightarrow$ (iii) We may assume that $A-F$ is an irreducible tridiagonal matrix. Now use Lemma 2.

(iii) $\Rightarrow$ (iv) By Lemma 162, $\{\theta'_i\}_{i=0}^d$ are the eigenvalues of $A-F$. These eigenvalues are mutually distinct since $A-F$ is multiplicity-free. By this and Lemma 163, we get (iv).

(iv) $\Rightarrow$ (i) Note by Lemma 164(i) that (23.134) is a parameter array over $\mathbb{F}$. Now use Lemmas 66 and 165. ∎

Definition 167. The Leonard pair A, A^* is said to be *reinforced* whenever $q^{2i} \neq -1$ for $1 \leq i \leq d-1$.

Corollary 168. *The Leonard pair A, A^* is near-bipartite if and only if A, A^* is reinforced. In this case, the bipartite contraction of A, A^* has reinforced dual q-Krawtchouk type with parameter array* (23.134).

Proof. By Lemma 164(iii) and Proposition 166. ∎

We have a comment.

Lemma 169. *Assume that q is not a root of unity. Then A, A^* is reinforced.*

24. Leonard pairs of Krawtchouk type are near-bipartite. In this section, we show that a Leonard pair of Krawtchouk type is near-bipartite, and we describe its bipartite contraction.

Throughout this section, let A, A^* denote a Leonard pair over $\mathbb{F}$ that has Krawtchouk type, with parameter array

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \tag{24.136}$$

and corresponding TD/D sequence

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d), . \tag{24.137}$$

Let

$$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau), \tag{24.138}$$

denote the primary data corresponding to the parameter array (24.136). In view of Lemma 126 and (16.103), we make a definition.

Definition 170. Define scalars δ', μ', h', τ' in $\mathbb{F}$ such that

$$\delta' = 0, \qquad h' = 0, \qquad \tau' = 0, \qquad \mu'^2 = \mu^2 - \frac{4\tau^2}{\mu^{*2}}. \tag{24.139}$$

Definition 171. Define scalars

$$\begin{aligned}
\theta'_i &= \mu'(i - d/2) && (0 \leq i \leq d),\\
\varphi'_i &= -\mu'\mu^* i(d-i+1)/2 && (1 \leq i \leq d),\\
\phi'_i &= \mu'\mu^* i(d-i+1)/2 && (1 \leq i \leq d).
\end{aligned}$$

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LEMMA 172. *The following hold:*

(i) *the sequence*

$$(\{\theta'_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi'_i\}_{i=1}^d; \{\phi'_i\}_{i=1}^d), \tag{24.140}$$

is a parameter array over $\mathbb{F}$*;*

(ii) *the sequence*

$$(\delta', \mu', h', \delta^*, \mu^*, h^*, \tau'), \tag{24.141}$$

is the primary data corresponding to the parameter array (24.140)*;*

(iii) *the parameter array* (24.140) *is bipartite and has Krawtchouk type.*

Proof. (i), (ii) Note by Lemma 149 that $\mu' \neq 0$. Now use Lemmas 96 and 97.

(iii) By Lemma 157. ∎

LEMMA 173. *We have* $\varphi_i \phi_i = \varphi'_i \phi'_i$ *for* $1 \leq i \leq d$.

Proof. Use Lemma 107. ∎

PROPOSITION 174. *The Leonard pair* A, A^* *is near-bipartite. The bipartite contraction of* A, A^* *has Krawtchouk type with parameter array* (24.140).

Proof. By Lemma 172, the sequence (24.140) is a parameter array over $\mathbb{F}$ that is bipartite and has Krawtchouk type. Now use Lemmas 66 and 173. ∎

25. The classification of near-bipartite Leonard pairs. In this section, we classify the near-bipartite Leonard pairs with diameter at least 3. In Section 8, we saw that an essentially bipartite Leonard pair is near-bipartite. In Section 23, we showed that a Leonard pair A, A^* of dual q-Krawtchouk type is near-bipartite if and only if A, A^* is reinforced, and in that case the bipartite contraction of A, A^* has reinforced dual q-Krawtchouk type. In Section 24, we showed that a Leonard pair of Krawtchouk type is near-bipartite, and its bipartite contraction has Krawtchouk type. We now state our classification result.

