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ON THE STRONG ARNOL'D HYPOTHESIS AND THE CONNECTIVITY OF GRAPHS*

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Abstract. In the definition of the graph parameters $\mu(G)$ and $\nu(G)$, introduced by Colin de Verdière, and in the definition of the graph parameter $\xi(G)$, introduced by Barioli, Fallat, and Hogben, a transversality condition is used, called the Strong Arnol'd Hypothesis. In this paper, we define the Strong Arnol'd Hypothesis for linear subspaces $L \subseteq \mathbb{R}^n$ with respect to a graph $G = (V, E)$, with $V = \{1, 2, \dots, n\}$. We give a necessary and sufficient condition for a linear subspace $L \subseteq \mathbb{R}^n$ with $\dim L \leq 2$ to satisfy the Strong Arnol'd Hypothesis with respect to a graph G, and we obtain a sufficient condition for a linear subspace $L \subseteq \mathbb{R}^n$ with $\dim L = 3$ to satisfy the Strong Arnol'd Hypothesis with respect to a graph G. We apply these results to show that if $G = (V, E)$ with $V = \{1, 2, \dots, n\}$ is a path, 2-connected outerplanar, or 3-connected planar, then each real symmetric $n \times n$ matrix $M = [m_{i,j}]$ with $m_{i,j} < 0$ if $ij \in E$ and $m_{i,j} = 0$ if $i \neq j$ and $ij \notin E$ (and no restriction on the diagonal), having exactly one negative eigenvalue, satisfies the Strong Arnol'd Hypothesis.

Key words. Symmetric matrices, Nullity, Graphs, Transversality, Planar, Outerplanar, Graph minor.

AMS subject classifications. 05C50, 15A18.

1. Introduction. In the definition of the graph parameters $\mu(G)$ and $\nu(G)$, introduced by Colin de Verdière in respectively [2, 3] and [4], and in the definition of the graph parameter $\xi(G)$, introduced by Barioli, Fallat, and Hogben in [1], a transversality condition is used, called the Strong Arnol'd Hypothesis. The addition of this Strong Arnol'd Hypothesis allows to show the minor-monotonicity of these graph parameters. For example, $\mu(G') \leq \mu(G)$ if G' is a minor of G; we refer to Diestel [5] for the notions used in graph theory. It is this minor-monotonicity that makes these graph parameters so useful.

Let us first recall the definition of the Strong Arnol'd Hypothesis. For a graph $G = (V, E)$ with vertex set $V = \{1, 2, \dots, n\}$, denote by $\mathcal{S}(G)$ the set of all real symmetric $n \times n$ matrices $M = [m_{i,j}]$ with

$$m_{i,j} \neq 0,\ i \neq j \quad \Leftrightarrow \quad ij \in E.$$

The tangent space, $T_M\mathcal{S}(G)$, of $\mathcal{S}(G)$ at M is the space of all real symmetric $n \times n$

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matrices $A = [a_{i,j}]$ with $a_{i,j} = 0$ if $i \neq j$ and i and j are nonadjacent. Denote by $\mathcal{R}_{n,k}$ the manifold of all real symmetric $n \times n$ matrices of nullity k. The tangent space, $T_M\mathcal{R}_{n,k}$, of $\mathcal{R}_{n,k}$ at M is the space of all real symmetric $n \times n$ matrices $B = [b_{i,j}]$ such that $x^T Bx = 0$ for all $x \in \ker(M)$. Here, $\ker(M)$ denotes the null space of M. A matrix $M \in \mathcal{S}(G)$ satisfies the *Strong Arnol'd Hypothesis* if the sum of $T_M\mathcal{S}(G)$ and $T_A\mathcal{R}_{n,k}$ equals the space of all real symmetric $n \times n$ matrices. So, a matrix $M \in \mathcal{S}(G)$ satisfies the Strong Arnol'd Hypothesis if and only if for each real symmetric $n \times n$ matrix B, there is a real symmetric matrix $A = [a_{i,j}]$ with $a_{i,j} = 0$ if $i \neq j$ and i and j nonadjacent, such that $x^T Bx = x^T Ax$ for each $x \in \ker(M)$.

Although stated above as a condition on the matrix M, it can be viewed as a condition on $\ker(M)$. In this paper, we extend the definition of the Strong Arnol'd Hypothesis to linear subspaces $L \subseteq \mathbb{R}^n$ with respect to a graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$. We give a necessary and sufficient condition for a linear subspace $L \subseteq \mathbb{R}^n$ with $\dim L \leq 2$ to satisfy the Strong Arnol'd Hypothesis with respect to a graph G, and we obtain a sufficient condition for a linear subspace $L \subseteq \mathbb{R}^n$ with $\dim L = 3$ to satisfy the Strong Arnol'd Hypothesis with respect to a graph G.

