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PARETO H-EIGENVALUES OF NONNEGATIVE TENSORS AND UNIFORM HYPERGRAPHS*

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Abstract. The Pareto H-eigenvalues of nonnegative tensors and (adjacency tensors of) uniform hypergraphs are studied. Particularly, it is shown that the Pareto H-eigenvalues of a nonnegative tensor are just the spectral radii of its weakly irreducible principal subtensors, and those hypergraphs that minimize or maximize the second largest Pareto H-eigenvalue over several well-known classes of uniform hypergraphs are determined.

Key words. Pareto H-eigenvalues, Nonnegative tensor, Uniform hypergraph.

AMS subject classifications. 15A48, 05C65.

1. Introduction. A vector $\mathbf{x} = (x_1, \ldots, x_n)^\top$ is nonnegative (positive, respectively) if $x_i \geq 0$ ($x_i > 0$, respectively) for all $i \in [n] := \{1, \ldots, n\}$. Let $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \text{ is nonnegative}\}$ and $\mathbb{R}^n_{++} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \text{ is positive}\}$.

For positive integers k and n with $k \geq 2$, a k-order n-dimensional tensor (or hypermatrix) $\mathcal{T}$ is a multidimensional array of n^k real entries of the form $\mathcal{T} = (t_{i_1 \ldots i_k})$, where $i_1, \ldots, i_k \in [n]$. A k-order n-dimensional real tensor is symmetric if its entries $t_{i_1 \ldots i_k}$ are invariant for any permutation of the indices $i_1, \ldots, i_k$. A k-order n-dimensional real tensor is said to be a nonnegative tensor if all its entries are nonnegative. For a k-order n-dimensional real tensor $\mathcal{T}$ and an n-dimensional vector $\mathbf{x} = (x_1, \ldots, x_n)^\top$, the product $\mathcal{T}\mathbf{x}^{k-1}$ is defined to be an n-dimensional vector so that for $i \in [n]$,

$$(\mathcal{T}\mathbf{x}^{k-1})_i = \sum_{i_2 \in [n]} \cdots \sum_{i_k \in [n]} t_{i,i_2,\ldots,i_k} x_{i_2} \ldots x_{i_k},$$

while $\mathcal{T}\mathbf{x}^k$ is defined as the following homogeneous polynomial

$$\mathcal{T}\mathbf{x}^k = \sum_{i_1 \in [n]} \cdots \sum_{i_k \in [n]} t_{i_1,\ldots,i_k} x_{i_1} \ldots x_{i_k}.$$

So $\mathcal{T}\mathbf{x}^k = \mathbf{x}^\top(\mathcal{T}\mathbf{x}^{k-1})$. Let $\mathbf{x}^{[k]} = (x_1^k, \ldots, x_n^k)^\top$.

Definition 1.1 ([13, 7]). *A complex number λ is called an eigenvalue of tensor $\mathcal{T}$ of order k and dimension n, if the system of homogeneous polynomial equations*

$$\mathcal{T}\mathbf{x}^{k-1} = \lambda \mathbf{x}^{[k-1]},$$

i.e.,

$$(\mathcal{T}\mathbf{x}^{k-1})_i = \lambda x_i^{k-1} \text{ for } i \in [n],$$

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has a nonzero solution $\mathbf{x}$. *The vector* $\mathbf{x}$ *is called an eigenvector of* $\mathcal{T}$ *corresponding to* λ. *Moreover, if both* λ *and* $\mathbf{x}$ *are real, then we call* λ *an H-eigenvalue and* $\mathbf{x}$ *an H-eigenvector of* $\mathcal{T}$. *The spectral radius of* $\mathcal{T}$ *is the largest modulus of its eigenvalues, denoted by* $\rho(\mathcal{T})$. *An H-eigenvalue of* $\mathcal{T}$ *is called an* H^+*-eigenvalue (*H^{++}*-eigenvalue, respectively) of* $\mathcal{T}$ *if its H-eigenvector* $\mathbf{x} \in \mathbb{R}^n_+$ *(*$\mathbf{x} \in \mathbb{R}^n_{++}$*, respectively).*

Pareto eigenvalues of tensors have been studied to some extent, see [9, 17, 18, 19].

Definition 1.2 ([17]). *A real number* λ *is called a Pareto H-eigenvalue of tensor* $\mathcal{T}$ *of order* k *and dimension* n *if there is a nonzero vector* $\mathbf{x} \in \mathbb{R}^n_+$ *satisfying*

$$\begin{cases} \mathcal{T}\mathbf{x}^k = \lambda \mathbf{x}^\top \mathbf{x}^{[k-1]}, \\ \mathcal{T}\mathbf{x}^{k-1} - \lambda \mathbf{x}^{[k-1]} \in \mathbb{R}^n_+. \end{cases}$$

The vector $\mathbf{x}$ *is called a Pareto H-eigenvector of* $\mathcal{T}$ *associated to* λ.

If $k = 2$ in Definition 1.2, then λ is a Pareto eigenvalue of $n \times n$ matrix $\mathcal{T}$, and $\mathbf{x}$ is a Pareto H-eigenvector of $\mathcal{T}$ associated to λ. Pareto eigenvalues for matrices are also known as complementarity eigenvalues. Fernandes et al. [3] and Seeger [15] studied the Pareto eigenvalues of adjacency matrix of a graph.

