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EXTREME RANKS OF (SKEW-)HERMITIAN SOLUTIONS TO A QUATERNION MATRIX EQUATION*

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Abstract. The extreme ranks, i.e., the maximal and minimal ranks, are established for the general Hermitian solution as well as the general skew-Hermitian solution to the classical matrix equation $AXA^* + BYB^* = C$ over the quaternion algebra. Also given in this paper are the formulas of extreme ranks of real matrices $X_i,\ Y_i,\ i = 1, \cdots, 4$, in a pair (skew-)Hermitian solution $X = X_1 + X_2 i + X_3 j + X_4 k$, $Y = Y_1 + Y_2 i + Y_3 j + Y_4 k$. Moreover, the necessary and sufficient conditions for the existence of a real (skew-)symmetric solution, a complex (skew-)Hermitian solution, and a pure imaginary (skew-)Hermitian solution to the matrix equation mentioned above are presented in this paper. Also established are expressions of such solutions to the equation when corresponding solvability conditions are satisfied. The findings of this paper widely extend the known results in the literature.

Key words. Quaternion matrix equation, Minimal rank, Maximal rank, Hermitian solution, Skew-Hermitian solution.

AMS subject classifications. 15A03, 15A09, 15A24, 15A33.

1. Introduction.

Throughout this paper, we denote the real number field by $\mathbb{R}$, the complex number field by $\mathbb{C}$, the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \mid i^2 = j^2 = k^2 = ijk = -1,\ a_0, a_1, a_2, a_3 \ \in \mathbb{R}\}$$

by $\mathbb{H}^{m \times n}$, the identity matrix with the appropriate size by I. For a quaternion matrix A, we denote the column right space, the row left space of A by $\mathcal{R}(A)$, $\mathcal{N}(A)$, respectively, the dimension of $\mathcal{R}(A)$ by $\dim \mathcal{R}(A)$. By [9], $\dim \mathcal{R}(A) = \dim \mathcal{N}(A)$, which is called the rank of A, and denoted by $r(A)$. The Moore-Penrose inverse of a matrix $A \in \mathbb{H}^{m \times n}$, denoted by $A^\dagger$, is defined to be the unique matrix $X \in \mathbb{H}^{n \times m}$ satisfying

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the following four matrix equations

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (AX)^* = AX,\ (4)\ (XA)^* = XA.$$

Moreover, L_A and R_A stand for the two projectors $L_A = I - A^\dagger A$, $R_A = I - AA^\dagger$ induced by A.

We know that matrix equation is one of the topics of very active research in matrix theory and applications, and a large number of papers have presented several methods for solving several matrix equations (e.g. [4]-[6], [10], [23]-[26], [29]-[31], [37], [39], [42]). As a very classical linear matrix equation,

$$AXA^* + BYB^* = C \tag{1.1}$$

has been investigated by many authors from different aspects. For example, using generalized singular value decomposition, Chang and Wang [2] derived the expressions for the general symmetric and minimum-2-norm symmetric solutions to (1.1) within the real setting. Xu, Wei, and Zheng [38] obtained the general form of all least-squares Hermitian (skew-Hermitian) solutions to (1.1). Liao and Bai [11] used generalized singular value decomposition to investigate the symmetric positive semidefinite solution to (1.1). Zhang [40] gave necessary and sufficient conditions for the existence of a Hermitian nonnegative-definite solution to (1.1). Wang and Zhang [32] gave a necessary and sufficient condition for the existence and an expression for the re-nonnegative definite solution to (1.1) over $\mathbb{H}$ by using the decomposition of pairwise matrices.

Research on extreme ranks, i.e., maximal and minimal ranks, of solutions to linear matrix equations have been actively ongoing for more than 30 years (see [15]-[17], [19], [20], [27], [28], [33]-[35]). It is worthy to say that minimal and maximal ranks of a general solution to a matrix equation are very useful in linear programming computations (see [15]-[17]). In 2009, Liu, Tian and Takane [13] presented formulas for the maximal and minimal ranks of a Hermitian solution and a skew-Hermitian solution to the special case of the matrix equation (1.1) in which $B = 0$. The following matrix equation:

$$AXA^* = C \tag{1.2}$$

over $\mathbb{C}$, has been well examined by many authors (see [1], [3], [7], [8], [32], [36], [41]).

Note that, to our knowledge, there has been little information on extreme ranks of the (skew-)Hermitian solution to the matrix equation (1.1). This paper aims to consider the formulas of extremal ranks of the general (skew-)Hermitian solution to (1.1).

The paper is organized as follows. In Section 2, we first give an expression of the general Hermitian solution to (1.1) by using the generalized inverses of the coefficient matrices of this equation, then derive the formulas of extremal ranks of the

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general Hermitian solution to (1.1). We also give the corresponding results on the skew-Hermitian solution to (1.1). In Section 3, we first present the maximal and minimal ranks of the real matrices X_1, X_2, X_3, X_4 and Y_1, Y_2, Y_3, Y_4 in a Hermitian solution $X = X_1 + X_2 i + X_3 j + X_4 k$, $Y = Y_1 + Y_2 i + Y_3 j + Y_4 k$ to (1.1), then give some necessary and sufficient conditions for (1.1) to have a real symmetric solution, a complex Hermitian solution, and a pure imaginary Hermitian solution. The corresponding results on the skew-Hermitian solution to (1.1) are also considered. Some special cases of (1.1) are also considered in Section 4.

2. Ranks of the general Hermitian solution to (1.1). In this section, we consider the maximal and minimal ranks of the general (skew-)Hermitian solution to (1.1) over $\mathbb{H}$.

We begin with the following lemma that is due to Tian [18], and can be generalized to $\mathbb{H}$.

LEMMA 2.1. *Suppose that the matrix equation*

$$AXB + CYD = E \tag{2.1}$$

is consistent over $\mathbb{H}$, *where* $X \in \mathbb{H}^{p\times q}, Y \in \mathbb{H}^{s\times t}$ *unknown. Then the general solution of* (2.1) *can be expressed by*

$$X = X_0 + S_1 L_G U R_H T_1 + L_A V_1 + V_2 R_B,$$
$$Y = Y_0 - S_2 L_G U R_H T_2 + L_C W_1 + W_2 R_D,$$

where X_0 *and* Y_0 *are a special pair solution of* (2.1),

$$S_1 = (I_p, 0)\,,\ S_2 = (0, I_s)\,,\ T_1 = (I_q, 0)^*\,,\ T_2 = (0, I_t)^*\,,\ G = (A, C),\ H = \begin{pmatrix} B \\ D \end{pmatrix};$$

U, V_1, V_2, W_1, *and* W_2 *are arbitrary quaternion matrices with suitable sizes.*

LEMMA 2.2. *Consider the matrix equation* (1.1) *where* $A \in \mathbb{H}^{m\times n}, B \in \mathbb{H}^{m\times p}$, $C \in \mathbb{H}^{m\times m}$ *are given, and* $X \in \mathbb{H}^{n\times n}, Y \in \mathbb{H}^{p\times p}$ *unknown.*

1. *If* $C = C^*$, *and* (1.1) *has a Hermitian solution, then the general Hermitian solution to* (1.1) *can be expressed as*

$$X = X_0 + S_1 L_G Z L_G S_1^* + L_A V + V^* L_A,$$
$$Y = Y_0 - S_2 L_G Z L_G S_2^* + L_B W + W^* L_B, \tag{2.2}$$

where X_0 *and* Y_0 *are a special pair Hermitian solution of* (1.1),

$$S_1 = (I_n, 0)\,,\ S_2 = (0, I_p)\,,\ G = (A, B)\,; \tag{2.3}$$

Z *is an arbitrary Hermitian quaternion matrix with consistent size, and* V *and* W *are arbitrary quaternion matrices with suitable sizes.*

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2. *If* $C = -C^*$, *and* (1.1) *has a skew-Hermitian solution, then the general skew-Hermitian solution can be expressed as*

$$X = X_0 + S_1 L_G Z L_G S_1^* + L_A V - V^* L_A,$$
$$Y = Y_0 - S_2 L_G Z L_G S_2^* + L_B W - W^* L_B,$$

where X_0 *and* Y_0 *are a special pair skew-Hermitian solution of* (1.1), *and* S_1, S_2, *and* G *are the same as* (2.3); Z *is an arbitrary skew-Hermitian quaternion matrix with consistent size, and* V *and* W *are arbitrary quaternion matrices with suitable sizes.*

Proof. We here only prove 1. The proof of 2 can be similarly finished.