For a graph $G = (V, E)$, let $\mathcal{O}(G)$ be the set of all $M = [m_{i,j}] \in \mathcal{S}(G)$ such that $m_{i,j} < 0$ for each adjacent pair of vertices i and j. Notice that for a matrix $M \in \mathcal{O}(G)$ with exactly one negative eigenvalue, the tangent space of $\mathcal{O}(G)$ at M is the same as the tangent space of $\mathcal{S}(G)$ at M. The parameter $\mu(G)$ is defined as the largest nullity of any $M = [m_{i,j}] \in \mathcal{O}(G)$ such that M has exactly one negative eigenvalue and satisfies the Strong Arnol'd Hypothesis. This graph parameter characterizes outerplanar graphs as those graphs G for which $\mu(G) \leq 2$, and planar graphs as those graphs G for which $\mu(G) \leq 3$; see van der Holst, Lovász, and Schrijver [9] for an introduction to this graph parameter. We show that in certain cases each $M \in \mathcal{O}(G)$ with exactly one negative eigenvalue (automatically) satisfies the Strong Arnol'd Hypothesis. More precisely, if G is a path, 2-connected outerplanar, or 3-connected planar, then each $M \in \mathcal{O}(G)$ with exactly one negative eigenvalue satisfies the Strong Arnol'd Hypothesis.

2. The Strong Arnol'd Property for linear subspaces. A *representation* of linearly independent vectors $x_1, x_2, \ldots, x_r \in \mathbb{R}^n$ is a function $\phi : \{1, 2, \ldots, n\} \to \mathbb{R}^r$ such that

$$\begin{bmatrix} \phi(1) & \phi(2) & \ldots & \phi(n) \end{bmatrix} = \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_r^T \end{bmatrix}.$$

A *representation* of a linear subspace L of $\mathbb{R}^n$ is a representation of some basis of L.

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Let $\phi : \{1, 2, \ldots, n\} \to \mathbb{R}^r$ be a representation of a basis $x_1, x_2, \ldots, x_r$ of a linear subspace L of $\mathbb{R}^n$, and let $G = (V, E)$ be a graph with vertex set $V = \{1, 2, \ldots, n\}$. If A is a nonsingular $r \times r$ matrix and the linear span of the symmetric $r \times r$ matrices $\phi(i)\phi(i)^T$, $i \in V$, and $\phi(i)\phi(j)^T + \phi(j)\phi(i)^T$, $ij \in E$, is equal to the space of all symmetric $r \times r$ matrices, then the same holds for the linear span of $A\phi(i)\phi(i)^T A^T$, $i \in V$, and $A\phi(i)\phi(j)^T A^T + A\phi(j)\phi(i)^T A^T$, $ij \in E$. This suggests to define the following property for linear subspaces of $\mathbb{R}^n$.

An r-dimensional linear subspace L of $\mathbb{R}^n$ satisfies the *Strong Arnol'd Hypothesis* with respect to G if for any representation $\phi : \{1, 2, \ldots, n\} \to \mathbb{R}^r$ of a basis of L, the linear span of all matrices of the form $\phi(i)\phi(i)^T$, $i \in V$, and $\phi(i)\phi(j)^T + \phi(j)\phi(i)^T$, $ij \in E$, is equal to the space of all symmetric $r \times r$ matrices. Equivalently, an r-dimensional linear subspace L of $\mathbb{R}^n$ satisfies the Strong Arnol'd Hypothesis if the $r \times r$ all-zero matrix is the only symmetric $r \times r$ matrix N such that $\phi(i)^T N\phi(j) = 0$, $ij \in E$, and $\phi(i)^T N\phi(i) = 0$, $i \in V$. If it is clear what graph G we are dealing with, we only write that L satisfies the Strong Arnol'd Hypothesis, omitting the part with respect to G.

The next lemma shows why we call this property the Strong Arnol'd Hypothesis.

Lemma 2.1. *Let $G = (V, E)$ be a graph with vertex set $V = \{1, 2, \ldots, n\}$. A matrix $M \in \mathcal{S}(G)$ has the Strong Arnol'd Hypothesis if and only if* $\ker(M)$ *has the Strong Arnol'd Hypothesis.*

Proof. Choose a basis $x_1, x_2, \ldots, x_r$ of $\ker(M)$, and let ϕ be a representation of $x_1, x_2, \ldots, x_r$.

