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MATRICES HAVING NONZERO OUTER INVERSES*

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Abstract. It is well known that every nonzero von Neumann regular $m \times n$-matrix A over an arbitrary ring R has a nonzero outer inverse $n \times m$-matrix B in the sense that $B = BAB$. Generalizing previous work on von Neumann regular matrices, the matrices having nonzero outer inverses over semiperfect rings are characterized as the matrices having some entry outside the Jacobson radical of R. Such matrices over finite semiperfect rings and finite commutative rings are counted, and several applications are given.

Key words. Von Neumann regular matrix, Outer inverse matrix, Semiperfect ring, Local ring.

AMS subject classifications. 15A09, 16E50, 13B30.

1. Introduction.

Having the root in the work of von Neumann [20], who studied rings now bearing his name, an $m \times n$-matrix A over a ring R is called *von Neumann regular* if it has an *inner inverse* (or *generalized inverse*) in the sense that there is an $n \times m$-matrix B over R such that $ABA = A$. An interesting problem in Linear Algebra is the study of von Neumann regular matrices and their generalized inverses over a ring. This was first investigated by Rao [16] over commutative rings and continued by numerous authors, such as Prasad [15], Lam and Swan [9], Ben-Israel and Greville [2], or Rao [17]. Some supplementary motivation for their study has been provided by their applications to other research fields, such as control theory, systems theory, operator algebras, or cryptography.

An $n \times m$-matrix B is called an *outer inverse* of an $m \times n$-matrix A if $BAB = B$. If A is a von Neumann regular $m \times n$-matrix with inner inverse $n \times m$-matrix B, then it is well known and easy to see that BAB is an outer inverse of A. Clearly, if A is nonzero von Neumann regular, then its outer inverse is nonzero. The problem of characterizing outer inverses of matrices over commutative rings was first stated and studied by Robinson [18], who also gave credit to a previous unpublished manuscript by Stanimirović. Recently, Chiru, Crivei and Olteanu [5, 6] established some intrinsic characterizations of (strongly) von Neumann regular matrices over commutative rings as well as some related counting results. As a continuation of our previous work, in this paper, we consider the more general class of matrices having nonzero outer inverses over arbitrary rings, and we look for some intrinsic descriptions of such matrices.

We first study nonzero outer inverses of elements of an arbitrary ring. It is clear that if an element of a ring R has a nonzero outer inverse, then it must be outside the Jacobson radical $J(R)$ of R. The main problem we discuss is to find a relevant class of rings R which satisfy the converse, that is, every element outside $J(R)$ has a nonzero outer inverse. In this direction, we prove that semiperfect rings (that is, rings R such that $R/J(R)$ is semisimple Artinian and idempotents lift modulo $J(R)$) satisfy this property. During the process, we show a result of possible independent interest, namely that elements having a nonzero outer inverse lift modulo one-sided ideals of exchange rings.

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Next we study nonzero outer inverses of matrices over an arbitrary ring. Thus, we show that the existence of an entry having a nonzero outer inverse ensures that the matrix A has a nonzero outer inverse, which in turn implies that A must have an entry outside the Jacobson radical of the ring. We prove that these conditions are equivalent, and they provide a constructive criterion for matrices having a nonzero outer inverse in the case of a large class of rings, namely the class of semiperfect rings. Examples of semiperfect rings include local rings, one-sided Artinian rings, semiprimary rings, and one-sided perfect rings. We also give an example showing that our result cannot be generalized to matrices over semilocal rings.

Finally, we count matrices having a nonzero outer inverse over finite semiperfect rings and finite commutative rings, and we give several applications to rings of residue classes, products of Galois rings, quaternion rings over rings of residue classes, and finite group algebras. Such results may also have applications to cryptography, by describing and counting the elements of the key space of some cryptosystems, in a similar way as for von Neumann regular matrices, e.g., see the key exchange protocol and the public key encryption with keyword search scheme from [11].