THEOREM 175. *Let* A, A^* *denote a Leonard pair over* $\mathbb{F}$ *with diameter* $d \geq 3$. *Then* A, A^* *is near-bipartite if and only if at least one of the following* (i)–(iii) *holds:*

(i) A, A^* *is essentially bipartite;*
(ii) A, A^* *has reinforced dual* q*-Krawtchouk type;*
(iii) A, A^* *has Krawtchouk type.*

Proof. First assume that at least one of (i)–(iii) holds. If (i) holds, then A, A^* is near-bipartite by Note 57. If (ii) holds, then A, A^* is near-bipartite by Corollary 168. If (iii) holds, then A, A^* is near-bipartite by Proposition 174. We are done in one logical direction. Next assume that A, A^* is near-bipartite, and let B, A^* denote the bipartite contraction of A, A^*. By Lemma 36, we may assume that A, A^* is in normalized TD/D form. Let

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \qquad (\{\theta'_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi'_i\}_{i=1}^d; \{\phi'_i\}_{i=1}^d), \tag{25.142}$$

denote a parameter array of A, A^* and B, A^*, respectively. By Lemma 121, B, A^* does not have type III^-. Recall that A, A^* has the same type as B, A^*. So the type of A, A^* is one of I, II, and III^+.

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First assume that A, A^* has type I. For the parameter arrays in (25.142), let

$$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau), \qquad (\delta', \mu', h', \delta^*, \mu^*, h^*, \tau')$$

denote the corresponding primary q-data. By Lemmas 65 and 106, we have (16.98)–(16.100). Since B, A^* is bipartite, we have by Lemma 122 that

$$\tau' = 0, \qquad \mu' + h' = 0.$$

By these comments,

$$\tau(\mu + h) = 0, \tag{25.143}$$
$$\tau^2 + (\mu + h)^2 \mu^* h^* = 0. \tag{25.144}$$

First assume that $\mu + h = 0$. Then $\tau = 0$ by (25.144). Now by Lemma 109, A, A^* is essentially bipartite. Next assume that $\mu + h \neq 0$. By (25.143), $\tau = 0$. By this and (25.144), $\mu^* h^* = 0$. Thus, A, A^* has dual q-Krawtchouk type by Lemma 138. By Corollary 168, A, A^* is reinforced. We have shown that A, A^* is either essentially bipartite or of reinforced dual q-Krawtchouk type.

Next assume that A, A^* has type II. For the parameter arrays in (25.142), let

$$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau), \qquad (\delta', \mu', h', \delta^*, \mu^*, h^*, \tau'),$$

denote the corresponding primary data. By Lemmas 65 and 107, we have (16.101)–(16.103). Since B, A^* is bipartite, we have by Lemma 126 that

$$\tau' = 0, \qquad h' = 0.$$

By these comments,

$$h = 0, \tag{25.145}$$
$$(\mu^2 - \mu'^2) h^* = 0, \tag{25.146}$$
$$4\tau^2 = (\mu^2 - \mu'^2)\big(\mu^{*2} + (d-1)^2 h^{*2}\big). \tag{25.147}$$

First assume that $h^* = 0$. Then, A, A^* has Krawtchouk type by Lemma 146. Next assume that $h^* \neq 0$. Then by (25.146), $\mu^2 - \mu'^2 = 0$. By this and (25.147), we get $\tau = 0$. By these comments and Lemma 113, A, A^* is essentially bipartite. We have shown that A, A^* is either essentially bipartite or of Krawtchouk type.