From Definitions 1.1 and 1.2, we know that, if λ is an H^+-eigenvalue of $\mathcal{T}$, then λ is also a Pareto H-eigenvalue of $\mathcal{T}$.

Definition 1.3. *Let* $\mathcal{T}$ *be a tensor of order* k *and dimension* n. *For* $\emptyset \neq I \subseteq [n]$, *the principal subtensor of* $\mathcal{T}$ *indexed by* I, *denoted by* $\mathcal{T}_I$, *is the tensor of order* k *and dimension* $|I|$ *with entries* $t_{i_1 \dots i_k}$ *with* $i_1, \dots, i_k \in I$.

We need the following necessary and sufficient conditions for Pareto H-eigenvalues established by Song and Qi.

Theorem 1.4 ([17]). *Let* $\mathcal{T}$ *be a tensor of order* k *and dimension* n. *Then* λ *is a Pareto H-eigenvalue of* $\mathcal{T}$ *if and only if there exists* I *with* $\emptyset \neq I \subseteq [n]$ *and* $\mathbf{y} \in \mathbb{R}^{|I|}_{++}$ *satisfying*

$$\mathcal{T}_I \mathbf{y}^{k-1} = \lambda \mathbf{y}^{[k-1]},$$

and

$$\sum_{i_2 \in I} \cdots \sum_{i_k \in I} t_{i i_2 \dots i_k} y_{i_2} \dots y_{i_k} \geq 0 \text{ for } i \in [n] \setminus I.$$

Furthermore, a Pareto H-eigenvector $\mathbf{x}$ *of* $\mathcal{T}$ *associated to* λ *is given by*

$$x_i = \begin{cases} y_i & \text{if } i \in I, \\ 0 & \text{if } i \in [n] \setminus I. \end{cases}$$

Given a positive integer $k \geq 2$, a k-uniform hypergraph G consists of a finite set of vertices $V(G)$ a set of hyperedges (or simply edges) and $E(G) \subseteq 2^{V(G)}$ such that each edge contains exactly k vertices, where $2^{V(G)}$ denotes the power set of $V(G)$. We call the numbers of vertices and edges of G as the order and size of G, respectively. A uniform hypergraph is a k-uniform hypergraph for some k. A linear hypergraph is one in which every two distinct edges intersect in at most one vertex. Let H be an ordinary graph (i.e., a 2-uniform hypergraph). For any $k \geq 3$, the kth power of H, denoted by H^k, is defined as the k-uniform

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hypergraph with edge set $E(H^k) = \{e \cup \{i_{e,1}, \ldots, i_{e,k-2}\} : e \in E(H)\}$ and vertex set $V(H^k) = V(H) \cup \{i_{e,j} : e \in E(H), j \in [k-2]\}$.

Let G be a k-uniform hypergraph. For $v \in V(G)$, denote by $E_G(v)$ the set of edges containing v, and the degree of v in G, denoted by $d_G(v)$ or simply d_v, is $|E_G(v)|$. A vertex is called a pendant vertex if its degree is one, and an edge e is a pendant edge (at v) if v is the only vertex of e with degree more than one. A hypergraph G is r-regular if the degree of each vertex is r. Let U be a proper nonempty subset of $V(G)$, $G - U$ denotes the hypergraph obtained from G by deleting the vertices of U and the edges containing at least one vertex of U. In particular, we write $G - u$ for $G - \{u\}$ if $U = \{u\}$.

A walk is an alternating sequence $v_1, e_1, v_2, e_2, \ldots, e_\ell, v_{\ell+1}$ such that edge e_i contains vertices v_i and v_{i+1} for $i = 1, \ldots, \ell$. The value ℓ is the length of this walk. A path is a walk with all v_i distinct and all e_i distinct. A cycle is a walk containing at least two edges, all e_i are distinct and all v_i are distinct except $v_1 = v_{\ell+1}$. If G is connected and acyclic, then G is called a hypertree. If G is connected and contains exactly one cycle, then G is called a unicyclic hypergraph. It is evident that a hypertree is a linear hypergraph, while a unicyclic hypergraph is linear if the length of its unique cycle is at least three.

Definition 1.5 ([2]). *Let G be a k-uniform hypergraph of order n. The adjacency tensor $\mathcal{A}(G) = (a_{i_1 \ldots i_k})$ of G is defined as*

$$a_{i_1 \ldots i_k} = \begin{cases} \frac{1}{(k-1)!} & \text{if } \{i_1, \ldots, i_k\} \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

The spectral radius of G is the the spectral radius of $\mathcal{A}(G)$, denoted by $\rho(G)$. That is, $\rho(G) = \rho(\mathcal{A}(G))$. The Pareto H-eigenvalues of G are just the Pareto H-eigenvalues of $\mathcal{A}(G)$.

In this paper, we study the Pareto H-eigenvalues of nonnegative tensors and uniform hypergraphs. Among others, we show that the Pareto H-eigenvalues of a nonnegative tensor are just the spectral radii of its weakly irreducible principal subtensors, and we determine those hypergraphs that minimize or maximize the second largest Pareto H-eigenvalue over some classes of uniform hypergraphs.