Throughout the paper, $m, n \geq 1$ will be integers, and R will be an associative ring with identity. Also, we denote by $M_{m,n}(R)$, $M_n(R)$, $\mathbb{Z}_n$, $J(R)$ and $U(R)$ the set of $m \times n$-matrices over R, the set of $n \times n$-matrices over R, the ring of residue classes modulo n, the Jacobson radical of R, and the set of units of R, respectively. Also, we denote by $\mathrm{rad}(M)$ the Jacobson radical of a right R-module M.

2. Nonzero outer inverses of elements of a ring. We will first look at 1×1-matrices, that is, elements of a ring R. In order to do that, we need some ring-theoretic notions.

Recall that a ring R is called *local* if it has a unique maximal one-sided ideal. Also, R is called *semiperfect* if $R/J(R)$ is semisimple Artinian and idempotents lift modulo $J(R)$. If R is a commutative semiperfect ring, then it is well known that it is a finite direct product of commutative local rings.

A one-sided ideal I of R is called *strongly lifting* if whenever $x^2 - x \in I$ for some $x \in R$ (i.e., x is idempotent modulo I), there is an idempotent $e \in xR$ such that $e - x \in I$ [13]. Note that this property is left–right symmetric, and the Jacobson radical J of a ring is a strongly lifting ideal provided idempotents lift modulo J.

A ring R is called an *exchange ring* (or a *suitable ring*) if there is an idempotent $e \in R$ such that $e - x \in (x^2 - x)R$, and this concept is left–right symmetric [12]. Note that exchange rings are exactly those rings for which idempotents lift modulo all one-sided ideals.

It will be useful in the following concept, which generalizes lifting of von Neumann regular elements modulo one-sided ideals in the sense of Khurana, Lam and Nielsen [8].

Definition 2.1. Let I be a one-sided ideal of R. We say that *elements having a nonzero outer inverse lift modulo I*, if whenever $x + I \in R/I$ has a nonzero outer inverse in R/I, there is an element $a \in R$ having a nonzero outer inverse in R such that $x - a \in I$.

Now we extend [8, Theorem 4.9] from von Neumann regular elements to elements having a nonzero outer inverse.

Proposition 2.2. *Let I be a strongly lifting right ideal of R. Then elements having a nonzero outer inverse lift modulo I.*

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Proof. Let $x+I \in R/I$ with a nonzero outer inverse $y+I \in R/I$. Then we have $(y+I)(x+I)(y+I) = y+I$, whence $yxy - y \in I$, and thus $(yx)^2 - yx \in I$. Since I is strongly lifting, there is an idempotent $e \in yxR$, say $e = yxr$, such that $e - yx \in I$. We have $ey - yxy = (e - yx)y \in I$, which together with $yxy - y \in I$ implies that $ey - y \in I$. Note that $e \neq 0$, because $y \notin I$. It follows easily that $(rey)x(rey) = rey$, which shows that rey is an outer inverse of x. We claim that $rey \neq 0$. Suppose to the contrary that $rey = 0$. Since $e = yxr$ and e is idempotent, we have $e = e^3 = yxreyxr = 0$, which is a contradiction. This shows that rey is a nonzero outer inverse of x. Hence, elements having a nonzero outer inverse lift modulo I. ∎

We have some useful consequence of Proposition 2.2 in the case of exchange rings and, in particular, von Neumann regular rings, π-regular rings or semiperfect rings, the latter being the case of interest for us.

Corollary 2.3. *Let I be a one-sided ideal of an exchange ring R. Then elements having a nonzero outer inverse lift modulo I.*

Proof. Note that one-sided ideals of exchange rings are strongly lifting [13, Theorem 4] and use Proposition 2.2. ∎

Now the following theorem may be stated and proved.

Theorem 2.4. *If an element $a \in R$ has a nonzero outer inverse, then $a \notin J(R)$. If R is semiperfect, then the converse is also true.*

Proof. Let $a \in R$ have a nonzero outer inverse $b \in R$. Then we have $b(1 - ab) = 0$. If $a \in J(R)$, then $1 - ab \in U(R)$, which implies $b = 0$, a contradiction. Hence, $a \notin J(R)$.