M satisfies the Strong Arnol'd Hypothesis if and only if for every symmetric $n \times n$ matrices A, there is a symmetric $n \times n$ matrix $B = [b_{i,j}]$ with $b_{i,j} = 0$ if $i \neq j$ and i and j are nonadjacent, such that for all $x \in \ker(M)$, $x^T Ax = x^T Bx$. Hence, M has the Strong Arnol'd Hypothesis if and only if for every symmetric $r \times r$ matrices C, there is a symmetric $n \times n$ matrix $B = [b_{i,j}]$ with $b_{i,j} = 0$ if $i \neq j$ and i and j are nonadjacent, such that

$$C = \begin{bmatrix} x_1 & \ldots & x_r \end{bmatrix}^T B \begin{bmatrix} x_1 & \ldots & x_r \end{bmatrix}.$$

This is equivalent to: M has the Strong Arnol'd Hypothesis if and only if the linear span of all matrices of the form $\phi(i)\phi(i)^T$, $i \in V$, and $\phi(i)\phi(j)^T + \phi(j)\phi(i)^T$, $ij \in E$, is equal to the space of all symmetric $r \times r$ matrices. ∎

Let $G = (V, E)$ be a graph. For $S \subseteq V$, we denote by $N(S)$ the set of all vertices in $V \setminus S$ adjacent to a vertex in S, and we denote by $G[S]$ the subgraph induced by S. For $x \in \mathbb{R}^n$, we denote $\mathrm{supp}(x) = \{i \mid x_i \neq 0\}$. Two subsets of the vertex set or two subgraphs of a graph *touch* if they have common vertex or are adjacent. If two

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subsets of the vertex set or two subgraphs of a graph do not touch, then we say that they are *separated*.

LEMMA 2.2. *Let L be a linear space of $\mathbb{R}^n$ of dimension r and let $\phi : V \to \mathbb{R}^r$ be a representation of the basis $x_1, x_2, \ldots, x_r$ of L. Then there is a symmetric $r \times r$ matrix $N = [n_{i,j}]$ with $n_{1,2} = n_{2,1} = 1$ and $n_{i,j} = 0$ elsewhere, such that $\phi(i)^T N \phi(i) = 0$ for all $i \in V$ and $\phi(i)^T N \phi(j) = 0$ for all $ij \in E$ if and only if $\operatorname{supp}(x_1)$ and $\operatorname{supp}(x_2)$ are separated.*

Proof. It is easily checked that $\phi(i)^T N \phi(i) = 0$ for all $i \in V$ and $\phi(i)^T N \phi(j) = 0$ for all $ij \in E$ if $\operatorname{supp}(x_1)$ and $\operatorname{supp}(x_2)$ are separated.

Conversely, from $\phi(i)^T N \phi(i) = 0$, $i \in V$, it follows that $\operatorname{supp}(x_1)$ and $\operatorname{supp}(x_2)$ have no common vertex, and from $\phi(i)^T N \phi(j) = 0$, $ij \in E$, it follows that $\operatorname{supp}(x_1)$ and $\operatorname{supp}(x_2)$ are not adjacent. Hence, $\operatorname{supp}(x_1)$ and $\operatorname{supp}(x_2)$ are separated. □

If a linear subspace $L \subseteq \mathbb{R}^n$ has $\dim L \leq 2$, then the following theorem gives a sufficient and necessary condition for L to satisfy the Strong Arnol'd Hypothesis with respect to G.

THEOREM 2.3. *Let $G = (V, E)$ be a graph with vertex set $V = \{1, 2, \ldots, n\}$ and let $k \leq 2$. A k-dimensional linear subspace L of $\mathbb{R}^n$ does not satisfy the Strong Arnol'd Hypothesis if and only if there are nonzero vectors $x_1, x_2 \in L$ such that $\operatorname{supp}(x_1)$ and $\operatorname{supp}(x_2)$ are separated.*

Proof. $k = 1$. This is easy as every 1-dimensional linear subspace L satisfies the Strong Arnol'd Hypothesis, and there are no two nonzero vectors $x_1, x_2 \in L$ such that $\operatorname{supp}(x_1)$ and $\operatorname{supp}(x_2)$ are separated.

$k = 2$. If there are nonzero vectors $x_1, x_2 \in L$ for which $\operatorname{supp}(x_1)$ and $\operatorname{supp}(x_2)$ are separated, then L does not satisfy the Strong Arnol'd Hypothesis, by Lemma 2.2.

Conversely, suppose that L does not satisfy the Strong Arnol'd Hypothesis. Since L has dimension 2, we can find two vertices u and v and a basis x, z of L with $x_u = 1, z_u = 0$ and $x_v = 0, z_v = 1$. Let $\phi : V \to \mathbb{R}^2$ be a representation of x, z. As L does not satisfy the Strong Arnol'd Hypothesis, there is a nonzero symmetric 2×2 matrix $N = [n_{i,j}]$ such that $\phi(i)^T N \phi(i) = 0$ for all $i \in V$ and $\phi(i)^T N \phi(j) = 0$ for all $ij \in E$. In particular, since $\phi(u) = [1, 0]^T$ and $\phi(v) = [0, 1]^T$, $n_{1,1} = n_{2,2} = 0$. Hence, by Lemma 2.2, $\operatorname{supp}(x_1)$ and $\operatorname{supp}(x_2)$ are separated. □

Theorem 2.3 need not hold when $\dim L = 3$, as the following example shows. Let $G = (V, E)$ be the graph with $V = \{1, 2, \ldots, 5\}$ and $E = \emptyset$, and let L be the linear

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subspace of $\mathbb{R}^5$ spanned by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}.$$

Every two nonzero vectors $x_1, x_2 \in L$ touch, but L does not satisfy the Strong Arnol'd Hypothesis, as can be easily verified.