Suppose that $X_1 = X_1^*$, $Y_1 = Y_1^*$ are an arbitrary pair Hermitian solution of (1.1). It follows from Lemma 2.1 and $R_{(A,B)^*} = L_{(A,B)}$, $R_{A^*} = L_A$, $R_{B^*} = L_B$ that

$$X_1 = \tilde{X}_0 + S_1 L_G U L_G S_1^* + L_A V_1 + V_2 L_A,$$
$$Y_1 = \tilde{Y}_0 - S_2 L_G U L_G S_2^* + L_B W_1 + W_2 L_B,$$

where $\tilde{X}_0$ and $\tilde{Y}_0$ are a special pair solution of (1.1), and U, V_1, V_2, W_1, and W_2 are arbitrary matrices with suitable sizes. Since $\left(\frac{X_1+X_1^*}{2}, \frac{Y_1+Y_1^*}{2}\right)$ is also a pair Hermitian solution of (1.1), we get that

$$\begin{aligned} X_1 &= \frac{1}{2}(X_1 + X_1^*) = \frac{1}{2}(\tilde{X}_0 + \tilde{X}_0^*) + \frac{1}{2}S_1 L_G (U + U^*) L_G S_1^* + \frac{1}{2}L_A(V_1 + V_2^*) \\ &\quad + \frac{1}{2}(V_1 + V_2^*)^* L_A, \\ Y_1 &= \frac{1}{2}(Y_1 + Y_1^*) = \frac{1}{2}(\tilde{Y}_0 + \tilde{Y}_0^*) - \frac{1}{2}S_2 L_G (U + U^*) L_G S_2^* + \frac{1}{2}L_B(W_1 + W_2^*) \\ &\quad + \frac{1}{2}(W_1 + W_2^*)^* L_B. \end{aligned}$$

Putting

$$X_0 = \frac{1}{2}(\tilde{X}_0 + \tilde{X}_0^*),\ Y_0 = \frac{1}{2}(\tilde{Y}_0 + \tilde{Y}_0^*),\ V = \frac{1}{2}(V_1 + V_2^*),$$
$$W = \frac{1}{2}(W_1 + W_2^*),\ Z = \frac{1}{2}(U + U^*),$$

and noting that X_0 and Y_0 are a special pair Hermitian solution of (1.1), Z is an arbitrary Hermitian matrix, we get that any pair Hermitian solution (X_1, Y_1) of (1.1) has the form of (2.2).

Conversely, it can be verified that a pair of matrices having the form of (2.2) are a pair Hermitian solution of (1.1). Therefore, (2.2) is an expression of the general Hermitian solution to (1.1). ∎

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Tian and Liu in [21] and [12] gave the following extremal ranks of matrix expressions $A - BXB^*$ and $A - BX - X^*B^*$ over a field. The results can be generalized to $\mathbb{H}$.

LEMMA 2.3. *Let* $A \in \mathbb{H}^{m\times m}$, $B \in \mathbb{H}^{m\times n}$.

1. *If* $A = A^*$, *then*

$$\max_{X=X^*\in\mathbb{H}^{n\times n}} r(A - BXB^*) = r[A, B],$$
$$\min_{X=X^*\in\mathbb{H}^{n\times n}} r(A - BXB^*) = 2r[A, B] - r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix};$$
$$\max_{X\in\mathbb{H}^{n\times m}} r(A + BX + X^*B^*) = \min\left\{m,\ r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}\right\},$$
$$\min_{X\in\mathbb{H}^{n\times m}} r(A + BX + X^*B^*) = r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - 2r(B).$$

2. *If* $A = -A^*$, *then*

$$\max_{X=-X^*\in\mathbb{H}^{n\times n}} r(A - BXB^*) = r[A, B],$$
$$\min_{X=-X^*\in\mathbb{H}^{n\times n}} r(A - BXB^*) = 2r[A, B] - r\begin{bmatrix} A & B \\ -B^* & 0 \end{bmatrix};$$
$$\max_{X\in\mathbb{H}^{n\times m}} r(A + BX - X^*B^*) = \min\left\{m,\ r\begin{bmatrix} A & B \\ -B^* & 0 \end{bmatrix}\right\},$$
$$\min_{X\in\mathbb{H}^{n\times m}} r(A + BX - X^*B^*) = r\begin{bmatrix} A & B \\ -B^* & 0 \end{bmatrix} - 2r(B).$$

The following lemma is due to Marsaglia and Styan [14], which can be generalized to $\mathbb{H}$.

LEMMA 2.4. *Let* $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{m\times k}$, $C \in \mathbb{H}^{l\times n}$, $D \in \mathbb{H}^{j\times k}$ *and* $E \in \mathbb{H}^{l\times i}$. *Then they satisfy the following rank equalities:*

(a) $r(CL_A) = r\begin{bmatrix} A \\ C \end{bmatrix} - r(A)$,

(b) $r\begin{bmatrix} B & AL_C \end{bmatrix} = r\begin{bmatrix} B & A \\ 0 & C \end{bmatrix} - r(C)$,

(c) $r\begin{bmatrix} C \\ R_BA \end{bmatrix} = r\begin{bmatrix} C & 0 \\ A & B \end{bmatrix} - r(B)$.

Now we give one of the main theorems in this paper.

THEOREM 2.5. *Suppose that the matrix equation* (1.1)*, where* $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{m\times p}$, $C \in \mathbb{H}^{m\times m}$, $C = C^*$, $X \in \mathbb{H}^{n\times n}$, *and* $Y \in \mathbb{H}^{p\times p}$, *has a Hermitian solution.*

(a) The maximal and minimal ranks of the general Hermitian solution to (1.1)

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are given by:

$$\max_{\substack{AXA^*+BYB^*=C\\ X=X^*}} r(X) = \min\{n, r[B,C] + 2n - r(A) - r(G)\}, \tag{2.4}$$

$$\min_{\substack{AXA^*+BYB^*=C\\ X=X^*}} r(X) = 2r[B,C] - r\begin{bmatrix} C & B \\ B^* & 0 \end{bmatrix}; \tag{2.5}$$

$$\max_{\substack{AXA^*+BYB^*=C\\ Y=Y^*}} r(Y) = \min\{p, r[A,C] + 2p - r(B) - r(G)\}, \tag{2.6}$$

$$\min_{\substack{AXA^*+BYB^*=C\\ Y=Y^*}} r(Y) = 2r[A,C] - r\begin{bmatrix} C & A \\ A^* & 0 \end{bmatrix}. \tag{2.7}$$