Next assume that A, A^* has type III$^+$. For the parameter arrays in (25.142), let

$$(\delta, s, h, \delta^*, s^*, h^*, \tau), \qquad (\delta', s', h', \delta^*, s^*, h^*, \tau'),$$

denote the corresponding primary data. By Lemmas 65 and 108, we have (16.104)–(16.106). Since B, A^* is bipartite, we have by Lemma 130 that

$$\tau' = 0, \qquad s' = 0.$$

By these comments,

$$\tau + s h^* = 0, \qquad \tau - s h^* = 0.$$

We have $h^* \neq 0$ by (15.92), so $\tau = 0$ and $s = 0$. By this and Lemma 117, A, A^* is essentially bipartite. $\square$

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26. Near-bipartite expansions of a bipartite Leonard pair that has reinforced dual q-Krawtchouk type. In Theorem 175, we classified the near-bipartite Leonard pairs with diameter $d \geq 3$. Our next goal is to describe the near-bipartite expansions A, A^* of a given bipartite Leonard pair B, A^* with diameter $d \geq 3$. In view of Theorem 175 and the discussion above it, we may assume that B, A^* has either reinforced dual q-Krawtchouk type or Krawtchouk type. In the present section, we assume that B, A^* has reinforced dual q-Krawtchouk type. In Section 27, we assume that B, A^* has Krawtchouk type.

For the rest of this section, we assume that B, A^* has reinforced dual q-Krawtchouk type. Let

$$(\{\theta'_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi'_i\}_{i=1}^d; \{\phi'_i\}_{i=1}^d) \tag{26.148}$$

denote a parameter array of B, A^*. Let

$$(\delta', \mu', h', \delta^*, \mu^*, h^*, \tau') \tag{26.149}$$

denote the primary q-data corresponding to the parameter array (26.148). By Lemma 152,

$$\delta' = 0, \qquad \mu' + h' = 0, \qquad \mu^* h^* = 0, \qquad \tau' = 0. \tag{26.150}$$

Note by Lemma 140 that $\mu' h' \neq 0$.

Theorem 176. *For scalars δ, μ, h, τ in $\mathbb{F}$ the following are equivalent:*

(i) *the sequence*

$$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau), \tag{26.151}$$

is a primary q-data of a near-bipartite expansion A, A^* of B, A^*;

(ii) *the following conditions hold:*

$$\mu \neq 0, \tag{26.152}$$

$$\mu \neq \pm\sqrt{-1}\mu' q^i \qquad (1 - d \leq i \leq d - 1), \tag{26.153}$$

$$\tau = 0, \tag{26.154}$$

$$h = \mu' h'/\mu. \tag{26.155}$$

Assume that (i) *and* (ii) *hold. Then A, A^* has reinforced dual q-Krawtchouk type.*

Proof. (i) $\Rightarrow$ (ii) Let

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \tag{26.156}$$

denote the parameter array of A, A^* corresponding to the primary q-data (26.151). By Lemma 67, $\varphi_i \phi_i = \varphi'_i \phi'_i$ for $1 \leq i \leq d$. By this and Lemma 106, (16.98)–(16.100) hold. By (16.98) and since $\mu' h' \neq 0$, we get (26.152). By this and (16.98), we get (26.155). By (16.99) and (26.150),

$$\tau(\mu + h) = 0. \tag{26.157}$$

If $\mu + h \neq 0$, then $\tau = 0$ by (26.157). If $\mu + h = 0$, then $\tau = 0$ by (16.100) and (26.150). Thus, we have (26.154). By Lemma 92, $\mu \neq hq^{2i}$ for $1 - d \leq i \leq d - 1$. By this and (26.150) and (26.155), we get (26.153). We have shown that (26.152)–(26.155) hold.