If a linear subspace $L \subseteq \mathbb{R}^n$ has $\dim L = 3$, then the following theorem gives a sufficient condition for L to satisfy the Strong Arnol'd Hypothesis.

Theorem 2.4. *Let $G = (V, E)$ be a graph with vertex set $V = \{1, 2, \ldots, n\}$, and let L be a linear subspace of $\mathbb{R}^n$ with $\dim L = 3$. Let $\phi : V \to \mathbb{R}^3$ be a representation of L. If there are adjacent vertices u and v in G such that $\phi(u)$ and $\phi(v)$ are linearly independent, and there are no nonzero vectors $x_1, x_2 \in L$ such that $\operatorname{supp}(x_1)$ and $\operatorname{supp}(x_2)$ are separated, then L satisfies the Strong Arnol'd Hypothesis.*

Proof. For the sake of contradiction, assume that L does not satisfy the Strong Arnol'd Hypothesis. Then there is a nonzero symmetric 3×3 matrix $N = [n_{i,j}]$ such that $\phi(i)^T N \phi(i) = 0$ for all $i \in V$ and $\phi(i)^T N \phi(j) = 0$ for all $ij \in E$. There exists a nonsingular matrix A such that $A^T N A$ is a diagonal matrix in which each of the diagonal entries belongs to $\{-1, 0, 1\}$. Thus, by multiplying ϕ with A we may assume that N is a diagonal matrix and that its diagonal entries belongs to $\{-1, 0, 1\}$. We will now show that each of the elements in $\{-1, 0, 1\}$ occurs as a diagonal entry.

Suppose that 0 occurs twice as a diagonal entry; without loss of generality, we may assume that $n_{2,2} = n_{3,3} = 0$. Since the dimension of L is three, there exists a vertex v for which the first coordinate of $\phi(v)$ is nonzero. Then $\phi(v)^T N \phi(v) \neq 0$, contradicting that $\phi(i)^T N \phi(i) = 0$ for all $i \in V$.

Suppose that 1 occurs twice as a diagonal entry; without loss of generality, we may assume that $n_{2,2} = n_{3,3} = 1$. Since $\phi(u)$ and $\phi(v)$ are linearly independent, there exists a linear combination $z = a\phi(u) + b\phi(v)$ for which the first coordinate equals 0. Then $0 \neq z^T N z = a^2 \phi(u)^T N \phi(u) + 2ab\phi(u)^T N \phi(v) + b^2 \phi(v)^T N \phi(v)$. Since $\phi(u)^T N \phi(u) = 0$, $\phi(v)^T N \phi(v) = 0$, and $\phi(u)^T N \phi(v) = 0$, we obtain a contradiction. The case where -1 occurs twice is analogous.

Hence, each of the elements in $\{-1, 0, 1\}$ occurs as a diagonal entry; we may assume that

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

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We now define $\psi : V \to \mathbb{R}^3$ by $\psi(i) = B\phi(i)$ for $i \in V$, where

$$B = \begin{bmatrix} 1 & -1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then ψ is a representation of L such that if

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then $\psi(i)^T Q\psi(i) = 0$ for all $i \in V$ and $\psi(i)^T Q\psi(j) = 0$ for all $ij \in E$. By Lemma 2.2, there are nonzero vectors $x_1, x_2 \in L$ such that $\mathrm{supp}(x_1)$ and $\mathrm{supp}(x_2)$ are separated, contradicting the assumption. Hence, L satisfies the Strong Arnol'd Hypothesis. □

LEMMA 2.5. *Let $G = (V, E)$ be a graph. Let $\phi : V \to \mathbb{R}^3$ be a representation of a linear subspace L of $\mathbb{R}^n$ with $\dim L = 3$. If there are nonzero vectors $x_1, x_2 \in L$ for which there are touching components C_1 and C_2 of $G[\mathrm{supp}(x_1)]$ and $G[\mathrm{supp}(x_2)]$, respectively, with $C_1 \neq C_2$, then there are adjacent vertices u and v such that $\phi(u)$ and $\phi(v)$ are independent.*

Proof. The vectors x_1, x_2 are clearly linearly independent. Let x_3 be a vector in L such that x_1, x_2, x_3 form a basis of L, and let $\psi : V \to \mathbb{R}^3$ be a representation of x_1, x_2, x_3. If for adjacent vertices u and v, $\psi(u)$ and $\psi(v)$ are linearly independent, then also $\phi(u)$ and $\phi(v)$ are linearly independent.