(b) The rank of the general Hermitian solution X to (1.1) *is invariant if and only if*

$$2r[B,C] - r\begin{bmatrix} C & B \\ B^* & 0 \end{bmatrix} = n$$

or

$$r[B,C] + r(A) + r(G) = r\begin{bmatrix} C & B \\ B^* & 0 \end{bmatrix} + 2n.$$

The rank of the general Hermitian solution Y to (1.1) *is invariant if and only if*

$$2r[A,C] - r\begin{bmatrix} C & A \\ A^* & 0 \end{bmatrix} = p$$

or

$$r[A,C] + r(B) + r(G) = r\begin{bmatrix} C & A \\ A^* & 0 \end{bmatrix} + 2p.$$

Proof. (a) Applying Lemma 2.2 and Lemma 2.3 to X of (2.2), we get that

$$\begin{aligned}
\max_{\substack{AXA^*+BYB^*=C\\ X=X^*}} r(X) &= \max_{Z=Z^*,V} r(X_0 + S_1L_GZL_GS_1^* + L_AV + V^*L_A) \\
&= \min\left\{n,\ \max_{Z=Z^*} r\begin{bmatrix} X_0 + S_1L_GZL_GS_1^* & L_A \\ L_A & 0 \end{bmatrix}\right\} \\
&= \min\left\{n,\ \max_{Z=Z^*} r\left(\begin{bmatrix} X_0 & L_A \\ L_A & 0 \end{bmatrix} + \begin{bmatrix} S_1L_G \\ 0 \end{bmatrix} Z \begin{bmatrix} L_GS_1^* & 0 \end{bmatrix}\right)\right\} \\
&= \min\left\{n,\ r\begin{bmatrix} X_0 & L_A & S_1L_G \\ L_A & 0 & 0 \end{bmatrix}\right\}; \qquad (2.8)
\end{aligned}$$

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$$\begin{aligned}
\min_{\substack{AXA^*+BYB^*=C\\ X=X^*}} r(X) &= \min_{Z=Z^*,V} r(X_0+S_1L_GZL_GS_1^*+L_AV+V^*L_A)\\
&= \min_{Z=Z^*} r\begin{bmatrix} X_0+S_1L_GZL_GS_1^* & L_A\\ L_A & 0\end{bmatrix} - 2r(L_A)\\
&= \min_{Z=Z^*} r\left(\begin{bmatrix} X_0 & L_A\\ L_A & 0\end{bmatrix} + \begin{bmatrix} S_1L_G\\ 0\end{bmatrix} Z\begin{bmatrix} L_GS_1^* & 0\end{bmatrix}\right) - 2r(L_A)\\
&= 2r\begin{bmatrix} X_0 & L_A & S_1L_G\\ L_A & 0 & 0\end{bmatrix} - r\begin{bmatrix} X_0 & L_A & S_1L_G\\ L_A & 0 & 0\\ L_GS_1^* & 0 & 0\end{bmatrix}\\
&\quad - 2r(L_A). \qquad (2.9)
\end{aligned}$$

By Lemma 2.4, block Gaussian elimination, and $AX_0A^*+BY_0B^*=C$, we have that $r(L_A)=n-r(A)$,

$$\begin{aligned}
r\begin{bmatrix} X_0 & S_1L_G & L_A\\ L_A & 0 & 0\end{bmatrix} &= r\begin{bmatrix} X_0 & I_n & S_1 & 0\\ I_n & 0 & 0 & A^*\\ 0 & A & 0 & 0\\ 0 & 0 & G & 0\end{bmatrix} - r(A) - r(A^*) - r(G)\\
&= r[B,C] + 2n - r(A) - r(G),\\
r\begin{bmatrix} X_0 & L_A & S_1L_G\\ L_A & 0 & 0\\ L_GS_1^* & 0 & 0\end{bmatrix} &= r\begin{bmatrix} X_0 & I_n & S_1 & 0 & 0\\ I_n & 0 & 0 & A^* & 0\\ S_1^* & 0 & 0 & 0 & G^*\\ 0 & A & 0 & 0 & 0\\ 0 & 0 & G & 0 & 0\end{bmatrix} - r(A) - r(A^*) - 2r(G)\\
&= r\begin{bmatrix} C & B\\ B^* & 0\end{bmatrix} + 2n - 2r(G).
\end{aligned}$$

Substituting above two equalities into (2.8) and (2.9) yields (2.4) and (2.5), respectively.

Similarly, we can get the corresponding results on Y.

(b) The ranks of X, Y, expressed as (2.2), in the general pair Hermitian solution to (1.1) are invariant if and only if

$$\max r(X) - \min r(X) = 0, \quad \max r(Y) - \min r(Y) = 0. \qquad (2.10)$$

Hence result (b) follows from (2.4)-(2.7), and (2.10). □

Similarly, we can get the following.

THEOREM 2.6. *Let* $A\in\mathbb{H}^{m\times n}$, $B\in\mathbb{H}^{m\times p}$, $C\in\mathbb{H}^{m\times m}$, $C=-C^*$, $X\in\mathbb{H}^{n\times n}$, *and* $Y\in\mathbb{H}^{p\times p}$. *Suppose that* (1.1) *has a skew-Hermitian solution over* $\mathbb{H}$.

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(a) The maximal and minimal ranks of the general skew-Hermitian solution to (1.1) are given by

$$\max_{\substack{AXA^*+BYB^*=C\\X=-X^*}} r(X) = \min\{n, r[B,C] + 2n - r(A) - r(G)\},$$

$$\min_{\substack{AXA^*+BYB^*=C\\X=-X^*}} r(X) = 2r[B,C] - r\begin{bmatrix} C & B \\ -B^* & 0 \end{bmatrix}.$$

$$\max_{\substack{AXA^*+BYB^*=C\\Y=-Y^*}} r(Y) = \min\{p, r[A,C] + 2p - r(B) - r(G)\};$$

$$\min_{\substack{AXA^*+BYB^*=C\\Y=-Y^*}} r(Y) = 2r[A,C] - r\begin{bmatrix} C & A \\ -A^* & 0 \end{bmatrix}.$$

(b) The rank of the general skew-Hermitian solution X to (1.1) *is invariant if and only if*

$$2r[B,C] - r\begin{bmatrix} C & B \\ -B^* & 0 \end{bmatrix} = n,$$

$$\textit{or } r[B,C] + r(A) + r(G) = r\begin{bmatrix} C & B \\ -B^* & 0 \end{bmatrix} + 2n.$$

The rank of the general skew-Hermitian solution Y to (1.1) *is invariant if and only if*

$$2r[A,C] - r\begin{bmatrix} C & A \\ -A^* & 0 \end{bmatrix} = p, \ \textit{or } r[A,C] + r(B) + r(G) = r\begin{bmatrix} C & A \\ -A^* & 0 \end{bmatrix} + 2p.$$

3. Extreme ranks of the real matrices in a Hermitian solution to (1.1) over $\mathbb{H}$. In this section, we consider the maximal and minimal ranks of real matrices X_i, Y_i in a pair Hermitian solution $X = X_1 + X_2 i + X_3 j + X_4 k \in \mathbb{H}^{n\times n}$ and $Y = Y_1 + Y_2 i + Y_3 j + Y_4 k \in \mathbb{H}^{p\times p}$ to (1.1) where

$$A = A_1 + A_2 i + A_3 j + A_4 k, \ B = B_1 + B_2 i + B_3 j + B_4 k, \ C = C_1 + C_2 i + C_3 j + C_4 k,$$

$A_i, B_i, C_i, i = 1, \cdots, 4$, are real matrices with suitable sizes.

