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ON INEQUALITIES INVOLVING THE HADAMARD PRODUCT OF MATRICES*

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Abstract. Recently, the authors established a number of inequalities involving integer powers of the Hadamard product of two positive definite Hermitian matrices. Here these results are extended in two ways. First, the restriction to integer powers is relaxed to include all real numbers not in the open interval $(-1, 1)$. Second, the results are extended to the Hadamard product of any finite number of Hermitian positive definite matrices.

Key words. matrix inequalities, Hadamard product

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1. Introduction. Let A and B be $n \times n$ matrices. $A \circ B$ denotes the Hadamard product and $A \otimes B$ the Kronecker product of A and B.

These two products are related by the following relation [2], [3].

There exists an $n^2 \times n$ selection matrix J such that $J^T J = I$ and

$$A \circ B = J^T (A \otimes B) J.$$

Note that J^T is the $n \times n^2$ matrix $[E_{11} E_{22} \dots E_{nn}]$, where E_{ii} is the $n \times n$ matrix of zeros except for a one in the (i, i)th position.

Using this result, in [4] the authors proved a number of inequalities involving integer powers of the Hadamard product of two positive definite Hermitian matrices. Here we extend these results in two ways. First, the restriction to integer powers is relaxed to include all real numbers not in the open interval $(-1, 1)$. Second, the results are extended to the Hadamard product of any finite number of $n \times n$ Hermitian positive definite matrices.

2. Notation and Preliminary Results. The Hadamard and Kronecker products of matrices A_i, $i = 1, \dots k$, will be denoted by $\overset{k}{\underset{i=1}{\circ}}$ and $\bigotimes_{i=1}^{k}$, respectively.

We shall make frequent use of the following property of the Kronecker product:

$$(AB) \otimes (CD) = (A \otimes C)(B \otimes D).$$

For a finite number of matrices A_i, B_i, $i = 1, \dots, k$, this becomes

$$\left(\prod_{i=1}^{k} A_i \right) \otimes \left(\prod_{i=1}^{k} B_i \right) = \prod_{i=1}^{k} (A_i \otimes B_i). \tag{1}$$

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Let A be a positive definite $n \times n$ Hermitian matrix. There exists a matrix U such that

$$A = U^*[\lambda_1, \lambda_2, \ldots, \lambda_n]U, \quad U^*U = I,$$

where $[\lambda_1, \lambda_2, \ldots, \lambda_n]$ is the diagonal matrix with λ_i, the positive eigenvalues of A, along the diagonal [1]. For any real number s, A^s is defined by

$$A^s = U^*[\lambda_1^s, \lambda_2^s, \ldots, \lambda_n^s]U.$$

LEMMA 2.1. *Let A and B be positive definite Hermitian $n \times n$ matrices and s a nonzero real number. Then*

$$A^s \otimes B^s = (A \otimes B)^s. \tag{2}$$

Proof. Assume

$$B = V^*[\gamma_1, \gamma_2, \ldots, \gamma_n]V, \quad V^*V = I,$$

where γ_i are the eigenvalues of B. Then

$$\begin{aligned} A^s \otimes B^s &= (U^*[\lambda_1^s, \ldots, \lambda_n^s]U) \otimes (V^*[\gamma_1^s, \ldots, \gamma_n^s]V) \\ &= (U^* \otimes V^*)([\lambda_1^s, \ldots, \lambda_n^s] \otimes [\gamma_1^s, \ldots, \gamma_n^s])(U \otimes V) \\ &= (U \otimes V)^*([\lambda_1^s, \ldots, \lambda_n^s] \otimes [\gamma_1^s, \ldots, \gamma_n^s])(U \otimes V) = (A \otimes B)^s. \quad \square \end{aligned}$$

Note that

$$\begin{aligned} (U \otimes V)^*(U \otimes V) &= (U^* \otimes V^*)(U \otimes V) \\ &= (U^*U) \otimes (V^*V) = I \otimes I = I_{n^2}. \end{aligned}$$

Equation (2) extends readily, for a finite number of $n \times n$ positive definite Hermitian matrices $A_i,\ i = 1, \ldots, k$, to

$$\bigotimes_{i=1}^{k} (A_i^s) = \left(\bigotimes_{i=1}^{k} A_i \right)^s. \tag{3}$$