Next suppose that R is semiperfect. We shall show that every element $a \notin J(R)$ has a nonzero outer inverse. First, note that local rings trivially have this property, because every element outside $J(R)$ has an inverse. Moreover, matrix rings over local rings R have this property, because if $A = (a_{ij}) \notin \mathrm{rad}(M_{m,n}(R)) = M_{m,n}(J(R))$, then there is some $a_{ij} \in R \setminus J(R) = U(R)$, which yields a nonzero outer inverse $B = (b_{ij})$ of A all of whose entries are zero except for $b_{ji} = a_{ij}^{-1}$. Furthermore, direct products $\prod_{i \in I} R_i$ of rings with this property also have this property. In fact, an element $a = (a_i)_{i \in I} \in \prod_{i \in I} R_i$ has a nonzero outer inverse if and only if a_i has a nonzero outer inverse for some $i \in I$.

Since R is semiperfect, we may write $R/J(R) \cong \bigoplus_{k=1}^{s} M_{n_k}(D_k)$ for some positive integers $n_1, \ldots, n_s$ and division rings $D_1, \ldots, D_s$ by the Wedderburn–Artin Theorem. By the above considerations, it follows that $a + J(R)$ has a nonzero outer inverse. Since every semiperfect ring is an exchange ring (or one may use the fact that the Jacobson radical of a semiperfect ring is strongly lifting), elements having a nonzero outer inverse lift modulo $J(R)$ by Corollary 2.3 (or Proposition 2.2). Precisely, by the proof of Proposition 2.2, a has a nonzero outer inverse, as required. ∎

3. Nonzero outer inverses of matrices over a ring. We begin this section with an easy, but useful result on outer inverses.

Lemma 3.1. *Let $A \in M_{m,n}(R)$ be a matrix having an outer inverse $B \in M_{n,m}(J(R))$. Then $B = 0_{n,m}$.*

Proof. Since $BAB = B$, the matrix AB is idempotent, hence AB is von Neumann regular. But $B \in M_{n,m}(J(R))$, which implies that $AB \in M_m(J(R))$, because $J(R)$ is an ideal of R. Then we have $AB = 0_m$ by [9, Lemma 3.1], whence it follows that $B = 0_{n,m}$. ∎

Now we may extend Theorem 2.4 from elements of a ring to matrices over a ring as follows. In the case of square matrices, this follows immediately from Theorem 2.4 applied to the ring $M_n(R)$, but next we give it for arbitrary matrices, that is, elements of the right R-module $M_{m,n}(R)$.

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Theorem 3.2. *Let $A = (a_{ij}) \in M_{m,n}(R)$. Consider the following statements:*

(i) There exists some a_{ij} having a nonzero outer inverse.
(ii) A has a nonzero outer inverse.
(iii) $A \notin M_{m,n}(J(R))$.

Then (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii). The implication (iii)$\Longrightarrow$(i) is true for every semiperfect ring.

In this case, a nonzero outer inverse of A is the matrix $B \in M_{n,m}(R)$ having all entries zero, except for the entry (j, i), which is a nonzero outer inverse of a_{ij}.

Proof. $(i) \Longrightarrow (ii)$ This is trivial. A nonzero outer inverse $B \in M_{n,m}(R)$ of A is the matrix having all entries zero, except for the entry (j, i), which is a nonzero outer inverse of a_{ij}.

$(ii) \Longrightarrow (iii)$ Suppose that A has a nonzero outer inverse $B \in M_{n,m}(R)$. Let us assume that $A \in M_{m,n}(J(R))$. As $J(R)$ is an ideal of R, we have $BA \in M_n(J(R))$. Since $BAB = B$, it follows that BA has outer inverse BA. By Lemma 3.1, we must have $BA = 0_n$, which implies $B = 0_{n,m}$, a contradiction. Thus, we have $A \notin M_{m,n}(J(R))$.