For an arbitrary quaternion matrix $M = M_1 + M_2 i + M_3 j + M_4 k$, we now define a map $\phi(\cdot)$, from $\mathbb{H}^{m\times n}$ to $\mathbb{R}^{4m\times 4n}$, by

$$\phi(M) = \begin{bmatrix} M_1 & -M_2 & -M_3 & -M_4 \\ M_2 & M_1 & -M_4 & M_3 \\ M_3 & M_4 & M_1 & -M_2 \\ M_4 & -M_3 & M_2 & M_1 \end{bmatrix}. \tag{3.1}$$

By (3.1), it is easy to verify that $\phi(\cdot)$ satisfies the following properties:

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(a) $M = N \iff \phi(M) = \phi(N)$.
(b) $\phi(M+N) = \phi(M) + \phi(N)$, $\phi(MN) = \phi(M)\phi(N)$, $\phi(kM) = k\phi(M)$, $k \in \mathbb{R}$.
(c) $\phi(M^*) = \phi^T(M)$, $\phi(M^\dagger) = \phi^\dagger(M)$.
(d) $\phi(M) = T_m^{-1}\phi(M)T_n = R_m^{-1}\phi(M)R_n = S_m^{-1}\phi(M)S_n$, where for $t = m, n$,

$$R_t = \begin{bmatrix} 0 & -I_{2t} \\ I_{2t} & 0 \end{bmatrix}, \ S_t = \begin{bmatrix} 0 & 0 & 0 & -I_t \\ 0 & 0 & I_t & 0 \\ 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \end{bmatrix},$$

$$T_t = \begin{bmatrix} 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \\ 0 & 0 & 0 & I_t \\ 0 & 0 & -I_t & 0 \end{bmatrix}.$$

(e) $r\left[\phi(M)\right] = 4r(M)$.
(f) $M^* = M \Leftrightarrow \phi^T(M) = \phi(M)$, and $M^* = -M \Leftrightarrow \phi^T(M) = -\phi(M)$.

In the following theorems and corollaries, X_0, Y_0, S_1, S_2, and G are defined as in Lemma 2.2.

Theorem 3.1. *The matrix equation* (1.1) *has a Hermitian solution over* $\mathbb{H}$ *if and only if the matrix equation*

$$\phi(A)\left(X_{ij}\right)_{4\times 4}\phi^T(A) + \phi(B)\left(Y_{ij}\right)_{4\times 4}\phi^T(B) = \phi(C), \ i,j = 1,2,3,4, \tag{3.2}$$

has a symmetric solution over $\mathbb{R}$. *In this case, the general Hermitian solution of* (1.1) *over* $\mathbb{H}$ *can be written as:*

$$\begin{aligned} X &= X_1 + X_2 i + X_3 j + X_4 k \\ &= \frac{1}{4}(X_{11} + X_{22} + X_{33} + X_{44}) + \frac{1}{4}(X_{12}^T - X_{12} + X_{34}^T - X_{34})i \\ &+ \frac{1}{4}(X_{13}^T - X_{13} + X_{24} - X_{24}^T)j + \frac{1}{4}(X_{14}^T - X_{14} + X_{23}^T - X_{23})k, \end{aligned} \tag{3.3}$$

$$\begin{aligned} Y &= Y_1 + Y_2 i + Y_3 j + Y_4 k \\ &= \frac{1}{4}(Y_{11} + Y_{22} + Y_{33} + Y_{44}) + \frac{1}{4}(Y_{12}^T - Y_{12} + Y_{34}^T - Y_{34})i \\ &+ \frac{1}{4}(Y_{13}^T - Y_{13} + Y_{24} - Y_{24}^T)j + \frac{1}{4}(Y_{14}^T - Y_{14} + Y_{23}^T - Y_{23})k, \end{aligned} \tag{3.4}$$

where $X_{tt} = X_{tt}^T, Y_{tt} = Y_{tt}^T$, $t = 1,2,3,4$; $X_{1j}^T = X_{j1}$, $Y_{1j}^T = Y_{j1}$, $j = 2,3,4$; $X_{2j}^T = X_{j2}$, $Y_{2j}^T = Y_{j2}$, $j = 3,4$; $X_{34}^T = X_{43}$, $Y_{34}^T = Y_{43}$ *are the general solutions of* (3.2) *over*

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$\mathbb{R}$. *Written in explicit forms, X_i, Y_i, $i = 1, 2, 3, 4$, in (3.3) and (3.4) are*

$$\begin{aligned}
X_1 = {}& \frac{1}{4}P_1\phi(X_0)P_1^T + \frac{1}{4}P_2\phi(X_0)P_2^T + \frac{1}{4}P_3\phi(X_0)P_3^T + \frac{1}{4}P_4\phi(X_0)P_4^T \\
&+ \frac{1}{4}\left[P_1, P_2, P_3, P_4\right] E_1 Z E_1^T \left[P_1, P_2, P_3, P_4\right]^T \\
&+ \left[P_1, P_2, P_3, P_4\right] L_{\phi(A)} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}^T L_{\phi(A)}^T \left[P_1, P_2, P_3, P_4\right]^T,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
X_2 = {}& \frac{1}{4}P_2\phi(X_0)P_1^T - \frac{1}{4}P_1\phi(X_0)P_2^T + \frac{1}{4}P_4\phi(X_0)P_3^T - \frac{1}{4}P_3\phi(X_0)P_4^T \\
&- \frac{1}{4}\left[P_1, -P_2, P_3, -P_4\right] E_1 Z E_1^T \left[P_2, P_1, P_4, P_3\right]^T \\
&- \left[-P_2, P_1, -P_4, P_3\right] L_{\phi(A)} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}^T L_{\phi(A)}^T \left[-P_2, P_1, -P_4, P_3\right]^T,
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
X_3 = {}& \frac{1}{4}P_3\phi(X_0)P_1^T - \frac{1}{4}P_1\phi(X_0)P_3^T + \frac{1}{4}P_2\phi(X_0)P_4^T - \frac{1}{4}P_4\phi(X_0)P_2^T \\
&- \frac{1}{4}[P_1, -P_2, -P_3, P_4]E_1 Z E_1^T \left[P_3, P_4, P_1, P_2\right]^T \\
&- \left[-P_3, P_1, -P_2, P_4\right] L_{\phi(A)} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}^T L_{\phi(A)}^T \left[-P_3, P_1, -P_2, P_4\right]^T,
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
X_4 = {}& \frac{1}{4}P_4\phi(X_0)P_1^T - \frac{1}{4}P_1\phi(X_0)P_4^T + \frac{1}{4}P_3\phi(X_0)P_2^T - \frac{1}{4}P_2\phi(X_0)P_3^T \\
&- \frac{1}{4}[-P_{1,} - P_2, P_3, P_4]E_1 Z E_1^T \left[P_4, P_3, P_2, P_1\right]^T \\
&- \left[-P_4, P_1, -P_3, P_2\right] L_{\phi(A)} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}^T L_{\phi(A)}^T \left[-P_4, P_1, -P_3, P_2\right]^T,
\end{aligned} \tag{3.8}$$

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$$
\begin{aligned}
Y_1 = {}& \frac{1}{4}Q_1\phi(Y_0)Q_1^T + \frac{1}{4}Q_2\phi(Y_0)Q_2^T + \frac{1}{4}Q_3\phi(Y_0)Q_3^T + \frac{1}{4}Q_4\phi(Y_0)Q_4^T \\
&- \frac{1}{4}[Q_1, Q_2, Q_{3,}Q_4]E_2ZE_2^T\,[Q_1, Q_2, Q_3, Q_4]^T \\
&+ [Q_1, Q_2, Q_3, Q_4]\,L_{\phi(B)}\begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix}^T L_{\phi(B)}^T\,[Q_1, Q_2, Q_3, Q_4]^T,
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
Y_2 = {}& \frac{1}{4}Q_2\phi(Y_0)Q_1^T - \frac{1}{4}Q_1\phi(Y_0)Q_2^T + \frac{1}{4}Q_4\phi(Y_0)Q_3^T - \frac{1}{4}Q_3\phi(Y_0)Q_4^T \\
&+ \frac{1}{4}[Q_1, -Q_2, Q_3, -Q_4]E_2ZE_2^T\,[Q_2, Q_1, Q_4, Q_3]^T \\
&- [-Q_2, Q_1, -Q_4, Q_3]\,L_{\phi(B)}\begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix}^T L_{\phi(B)}^T\,[-Q_2, Q_1, -Q_4, Q_3]^T,
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
Y_3 = {}& \frac{1}{4}Q_3\phi(Y_0)Q_1^T - \frac{1}{4}Q_1\phi(Y_0)Q_3^T + \frac{1}{4}Q_2\phi(Y_0)Q_4^T - \frac{1}{4}Q_4\phi(Y_0)Q_2^T \\
&+ \frac{1}{4}[Q_1, -Q_2, -Q_3, Q_4]E_2ZE_2^T\,[Q_3, Q_4, Q_1, Q_2]^T \\
&- [-Q_3, Q_1, -Q_2, Q_4]\,L_{\phi(B)}\begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix}^T L_{\phi(B)}^T\,[-Q_3, Q_1, -Q_2, Q_4]^T,
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
Y_4 = {}& \frac{1}{4}Q_4\phi(Y_0)Q_1^T - \frac{1}{4}Q_1\phi(Y_0)Q_4^T + \frac{1}{4}Q_3\phi(Y_0)Q_2^T - \frac{1}{4}Q_2\phi(Y_0)Q_3^T \\
&+ \frac{1}{4}[-Q_1, -Q_2, Q_3, Q_4]E_2ZE_2^T\,[Q_4, Q_3, Q_2, Q_1]^T \\
&- [-Q_4, Q_1, -Q_3, Q_2]\,L_{\phi(B)}\begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix}^T L_{\phi(B)}^T\,[-Q_4, Q_1, -Q_3, Q_2]^T,
\end{aligned}
\tag{3.12}
$$

