Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 307-326, May 2023.

POSITIVE AND NEGATIVE SQUARE ENERGIES OF GRAPHS[*]

AIDA ABIAD[†], LEONARDO DE LIMA[‡], DHEER NOAL DESAI[§], KRYSTAL GUO[¶], LESLIE HOGBEN[‖], AND JOSÉ MADRID[#]

Abstract. The energy of a graph G is the sum of the absolute values of the eigenvalues of the adjacency matrix of G. Let $s^+(G), s^-(G)$ denote the sum of the squares of the positive and negative eigenvalues of G, respectively. It was conjectured by [Elphick, Farber, Goldberg, Wocjan, *Discrete Math.* (2016)] that if G is a connected graph of order n, then $s^+(G) \geq n-1$ and $s^-(G) \geq n-1$. In this paper, we show partial results towards this conjecture. In particular, numerous structural results that may help in proving the conjecture are derived, including the effect of various graph operations. These are then used to establish the conjecture for several graph classes, including graphs with certain fraction of positive eigenvalues and unicyclic graphs.

Key words. Graph eigenvalues, Adjacency matrix, Inertia of a graph, Energy.

AMS subject classifications. 05C50, 15A18, 15A42.

1. Introduction and preliminaries.

Suppose $G = (V(G), E(G))$ is a graph on $n = |V(G)|$ vertices with adjacency matrix $A(G)$. Let π_G be the number of positive eigenvalues and ν_G be the number of negative eigenvalues and number of the eigenvalues of $A(G)$ in decreasing order, so $\lambda_1 \geq \cdots \geq \lambda_{\pi_G} > 0$ are the positive eigenvalues and $0 > \lambda_{n-\nu_G+1} \geq \cdots \geq \lambda_n$ are the negative eigenvalues. The *energy* $\mathcal{E}(G)$ of G is defined by $\mathcal{E}(G) = \sum_{i=1}^{\pi_G} \lambda_i - \sum_{j=n-\nu_G+1}^{n} \lambda_j$. Since its introduction in the study of molecular chemistry more than 60 years ago, the energy of a graph has attracted a considerable amount of attention; for an overview see [7].

In this paper, we investigate the positive and negative square energies of a graph, introduced by Wocjan and Elphick in [13] to provide bounds on the chromatic number. The *square positive energy* of G is defined to be $s^+(G) = \sum_{i=1}^{\pi_G} \lambda_i^2$. Similarly, the *square negative energy* of G is defined by $s^-(G) = \sum_{j=n-\nu_G+1}^{n} \lambda_j^2$. The following conjecture is due to Elphick, Farber, Goldberg, and Wocjan.

Conjecture 1.1. [4] *If G is a connected graph of order n, then $s^+(G) \geq n-1$ and $s^-(G) \geq n-1$.*

We should note that in [4], the above conjecture is stated as a minimum, but we will consider it as two separate conjectures, namely that $s^+(G) \geq n-1$ and $s^-(G) \geq n-1$, which may be proved separately.

---

[*] Received by the editors on March 18, 2023. Accepted for publication on May 18, 2023. Handling Editor: Michael Tait. Corresponding Author: Leonardo de Lima.

[†] Department of Mathematics and Computer Science, Eindhoven University of Technology, Eindhoven, The Netherlands (a.abiad.monge@tue.nl). Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Ghent, Belgium. Department of Mathematics and Data Science, Vrije Universiteit Brussel, Brussels, Belgium. Partially supported by the Research Foundation Flanders (FWO) grant 1285921N.

[‡] Departamento de Administração Geral e Aplicada, Federal University of Parana, Curitiba, Brazil (leonardo.delima@ufpr.br).

[§] Department of Mathematical Sciences, University of Delaware, DE, USA. (dheernsd@udel.edu). Department of Mathematics and Statistics, University of Wyoming, WY, USA. (ddesai1@uwyo.edu).

[¶] Korteweg-de Vries Institute, University of Amsterdam, Amsterdam, The Netherlands (k.guo@uva.nl).