LEMMA 2.2. *Let $A_i,\ i = 1, \ldots, k$, be $n \times n$ matrices. There exists an $n^k \times n$ selection matrix P such that $P^TP = I$*
and

$$\mathop{\circ}_{i=1}^{k} A_i = P^T \left(\bigotimes_{i=1}^{k} A_i \right) P. \tag{4}$$

We prove this for three matrices.[1] The extension from m to $m+1$ is similar.

$$\begin{aligned} A \circ B \circ C &= A \circ J^T(B \otimes C)J \\ &= J^T(A \otimes (J^T(B \otimes C)J))J \\ &= J^T((IAI) \otimes (J^T(B \otimes C)J))J \\ &= J^T(I \otimes J^T)(A \otimes B \otimes C)(I \otimes J)J \qquad \text{by (1)} \\ &= J^T(I \otimes J)^T(A \otimes B \otimes C)(I \otimes J)J \\ &= \hat{J}^T(A \otimes B \otimes C)\hat{J}, \end{aligned}$$

[1]This proof was provided by George Visick in a private communication.

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where $\hat{J}$ is the $n^3 \times n$ matrix $\hat{J} = (I \otimes J)J$. Note that $\hat{J}^T \hat{J} = I$.

3. Results. In this section, $A_i,\ i = 1, \ldots, k$, will denote $n \times n$ positive definite Hermitian matrices. $A_i \geq A_j$ means that $A_i - A_j$ is positive semidefinite.

THEOREM 3.1. *Let r and s be real numbers $r < s$, and either $r \notin (-1, 1)$ and $s \notin (-1, 1)$ or $s \geq 1 \geq r \geq \frac{1}{2}$ or $r \leq -1 \leq s \leq -\frac{1}{2}$. Then*

$$\left(\mathop{\circ}_{i=1}^{k} A_i^s \right)^{1/s} \geq \left(\mathop{\circ}_{i=1}^{k} A_i^r \right)^{1/r}. \tag{5}$$

Proof. We make use of the following result [5].

Let A be an $n \times n$ positive definite Hermitian matrix and let V be an $n \times t$ matrix such that $V^*V = I$. Then

$$(V^* A^s V)^{1/s} \geq (V^* A^r V)^{1/r}$$

for all real r and s, $r < s$, such that either $r \notin (-1, 1)$ and $s \notin (-1, 1)$ or $s \geq 1 \geq r \geq \frac{1}{2}$ or $r \leq -1 \leq s \leq -\frac{1}{2}$.

Here instead of V, we use the $n^k \times n$ selection matrix P given by (4). Noting (3), we have

$$\begin{aligned} \left(\mathop{\circ}_{i=1}^{k} A_i^s \right)^{1/s} &= \left(P^t \left(\bigotimes_{i=1}^k A_i^s \right) P \right)^{1/s} \\ = \quad & \left(P^t \left(\bigotimes_{i=1}^k A_i \right)^s P \right)^{1/s} \geq \left(P^T \left(\bigotimes_{i=1}^k A_i \right)^r P \right)^{1/r} \\ = \quad & \left(P^t \left(\bigotimes_{i=1}^k A_i^r \right) P \right)^{1/r} = \left(\mathop{\circ}_{i=1}^{k} A_i^r \right)^{1/r}. \qquad \square \end{aligned}$$

Some special cases of (5) are the following:

$$\left(\mathop{\circ}_{i=1}^{k} A_i^{-1} \right)^{-1} \leq \left(\mathop{\circ}_{i=1}^{k} A_i \right)$$

or, equivalently

$$\left(\mathop{\circ}_{i=1}^{k} A_i \right)^{-1} \leq \left(\mathop{\circ}_{i=1}^{k} A_i^{-1} \right).$$

For $r > 1$, we have

$$\left(\mathop{\circ}_{i=1}^{k} A_i \right) \leq \left(\mathop{\circ}_{i=1}^{k} A_i^r \right)^{1/r}$$

or, equivalently,

$$\left(\mathop{\circ}_{i=1}^{k} A_i^{1/r} \right) \leq \left(\mathop{\circ}_{i=1}^{k} A_i \right)^{1/r}.$$