2. Preliminaries. In this section, we introduce some basic definitions and important lemmas that will be used.

Definition 2.1. *Let $\mathcal{T}$ be a k-order n-dimensional nonnegative tensor. If there exists some I with $\emptyset \neq I \subset [n]$ such that $t_{i_1 \ldots i_k} = 0$ whenever $i_1 \in I$ and $i_j \in [n] \setminus I$ for some $j = 2, \ldots, k$, then, $\mathcal{T}$ is weakly reducible. Otherwise, $\mathcal{T}$ is weakly irreducible.*

The following lemma is the Perron–Frobenius Theorem for nonnegative tensors, see [1, Theorem 1.4], [20, Theorem 2.3], and [4, Theorem 4.1].

Lemma 2.2. *Let $\mathcal{T}$ be a k-order n-dimensional nonnegative tensor. Then*

(i) $\rho(\mathcal{T}) \geq 0$ is an H^+-eigenvalue.
(ii) If $\mathcal{T}$ is weakly irreducible, then $\rho(\mathcal{T})$ is an H^{++}-eigenvalue with a unique positive eigenvector, up to a positive scalar.
(iii) If $\mathcal{T}$ is weakly irreducible and λ is an H-eigenvalue of $\mathcal{T}$ with a positive eigenvector, then $\lambda = \rho(\mathcal{T})$.

A nonnegative vector $\mathbf{x} \in \mathbb{R}^n$ is called k-unit if $\sum_{i=1}^n x_i^k = 1$.

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For a weakly irreducible k-order n-dimensional nonnegative tensor $\mathcal{T}$, Lemma 2.2 (ii) implies that there is a unique k-unit positive eigenvector corresponding to $\rho(\mathcal{T})$, which is called the Perron vector.

The first part of the following lemma was given in [5] and the second part follows from Theorem 2.2.

LEMMA 2.3. *Let $\mathcal{T}$ be a nonnegative tensor of order k and dimension n. Let $\mathcal{T}_1$ be a principal subtensor of $\mathcal{T}$. Then $\rho(\mathcal{T}_1) \leq \rho(\mathcal{T})$. Moreover, if $\mathcal{T}$ is irreducible and $\mathcal{T}_1 \neq \mathcal{T}$, then $\rho(\mathcal{T}_1) < \rho(\mathcal{T})$.*

LEMMA 2.4 ([12]). *Let G be a k-uniform hypergraph on n vertices. Then $\mathcal{A}(G)$ is weakly irreducible if and only if G is connected.*

LEMMA 2.5 ([22]). *If λ is an eigenvalue of a graph G, then $\lambda^{\frac{2}{k}}$ is an eigenvalue of G^k. Moreover, $\rho(G^k) = \sqrt[k]{\rho^2(G)}$.*

For a k-uniform hypergraph G, denote by $\sigma(G)$ the set of Pareto H-eigenvalues of G. If G is an ordinary graph, then $\sigma(G)$ the set of Pareto eigenvalues of G.

LEMMA 2.6 ([15]). *Let G be a connected graph of order n. Then*

$$|\sigma(G)| \geq n,$$

with equality if and only if G is either a star, a path, a cycle, or a clique.

Denote by $S_{m,k}$ the k-uniform hyperstar with m edges, which is a k-uniform hypertree with m edges and there is a common vertex in any edge. In particular, $S_{0,k}$ is a single vertex, while $S_{1,k}$ is a single edge. The ordinary star on $n \geq 1$ vertices is $S_{n-1,2}$, denoted by S_n. For $k \geq 3$, let $U_{m,k}$ be the k-uniform hypergraph consisting of two edges e_1, e_2 with precisely two vertices v_1, v_2 in common if $m = 2$, and the k-uniform hypergraph obtained from $U_{2,k}$ by attaching $m-2$ pendant edges $e_3, \ldots, e_m$ at v_1 if $m \geq 3$, see Figure 1.

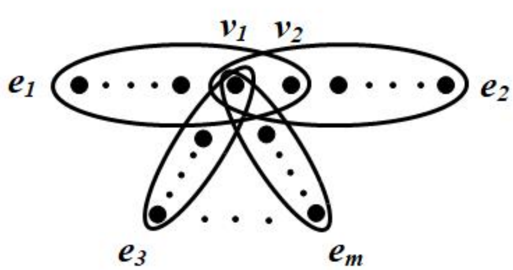

FIGURE 1. *Unicyclic hypergraph $U_{m,k}$.*

LEMMA 2.7 ([6, 11]). *If G is a k-uniform hypertree with m edges, then $\rho(G) \leq \sqrt[k]{m}$, with equality if and only if $G \cong S_{m,k}$. If G is a k-uniform unicyclic hypergraph with $m \geq 2$ edges, then $\rho(G) \leq \rho(U_{m,k})$, with equality if and only if $G \cong U_{m,k}$ when $k \geq 3$.*

From [14, Theorem 2] and its proof, we have the following lemma.

LEMMA 2.8 ([14]). *Let $\mathcal{T}$ be a symmetric nonnegative tensor of order k and dimension n and $\mathbf{x}$ a k-unit vector in $\mathbb{R}^n_+$. Then $\rho(\mathcal{T}) \geq \mathcal{T}x^k$, with equality if and only if $\mathbf{x}$ is an H-eigenvector of $\mathcal{T}$ associated with $\rho(\mathcal{T})$.*

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3. Pareto H-eigenvalue of a nonnegative tensor. In this section, we give necessary and sufficient conditions for Pareto H-eigenvalues of nonnegative tensors.