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where

$$\begin{aligned}
&P_1=[I_n,0,0,0]\,,\ P_2=[0,I_n,0,0]\,,\ P_3=[0,0,I_n,0]\,,\ P_4=[0,0,0,I_n]\,,\\
&Q_1=[I_p,0,0,0]\,,\ Q_2=[0,I_p,0,0]\,,\ Q_3=[0,0,I_p,0]\,,\ Q_4=[0,0,0,I_p]\,,\\
&E_1=\phi(S_1)L_{\phi(G)},\ E_2=\phi(S_2)L_{\phi(G)};
\end{aligned}$$

Z is arbitrary real symmetric matrix, and V_1, V_2, V_3, V_4, W_1, W_2, W_3, and W_4 are arbitrary real matrices with compatible sizes.

Proof. Suppose that (1.1) has a Hermitian solution

$$X=X_1+X_2i+X_3j+X_4k,\ Y=Y_1+Y_2i+Y_3j+Y_4k$$

over $\mathbb{H}$. Applying properties (a) and (b) of $\phi(\cdot)$ to (1.1) yields

$$\phi(A)\phi(X)\phi^T(A)+\phi(B)\phi(Y)\phi^T(B)=\phi(C),\ \phi^T(X)=\phi(X),\ \phi^T(Y)=\phi(Y),$$

which implies that $\phi(X)$, $\phi(Y)$ are real symmetric solutions to (3.2). Conversely, suppose that (3.2) has a real symmetric solution

$$X=\begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14}\\ X_{21} & X_{22} & X_{23} & X_{24}\\ X_{31} & X_{32} & X_{33} & X_{24}\\ X_{41} & X_{42} & X_{43} & X_{44}\end{bmatrix},\ Y=\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14}\\ Y_{21} & Y_{22} & Y_{23} & Y_{24}\\ Y_{31} & Y_{32} & Y_{33} & Y_{24}\\ Y_{41} & Y_{42} & Y_{43} & Y_{44}\end{bmatrix},$$

i.e.

$$\phi(A)X\phi^T(A)+\phi(B)Y\phi^T(B)=\phi(C),\ X^T=X,\ Y^T=Y.$$

By property (d) of $\phi(\cdot)$, we have that

$$\begin{aligned}
T_m^{-1}\phi(A)T_nXT_n^{-1}\phi^T(A)T_m+T_m^{-1}\phi(B)T_pYT_p^{-1}\phi^T(B)T_m&=T_m^{-1}\phi(C)T_m,\\
R_m^{-1}\phi(A)R_nXR_n^{-1}\phi^T(A)R_m+R_m^{-1}\phi(B)R_pYR_p^{-1}\phi^T(B)R_m&=R_m^{-1}\phi(C)R_m,\\
S_m^{-1}\phi(A)S_nXS_n^{-1}\phi^T(A)S_m+S_m^{-1}\phi(B)S_pYS_p^{-1}\phi^T(B)S_m&=S_m^{-1}\phi(C)S_m.
\end{aligned}$$

Hence

$$\begin{aligned}
\phi(A)T_nXT_n^{-1}\phi^T(A)+\phi(B)T_pYT_p^{-1}\phi^T(B)&=\phi(C),\\
\phi(A)R_nXR_n^{-1}\phi^T(A)+\phi(B)R_pYR_p^{-1}\phi^T(B)&=\phi(C),\\
\phi(A)S_nXS_n^{-1}\phi^T(A)+\phi(B)S_pYS_p^{-1}\phi^T(B)&=\phi(C),
\end{aligned}$$

implying $T_nXT_n^{-1}$, $T_pYT_p^{-1}$, $R_nXR_n^{-1}$, $R_pYR_p^{-1}$, $S_nXS_n^{-1}$, and $S_pYS_p^{-1}$ are also symmetric solutions of (3.2) over $\mathbb{R}$. Thus,

$$\frac{1}{4}(X+T_nXT_n^{-1}+R_nXR_n^{-1}+S_nXS_n^{-1}),\ \frac{1}{4}(Y+T_pYT_p^{-1}+R_pYR_p^{-1}+S_pYS_p^{-1})$$

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are symmetric solutions of (3.2), where

$$X + T_pXT_p^{-1} + R_pXR_p^{-1} + S_pXS_p^{-1} = (\widetilde{X_{ij}})_{4\times 4},\ i,j = 1,2,3,4$$

and

$$\begin{aligned}
&\widetilde{X_{11}} = X_{11} + X_{22} + X_{33} + X_{44},\ \widetilde{X_{12}} = X_{12} - X_{12}^T + X_{34} - X_{34}^T,\\
&\widetilde{X_{13}} = X_{13} - X_{13}^T + X_{24}^T - X_{24},\ \widetilde{X_{14}} = X_{14} - X_{14}^T + X_{23} - X_{23}^T,\\
&\widetilde{X_{21}} = X_{12}^T - X_{12} + X_{34}^T - X_{34},\ \widetilde{X_{22}} = X_{11} + X_{22} + X_{33} + X_{44},\\
&\widetilde{X_{23}} = X_{14} - X_{14}^T + X_{23} - X_{23}^T,\ \widetilde{X_{24}} = X_{13}^T - X_{13} + X_{24} - X_{24}^T,\\
&\widetilde{X_{31}} = X_{13}^T - X_{13} + X_{24} - X_{24}^T,\ \widetilde{X_{32}} = X_{14}^T - X_{14} + X_{23}^T - X_{23},\\
&\widetilde{X_{33}} = X_{11} + X_{22} + X_{33} + X_{44},\ \widetilde{X_{34}} = X_{12} - X_{12}^T + X_{34} - X_{34}^T,\\
&\widetilde{X_{41}} = X_{14}^T - X_{14} + X_{23}^T - X_{23},\ \widetilde{X_{42}} = X_{13} - X_{13}^T + X_{24}^T - X_{24},\\
&\widetilde{X_{43}} = X_{12}^T - X_{12} + X_{34}^T - X_{34},\ \widetilde{X_{44}} = X_{11} + X_{22} + X_{33} + X_{44}.
\end{aligned}$$

$Y + T_pYT_p^{-1} + R_pYR_p^{-1} + S_pYS_p^{-1}$ has a form similar to $(\widetilde{X_{ij}})_{4\times 4}$. We omit it here for simplicity.