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(ii) $\Rightarrow$ (i) We first show that (13.69)–(13.73) hold. We have (13.69) and (13.71) since (26.149) is a primary q-data. By (26.150), we have $h' = -\mu'$. By this and (26.153) and (26.155), we get (13.70). By (26.150), we have $\mu^* h^* = 0$. By Lemma 154, μ^* and h^* are not both zero. By (26.152), $\mu \neq 0$. We have $h \neq 0$ by (26.155). By (26.154), $\tau = 0$. By these comments, we get (13.72) and (13.73). We have shown that (13.69)–(13.73) hold. Therefore, (26.151) is a primary q-data. Define scalars $\{\theta_i\}_{i=0}^d$, $\{\varphi_i\}_{i=1}^d$, $\{\phi_i\}_{i=1}^d$ by (13.64), (13.66), (13.67). By Lemmas 91 and 92, the sequence

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \tag{26.158}$$

is a parameter array over $\mathbb{F}$ corresponding to (26.151). By (26.150) and (26.152)–(26.155), we find that (16.98)–(16.100) hold. By this and Lemma 106, $\varphi_i \phi_i = \varphi_i' \phi_i'$ for $1 \leq i \leq d$. By this and Lemma 67, there exists a near-bipartite expansion A, A^* of B, A^* that has parameter array (26.158). By construction, the sequence (26.151) is a primary q-data of A, A^*.

Assume (i) and (ii) hold. By Lemma 138 and (26.150) and (26.154), we find that A, A^* has dual q-Krawtchouk type. Moreover ,A, A^* is reinforced since B, A^* is reinforced. ∎

Our next goal is to describe the near-bipartite expansions of B, A^* in terms of matrices. In this description, we will use the following matrix.

Definition 177. Let $K \in \mathrm{Mat}_{d+1}(\mathbb{F})$ denote the diagonal matrix that has (i,i)-entry q^{2i-d} for $0 \leq i \leq d$.

By Lemma 36, we may assume that B, A^* is in normalized TD/D form:

$$B = \begin{pmatrix} 0 & x_1' & & & & \mathbf{0} \\ 1 & 0 & x_2' & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & x_d' \\ \mathbf{0} & & & & 1 & 0 \end{pmatrix}, \qquad A^* = \mathrm{diag}(\theta_0^*, \theta_1^*, \ldots, \theta_d^*).$$

Theorem 178. *Given scalars δ, μ, h, τ in $\mathbb{F}$ that satisfy the equivalent conditions* (i) *and* (ii) *in Theorem 176. Let A, A^* denote the near-bipartite expansion of B, A^* that has primary q-data $(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau)$.*

(i) *Assume that $\mu^* = 0$. Then*
$$A = B + (\mu + h)K + \delta I.$$

(ii) *Assume that $h^* = 0$. Then*
$$A = B + (\mu + h)K^{-1} + \delta I.$$

Proof. By Lemma 64 the Leonard pair A, A^* is in normalized TD/D form. Let

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta_i^*\}_{i=0}^d),$$

denote the TD/D sequence of A, A^*. By Lemma 141, we see that

$$a_i = \begin{cases} \delta + (\mu + h) q^{2i-d} & \text{if } \mu^* = 0, \\ \delta + (\mu + h) q^{d-2i} & \text{if } h^* = 0 \end{cases} \qquad (0 \leq i \leq d).$$

By these comments and Definition 177, we get the result. ∎

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27. Near-bipartite expansions of a bipartite Leonard pair that has Krawtchouk type. In this section, we describe the near-bipartite expansions A, A^* of a given bipartite Leonard pair B, A^* that has Krawtchouk type. Let

$$(\{\theta'_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi'_i\}_{i=1}^d; \{\phi'_i\}_{i=1}^d), \tag{27.159}$$

denote a parameter array of B, A^*. Let

$$(\delta', \mu', h', \delta^*, \mu^*, h^*, \tau'), \tag{27.160}$$

denote the primary data corresponding to the parameter array (27.159). By Lemma 156,

$$\delta' = 0, \qquad h' = 0, \qquad h^* = 0, \qquad \tau' = 0. \tag{27.161}$$

Theorem 179. *For scalars δ, μ, h, τ in $\mathbb{F}$ the following are equivalent:*