LEMMA 3.1. *Let $\mathcal{T}$ be a k-order n-dimensional nonnegative tensor with an H^{++}-eigenvalue ρ_0. Then $\rho_0 = \rho(\mathcal{T}_I)$ for some I with $\emptyset \neq I \subseteq [n]$ and $\mathcal{T}_I$ is weakly irreducible.*

Proof. If $\mathcal{T}$ is weakly irreducible, then it follows from Lemma 2.2 that $\rho_0 = \rho(\mathcal{T})$ and hence the result follows by setting $I = [n]$.

Suppose that $\mathcal{T}$ is weakly reducible. Then there exists some J with $\emptyset \neq J \subset [n]$ such that $t_{i_1 \dots i_k} = 0$ whenever $i_1 \in J$ and $i_s \in [n] \setminus J$ for some $s = 2, \dots, k$. Let $\mathbf{x}$ be the positive eigenvector of $\mathcal{T}$ associated to ρ_0. Then, for $i_1 \in J$, one has

$$\rho_0 x_{i_1}^{k-1} = \sum_{i_2,\dots,i_k \in [n]} t_{i_1 \dots i_k} x_{i_2} \dots x_{i_k} = \sum_{i_2,\dots,i_k \in J} t_{i_1 \dots i_k} x_{i_2} \dots x_{i_k} = \left(\mathcal{T}_J \mathbf{x}_J^{k-1}\right)_{i_1},$$

so

$$\rho_0 \mathbf{x}_J^{[k-1]} = \mathcal{T}_J \mathbf{x}_J^{k-1}.$$

This means that ρ_0 is an H^{++}-eigenvalue $\mathcal{T}_J$. If $\mathcal{T}_J$ is weakly irreducible, then by Lemma 2.2, $\rho_0 = \rho(\mathcal{T}_J)$, so we are done by setting $I = J$. Otherwise, by repeating the above process to T_J, we may finally find some I with $\emptyset \neq I \subset J \subset [n]$ such that $\rho_0 = \rho(\mathcal{T}_I)$ and $\mathcal{T}_I$ is weakly irreducible. ∎

Consider the case when $k = 2$ in Lemma 3.1. Note that ρ_0 is not necessarily the spectral radius of each maximal irreducible principal submatrix of $\mathcal{T}$. For example, let

$$\mathcal{T} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}.$$

Evidently, $\mathcal{T}\mathbf{j} = 4\mathbf{j}$ with $\mathbf{j} = (1,1,1)^\top$, and 4 is not the spectral radius of the principal submatrix (3).

THEOREM 3.2. *Let $\mathcal{T}$ be a nonnegative tensor of order k and dimension n. Then, λ is a Pareto H-eigenvalue of $\mathcal{T}$ if and only if there exists some I with $\emptyset \neq I \subseteq [n]$ such that $\mathcal{T}_I$ is weakly irreducible and $\lambda = \rho(\mathcal{T}_I)$.*

Proof. By Theorem 1.4, λ is a Pareto H-eigenvalue of $\mathcal{T}$ if and only if λ is an H^{++}-eigenvalue of A_J for some J with $\emptyset \neq J \subseteq [n]$.

By Lemma 3.1, λ is an H^{++}-eigenvalue of A_J if and only if $\lambda = \rho(\mathcal{T}_I)$ for some I with $\emptyset \neq I \subseteq J$ and $\mathcal{T}_I$ is weakly irreducible. So the result follows. ∎

By Theorem 3.2, any diagonal entry of a nonnegative tensor of order k and dimension n is a Pareto H-eigenvalue of $\mathcal{T}$.

COROLLARY 3.3. *Let $\mathcal{T}$ be a nonnegative tensor of order k and dimension n.*

(i) All Pareto H-eigenvalues of $\mathcal{T}$ are nonnegative.
(ii) $\rho(\mathcal{T})$ is the largest Pareto H-eigenvalue of $\mathcal{T}$.
(iii) 0 is the smallest Pareto H-eigenvalue of $\mathcal{T}$.

Proof. (i) follows from Theorem 1.4 trivially.

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By Lemma 2.2 (i), $\rho(\mathcal{T})$ is an H$^+$-eigenvalue of $\mathcal{T}$, so $\rho(T)$ is a Pareto H-eigenvalue of $\mathcal{T}$. Let λ be any Pareto H-eigenvalue of $\mathcal{T}$. By Theorem 3.2, $\lambda = \rho(\mathcal{T}_I)$ for some I with $\emptyset \neq I \subseteq [n]$. By Lemma 2.3, $\lambda = \rho(\mathcal{T}_I) \leq \rho(\mathcal{T})$. This proves (ii).