If C_1 and C_2 have no vertex in common, then they must be joined by an edge uv. As a consequence, $\psi(u)$ and $\psi(v)$ are linear independent, and so $\phi(u)$ and $\phi(v)$ are linearly independent.

We may therefore assume that C_1 and C_2 have a vertex c in common. Since $C_1 \neq C_2$, $V(C_1)\Delta V(C_2) \neq \emptyset$; choose a vertex d from $V(C_1)\Delta V(C_2)$. By symmetry, we may assume that $d \in V(C_1)$ and $d \notin V(C_2)$. Since C_1 and C_2 are connected, there is a path in C_1 connecting c and d. On this path there is an edge uv such that $u \in V(C_1)$, $u \notin V(C_2)$ and $v \in V(C_1)$, $v \in V(C_2)$. Then $\psi(u)$ and $\psi(v)$ are linear independent. Hence, $\phi(u)$ and $\phi(v)$ are linearly independent. □

Using Theorem 2.4 and Lemma 2.5, we obtain:

THEOREM 2.6. *Let $G = (V, E)$ be a graph. Let $\phi : V \to \mathbb{R}^3$ be a representation of a linear subspace L of $\mathbb{R}^n$ with $\dim L = 3$. If there are nonzero vectors $x_1, x_2 \in L$ for which there are touching components C_1 and C_2 of $G[\mathrm{supp}(x_1)]$ and $G[\mathrm{supp}(x_2)]$, respectively, with $C_1 \neq C_2$, then L satisfies the Strong Arnol'd Hypothesis.*

LEMMA 2.7. *Let $G = (V, E)$ be a graph with vertex set $V = \{1, 2, \ldots, n\}$, and let L be a linear subspace of $\mathbb{R}^n$ with $\dim L \leq 3$, which has a nonzero vector x such that*

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$G[\mathrm{supp}(x)]$ is connected. If L does not satisfy the Strong Arnol'd Hypothesis, then there exists a nonzero vector $y \in L$ such that $\mathrm{supp}(x)$ and $\mathrm{supp}(y)$ are separated.

Proof. If each nonzero vector $y \in L$ satisfies $\mathrm{supp}(y) = \mathrm{supp}(x)$, then L is 1-dimensional; each 1-dimensional linear subspace L of $\mathbb{R}^n$ satisfies the Strong Arnol'd Hypothesis.

Thus, there exists a nonzero vector $y \in L$ such that $\mathrm{supp}(y) \neq \mathrm{supp}(x)$. We may assume that $\mathrm{supp}(x)$ and $\mathrm{supp}(y)$ touch, for otherwise $\mathrm{supp}(x)$ and $\mathrm{supp}(y)$ are separated. Hence, there is a component C of $G[\mathrm{supp}(y)]$ such that $G[\mathrm{supp}(x)]$ and C touch. If $C \neq G[\mathrm{supp}(x)]$, then L would satisfy the Strong Arnol'd Hypothesis by Theorem 2.6. This contradiction shows that $C = G[\mathrm{supp}(x)]$. Now choose a vertex $v \in \mathrm{supp}(x)$. There exists a scalar α such that $z = \alpha x + y$ satisfies $z_v = 0$. If there is a vertex $w \in \mathrm{supp}(x)$ such that $z_w \neq 0$, then there is a component D of $G[\mathrm{supp}(z)]$ such that D and $G[\mathrm{supp}(x)]$ touch and $D \neq G[\mathrm{supp}(x)]$. By Theorem 2.6, L would satisfy the Strong Arnol'd Hypothesis. This contradiction shows that $z_u = 0$ for all $u \in G[\mathrm{supp}(x)]$. Then $\mathrm{supp}(x)$ and $\mathrm{supp}(z)$ are separated. $\square$

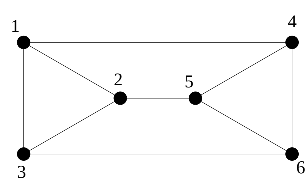

Fig. 2.1. *Complement of C_6.*

In Theorem 2.4, the restriction $k \leq 3$ cannot be removed. For $k = 4$, there is the following example. Let $G = (V, E)$ be the complement of the 6-cycle C_6, which is the graph with $V = \{1, 2, \ldots, 6\}$ obtained from taking two disjoint triangles and connecting each vertex of one triangle to a vertex of the other triangle by an edge in a one-to-one way; see Figure 2.1. Let L be generated by the columns of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

and, for $i \in V$, let $\phi(i)$ be the ith column of A^T. Then for every vector $x \in L$, $\mathrm{supp}(x)$ induces a connected subgraph of G, and hence, for every two vectors $x_1, x_2 \in L$, $\mathrm{supp}(x_1)$ and $\mathrm{supp}(x_2)$ touch. But L does not satisfy the Strong Arnol'd Hypothesis,

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as $\phi(i)^T Q\phi(i) = 0$ for $i \in V$ and $\phi(i)^T Q\phi(j) = 0$ for $ij \in E$ if

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{2.1}$$

However, this is essentially the only type of matrix that can occur as we will see in the next result.