[‖] Department of Mathematics, Iowa State University, Ames, IA, USA. American Institute of Mathematics, San Jose, CA, USA. (hogben@aimath.org).

[#] Section de Mathematiques, Universite de Geneve, Geneva, Switzerland (Jose.Madrid@unige.ch).

Conjecture 1.1 has been established for several graph classes in [4]. Using a bound on the chromatic number of a graph using s^+ and s^- established by Ando and Lin in [2], the conjecture was shown to be true for regular graphs (excluding odd cycles) in [4]. Other classes for which the conjecture is known to be true include bipartite graphs, complete q-partite graphs, barbell graphs, hyper-energetic graphs, and graphs with exactly two negative eigenvalues and minimum degree two [4]. As noted in [4], Nikiforov showed in [11] that almost all graphs are hyper-energetic, so almost all graphs satisfy the conjecture. Moreover, a computational search among small graphs was done in [4] and no counterexample was found.

In this paper, we present several results that support Conjecture 1.1. In Section 2, we provide some structural results, including establishing bounds on the effect of several graph operations and graph products. We use a variety of linear algebraic techniques to establish these results, including Perron–Frobenius theory, Rayleigh quotients, eigenvalue interlacing for edge deletion, and quotient matrices. In Section 3, we apply the tools developed in Section 2 to prove that the conjecture holds for various families of graphs, including odd cycles and some other unicyclic graphs, some cactus graphs, and extended barbell graphs. We also find additional bounds. Section 4 contains concluding remarks and directions for future research. Preliminary results and additional background are stated in the remainder of this introduction.

We begin with some graph notation and terminology that will be used throughout. For a graph G, $V(G)$ denotes the set of vertices and $E(G)$ denotes the set of edges. Recall that a graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and H is an *induced subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) = \{uv \in E(G) : u, v \in V(H)\}$. Two vertices u and v of G are *neighbors* (or u and v are *adjacent*) if $uv \in E(G)$. The (open) *neighborhood* of a vertex v, denoted by $N_G(v)$ or $N(v)$ is the set of neighbors of v, and the *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of $v \in V(G)$ is $\deg(v) = |N(v)|$. Let $\bar{d}(G)$ denote the average degree of G. Note $\bar{d}(G) = \frac{2}{n}|E(G)|$ where n is the order of G.

As noted in [4], there is a slightly stronger version of Conjecture 1.1: $s^+(G), s^-(G) \geq n-k$ for a graph G of order n with k connected components. However, in this paper, we will focus on connected graphs, because it is shown in [4] that Conjecture 1.1 implies $s^+(G), s^-(G) \geq n-k$ when G has k connected components. Suppose G is a connected graph of order n. Since the sum of the squares of all of the eigenvalues is equal to $2|E(G)|$ and $|E(G)| \geq n-1$, it immediately follows that

$$\max(s^+(G), s^-(G)) \geq n-1.$$

If G is bipartite, then $s^+(G) = s^-(G) = |E(G)| \geq n-1$, with equality if and only if G is a tree. Furthermore, $s^+(G) + s^-(G) = 2|E(G)|$ implies that Conjecture 1.1 is equivalent to

$$\max(s^+(G), s^-(G)) \leq 2|E(G)| - n + 1.$$

In [14], Wu and Elphick prove that $\max(s^+(G), s^-(G)) \leq \frac{1}{4}\left(\sqrt{8|E(G)|+1}-1\right)^2$. However, $2|E(G)|-n+1 \leq \frac{1}{4}\left(\sqrt{8|E(G)|+1}-1\right)^2$ (with equality for the complete graph).

We observe that the proof of the conjecture for regular graphs (excluding odd cycles) in [4] gives $s^+(G), s^-(G) \geq \frac{2|E(G)|}{\chi(G)}$; loosely speaking, this is useful for a graph with many edges and low chromatic number. By the four color theorem, the conjecture is true for planar graphs with at least $2n-2$ edges, which includes all maximal planar graphs when $n \geq 4$. Another example is the wheel graph W_n (a dominating vertex joined to an $(n-1)$-cycle), because $|E(W_n)| = 2n-2$, so these graphs also satisfy Conjecture 1.1.