Let

$$\begin{aligned}
\hat{X} &= \frac{1}{4}(X_{11} + X_{22} + X_{33} + X_{44}) + \frac{1}{4}(X_{12}^T - X_{12} + X_{34}^T - X_{34})i\\
&\quad + \frac{1}{4}(X_{13}^T - X_{13} + X_{24} - X_{24}^T)j + \frac{1}{4}(X_{14}^T - X_{14} + X_{23}^T - X_{23})k,
\end{aligned}$$

$$\begin{aligned}
\hat{Y} &= \frac{1}{4}(Y_{11} + Y_{22} + Y_{33} + Y_{44}) + \frac{1}{4}(Y_{12}^T - Y_{12} + Y_{34}^T - Y_{34})i\\
&\quad + \frac{1}{4}(Y_{13}^T - Y_{13} + Y_{24} - Y_{24}^T)j + \frac{1}{4}(Y_{14}^T - Y_{14} + Y_{23}^T - Y_{23})k.
\end{aligned}$$

Then by (3.1),

$$\begin{aligned}
\phi(\hat{X}) &= \frac{1}{4}(X + T_nXT_n^{-1} + R_nXR_n^{-1} + S_nXS_n^{-1}),\\
\phi(\hat{Y}) &= \frac{1}{4}(Y + T_pYT_p^{-1} + R_pYR_p^{-1} + S_pYS_p^{-1}),
\end{aligned}$$

we have that $\hat{X}, \hat{Y}$ are a pair Hermitian solution of (1.1) by the property (a). Observe that X_{ij} and Y_{ij}, $i,j = 1,2,3,4$ in (3.2) can be written as

$$X_{ij} = P_iXP_j^T,\ Y_{ij} = Q_iYQ_j^T.$$

From Lemma 2.2, the general solutions to (3.2) can be written as

$$\begin{aligned}
X &= \phi(X_0) + \phi(S_1)L_{\phi(G)}ZL_{\phi(G)}\phi^T(S_1) + 4L_{\phi(A)}\left[V_1, V_2, V_3, V_4\right]\\
&\quad + 4\left[V_1, V_2, V_3, V_4\right]^T L_{\phi(A)},\\
Y &= \phi(Y_0) - \phi(S_2)L_{\phi(G)}ZL_{\phi(G)}\phi^T(S_2) + 4L_{\phi(B)}\left[W_1, W_2, W_3, W_4\right]\\
&\quad + 4\left[W_1, W_2, W_3, W_4\right]^T L_{\phi(B)},
\end{aligned}$$

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where Z is an arbitrary real symmetric matrix and V_1, V_2, V_3, V_4, W_1, W_2, W_3, and W_4 are arbitrary with suitable sizes. Hence, for $i, j = 1, 2, 3, 4$,

$$X_{ij} = P_i\phi(X_0)P_j^T + P_i\phi(S_1)L_{\phi(G)}ZL_{\phi(G)}\phi^T(S_1)P_j^T + 4P_iL_{\phi(A)}V_j + 4V_i^TL_{\phi(A)}P_j^T,$$

$$Y_{ij} = Q_i\phi(Y_0)Q_j^T - Q_i\phi(S_2)L_{\phi(G)}ZL_{\phi(G)}\phi^T(S_2)Q_j^T + 4Q_iL_{\phi(B)}W_j + 4W_i^TL_{\phi(B)}Q_j^T.$$

Substituting them into (3.3) and (3.4) yields real matrices X_i and Y_i, $i = 1, 2, 3, 4$ in (3.5)-(3.12). □

Now we consider the maximal and minimal ranks of real matrices X_i, Y_i in Hermitian solutions $X = X_1 + X_2i + X_3j + X_4k$ and $Y = Y_1 + Y_2i + Y_3j + Y_4k$ to (1.1).

Theorem 3.2. *Suppose the matrix equation* (1.1) *has a Hermitian solution over* $\mathbb{H}$*, and for* $i, j = 1, 2, 3, 4$,

$$J_i = \left\{ X_i \in \mathbb{R}^{n\times n} \;\middle|\; \begin{array}{c} A(X_1 + X_2i + X_3j + X_4k)A^* \\ +B(Y_1 + Y_2i + Y_3j + Y_4k)B^* = C \end{array} \right\},$$

$$T_j = \left\{ Y_j \in \mathbb{R}^{p\times p} \;\middle|\; \begin{array}{c} A(X_1 + X_2i + X_3j + X_4k)A^* \\ +B(Y_1 + Y_2i + Y_3j + Y_4k)B^* = C \end{array} \right\}.$$

Then we have the following:

(a) *The maximal and minimal ranks of* X_i *in the general Hermitian solution* $X = X_1 + X_2i + X_3j + X_4k$ *to* (1.1) *are given by*

$$\max_{X_i\in J_i} r(X_i) = \min\left\{ n, r\begin{bmatrix} 0 & 0 & \overline{A}_i \\ \widetilde{A}_i & \phi(B) & \phi(C) \end{bmatrix} + 2n - 4r(A) - 4r(G) \right\};$$

$$\min_{X_i\in J_i} r(X_i) = 2r\begin{bmatrix} 0 & 0 & \overline{A}_i \\ \widetilde{A}_i & \phi(B) & \phi(C) \end{bmatrix} - r\begin{bmatrix} 0 & 0 & \overline{A}_i \\ 0 & 0 & \phi^T(B) \\ \widetilde{A}_i & \phi(B) & \phi(C) \end{bmatrix} -2r\begin{bmatrix} -A_2 & -A_3 & -A_4 \\ A_1 & -A_4 & A_3 \\ A_4 & A_1 & -A_2 \\ -A_3 & A_2 & A_1 \end{bmatrix},$$

where

$$\widetilde{A}_1 = \begin{bmatrix} A_2 & A_3 & -A_4 \\ -A_1 & -A_4 & -A_3 \\ A_4 & -A_1 & A_2 \\ A_3 & -A_2 & -A_1 \end{bmatrix},\ \widetilde{A}_2 = \begin{bmatrix} -A_1 & A_3 & -A_4 \\ -A_2 & -A_4 & -A_3 \\ -A_3 & -A_1 & A_2 \\ A_4 & -A_2 & -A_1 \end{bmatrix},$$

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$$\widetilde{A}_3 = \begin{bmatrix} -A_1 & A_2 & -A_4 \\ -A_2 & -A_1 & -A_3 \\ -A_3 & A_4 & A_2 \\ A_4 & A_3 & -A_1 \end{bmatrix}, \ \widetilde{A}_4 = \begin{bmatrix} -A_1 & A_2 & A_3 \\ -A_2 & -A_1 & -A_4 \\ -A_3 & A_4 & -A_1 \\ A_4 & A_3 & -A_2 \end{bmatrix},$$

$$\overline{A}_1 = \begin{bmatrix} -A_2 & A_3 & A_4 \\ -A_1 & -A_4 & -A_3 \\ -A_4 & -A_1 & -A_2 \\ A_3 & A_2 & -A_1 \end{bmatrix}^T, \ \overline{A}_2 = \begin{bmatrix} -A_1 & A_3 & A_4 \\ A_2 & -A_4 & -A_3 \\ -A_3 & -A_1 & -A_2 \\ A_4 & A_2 & -A_1 \end{bmatrix}^T,$$

$$\overline{A}_3 = \begin{bmatrix} -A_1 & -A_2 & A_4 \\ A_2 & -A_1 & -A_3 \\ -A_3 & -A_4 & -A_2 \\ -A_4 & A_3 & -A_1 \end{bmatrix}^T, \ \overline{A}_4 = \begin{bmatrix} -A_1 & -A_2 & A_3 \\ A_2 & -A_1 & A_4 \\ -A_3 & -A_4 & -A_1 \\ -A_4 & A_3 & A_2 \end{bmatrix}^T.$$