THEOREM 2.8. *Let $G = (V, E)$ be a graph with vertex set $V = \{1, \ldots, n\}$, and let L be a linear subspace of $\mathbb{R}^n$ with $\dim L = 4$. Let $\phi : V \to \mathbb{R}^4$ be a representation of L. Suppose L has the following properties:*

1. *L does not satisfy the Strong Arnol'd Hypothesis,*
2. *there are adjacent vertices u and w in G such that $\phi(u)$ and $\phi(w)$ are linearly independent, and*
3. *there are no nonzero vectors $x_1, x_2 \in L$ such that $\mathrm{supp}(x_1)$ and $\mathrm{supp}(x_2)$ are separated.*

Then there is a representation $\psi : V \to \mathbb{R}^4$ of L such that if

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

then $\psi(i)^T Q\psi(i) = 0$ for all $i \in V$ and $\psi(i)^T Q\psi(j) = 0$ for all $ij \in E$.

Proof. Since L does not satisfy the Strong Arnol'd Hypothesis, there is a nonzero symmetric 4×4 matrix $N = [n_{i,j}]$ such that $\phi(v)^T N\phi(v) = 0$ for each $v \in V$ and $\phi(v)^T N\phi(w) = 0$ for each $vw \in E$. By multiplying ϕ with a nonsingular 4×4 matrix A, we may assume that N is a diagonal matrix and that each of its diagonal entries belongs to $\{-1, 0, 1\}$.

Suppose first that three of the diagonal entries are equal to zero; without loss of generality, we may assume that $n_{1,1} = n_{2,2} = n_{3,3} = 0$. Since $\dim L = 4$, there exists a vertex v such that the last coordinate of $\phi(v)$ is nonzero. Then $\phi(v)^T N\phi(v) \neq 0$. This contradiction shows that at most two of the diagonal entries are equal to zero.

Suppose next that two of the diagonal entries are equal to zero; without loss of generality, we may assume that $n_{1,1} = n_{2,2} = 0$. If $n_{3,3} = n_{4,4}$, then $\phi(v)^T N\phi(v) \neq 0$.

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Hence, $n_{3,3} = -n_{4,4}$; we may assume that $n_{3,3} = 1$. Taking

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & -1 & \frac{1}{2} \end{bmatrix},$$

we obtain

$$A^T N A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Let $\psi : V \to \mathbb{R}^4$ be defined by $\psi(i) = A^{-1}\phi(i)$ for $i = 1, 2, \ldots, n$. Then, by Lemma 2.2, there exist vectors $x_1, x_2 \in L$ such that $\text{supp}(x_1)$ and $\text{supp}(x_2)$ are separated, contradicting the assumption.

Suppose next that exactly one of the diagonal entries is equal to zero; without loss of generality, we may assume that $n_{4,4} = 0$. Each of the other diagonal entries is -1 or 1. Let z_1, z_2, z_3, z_4 be the basis corresponding to ϕ and let ψ be the representation corresponding to z_1, z_2, z_3. If $R = [r_{i,j}]$ is the diagonal matrix defined by $r_{j,j} = n_{j,j}$ for $j = 1, 2, 3$, then $\psi(v)^T R\psi(v) = 0$ for all $v \in V$ and $\psi(v)^T R\psi(w) = 0$ for all $vw \in E$. By Theorem 2.4, there exist vectors y_1, y_2 in the linear span of z_1, z_2, z_3 such that $\text{supp}(y_1)$ and $\text{supp}(y_2)$ are separated. This contradiction shows that all diagonal are nonzero.