$(iii) \Longrightarrow (i)$ Assume that $A \notin M_{m,n}(J(R))$. Denote by $p : R \to R/J(R)$ the natural ring epimorphism, and by $h : M_{m,n}(R) \to M_{m,n}(R/J(R))$ the induced homomorphism of right R-modules defined by $h((a_{ij})) = (p(a_{ij}))$. Since R is semiperfect, we may write $R/J(R) \cong \bigoplus_{k=1}^{s} R_k$, where $R_k = M_{n_k}(D_k)$ for some positive integers $n_1, \ldots, n_s$ and division rings $D_1, \ldots, D_s$ by the Wedderburn–Artin Theorem. For every $k \in \{1, \ldots, s\}$, denote by $h_k : M_{m,n}(R) \to M_{m,n}(R_k)$ the canonical projection. Since $h(A) \neq 0_{m,n}$, there is $k \in \{1, \ldots, s\}$ such that $h_k(A) \neq 0_{m,n}$. Then $h_k(A)$ has an invertible entry, which implies that $h(A)$ has an entry, say a'_{ij}, having a nonzero outer inverse, say b'. Since every semiperfect ring is an exchange ring, one may use Corollary 2.3 in order to lift the element $a'_{ij} \in R/J(R)$ which has a nonzero outer inverse b' to an element $a_{ij} \in R$ which has a nonzero outer inverse b. A nonzero outer inverse of A is the matrix $B \in M_{n,m}(R)$ having all entries zero, except for the entry (j, i), which is b. ◻

Noting that any finite direct product of local rings is semiperfect, we have the following corollary.

Corollary 3.3. *Let R be a finite direct product of local rings, and let $A \in M_{m,n}(R)$. Then the following are equivalent:*

(i) A has an entry with a nonzero outer inverse.
(ii) A has a nonzero outer inverse.
(iii) $A \notin M_{m,n}(J(R))$.

Next let us illustrate Theorem 2.4.

Example 3.4. Let $A = (a_{ij}) = \begin{pmatrix} 2 & 3 & 4 & 6 \\ 8 & 9 & 10 & 0 \\ 2 & 3 & 8 & 9 \end{pmatrix} \in M_{3,4}(\mathbb{Z}_{12})$. Note that $\mathbb{Z}_{12}$ is a semiperfect ring, $I = J(\mathbb{Z}_{12}) = \{0, 6\}$, and there is a ring isomorphism $\mathbb{Z}_{12}/J(\mathbb{Z}_{12}) \cong \mathbb{Z}_6$. With the above notation, we have $h(A) = (a'_{ij}) = \begin{pmatrix} 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 \\ 2 & 3 & 2 & 3 \end{pmatrix} \in M_{3,4}(\mathbb{Z}_6)$. Let $x = a_{23} = 10 \in \mathbb{Z}_{12}$, and note that $x + I$ has the nonzero outer inverse $y + I$ with $y = 4 \in \mathbb{Z}_{12}$. Following the proof of Proposition 2.2, we are looking for an idempotent $e = yxr \in yx\mathbb{Z}_{12} = 4\mathbb{Z}_{12}$ such that $e - yx = e - 4 \in I = J(\mathbb{Z}_{12})$. Choosing $e = 4$ for

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$r = 1$, the element $rey = 4$ is a nonzero outer inverse of $x = 10$. Then a nonzero outer inverse of A is $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M_{4,3}(\mathbb{Z}_{12})$ by Theorem 2.4.

Let us also see an example in the case of matrices over a noncommutative ring.

Example 3.5. Consider the formal triangular matrix ring $R = \begin{pmatrix} \mathbb{Z}_{12} & \mathbb{Z}_{12} \\ 0 & \mathbb{Z}_{12} \end{pmatrix}$. Since $\mathbb{Z}_{12}$ is a semiperfect ring, so is R by [7, Corollary 2.5]. Note that $I = J(R) = \begin{pmatrix} J(\mathbb{Z}_{12}) & \mathbb{Z}_{12} \\ 0 & J(\mathbb{Z}_{12}) \end{pmatrix} = \begin{pmatrix} \{0,6\} & \mathbb{Z}_{12} \\ 0 & \{0,6\} \end{pmatrix}$ [7, Corollary 2.2], and there is a ring isomorphism $R/J(R) \cong \mathbb{Z}_6 \times \mathbb{Z}_6$ [7, Corollary 2.3]. Let

$$A = (a_{ij}) = \begin{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 9 \end{pmatrix} & \begin{pmatrix} 4 & 6 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 8 & 9 \\ 0 & 3 \end{pmatrix} & \begin{pmatrix} 10 & 0 \\ 0 & 9 \end{pmatrix} \end{pmatrix} \in M_2(R).$$