Take $i \in [n]$. Let $\mathbf{x} \in \mathbb{R}^n$ with $x_i = 1$ and $x_j = 0$ for $j \in [n] \setminus \{i\}$. Then,

$$\left(\mathcal{T}\mathbf{x}^{k-1}\right)_i = \sum_{i_2 \in [n]} \cdots \sum_{i_k \in [n]} t_{i,i_2,\ldots,i_k} x_{i_2} \ldots x_{i_k} = 0,$$

and

$$\left(\mathcal{T}\mathbf{x}^{k-1}\right)_j = \sum_{i_2 \in [n]} \cdots \sum_{i_k \in [n]} t_{j,i_2,\ldots,i_k} x_{i_2} \ldots x_{i_k} \geq 0.$$

Applying Theorem 1.4 by setting $I = \{i\}$ and (i), we know that 0 is the smallest Pareto H-eigenvalue of $\mathcal{T}$. This proves (iii). ∎

4. Pareto H-eigenvalue of a uniform hypergraph. Let G be a k-uniform hypergraph. By Lemma 2.2, $\rho(G)$ is the largest H$^+$-eigenvalue. If G is connected, then by Lemmas 2.4 and 2.2, $\rho(G)$ is the largest H^{++}-eigenvalue. Recall that $\sigma(G)$ denotes the set of Pareto H-eigenvalues of G.

Theorem 4.1. *Let G be a connected k-uniform hypergraph on n vertices. Then,*

$$\sigma(G) = \{\rho(H) : H \text{ is a connected induced subhypergraph of } G\}.$$

Proof. Let λ be any Pareto H-eigenvalue of G. By Theorem 3.2, $\lambda = \rho(\mathcal{A}(G)_I)$ for some I with $\emptyset \neq I \subseteq [n]$ such that $\mathcal{A}(G)_I$ is weakly irreducible. Let H be the subhypergraph of G induced by I. By Lemma 2.4, H is connected. Note that $\mathcal{A}(H) = \mathcal{A}(G)_I$. So $\lambda = \rho(\mathcal{A}(G)_I) = \rho(H)$. Thus,

$$\sigma(G) \subseteq \{\rho(H) : H \text{ is a connected induced subhypergraph of } G\}.$$

Conversely, if H is a connected induced subhypergraph of G, then $\mathcal{A}(H)$ is a principal subtensor of $\mathcal{A}(G)$, and by Proposition 2.4, $\mathcal{A}(H)$ is weakly irreducible, so by Theorem 3.2, $\rho(H) = \rho(\mathcal{A}(H))$ is a Pareto H-eigenvalue of G. So

$$\sigma(G) \supseteq \{\rho(H) : H \text{ is a connected induced subhypergraph of } G\}.$$

This completes the proof. ∎

Let G be a k-uniform hypergraph. By Theorem 4.1 and Lemma 2.4, $\rho(G)$ is the largest Pareto H-eigenvalue of G. For a k-uniform hypergraph with at least one edge, we denote by $\lambda_2(G)$ the second largest Pareto H-eigenvalue of G.

Proposition 4.2. *Let G be a connected k-uniform hypergraph with at least one edge. Then,*

$$\lambda_2(G) = \max\{\rho(G - v) : v \in V(G)\}.$$

Proof. Let $r = \max\{\rho(G - v) : v \in V(G)\}$, say $r = \rho(G - w)$ with $w \in V(G)$. By Lemma 2.3, r is the largest spectral radius among all proper induced subhypergraphs of G whether $G - w$ is connected or not. By Theorem 4.1, $\lambda_2(G) = r$. ∎

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For a hypergraph G, denote by $S(G)$ the set of representatives of all isomorphic connected induced subhypergraphs of G. A connected k-uniform hypergraph G is said to be spectrally scattered if for any connected induced subhypergraph H of G, the spectral radius of $\mathcal{A}(H)$ are all different. Let G be a connected k-uniform hypergraph, then from Theorem 4.1,

$$|\sigma(G)| \leq |S(G)|,$$

with equality if and only if G is spectrally scattered.

Let $P_{m,k}$ be the k-uniform hyperpath with m edges. Particularly, $P_{0,k}$ is a single vertex. The ordinary path on n vertices is $P_n = P_{n-1,2}$. Let $C_{m,k}$ be the k-uniform hypercycle with m edges, where $m \geq 2$ if $k \geq 3$ and $m \geq 3$ if $k = 2$. The ordinary cycle on $n \geq 3$ vertices is $C_n = C_{n,2}$. Let K_n be the complete graph on vertices.

Theorem 4.3. *Let G be a connected graph with $n \geq 3$ vertices. For $k \geq 3$,*

$$|\sigma(G^k)| \geq n,$$

with equality if and only if G is a path or a star.

Proof. Note that

$$\{H^k : H \in S(G)\} \subseteq S(G^k).$$

By Theorem 4.1, we have

$$\sigma(G^k) = \{\rho(H) : H \in S(G^k)\} \supseteq \{\rho(H^k) : H \in S(G)\}.$$

Now, by Lemmas 2.5 and 2.6, we have $|\sigma(G^k)| \geq |\sigma(G)| \geq n$.