Let $\rho(A)$ denote the spectral radius of the matrix A and $\rho(G) = \rho(A(G))$. Note that $\rho(G) = \lambda_1(G)$ by Perron–Frobenius theory since $A(G)$ is non-negative.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 307-326, May 2023.

Remark 1.2. Using Rayleigh quotients, it is easy to see that $\rho(G) = \lambda_1(G) \geq \bar{d}(G)$, so $s^+(G) \geq \bar{d}(G)^2$. Thus, if $\bar{d}(G)^2 \geq n-1$, then $s^+(G) \geq n-1$.

Remark 1.3. It is known that $\rho(G+uv) \geq \rho(G)$, which follows from Rayleigh quotients for non-negative matrices. So if G has a spanning subgraph H for which $\rho(H)^2 \geq n-1$, then $s^+(G) \geq \rho(G)^2 \geq \rho(H)^2 \geq n-1$. In particular,

1. If G has a dominating vertex (a vertex adjacent to every other vertex), then $s^+(G) \geq n-1$, because $\rho(K_{1,n-1}) = \sqrt{n-1}$.
2. If G has a $K_{r,n-r}$ subgraph for $1 \leq r \leq n-1$, then $s^+(G) \geq n-1$, because every bipartite graph satisfies Conjecture 1.1 (see [4]) and a complete bipartite graph has exactly one positive eigenvalue (the spectral radius $\rho(K_{r,n-r})$) and exactly one negative eigenvalue ($-\rho(K_{r,n-r})$).
3. If G has a clique K_{r+1} with $r \geq \sqrt{n-1}$, then $s^+(G) \geq n-1$, because $\rho(K_{r+1}) = r \geq \sqrt{n-1}$.

2. Structural techniques and tools. In this section, we establish bounds on the changes in $s^+(G)$ and $s^-(G)$ caused by graph operations and products and apply quotient matrices to the study of $s^+(G)$ and $s^-(G)$. In particular, in Section 2.1, we show that if G and H satisfy Conjecture 1.1, then so does $G \otimes H$. In Section 2.2, we bound the changes in $s^+(G)$ and $s^-(G)$ caused by removing an edge. In Section 2.3, we bound the changes in $s^+(G)$ and $s^-(G)$ caused by moving neighbors from one vertex to another. Sections 2.4 and 2.5 apply interlacing to induced subgraphs and to vertex partitions via quotient matrices. Section 2.6 uses quotients of equitable partitions to determine spectra of graphs with twins.

We begin with a simple result for joins and its application to threshold graphs. For disjoint graphs G and H, the *join* of G and H, denoted by $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{gh : g \in V(G), h \in V(H)\}$. A *threshold graph* is a graph that can be constructed from a one-vertex graph by repeated applications of the following two operations: (1) addition of a single isolated vertex to the graph; (2) addition of a single dominating vertex to the graph, i.e., a single vertex that is connected to all other vertices.

Lemma 2.1. *Let G, H be two nonempty disjoint graphs with $|V(G)| + |V(H)| = n$. Then,*

$$s^+(G \vee H) \geq n-1.$$

Proof. Note that $G \vee H$ contains $K_{s,n-s}$ as a subgraph for some $1 \leq s \leq n-1$, so $s^+(G \vee H) \geq n-1$ by Remark 1.3. □

The method in Lemma 2.1 does not resolve the conjecture for $s^-(G \vee H)$. Note that there is a tight example from taking $G = K_1$ and $H = K_{n-1}$ shows that we can have $s^-(G \vee H) = n-1$.

The following result is immediate from Lemma 2.1 and the definition of threshold graph.

Corollary 2.2. *If G is a connected threshold graph of order n, then $s^+(G) \geq n-1$.*

2.1. Kronecker product. The *Kronecker product* (also called the *tensor product*) of two graphs G and H, denoted $G \otimes H$, is the graph with vertex set $V(G) \times V(H)$ where (g, h) and (g', h') are adjacent whenever $g \sim g'$ in G and $h \sim h'$ in H. The adjacency matrix of $G \otimes H$ is $A(G) \otimes A(H)$, where $\otimes$ denotes the Kronecker product of matrices.