(i) *the sequence*

$$(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau), \tag{27.162}$$

is a primary data of a near-bipartite expansion A, A^* of B, A^*;

(ii) *the following conditions hold:*

$$\mu \neq 0, \tag{27.163}$$

$$h = 0, \tag{27.164}$$

$$4\tau^2 = (\mu^2 - \mu'^2)\mu^{*2}. \tag{27.165}$$

Assume that (i) *and* (ii) *hold. Then A, A^* has Krawtchouk type.*

Proof. (i) ⇒ (ii) Let

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \tag{27.166}$$

denote the parameter array of A, A^* corresponding to (27.162). By Lemma 67, $\varphi_i \phi_i = \varphi'_i \phi'_i$ for $1 \leq i \leq d$. By this and Lemma 107, (16.101)–(16.103) hold. By (16.101) and (27.161), we get (27.164). By this and (14.81), we get (27.163). By (16.103), (27.161), and (27.164), we get (27.165).

(ii) ⇒ (i) We first show that (14.80)–(14.84) hold. We have (14.80) and (14.82) since (27.160) is a primary data of type II. By (27.163) and (27.164), we get (14.81). By (27.161), (27.164), and (27.165), we get (14.83) and (14.84). We have shown that (14.80)–(14.84) hold. Therefore, (27.162) is a primary data of type II. Define scalars $\{\theta_i\}_{i=0}^d$, $\{\varphi_i\}_{i=1}^d$, and $\{\phi_i\}_{i=1}^d$ by (14.75), (14.77), (14.78). By Lemmas 96 and 97, the sequence

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d), \tag{27.167}$$

is the parameter array corresponding to the primary data (27.162) of type II. By (27.161) and (27.163)–(27.165), we get (16.101)–(16.103). By this and Lemma 107, $\varphi_i \phi_i = \varphi'_i \phi'_i$ for $1 \leq i \leq d$. By this and Lemma 67, there exists a near-bipartite expansion A, A^* of B, A^* that has parameter array (27.167). By construction, (27.162) is a primary data of A, A^*.

Assume that (i) and (ii) hold. By (27.161) and (27.164), we have $h = 0$ and $h^* = 0$. By this and Lemma 146, the Leonard pair A, A^* has Krawtchouk type. ∎

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Our next goal is to describe the near-bipartite expansions of B, A^* in terms of matrices. In this description, we will use the following matrix.

DEFINITION 180. Define a diagonal matrix $H \in \mathrm{Mat}_{d+1}(\mathbb{F})$ that has (i,i)-entry $2i-d$ for $0 \le i \le d$.

By Lemma 36, we may assume that B, A^* is in normalized TD/D form:

$$B = \begin{pmatrix} 0 & x'_1 & & & & \mathbf{0} \\ 1 & 0 & x'_2 & & & \\ & 1 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & x'_d \\ \mathbf{0} & & & & 1 & 0 \end{pmatrix}, \qquad\qquad A^* = \mathrm{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d).$$

THEOREM 181. *Given scalars δ, μ, h, τ in $\mathbb{F}$ that satisfy the equivalent conditions* (i) *and* (ii) *in Theorem 179. Let A, A^* denote the near-bipartite expansion of B, A^* that has primary data $(\delta, \mu, h, \delta^*, \mu^*, h^*, \tau)$. Then*

$$A = B + (\tau/\mu^*)H + \delta I.$$

Proof. By Lemma 64, the Leonard pair A, A^* is in normalized TD/D form. Let

$$(\{a_i\}_{i=0}^d; \{x_i\}_{i=1}^d; \{\theta^*_i\}_{i=0}^d),$$

denote the TD/D sequence of A, A^*. By Lemma 150,

$$a_i = \delta + (2i-d)\tau/\mu^* \qquad\qquad (0 \le i \le d).$$

By these comments and Definition 180, we get the result. ∎

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