For $r = 2$, the last two inequalities become

$$\left(\mathop{\circ}_{i=1}^{k} A_i \right) \leq \left(\mathop{\circ}_{i=1}^{k} A_i^2 \right)^{1/2}$$

and

$$\left(\mathop{\circ}_{i=1}^{k} A_i^{1/2}\right) \leq \left(\mathop{\circ}_{i=1}^{k} A_i\right)^{1/2}.$$

THEOREM 3.2. *Let r and s be nonzero real numbers such that $s > r$ and $s \notin (-1,1)$ or $r \notin (-1,1)$. Then*

$$\text{(a)} \qquad \left(\mathop{\circ}_{i=1}^{k} A_i^s\right)^{1/s} \leq \bar{\triangle} \left(\mathop{\circ}_{i=1}^{k} A_i^r\right)^{1/r},$$

where

$$\bar{\triangle} = \left\{\frac{r(\gamma^s-\gamma^r)}{(s-r)(\gamma^r-1)}\right\}^{1/s} \left\{\frac{s(\gamma^r-\gamma^s)}{(r-s)(\gamma^s-1)}\right\}^{-1/r}, \tag{6}$$

$\gamma = M/m$, and M and m are, respectively, the largest and smallest eigenvalues of $\bigotimes_{i=1}^{k} A_i$. Also,

$$\text{(b)} \qquad \left(\mathop{\circ}_{i=1}^{k} A_i^s\right)^{1/s} - \left(\mathop{\circ}_{i=1}^{k} A_i^r\right)^{1/r} \leq \triangle I,$$

where

$$\triangle = \max_{\theta\in[0,1]} \{[\theta M^s + (1-\theta)m^s]^{1/s} - [\theta M^r + (1-\theta)m^r]^{1/r}\}. \tag{7}$$

Proof. Let A be an $n \times n$ positive definite Hermitian matrix with eigenvalues contained in the interval $[m, M]$, where $0 < m < M$, and let V be an $n \times t$ matrix such that $V^*V = I$. If r and s are nonzero real numbers such that $r < s$ and either $s \notin (-1,1)$ or $r \notin (-1,1)$, then [6]

$$(V^*A^sV)^{1/s} \leq \bar{\triangle}(V^*A^rV)^{1/r} \tag{8}$$

where $\bar{\triangle}$ is given by (6), and

$$(V^*A^sV)^{1/s} - (V^*A^rV)^{1/r} \leq \triangle I \tag{9}$$

where $\triangle$ is given by (7). Thus for part (a), from (8) and noting (3) and (4), we have

$$\left(\mathop{\circ}_{i=1}^{k} A_i^s\right)^{1/s} = \left[P^T\left(\bigotimes_{i=1}^{k} A_i^s\right)P\right]^{1/s} = \left[P^T\left(\bigotimes_{i=1}^{k} A_i\right)^s P\right]^{1/s}$$
$$\leq \bar{\triangle}\left[P^T\left(\bigotimes_{i=1}^{k} A_i\right)^r P\right]^{1/r} = \bar{\triangle}\left[P^T\left(\bigotimes_{i=1}^{k} A_i^r\right)P\right]^{1/r} = \bar{\triangle}\left(\mathop{\circ}_{i=1}^{k} A_i^r\right)^{1/r}.$$

For part (b), from (9),

$$\left(\overset{k}{\underset{i=1}{\circ}} A_i^s\right)^{1s} - \left(\overset{k}{\underset{i=1}{\circ}} A_i^r\right)^{1/r} = \left[P^T\left(\bigotimes_{i=1}^k A_i^s\right)P\right]^{1/s} - \left[P^T\left(\bigotimes_{i=1}^k A_i^r\right)P\right]^{1/r}$$
$$= \left[P^T\left(\bigotimes_{i=1}^k A_i\right)^s P\right]^{1/s} - \left[P^T\left(\bigotimes_{i=1}^k A_i\right)^r P\right]^{1/r} \leq \triangle I. \qquad \square$$

Remark 3.3. The cases $k = 2$ of the above results were also considered in [7].