With the above notation, we have $h(A) = \begin{pmatrix} (2,3) & (4,0) \\ (2,3) & (4,3) \end{pmatrix} \in M_2(\mathbb{Z}_6 \times \mathbb{Z}_6)$. Let $x = a_{22} = \begin{pmatrix} 10 & 0 \\ 0 & 9 \end{pmatrix} \in R$. After the identification given by the above ring isomorphism, $x + I = (4,3) \in \mathbb{Z}_6 \times \mathbb{Z}_6$ has the nonzero outer inverse $y + I = (4,0) \in \mathbb{Z}_6 \times \mathbb{Z}_6$ with $y = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \in R$. As in the proof of Proposition 2.2, we are looking for an idempotent $e = yxr \in yxR = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} R$ such that $e - yx = e - \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \in I = J(R)$. Choosing $e = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ for $r = I_2 \in R$, the element $rey = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ is a nonzero outer inverse of $x = \begin{pmatrix} 10 & 0 \\ 0 & 9 \end{pmatrix}$. By Theorem 2.4, a nonzero outer inverse of A is

$$B = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in M_2(R).$$

In general, Theorem 2.4 does not hold for semilocal rings, as the following example shows. Recall that a ring R is called *semilocal* if $R/J(R)$ is semisimple Artinian.

Example 3.6. Consider the localizations $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_{(q)}$ of the ring of integers to some distinct primes p and q. Then the ring $R = \mathbb{Z}_{(p)} \cap \mathbb{Z}_{(q)}$ is a semilocal ring with two maximal ideals (p) and (q) generated by p and q, respectively, because $R/J(R) = R/(pq) \cong R/(p) \times R/(q)$. But R is not semiperfect, because idempotents do not lift modulo $J(R)$. Note that $x = \frac{a}{b} \in R$ has a nonzero outer inverse if and only if $x \in U(R)$ if and only if a and pq are relatively prime. Now let us choose $p = 3$ and $q = 5$, hence $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$. Let $A = \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix} \in M_2(R)$. No entry of A has a nonzero outer inverse, but A is invertible, and thus it has a nonzero outer inverse, namely $A^{-1} = \frac{1}{2}\begin{pmatrix} 9 & -5 \\ -5 & 3 \end{pmatrix} \in M_2(R)$. Also, $A' = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(R)$ and $A' \notin M_2(J(R)) = M_2(15R)$, but it is easily seen that A' has no nonzero outer inverse.

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4. Counting matrices having nonzero outer inverses. For a finite semiperfect ring, Theorem 2.4 allows us to easily determine the number of $m \times n$-matrices over R having a nonzero outer inverse, which will be denoted by $VO(M_{m,n}(R))$.

PROPOSITION 4.1. *Let R be finite semiperfect. Then* $VO(M_{m,n}(R)) = |M_{m,n}(R)| - |M_{m,n}(J(R))|$.

We may give more detailed consequences in the case of matrices over commutative rings. Recall that every finite commutative ring is a direct product of local finite commutative rings (e.g., see [10, (VI.2)].

PROPOSITION 4.2. *Let R be finite commutative, say $R = R_1 \times \cdots \times R_s$ for some local finite commutative rings. Then* $VO(M_{m,n}(R)) = \prod_{k=1}^{s} |M_{m,n}(R_k)| - \prod_{k=1}^{s} |M_{m,n}(J(R_k))|$.

Proof. Since $M_{m,n}(R) = \prod_{k=1}^{s} M_{m,n}(R_k)$ and $M_{m,n}(J(R)) = \prod_{k=1}^{s} M_{m,n}(J(R_k))$, by Corollary 3.3 we have $VO(M_{m,n}(R)) = \prod_{k=1}^{s} |M_{m,n}(R_k)| - \prod_{k=1}^{s} |M_{m,n}(J(R_k))|$, which proves the required formula. □

COROLLARY 4.3. *Let $R = D_{q_1} \times \cdots \times D_{q_s}$ be a direct product of fields. Then* $VO(M_{m,n}(R)) = \prod_{k=1}^{s} q_k^{mn} - 1$.