Proposition 2.3. *Suppose G and H are graphs of orders n and m, respectively, that satisfy $s^+(G)$, $s^-(G) \geq n-1$, $s^+(H), s^-(H) \geq m-1$, and $m, n \geq 3$. Then $s^+(G \otimes H), s^-(G \otimes H) \geq mn-1$.*

Proof. Let λ_i, μ_j denote the eigenvalues of G, H with $\lambda_{\pi_G} > 0, \lambda_{\pi_G+1} \leq 0$ and $\mu_{\pi_H} > 0,\ \mu_{\pi_H+1} \leq 0$. The eigenvalues of $G \otimes H$ are $\lambda_i \mu_j$. Thus

$$\begin{aligned} s^+(G \otimes H) &= \sum_{i \leq \pi_G, j \leq \pi_H} \lambda_i^2 \mu_j^2 + \sum_{i > \pi_G, j > \pi_H} \lambda_i^2 \mu_j^2 \\ &= s^+(G)s^+(H) + s^-(G)s^-(H) \\ &\geq 2(n-1)(m-1) \\ &= 2nm - 2m - 2n + 2 \geq nm - 1, \end{aligned}$$

where in the last step we used that $nm + 1 \geq 2(m + n - 1)$, which is equivalent to $(m-2)(n-2) \geq 1$ and which is valid for $m, n \geq 3$. A similar computation shows that the same holds for $s^-(G \otimes H)$. ☐

Note that the conclusion of Proposition 2.3 can be false if, for example, $G = K_2$ and H is a tree (so H satisfies the conjecture with equality) because $G \otimes H$ is two copies of H. However, this is not a counterexample to the more general conjecture (since $G \otimes H$ has 2 components and does satisfy $s^+(G \otimes H), s^-(G \otimes H) = mn - 2$) nor to the proposition (since we require $m, n \geq 3$).

We note that this naive method did not work for the Cartesian, categorical, or strong products of graphs.

2.2. Removing an edge. In this section, we use a result on edge interlacing for the adjacency matrix established by Hall, Patel, and Stewart in [8] to investigate what we can say regarding the conjecture when we consider the subgraph obtained on deleting an edge in the original graph.

THEOREM 2.4. [8] *Let G be a graph and let $H = G - e$, where e is an edge of G. If $\lambda_1 \geq \cdots \geq \lambda_n$ and $\theta_1 \geq \cdots \geq \theta_n$ denote the eigenvalues of $A(G)$ and $A(H)$, respectively, then*

$$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1} \qquad \text{for } i = 2, 3, \ldots, n-1,$$

and $\theta_1 \geq \lambda_2$ and $\theta_n \leq \lambda_{n-1}$.

We use Theorem 2.4 to obtain the next result, which gives lower bounds for $s^+(G)$ and $s^-(G)$ in terms of $s^+(G-e)$ and $s^-(G-e)$. The slightly stronger bound for $s^+(G)$ is obtained by further combining Theorem 2.4 with the known fact that the spectral radius of G is at least as much as the spectral radius of $G-e$. Recall that it is known that Conjecture 1.1 is true for a graph that has exactly one positive eigenvalue or exactly one negative eigenvalue [4], so the restriction to having at least two positive and two negative eigenvalues does not reduce the usefulness of the result.

THEOREM 2.5. *Let G be a graph, let $H = G - e$ (where e is an edge of G), and let $\lambda_1 \geq \cdots \geq \lambda_n$ and $\theta_1 \geq \cdots \geq \theta_n$ denote the eigenvalues of $A(G)$ and $A(H)$, respectively. If H has at least two positive eigenvalues and at least two negative eigenvalues, then $s^+(G) \geq s^+(H) - \theta_2^2$ and $s^-(G) \geq s^-(H) - \theta_n^2$.*

Proof. Let $\pi_H \geq 2$ be the number of positive eigenvalues of H, and $\nu_H \geq 2$ be the number of negative eigenvalues of H. Theorem 2.4 gives us that

$$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1} \qquad \text{for } i = 2, \ldots, n-1,$$

$\theta_1 \geq \lambda_2$ and $\theta_n \leq \lambda_{n-1}$. Further, we may also observe that $\lambda_1 \geq \theta_1$ since $H \subset G$.