Suppose that $|\sigma(G^k)| = n$. By the above argument, $|\sigma(G)| = n$, so by Lemma 2.6, $G \cong S_n, P_n, C_n, K_n$. Obviously, $S(S_n^k) = \{S_1^k, S_2^k, \dots, S_n^k\}$ and $S(P_n^k) = \{P_1^k, P_2^k, \dots, P_n^k\}$. Let U be the set of the $k-2$ vertices of degree one in an arbitrary but fixed edge of C_n^k. Then, $C_n^k - U \cong P_n^k$. So $S(C_n^k) = \{P_1^k, P_2^k, \dots, P_n^k, C_n^k\}$. Note that $\rho(S_t) = \sqrt{t-1}$, $\rho(P_t) = 2\cos\frac{\pi}{t+1}$ and if $t \geq 3$, $\rho(C_t) = 2$. By Lemma 2.5, $\rho(S_t)^k = \sqrt[k]{t-1}$, $\rho(P_t^k) = \sqrt[k]{4\cos^2\frac{\pi}{t+1}}$ and if $t \geq 3$, $\rho(C_t^k) = \sqrt[k]{4}$. Thus, we have

$$\sigma(S_n^k) = \{\sqrt[k]{t-1} : t = 1, \dots, n\},$$

$$\sigma(P_n^k) = \left\{\sqrt[k]{4\cos^2\frac{\pi}{t+1}} : t = 1, \dots, n\right\},$$

and

$$\sigma(C_n^k) = \sigma(P_n^k) \cup \{\sqrt[k]{4}\}.$$

Hence, $|\sigma(P_n^k)| = |\sigma(S_n^k)| = n$ and $|\sigma(C_n^k)| = n+1$. It remains to check the size of $\sigma(K_n^k)$ with $n \geq 3$. For any edge $e \in E(K_n^k)$, deleting the $k-2$ vertices of e from K_n^k results in $(K_n - e)^k$, which is is a connected, so

$$\{K_1^k, K_2^k, \dots, K_n^k, (K_n - e)^k\} \subseteq S(K_n^k).$$

For $i = 1, \dots, n-1$, K_i is a proper subgraph of $K_n - e$ and $K_n - e$ is a proper subgraph of K_n, so, for $i = 1, \dots, n$, we have $\rho(K_i) < \rho(K_n - e)$. Now by Lemma 2.5 that $|\sigma(K_n^k)| > n$. It follows that G is a star or a path.

Conversely, if G is a star or a path, then, as above, it is easy to see that $|\sigma(G^k)| = n$. ∎

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THEOREM 4.4. *Let G be a connected linear k-uniform hypergraph with diameter d. Then, $a_0 = 0$, $a_1 = 1$, $a_2 = \sqrt[k]{2}, \ldots, a_d = \sqrt[k]{4\cos^2 \frac{\pi}{d+2}}$ are always the Pareto H-eigenvalues of G.*

Proof. Note that G has an induced subhypergrraph that is isomorphic to $P_{d,k}$. By Theorem 4.1, we have $\sigma(P_{d,k}) \subseteq \sigma(G)$. ∎

Denote by $D_{m,k;1}$ the k-uniform hypertree obtained from $S_{m-1,k}$ by attaching a pendant edge at a vertex of degree one, where $m \geq 3$.

THEOREM 4.5. *Let G be a k-uniform hypertree with $m \geq 1$ edges. Then, $\lambda_2(G) \leq \sqrt[k]{m-1}$, with equality if and only if $G \cong S_{m,k}$, or $m \geq 3$ and $G \cong D_{m,k;1}$.*

Proof. If $m = 1, 2$, then $G \cong S_{m,k}$ with $\lambda_2(G) = \rho(S_{m-1,k}) = \sqrt[k]{m-1}$, so the result holds. Suppose that $m \geq 3$. By Theorem 4.1, $\lambda_2(G) \leq \rho(H)$ for some connected proper induced subhypergraph H of G. It is evident that H is a k-uniform hypertree with at most $m-1$ edges. By Lemma 2.7, $\rho(H) \leq \sqrt[k]{m-1}$, with equality if and only if $H \cong S_{m-1,k}$. It thus follows that $\lambda_2(G) \leq \sqrt[k]{m-1}$, with equality if and only if $H \cong S_{m-1,k}$, or equivalently, $G \cong S_{m,k}$ or $D_{m,k;1}$. ∎

LEMMA 4.6. *Let T be a tree with $m \geq 2$ edges. Then,*

$$\lambda_2(T^k) \geq \sqrt[k]{4\cos^2 \frac{\pi}{m+1}},$$

with equality if and only if T is either a path or a star with three edges.

Proof. Let w be a vertex with degree at least two. Let H be a component of $T - w$ with maximum spectral radius. Let v be a pendant vertex of T that belongs to a component of $T - w$ different from H. Then, H is a proper induced subgraph of $T - v$. By Lemmas 2.4 and 2.3, $\rho(H) < \rho(T - v)$. So, we have by Lemma 2.5 that $\rho(T^k - w) = \rho((T - w)^k) = \rho(H^k) < \rho((T - v)^k)$.