If the diagonal entries are all 1 or all -1, then $\phi(v)^T N\phi(v) \neq 0$ if $\phi(v) \neq 0$. Suppose three of the diagonal entries are 1 and one of them is -1; without loss of generality, we may assume that $n_{1,1} = -1$ and $n_{i,i} = 1$ for $i = 2, 3, 4$. Let uw be an edge in G such $\phi(u)$ and $\phi(w)$ are linearly independent. Let $a, b \in \mathbb{R}$ be such that $a\phi(u) + b\phi(w)$ is a vector in $\mathbb{R}^n$ whose first coordinate is equal to 0. Since $\phi(u)^T N\phi(u) = 0$, $\phi(w)^T N\phi(w) = 0$, and $\phi(u)^T N\phi(w) = 0$, $(a\phi(u) + b\phi(w))^T N(a\phi(u) + b\phi(w)) = 0$. However, since $n_{i,i} = 1$ for $i = 2, 3, 4$ and the first coordinate of $a\phi(u) + b\phi(w)$ equals 0, $(a\phi(w) + b\phi(w))^T N(a\phi(u) + b\phi(w)) \neq 0$; a contradiction. The case where three of the diagonal entries are -1 and one of them is 1 is similar. Thus, two of the diagonal entries are -1 and two of the diagonal entries are 1; we may assume that $n_{1,1} = n_{2,2} = 1$ and $n_{3,3} = n_{4,4} = -1$. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ -1 & 0 & 0 & \frac{1}{2} \end{bmatrix};$$

then $A^T N A = Q$. Defining $\psi : V \to \mathbb{R}^4$ by $\psi(i) = A^{-1}\phi(i)$ for $i = 1, 2, \ldots, n$, we obtain that $\psi(i)^T Q\psi(i) = 0$ for all $i \in V$ and $\psi(i)^T Q\psi(j) = 0$ for all $ij \in E$. $\square$

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3. The parameter $\mu(G)$ and the Strong Arnol'd Hypothesis. In this section we apply Theorems 2.3 and 2.4 to show that if G is a path, 2-connected outerplanar, or 3-connected planar, then each matrix in $\mathcal{O}(G)$ with exactly one negative eigenvalue satisfies the Strong Arnol'd Hypothesis. Different proofs can be found in [8].

For $x \in \mathbb{R}^n$, we denote $\mathrm{supp}_-(x) = \{i \mid x_i < 0\}$ and $\mathrm{supp}_+(x) = \{i \mid x_i > 0\}$. If $G = (V, E)$ is a connected graph with $V = \{1, 2, \ldots, n\}$, then the Perron-Frobenius Theorem says that each eigenvector z belonging to the smallest eigenvalue of $M \in \mathcal{O}(G)$ has multiplicity 1 and satisfies $z > 0$ or $z < 0$. Since any $x \in \ker(M)$ is orthogonal to z, $\mathrm{supp}_+(x) \neq \emptyset$ and $\mathrm{supp}_-(x) \neq \emptyset$ for every nonzero $x \in \ker(M)$.

LEMMA 3.1. [9, Theorem 2.17 (v)] *Let G be a connected graph and let $M \in \mathcal{O}(G)$ with exactly one negative eigenvalue. Let $x \in \ker(M)$ be such that $G[\mathrm{supp}_+(x)]$ or $G[\mathrm{supp}_-(x)]$ has at least two components. Then there is no edge connecting $\mathrm{supp}_+(x)$ and $\mathrm{supp}_-(x)$ and $N(K) = N(\mathrm{supp}(x))$ for each component K of $G[\mathrm{supp}(x)]$.*

LEMMA 3.2. *Let $G = (V, E)$ be a graph and let $M \in \mathcal{O}(G)$ with exactly one negative eigenvalue. If M has nullity at most three and there exists a nonzero $x \in \ker(M)$ such that $\mathrm{supp}(x)$ induces a connected subgraph of G, then M satisfies the Strong Arnol'd Hypothesis.*

Proof. For the sake of contradiction, assume that there is an $M \in \mathcal{O}(G)$ that does not satisfy the Strong Arnol'd Hypothesis. By Lemma 2.7, there exists a nonzero vector $y \in \ker(M)$ such that $\mathrm{supp}(x)$ and $\mathrm{supp}(y)$ are separated. The vector $z = x+y$ has the property that $G[\mathrm{supp}_+(z)]$ and $G[\mathrm{supp}_-(z)]$ are disconnected. By Lemma 3.1, $N(C) = N(\mathrm{supp}(z))$ for each component C in $G[\mathrm{supp}_-(z)] \cup G[\mathrm{supp}_+(z)]$ and there is no edge between $\mathrm{supp}_-(z)$ and $\mathrm{supp}_+(z)$. However, this would mean that $G[\mathrm{supp}_-(x)]$ and $G[\mathrm{supp}_+(x)]$ are separated, contradicting the connectedness of $G[\mathrm{supp}(x)]$. ☐

For a graph $G = (V, E)$ and an $S \subseteq V$, we denote by $G - S$ the subgraph of G induced by the vertices in $V \setminus S$.

THEOREM 3.3. *Let $G = (V, E)$ be a graph which has no vertex cut S such that $G - S$ has at least four components, each of which is adjacent to every vertex in S. Then every $M \in \mathcal{O}(G)$ with nullity at most three and with exactly one negative eigenvalue satisfies the Strong Arnol'd Hypothesis.*

Proof. For the sake of contradiction, assume that there is an $M \in \mathcal{O}(G)$ that does not satisfy the Strong Arnol'd Hypothesis.