3.1. Special Cases. For $s = 2$ and $r = 1$, we get

$$\left(\overset{k}{\underset{i=1}{\circ}} A_i^2\right)^{1/2} \leq \frac{(M+m)}{2\sqrt{Mm}}\left(\overset{k}{\underset{i=1}{\circ}} A_i\right)$$

and

$$\left(\overset{k}{\underset{i=1}{\circ}} A_i^2\right)^{1/2} - \left(\overset{k}{\underset{i=1}{\circ}} A_i\right) \leq \frac{(M-m)^2}{4(M+m)}I.$$

For $s = 1$ and $r = -1$, we get

$$\left(\overset{k}{\underset{i=1}{\circ}} A_i\right) \leq \frac{(m+M)^2}{4Mm}\left(\overset{k}{\underset{i=1}{\circ}} A_i^{-1}\right)^{-1}$$

and

$$\left(\overset{k}{\underset{i=1}{\circ}} A_i\right) - \left(\overset{k}{\underset{i=1}{\circ}} A_i^{-1}\right)^{-1} \leq (\sqrt{M} - \sqrt{m})^2 I.$$

We note that the eigenvalues of $\bigotimes_{i=1}^k A_i$ are the n^k products of the eigenvalues of $A_i,\ i = 1, \ldots, k$. Thus, if the eigenvalues of $A_i,\ i = 1, \ldots, k$, are ordered as

$$\lambda_1^i \geq \lambda_2^i \geq \ldots \geq \lambda_n^i > 0, \quad i = 1, \ldots, k,$$

then the maximum and minimum eigenvalues of $\bigotimes_{i=1}^k A_i$ are $M = \prod_{i=1}^k \lambda_1^i$ and $m = \prod_{i=1}^k \lambda_n^i$. This leads to the following four inequalities:

$$\left(\overset{k}{\underset{i=1}{\circ}} A_i^2\right)^{1/2} \leq \left(\frac{\prod_{i=1}^k \lambda_1^i + \prod_{i=1}^k \lambda_n^i}{2\sqrt{\prod_{i=1}^k \lambda_1^i \lambda_n^i}}\right)\left(\overset{k}{\underset{i=1}{\circ}} A_i\right),$$

$$\left(\mathop{\circ}_{i=1}^{k} A_i^2\right)^{1/2} - \left(\mathop{\circ}_{i=1}^{k} A_i\right) \le \frac{\left(\prod_{i=1}^{k}\lambda_1^i - \prod_{i=1}^{k}\lambda_n^i\right)^2}{4\left(\prod_{i=1}^{k}\lambda_1^i + \prod_{i=1}^{k}\lambda_n^i\right)} I,$$

$$\left(\mathop{\circ}_{i=1}^{k} A_i\right) \le \frac{\left(\prod_{i=1}^{k}\lambda_1^i + \prod_{i=1}^{k}\lambda_n^i\right)^2}{4\prod_{i=1}^{k}\lambda_1^i\lambda_n^i} \left(\mathop{\circ}_{i=1}^{k} A_i^{-1}\right)^{-1}, \tag{10}$$

$$\left(\mathop{\circ}_{i=1}^{k} A_i\right) - \left(\mathop{\circ}_{i=1}^{k} A_i^{-1}\right)^{-1} \le \left(\sqrt{\prod_{i=1}^{k}\lambda_1^i} - \sqrt{\prod_{i=1}^{k}\lambda_n^i.}\right)^2 I.$$

Finally, by taking A_i^{-1} for A_i in (10), we obtain

$$\left(\mathop{\circ}_{i=1}^{k} A_i^{-1}\right) \le \frac{\left(\prod_{i=1}^{k}\lambda_1^i + \prod_{i=1}^{k}\lambda_n^i\right)^2}{4\prod_{i=1}^{k}\lambda_1^i\lambda_n^i} \left(\mathop{\circ}_{i=1}^{k} A_i\right)^{-1}.$$

The inequalities here are generalizations of those given in [4]. Additional inequalities of a similar kind are possible and will be considered elsewhere.

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