COROLLARY 4.4. *Let $l = p_1^{r_1} \cdots p_s^{r_s}$ for some distinct primes $p_1, \dots, p_s$ and positive integers $r_1, \dots, r_s$. Then* $VO(M_{m,n}(\mathbb{Z}_l)) = \prod_{k=1}^{s} p_k^{r_k mn} - \prod_{k=1}^{s} p_k^{(r_k-1)mn}$.

Proof. Use Proposition 4.2 for the product $\mathbb{Z}_l \cong \mathbb{Z}_{p_1^{r_1}} \times \cdots \times \mathbb{Z}_{p_s^{r_s}}$ of local rings. □

Let p be a prime, and let k, d be integers. Following [10, Chapter XVI], recall that a *Galois ring* is a ring $R \cong \mathbb{Z}_{p^r}[x]/(h(x))$ for some monic polynomial $h(x) \in \mathbb{Z}_{p^r}[x]$ of degree d which is irreducible in $\mathbb{Z}_p[x]$. Such a Galois ring, which is denoted by $GR(p^r, d)$, has p^{rd} elements and characteristic p^r, it is a local commutative ring with maximal ideal $(p) = pGR(p^r, d)$ and $GR(p^r, d)/(p) \cong F_{p^d}$. Note that $GR(p, d) = F_{p^d}$ and $GR(p^r, 1) = \mathbb{Z}_{p^r}$.

Now Proposition 4.2 yields the following corollary, which extends Corollary 4.4.

COROLLARY 4.5. *Let $R = R_1 \times \cdots \times R_s$ be a direct product of Galois rings $R_k = GR(p_k^{r_k}, d_k)$ for $k \in \{1, \dots, s\}$. Then* $VO(M_{m,n}(R)) = \prod_{k=1}^{s} p_k^{r_k d_k mn} - \prod_{k=1}^{s} p_k^{(r_k-1)d_k mn}$.

As a further application, we may also determine the number of elements having a nonzero outer inverse in the quaternion ring over R. Denote $H(R) = \{r_0 + r_1 i + r_2 j + r_3 k \mid r_0, r_1, r_2, r_3 \in R\} = R \oplus Ri \oplus Rj \oplus Rk$, where $i^2 = j^2 = k^2 = ijk = -1$, $ij = -ji$ and i, j, k commute with R elementwise. With componentwise addition and multiplication using the above relations, one obtains the *quaternion ring over R* (e.g., see [4]).

COROLLARY 4.6. *Let $l = p_1^{r_1} \cdots p_s^{r_s}$ be an odd integer for some distinct primes $p_1, \dots, p_s$ and positive integers $r_1, \dots, r_s$. Then* $VO(H(\mathbb{Z}_l)) = \prod_{k=1}^{s} p_k^{4r_k} - \prod_{k=1}^{s} p_k^{4(r_k-1)}$.

Proof. We have $H(\mathbb{Z}_l) \cong M_2(\mathbb{Z}_l)$ by [4, Corollary 3.13]. Then use Corollary 4.4. □

Next we count matrices having a nonzero outer inverse over some finite group algebras $F_q[G]$, where F_q is a field with q elements and G is a group with l elements. By Maschke's Theorem, the group algebra $F_q[G]$ is semisimple if and only if $\mathrm{char}(F_q)$ does not divide l. In this case, one may use the Wedderburn–Artin Theorem in order to get an isomorphism of F_q-algebras $F_q[G] \cong \bigoplus_{k=1}^{s} M_{n_k}(D_k)$ for some positive integers $n_1, \dots, n_s$ and finite fields $D_1, \dots, D_s$ [14, Theorem 3.4.9]. For every $k \in \{1, \dots, s\}$, denote $q_k = |D_k| = q^{d_k}$, where $d_k = [D_k : F_q]$ is the degree of the field extension D_k over F_q.