By Lemma 3.2, $G[\mathrm{supp}(x)]$ is disconnected for each nonzero $x \in \ker(M)$. For every $x \in \ker(M)$, there are at most three components in $G[\mathrm{supp}(x)]$, by assumption and by Lemma 3.1. By Theorem 2.6, for every nonzero vectors $x, y \in \ker(M)$, any

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component C of $G[\mathrm{supp}(x)]$ and any component D of $G[\mathrm{supp}(y)]$, either $C = D$, or C and D are separated, for otherwise M would satisfy the Strong Arnol'd Hypothesis. Hence, we can conclude that there are at most three mutually disjoint connected subgraphs K_1, K_2, K_3 of G such that for every $x \in \ker(M)$, $G[\mathrm{supp}_+(x)]$ can be written as the union of some of K_1, K_2, K_3. We now show that $\ker(M)$ has dimension at most two.

For any $x \in \ker(M)$ and any K_i, $M_{K_i} x_{K_i} = 0$, and hence, by the Perron-Frobenius Theorem, $x_{K_i} < 0$, $x_{K_i} = 0$, or $x_{K_i} > 0$. Furthermore, the eigenvalue 0 has multiplicity 1 in M_{K_i}. Let z be a positive eigenvector belonging to the negative eigenvalue of M. Since $x^T z$ for any $x \in \ker(M)$, $\ker(M)$ has dimension at most two. If M does not satisfy the Strong Arnol'd Hypothesis, then, by Theorem 2.3, there are two nonzero vectors $x, y \in \ker(M)$ such that $G[\mathrm{supp}(x)]$ and $G[\mathrm{supp}(y)]$ are separated. Let $w = x + y$. Since $G[\mathrm{supp}(x)]$ and $G[\mathrm{supp}(y)]$ are disconnected, $G[\mathrm{supp}(w)]$ consists of at least four components. This contradicts the assumption in the theorem. $\square$

For a matrix M, we denote by *nullity*(M) the nullity of M.

Corollary 3.4. *Let $G = (V, E)$ be a graph and let $M \in \mathcal{O}(G)$ have $k :=$ nullity$(M) \leq 3$. If G has no $K_{4,k}$-minor, then M satisfies the Strong Arnol'd Hypothesis.*

We use this corollary to show that if G is a path, 2-connected outerplanar, or 3-connected planar, then each matrix in $\mathcal{O}(G)$ with exactly one negative eigenvalue satisfies the Strong Arnol'd Hypothesis.

Theorem 3.5. [6] *If G is a path, then each $M \in \mathcal{O}(G)$ has nullity$(M) \leq 1$.*

Since each 1-dimensional linear subspace $L \subseteq \mathbb{R}^n$ satisfies the Strong Arnol'd Hypothesis, we obtain:

Corollary 3.6. *If G is a path, then every matrix in $\mathcal{O}(G)$ satisfies the Strong Arnol'd Hypothesis.*

A graph G is *outerplanar* if it has an embedding in the plane such that each vertex is incident with the infinite face. Outerplanar graphs can be characterized as those graphs that have no K_4- or $K_{2,3}$-minor.

Theorem 3.7. [7, Corollary 13.10.4] *Let G be a graph and let $M \in \mathcal{O}(G)$ with exactly one negative eigenvalue. If G is 2-connected outerplanar, then nullity$(M) \leq 2$.*

Corollary 3.8. *Let G be a 2-connected outerplanar graph. Then every matrix in $\mathcal{O}(G)$ with exactly one negative eigenvalue satisfies the Strong Arnol'd Hypothesis.*

Planar graphs can be characterized as those graphs that have no K_5- or $K_{3,3}$-

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minor.

THEOREM 3.9. [7, Corollary 13.10.2] *Let G be a graph and let $M \in \mathcal{O}(G)$ with exactly one negative eigenvalue. If G is 3-connected planar, then nullity$(M) \leq 3$.*

COROLLARY 3.10. *Let G be a 3-connected planar graph. Then every matrix in $\mathcal{O}(G)$ with exactly one negative eigenvalue satisfies the Strong Arnol'd Hypothesis.*

An embedding of a graph in 3-space is linkless if each pair of disjoint circuits has zero linking number under the embedding; see Robertson, Seymour, and Thomas [10]. In the same paper they characterized graphs that have a linkless embedding as those graphs that have no minor isomorphic to a graph in the Petersen family, a family of seven graphs, one of which is the Petersen graph. We conclude with a conjecture.

CONJECTURE 3.11. *Let G be a 4-connected graph that has a linkless embedding. Then every matrix in $\mathcal{O}(G)$ with exactly one negative eigenvalue satisfies the Strong Arnol'd Hypothesis.*

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