Let $e = xy$ be an edge of T that is not a pendant edge. For any vertex $i_{e,j}$ of T^k, we denote by G the component of $T^k - i_{e,j}$ with maximum spectral radius. We may assume that $G = T_1^k$, where T_1 is a component of $T - x$ or $T - y$. Letting v be a pendant vertex of T lying outside T_1, T_1 is a proper induced subgraph of $T - v$. So, by Lemmas 2.4 and 2.3, $\rho(T_1) < \rho(T - v)$. By Lemma 2.5, $\rho(T^k - i_{e,j}) = \rho(T_1^k) < \rho((T - v)^k)$. Therefore, we have by Proposition 4.2 that

$$\begin{aligned}\lambda_2(T^k) &= \max\{\rho(T^k - z) : z \in V(T^k)\}\\ &= \max\{\rho((T - z)^k) : z \text{ is a pendant vertex of } T\}.\end{aligned}$$

Let z be a pendant vertex of T. By a classical result due to Lovász and Pelikán [10], $\rho(T - z) \geq 2\cos \frac{\pi}{m+1}$, with equality if and only if $T - z$ is a path. By Lemma 2.5, $\lambda_2(T^k) \geq \sqrt[k]{4\cos^2 \frac{\pi}{m+1}}$, with equality if and only if for any pendant vertex z of T, $T - z$ is a path, or equivalently, T is a either path or a star with three edges. ∎

Let $e_1, \ldots, e_m$ be the edges of a k-uniform hyperpath $P_{m,k}$ with $m \geq 3$ such that $v_i, v_{i+1} \in e_i$ for $i = 1, \ldots, m$, where v_1, v_{m+1} are pendant vertices and the degree of $v_2, \ldots, v_m$ are all of degree two. Then e_1 and e_m are pendant edges. We call e_1 (or e_m) the first edge and e_2 (or e_{m-1}) the second edge of $P_{m,k}$.

THEOREM 4.7. *Let G be a k-uniform hypertree with $m \geq 2$ edges. Then,*

$$\lambda_2(G) \geq \sqrt[k]{4\cos^2 \frac{\pi}{m+1}},$$

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with equality if and only if $G \cong P^k_{m+1}$, S^k_3, *or* G *is obtainable from a* $P_{3,k}$ *by attaching a pendant edge at a vertex of degree one of the second edge.*

Proof. If $G \cong T^k$ for some tree T, then it follows from Lemma 4.6 that

$$\lambda_2(G) \geq \sqrt[k]{4\cos^2\frac{\pi}{m+1}},$$

with equality if and only if $G \cong P^k_{m+1}$, S^k_3.

Now suppose that $G \ncong T^k$ for any tree T. Then, $m \geq 4$. By Proposition 4.2, $\lambda_2(G) = \max\{\rho(G - z) : z \in V(G)\}$, say $\lambda_2(G) = \rho(G - v)$ with $v \in V(G)$. It is easy to see that v is a pendant vertex in a pendant edge. So $\lambda_2(G) = \rho(H)$, where H is a k-uniform hypertree with $m - 1$ edges that is a proper subhypergaph of G. By [21, Theorem 2], $P_{m-1,k}$ uniquely minimizes the spectral radius among all k-uniform hypertrees with $m - 1$ edges. So

$$\rho(H) \geq \sqrt[k]{4\cos^2\frac{\pi}{m+1}},$$

with equality if and only if $H \cong P_{m-1,k}$. Therefore, $\lambda_2(G) \geq \sqrt[k]{4\cos^2\frac{\pi}{m+1}}$, with equality if and only if $H \cong P_{m-1,k}$ and for any proper subhypergraph H' of G, $\rho(H') \leq \rho(H)$, that is, G is obtainable from a k-uniform hyperpath with 3 edges by attaching a pendant edge at a vertex of degree one in the second edge. □

Theorem 4.8. *Let* G *be a* k*-uniform unicyclic hypergraph with* $m \geq 3$ *edges, where* $k \geq 3$*. Then,* $\lambda_2(G) \leq \sqrt[k]{m+1}$*, with equality if and only if* $G \cong U_{m,k}$, G_1, G_2*, or* $m \geq 4$ *and* $G \cong G_3$*, where* G_1*,* G_2*, and* G_3 *are obtained from* $U_{m-1,k}$ *by attaching a pendant edge at a vertex of degree 2 , a pendant vertex of* e_2 *and a pendent vertex of* e_3*, respectively, see Figure 2.*

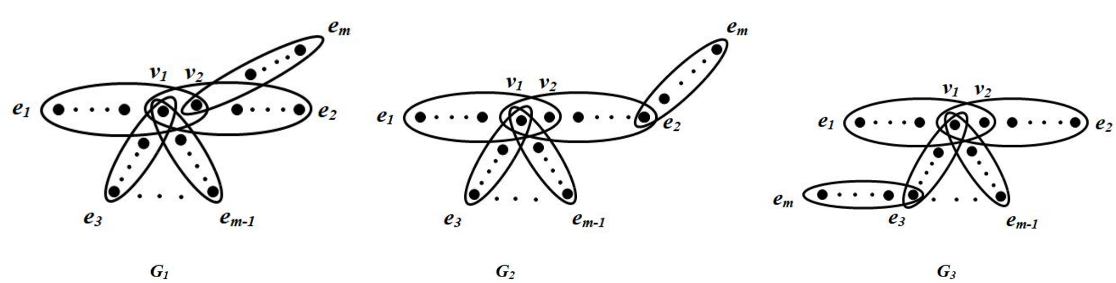

Figure 2. *Unicyclic hypergraphs* G_1*,* G_2 *and* G_3*.*

Proof. By Theorem 4.1, $\lambda_2(G) \leq \rho(H)$ for some connected proper induced subhypergraph H of G. Evidently, $|E(H)| \leq m - 1$, and H is either a k-uniform hypertree or a k-uniform unicyclic hypergraph.

Claim 1. $\rho(U_{m-1,k}) = \sqrt[k]{m+1} > \rho(S_{m-1,k})$.