(b) *The maximal and minimal ranks of* Y_j *in a Hermitian solution* $Y_1 + Y_2 i + Y_3 j + Y_4 k$ *to* (1.1) *are given by*

$$\max_{Y_j \in T_j} r(Y_j) = \min\left\{ p, r\begin{bmatrix} 0 & 0 & \overline{B}_j \\ \widetilde{B}_j & \phi(A) & \phi(C) \end{bmatrix} + 2p - 4r(B) - 4r(G) \right\};$$

$$\min_{Y_j \in T_j} r(Y_j) = 2r\begin{bmatrix} 0 & 0 & \overline{B}_j \\ \widetilde{B}_j & \phi(A) & \phi(C) \end{bmatrix} - r\begin{bmatrix} 0 & 0 & \overline{B}_j \\ 0 & 0 & \phi^T(A) \\ \widetilde{B}_j & \phi(A) & \phi(C) \end{bmatrix}$$
$$-2r\begin{bmatrix} -B_2 & -B_3 & -B_4 \\ B_1 & -B_4 & B_3 \\ B_4 & B_1 & -B_2 \\ -B_3 & B_2 & B_1 \end{bmatrix},$$

where

$$\widetilde{B}_1 = \begin{bmatrix} B_2 & B_3 & -B_4 \\ -B_1 & -B_4 & -B_3 \\ B_4 & -B_1 & B_2 \\ B_3 & -B_2 & -B_1 \end{bmatrix}, \ \widetilde{B}_2 = \begin{bmatrix} -B_1 & B_3 & -B_4 \\ -B_2 & -B_4 & -B_3 \\ -B_3 & -B_1 & B_2 \\ B_4 & -B_2 & -B_1 \end{bmatrix},$$

$$\widetilde{B}_3 = \begin{bmatrix} -B_1 & B_2 & -B_4 \\ -B_2 & -B_1 & -B_3 \\ -B_3 & B_4 & B_2 \\ B_4 & B_3 & -B_1 \end{bmatrix}, \ \widetilde{B}_4 = \begin{bmatrix} -B_1 & B_2 & B_3 \\ -B_2 & -B_1 & -B_4 \\ -B_3 & B_4 & -B_1 \\ B_4 & B_3 & -B_2 \end{bmatrix},$$

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$$\overline{B}_1 = \begin{bmatrix} -B_2 & B_3 & B_4 \\ -B_1 & -B_4 & -B_3 \\ -B_4 & -B_1 & -B_2 \\ B_3 & B_2 & -B_1 \end{bmatrix}^T, \ \overline{B}_2 = \begin{bmatrix} -B_1 & B_3 & B_4 \\ B_2 & -B_4 & -B_3 \\ -B_3 & -B_1 & -B_2 \\ -B_4 & B_2 & -B_1 \end{bmatrix}^T,$$

$$\overline{B}_3 = \begin{bmatrix} -B_1 & -B_2 & B_4 \\ B_2 & -B_1 & -B_3 \\ -B_3 & -B_4 & -B_2 \\ -B_4 & B_3 & -B_1 \end{bmatrix}^T, \ \overline{B}_4 = \begin{bmatrix} -B_1 & -B_2 & B_3 \\ B_2 & -B_1 & B_4 \\ -B_3 & -B_4 & -B_1 \\ -B_4 & B_3 & B_2 \end{bmatrix}^T.$$

Proof. We only derive the maximal and minimal ranks of the matrix X_1. The others can be established similarly. Applying Lemma 2.3 to (3.5), we get the following

$$\begin{aligned}
\max_{X_1 \in J_1} r(X_1) &= \max_{Z=Z^T, V} r(M + \frac{1}{4}\hat{P}Z\hat{P}^T + PV + V^TP^T) \\
&= \min\left\{n, \max_{Z=Z^T} r(U)\right\} \\
&= \min\left\{n, r\begin{bmatrix} M & P & \hat{P} \\ P^T & 0 & 0 \end{bmatrix}\right\}, \\
\min_{X_1 \in J_1} r(X_1) &= \min_{Z=Z^T, V} r(M + \frac{1}{4}\hat{P}Z\hat{P}^T + PV + V^TP^T) \\
&= \min_{Z=Z^T} r(U) - 2r(P) \\
&= 2r\begin{bmatrix} M & P & \hat{P} \\ P^T & 0 & 0 \end{bmatrix} - r\begin{bmatrix} M & P & \hat{P} \\ P^T & 0 & 0 \\ \hat{P}^T & 0 & 0 \end{bmatrix} - 2r(P),
\end{aligned}$$

where

$$\begin{aligned}
U &= \begin{bmatrix} M + \frac{1}{4}\hat{P}Z\hat{P}^T & P \\ P^T & 0 \end{bmatrix} = \begin{bmatrix} M & P \\ P^T & 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \hat{P} \\ 0 \end{bmatrix} Z \begin{bmatrix} \hat{P}^T & 0 \end{bmatrix}, \\
M &= \frac{1}{4}P_1\phi(X_0)P_1^T + \frac{1}{4}P_2\phi(X_0)P_2^T + \frac{1}{4}P_3\phi(X_0)P_3^T + \frac{1}{4}P_4\phi(X_0)P_4^T, \\
P &= [P_1, P_2, P_3, P_4]L_{\phi(A)}, \ \hat{P} = [P_1, P_2, P_3, P_4]E_1.
\end{aligned}$$

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By Lemma 2.4, block Gaussian elimination, and $AX_0A^* + BY_0B^* = C$, we have that

$$r\begin{bmatrix} M & P & \hat{P} \\ P^T & 0 & 0 \end{bmatrix} = r\begin{bmatrix} M & D_1 & D_2 & 0 \\ D_1^T & 0 & 0 & \Psi(A^*) \\ 0 & \Psi(A) & 0 & 0 \\ 0 & 0 & \Psi(G) & 0 \end{bmatrix} - 4r(\phi(A))$$
$$- 4r(\phi(A^*)) - 4r(\phi(G))$$
$$= r\begin{bmatrix} 0 & 0 & \overline{A}_1 \\ \widetilde{A}_1 & \phi(B) & \phi(C) \end{bmatrix} + 2n - 4r(A) - 4r(G),$$

where

$$D_1 = [P_1, P_2, P_3, P_4]\,,\ D_2 = [P_1, P_2, P_3, P_4]\,\phi(S_1),$$
$$\Psi(V) = \begin{bmatrix} \phi(V) & 0 & 0 & 0 \\ 0 & \phi(V) & 0 & 0 \\ 0 & 0 & \phi(V) & 0 \\ 0 & 0 & 0 & \phi(V) \end{bmatrix},\ V = A, A^*, G.$$

Similarly, we can simplify the following

$$r\begin{bmatrix} M & P & \hat{P} \\ P^T & 0 & 0 \\ \hat{P}^T & 0 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & 0 & \overline{A}_1 \\ 0 & 0 & \phi^T(B) \\ \widetilde{A}_1 & \phi(B) & \phi(C) \end{bmatrix} + 2n - 4r(G) - 4r(G^*),$$

$$r(P) = r\begin{bmatrix} -A_2 & -A_3 & -A_4 \\ A_1 & -A_4 & A_3 \\ A_4 & A_1 & -A_2 \\ -A_3 & A_2 & A_1 \end{bmatrix} + n - 4r(A).$$

Thus we have the results for extreme ranks of the matrix X_1 in (a). Similarly, applying Lemma 2.3 and Lemma 2.4 to (3.6)–(3.12) yields the other results in (a) and (b). ∎

Corollary 3.3. *Suppose that the matrix equation* (1.1) *has a Hermitian solution over* $\mathbb{H}$, *then we have the following:*

(a) (1.1) *has a real symmetric solution* X *if and only if, for* $i = 2, 3, 4$,

$$2r\begin{bmatrix} 0 & 0 & \overline{A}_i \\ \widetilde{A}_i & \phi(B) & \phi(C) \end{bmatrix} = r\begin{bmatrix} 0 & 0 & \overline{A}_i \\ 0 & 0 & \phi^T(B) \\ \widetilde{A}_i & \phi(B) & \phi(C) \end{bmatrix}$$
$$+ 2r\begin{bmatrix} -A_2 & -A_3 & -A_4 \\ A_1 & -A_4 & A_3 \\ A_4 & A_1 & -A_2 \\ -A_3 & A_2 & A_1 \end{bmatrix}.$$

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In that case, the real symmetric solution X can be expressed as $X = X_1$ in (3.5).