We thus have that

$$\lambda_{i-1}^2 \geq \theta_i^2, \quad \text{for } i = 2, \ldots, \pi_H,$$

and $\lambda_1^2 \geq \theta_1^2 \geq \lambda_2^2$.

Analogously, we have that

$$\lambda_{i+1}^2 \geq \theta_i^2, \quad \text{for } i = n - \nu_H + 1, \ldots, n-1,$$

and $\theta_n^2 \geq \lambda_{n-1}^2$.

Thus, we have that

$$s^+(G) \geq \sum_{i=2}^{\pi_H} \lambda_{i-1}^2 = \lambda_1^2 + \sum_{i=3}^{\pi_H} \lambda_{i-1}^2 \geq \theta_1^2 + \sum_{i=3}^{\pi_H} \theta_i^2 = s^+(H) - \theta_2^2,$$

and

$$s^-(G) \geq \sum_{i=n-\nu_H+1}^{n-1} \lambda_{i+1}^2 \geq \sum_{i=n-\nu_H+1}^{n-1} \theta_i^2 = s^-(H) - \theta_n^2,$$

as claimed. □

One might think that deleting an edge from G cannot raise s^+ or s^-, in which case upper bounds on the possible increase in s^+ and s^- caused by deleting an edge would not be needed (and the conjecture could be proved by reducing to a spanning tree). But this is not the case: Observe that $s^-(K_n - e) > s^-(K_n)$ for $n \geq 4$. Finding an example of a graph G with $s^+(G-e) > s^+(G)$ is more challenging. Clive Elphick raised this issue in 2011 and Chris Godsil found 5 graphs of order 9 with this property. The details for one of these graphs are given in the next example.

Example 2.6. Let graph G and edge e be as shown in Figure 1. Then $s^+(G) = 44.8424707858$ and $s^+(G-e) = 44.8475921409$, so s^+ is increased by more than 0.005 by deleting edge e.

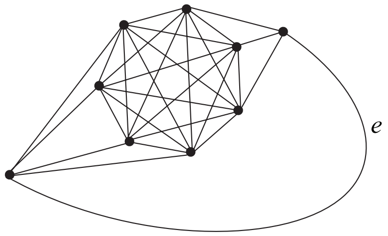

Figure 1. *A graph G and edge e where deletion of e increases s^+.*

We obtain the next result as an immediate corollary to Theorem 2.5.

Corollary 2.7. *If G is a graph on n vertices and $H = G - e$ (where e is an edge of G) satisfies $s^+(H) - \theta_2^2 \geq n-1$ and $s^-(H) - \theta_n^2 \geq n-1$, then $s^+(G), s^-(G) \geq n-1$.*

2.3. Moving neighbors from one vertex to another. In this section, we will focus on the following operation on graphs discussed in [15]. For a graph G with two vertices u and v and a set of vertices $\{w_1, w_2, \ldots w_r\} \subseteq N_G(v) \setminus (N_G(u) \cup \{u\})$, let $G_{u,v}$ denote the graph with vertex set $V(G_{u,v}) = V(G)$ and edge set $E(G_{u,v}) = (E(G) \setminus \{vw_i \text{ for } 1 \leq i \leq r\}) \cup \{uw_i \text{ for } 1 \leq i \leq r\}$. We say that $G_{u,v}$ is the graph obtained by *moving the neighbors* $\{w_1, w_2, \ldots, w_r\}$ *of* v *to* u (note that defining $G_{u,v}$ as moving the neighbors $\{w_1, w_2, \ldots, w_r\}$ of v to u implies $\{w_1, w_2, \ldots, w_r\}$ satisfy the condition of the definition). An example of this operation is shown in Figure 2. Note that the symbol $G_{u,v}$ can denote more than one graph, since the set of vertices moved is not embedded in the notation.