Let $\varrho = \rho(U_{m-1,k})$ and $\mathbf{x}$ be the unique k-unit positive eigenvector of $\mathcal{A}(U_{m-1,k})$ associated with ϱ. Let a and b be the entries of $\mathbf{x}$ corresponding to the vertices of degree $m - 1$ and 2, respectively. By symmetry, the entries $\mathbf{x}$ corresponding to the pendant vertices outside e_1 and e_2 are all equal, which we denote by c, and the entries $\mathbf{x}$ corresponding to the pendant vertices in $e_1 \cup e_2$ are all equal, which we denote by d. Then, ϱ satisfies the following equations:

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$$\varrho a^{k-1} = 2bd^{k-2} + (m-3)c^{k-1},$$
$$\varrho b^{k-1} = 2ad^{k-2},$$
$$\varrho c^{k-1} = ac^{k-2},$$
$$\varrho d^{k-1} = abd^{k-3}.$$

So

$$a = \frac{\varrho}{\sqrt[k]{2}}d, \; b = \sqrt[k]{2}d, \; c = \frac{1}{\sqrt[k]{2}}d.$$

Thus, ϱ is the largest real root of the equation $\rho^k - m - 1 = 0$. It follows that $\varrho = \sqrt[k]{m+1}$. Note that $\rho(S_{m-1,k}) = \sqrt[k]{m-1}$. So Claim 1 follows.

As H is either a k-uniform hypertree or a k-uniform unicyclic hypergraph, and $|E(H)| \leq m-1$, we have by Lemma 2.7 and Claim 1 that $\rho(H) \leq \sqrt[k]{m+1}$, with equality if and only if $H \cong U_{m-1,k}$. So $\lambda_2(G) \leq \sqrt[k]{m+1}$, with equality if and only if $H \cong U_{m-1,k}$, or equivalently $G \cong U_{m,k}, G_1, G_2$, or G_3. □

THEOREM 4.9. *Let G be a connected r-regular k-uniform hypergraph. Then,*

$$\lambda_2(G) \geq \frac{n-k}{n-1} r,$$

with equality if and only if $G - v$ is regular for some $v \in V(G)$.

Proof. As G is connected and r-regular, $\mathbf{x}$ with $x_w = \frac{1}{\sqrt[k]{n}}$ for any $w \in V(G)$ is the k-unit positive eigenvector associated with $\rho(G) = r$.

Let $v \in V(G)$. Let $\mathbf{y}$ the restriction of $\mathbf{x}$ on $V(G)\backslash\{v\}$. As $\rho(G)x_v^{k-1} = (\mathcal{A}(G)x^{k-1})_v = \sum_{e \in E_G(v)} x^{e\backslash\{v\}}$, we have

$$\rho(G)x_v^k = x_v \sum_{e \in E_G(v)} x^{e\backslash\{v\}} = \sum_{e \in E_v(G)} x^e.$$

Thus,

$$\rho(G) = k \sum_{\substack{e \in E(G) \\ v \notin e}} \mathbf{x}^e + k \sum_{e \in E_v(G)} \mathbf{x}^e = \mathcal{A}(G-v)\mathbf{y}^k + k\rho(G)x_v^k.$$

That is, $\mathcal{A}(G-v)\mathbf{y}^k = \rho(G)(1 - kx_v^k)$. By Lemma 2.8, we have

$$\rho(G-v) \geq \frac{\mathcal{A}(G-v)\mathbf{y}^k}{\|\mathbf{y}^k\|} = \frac{\rho(G)(1-kx_n^k)}{1-x_v^k} = \frac{n-k}{n-1}r,$$

and equality holds in the above inequality if and only if $\mathbf{y}$ is an eigenvector of $G-v$ associated to $\rho(G-v)$, i.e., $G-v$ is regular. Now the result follows from Proposition 4.2. □

We mention that a hypergraph that attains the bound in Theorem 4.9 is not necessarily a complete hypergraph. For example, let $V(G) = [7]$ and

$$E(G) = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,3,4\},\{2,3,5\},\{4,6,7\},\{5,6,7\}\}.$$

Then, G is a connected 3-regular 3-uniform hypergraph and $G-1$ is 2-regular. Note that G is not complete as $\{1,3,4\} \notin E(G)$. A more general example is as follows. A Steiner system $S(t,k,n)$ with $n > k \geq t \geq 2$ is a k-uniform hypergraph on n vertices, such that every t-subset of the vertices is contained in precisely one

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edge [16]. Let G be a Steiner system $S(t,k,n)$. Then, G is connected and $\frac{n-1}{k-1}$-regular. Let $\{u,v\} \subset V(G)$. The number of edges containing u and v is $a_2 = \frac{\binom{n-2}{t-2}}{\binom{k-2}{t-2}}$. So $G-u$ is $\left(\frac{n-1}{k-1} - a_2\right)$-regular.

Finally, we mention a related result from [8], where the bound in Theorem 4.9 is also given. Let G be a connected k-uniform linear hypergraph on n vertices with minimum degree δ, where $n > k \geq 2$. Then $\lambda_2(G) \geq \rho(G) - \sqrt[k-1]{\frac{\delta}{\rho(G)}}$, with equality if and only if G is a Steiner system $S(2,k,n)$.

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