When G is an *Abelian* group with l elements, the Wedderburn–Artin Theorem reduces to the Perlis–Walker Theorem [14, Theorem 3.5.4], which yields an isomorphism of F_q-algebras $F_q[G] \cong \bigoplus_{d|l} a_d F_q(\zeta_d)$,

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where ζ_d is a primitive root of unity of order d, $e_d = [F_q(\zeta_d) : F_q]$, n_d is the number of elements of order d of G, $a_d = \frac{n_d}{e_d}$, and $a_d F_q(\zeta_d)$ denotes the direct sum of a_d different fields all of which are isomorphic to the field extension $F_q(\zeta_d)$ of F_q, where $|F_q(\zeta_d)| = q^{e_d}$. Then Corollary 4.3 immediately gives the following result, in which we use the above notation.

PROPOSITION 4.7. *Let G be a group with l elements, and let F_q be a field with q elements such that* $\mathrm{char}(F_q)$ *does not divide l. Then:*

(i) $VO(M_{m,n}(F_q[G])) = \prod_{k=1}^{s} q_k^{mnn_k^2} - 1$.
(ii) If G is Abelian, then $VO(M_{m,n}(F_q[G])) = \prod_{k=1}^{s} q^{mne_d a_d} - 1$.

Finally, we consider the more interesting case of a *cyclic* group G with l elements, that is, $G \cong \mathbb{Z}_l$. This time we have a formula for the number of $m \times n$-matrices over $F_q[G]$ having a nonzero outer inverse even if the group algebra $F_q[G]$ is not semisimple.

PROPOSITION 4.8. *Let $l \geq 2$ be an integer and let F_q be a finite field with q elements. Write $x^l - 1 = p_1(x)^{r_1} \cdots p_s(x)^{r_s}$ for some distinct irreducible polynomials $p_1(x), \ldots, p_s(x) \in F_q[x]$ with degrees $d_1, \ldots, d_s$, respectively, and positive integers $r_1, \ldots, r_s$. Then $VO(M_{m,n}(F_q[\mathbb{Z}_l])) = \prod_{k=1}^{s} q_k^{r_k mn} - \prod_{k=1}^{s} q_k^{(r_k-1)mn}$, where $q_k = q^{d_k}$ for every $k \in \{1, \ldots, s\}$.*

Proof. We have the following classical isomorphism of F_q-algebras:

$$F_q[\mathbb{Z}_l] \cong F_q[x]/(x^l - 1) \cong F_q[x]/(p_1(x)^{r_1}) \times \cdots \times F_q[x]/(p_s(x)^{r_s})$$

(e.g., see [14, p. 145]). For every $k \in \{1, \ldots, s\}$, $R_k = F_q[x]/(p_k(x)^{r_k})$ is a local ring with maximal ideal $M_k = (p_k(x))/(p_k(x)^{r_k})$. We also have $|R_k| = q^{d_k r_k}$, $|R_k/M_k| = q^{d_k}$ and $|M_k| = \frac{|R_k|}{|R_k/M_k|} = q^{d_k(r_k-1)}$. Finally, use Proposition 4.2. ∎

REMARK 4.9. (1) The above results also hold for $m = n = 1$. For instance, if $R = R_1 \times \cdots \times R_s$ is a direct product of local finite rings, then $VO(R) = \prod_{k=1}^{s} |R_k| - \prod_{k=1}^{s} |J(R_k)|$. They extend counting results on von Neumann regular elements of $\mathbb{Z}_l$ [1, 19] and of certain group algebras [3].

(2) Unlike the numbers of (strongly) von Neumann regular matrices over finite rings (see [5, 6]), the number of matrices having a nonzero outer inverse over finite rings does not yield a multiplicative function. For instance, we have:

$$VO(M_2(\mathbb{Z}_2) \times M_2(\mathbb{Z}_3)) = VO(M_2(\mathbb{Z}_6)) = 1295 \neq 15 \cdot 80 = VO(M_2(\mathbb{Z}_2)) \cdot VO(M_2(\mathbb{Z}_3)).$$

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