(b) All the solutions of (1.1) *for X are real symmetric if and only if*

$$r\begin{bmatrix} 0 & 0 & \overline{A}_i \\ \widetilde{A}_i & \phi(B) & \phi(C) \end{bmatrix} + 2n = 4r(A) + 4r(G),\ \ i = 2, 3, 4.$$

In that case, the real symmetric solution X can be expressed as $X = X_1$ in (3.5).

(c) (1.1) *has a complex Hermitian solution X if and only if, for $i = 3, 4$,*

$$2r\begin{bmatrix} 0 & 0 & \overline{A}_i \\ \widetilde{A}_i & \phi(B) & \phi(C) \end{bmatrix} = r\begin{bmatrix} 0 & 0 & \overline{A}_i \\ 0 & 0 & \phi^T(B) \\ \widetilde{A}_i & \phi(B) & \phi(C) \end{bmatrix} + 2r\begin{bmatrix} -A_2 & -A_3 & -A_4 \\ A_1 & -A_4 & A_3 \\ A_4 & A_1 & -A_2 \\ -A_3 & A_2 & A_1 \end{bmatrix}.$$

In that case, the complex Hermitian solution X can be expressed as $X = X_1 + X_2 i$, where X_1, X_2 are expressed as (3.5) *and* (3.6).

(d) All the solutions of (1.1) *for X are complex Hermitian if and only if*

$$r\begin{bmatrix} 0 & 0 & \overline{A}_i \\ \widetilde{A}_i & \phi(B) & \phi(C) \end{bmatrix} + 2n = 4r(A) + 4r(G),\ \ i = 3, 4.$$

In that case, the complex Hermitian solution X can be expressed as $X = X_1 + X_2 i$, where X_1, X_2 are expressed as (3.5) *and* (3.6).

(e) (1.1) *has a pure imaginary Hermitian solution X if and only if*

$$2r\begin{bmatrix} 0 & 0 & \overline{A}_1 \\ \widetilde{A}_1 & \phi(B) & \phi(C) \end{bmatrix} = r\begin{bmatrix} 0 & 0 & \overline{A}_1 \\ 0 & 0 & \phi^T(B) \\ \widetilde{A}_1 & \phi(B) & \phi(C) \end{bmatrix} + 2r\begin{bmatrix} -A_2 & -A_3 & -A_4 \\ A_1 & -A_4 & A_3 \\ A_4 & A_1 & -A_2 \\ -A_3 & A_2 & A_1 \end{bmatrix}.$$

In that case, the pure imaginary Hermitian solution X can be expressed as $X = X_2 i + X_3 j + X_4 k$, where X_2, X_3, and X_4 are expressed as (3.6), (3.7), *and* (3.8).

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(f) *All the solutions of (1.1) for X are pure imaginary Hermitian if and only if*

$$r\begin{bmatrix} 0 & 0 & \overline{A}_1 \\ \widetilde{A}_1 & \phi(B) & \phi(C) \end{bmatrix} + 2n = 4r(A) + 4r(G).$$

In that case, the pure imaginary Hermitian solution X can be expressed as $X = X_2 i + X_3 j + X_4 k$, where X_2, X_3, and X_4 are expressed as (3.6), (3.7), *and* (3.8).

Using the same method, we can get the corresponding results on Y.

Remark 3.4. Similarly, we can get the corresponding results on the skew-Hermitian solution of (1.1).

4. Ranks of Hermitian solution to some special cases of (1.1). In this section, we consider some special cases of (1.1) over $\mathbb{C}$. When B vanishes, (1.1) becomes (1.2) where $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{m\times m}$. We get the corresponding results on (1.2) as follows.

Corollary 4.1. *Let $A = A_1 + A_2 i \in \mathbb{C}^{m\times n}$, $C = C^* = C_1 + C_2 i \in \mathbb{C}^{m\times m}$ be given. Then*

(a) *The matrix equation* (1.2) *has a Hermitian solution if and only if the real matrix equation*

$$\begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix}\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}\begin{bmatrix} A_1^T & A_2^T \\ -A_2^T & A_1^T \end{bmatrix} = \begin{bmatrix} C_1 & -C_2 \\ C_2 & C_1 \end{bmatrix} \tag{4.1}$$

has a symmetric solution over $\mathbb{R}$. In this case, the general Hermitian solution of (1.2) *over $\mathbb{C}$ can be written as*

$$X = X_1 + X_2 i = \frac{1}{2}(X_{11} + X_{22}) + \frac{1}{2}(X_{12}^T - X_{12})i, \tag{4.2}$$

where $X_{tt} = X_{tt}^T$, $t = 1, 2$; and $X_{12}^T = X_{21}$ are the general solutions of (4.1) *over $\mathbb{R}$. Written in an explicit form, X_1, X_2 in* (4.2) *are*

$$X_1 = \frac{1}{2}P_1\phi(X_0)P_1^T + \frac{1}{2}P_2\phi(X_0)P_2^T + [P_1, P_2]\, L_{\phi(A)}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$
$$+ \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^T L_{\phi(A)}^T\, [P_1, P_2]^T,$$
$$X_2 = \frac{1}{2}P_2\phi(X_0)P_1^T - \frac{1}{2}P_1\phi(X_0)P_2^T - [-P_2, P_1]\, L_{\phi(A)}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$
$$+ \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}^T L_{\phi(A)}^T\, [-P_2, P_1]^T,$$

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where $X_0 = A^\dagger C(A^\dagger)^*$, $P_1 = [I_n, 0]$, $P_2 = [0, I_n]$, *and* V_1 *and* V_2 *are arbitrary real matrices with compatible sizes.*

(b) Put

$$J_1 = \left\{ X_1 \in \mathbb{R}^{n\times n} \ \mid A(X_1 + X_2 i)A^* = C \right\},$$
$$J_2 = \left\{ X_2 \in \mathbb{R}^{n\times n} \ \mid A(X_1 + X_2 i)A^* = C \right\}.$$

Then we have the following:

(i) The maximal and minimal ranks of X_1 *in the Hermitian solution* $X = X_1 + X_2 i$ *to* (1.2) *are given by*

$$\max_{X_1\in J_1} r(X_1) = \min\left\{ n, r\begin{bmatrix} C_1 & -C_2 & A_1 \\ C_2 & C_1 & A_2 \\ A_1^T & A_2^T & 0 \end{bmatrix} + 2n - 4r(A) \right\},$$

$$\min_{X_1\in J_1} r(X_1) = r\begin{bmatrix} C_1 & -C_2 & A_1 \\ C_2 & C_1 & A_2 \\ A_1^T & A_2^T & 0 \end{bmatrix} - 2r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.$$

(ii) The maximal and minimal ranks of X_2 *in the Hermitian solution* $X = X_1 + X_2 i$ *to* (1.2) *are given by*

$$\max_{X_2\in J_2} r(X_2) = \min\left\{ n, r\begin{bmatrix} C_1 & -C_2 & A_1 \\ C_2 & C_1 & A_2 \\ A_2^T & -A_1^T & 0 \end{bmatrix} + 2n - 4r(A) \right\},$$

$$\min_{X_2\in J_2} r(X_2) = r\begin{bmatrix} C_1 & -C_2 & A_1 \\ C_2 & C_1 & A_2 \\ A_2^T & -A_1^T & 0 \end{bmatrix} - 2r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.$$

REMARK 4.2. Corollary 4.1 is Theorem 2.2 of [13]. Similarly, Theorem 3.2 of [13] can be regarded as a special case of (1.1) with skew-Hermitian solutions.

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