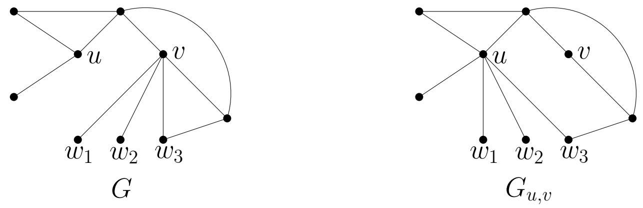

FIGURE 2. *Graph $G_{u,v}$ is obtained from G by moving the neighbors $\{w_1, w_2, w_3\}$ of v to u.*

Wu, Shao, and Liu proved the next result describing interlacing of the spectra of G and $G_{u,v}$ in [15].

THEOREM 2.8. [15] *Let G be a graph, let u, v be two vertices of G, and let $G_{u,v}$ be the graph obtained from G by moving the neighbors $\{w_1, w_2, \dots, w_r\}$ of v to u. Denote the eigenvalues of $A(G)$ and $A(G_{u,v})$ by $\lambda_1 \geq \cdots \geq \lambda_n$ and $\theta_1 \geq \cdots \geq \theta_n$, respectively. Then*

$$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1}, \text{ for } i = 2, 3, \dots, n-1, \ \theta_1 \geq \lambda_2, \text{ and } \lambda_{n-1} \geq \theta_n.$$

In the next result, we apply the preceding theorem to two special cases to improve the bound on the spectral radius.

LEMMA 2.9. *Let G be a graph with spectral radius $\rho(G)$ and Perron vector $\mathbf{x} = [x_i]$ and let $G_{u,v}$ be the graph obtained by moving $\{w_1, w_2, \dots, w_r\}$ from v to u. If*

1. $x_u \geq x_v$, *or*
2. $N_G(v) \cap (N_G(u) \cup \{u\}) = \emptyset$ *and* $\{w_1, w_2, \dots, w_r\} = N_G(v)$,

then $\rho(G_{u,v}) \geq \rho(G)$. Furthermore, if G is connected, then $\rho(G_{u,v}) > \rho(G)$.

Proof. Let $\theta_1 = \rho(G_{u,v})$ and $\lambda_1 = \rho(G)$. For a graph $H = (V(H), E(H))$ with spectral radius $\rho(H)$, and Perron vector $\mathbf{h}$, the Rayleigh quotient characterization of the spectral radius gives

$$\rho(H) = \max_{\mathbf{z} \neq \mathbf{0}} \frac{\sum_{ij \in E(H)} 2 z_i z_j}{\sum_{i \in V(H)} z_i^2} = \frac{\sum_{ij \in E(H)} 2 h_i h_j}{\sum_{i \in V(H)} h_i^2},$$

where $\mathbf{z}$ runs over all non-zero vectors. To prove the result, we show that with either of the hypotheses,

$$\theta_1 - \lambda_1 = \left(\max_{\mathbf{z} \neq \mathbf{0}} \frac{\sum_{ij \in E(G_{u,v})} 2 z_i z_j}{\sum_{i \in V(G_{u,v})} z_i^2} \right) - \frac{\sum_{ij \in E(G)} 2 x_i x_j}{\sum_{i \in V(G)} x_i^2} \geq 0,$$

with strict inequality if G is connected. Recall that $V(G_{u,v}) = V(G)$ and all the entries of the Perron vector of any graph are nonnegative. We consider the two cases separately.

Case 1. Let $\mathrm{x}_u \geq \mathrm{x}_v$. Consider the vector $\mathbf{z} = \mathbf{x}$. Then,

$$\theta_1 - \lambda_1 \geq \frac{\sum_{ij \in E(G_{u,v})} 2 x_i x_j - \sum_{ij \in E(G)} 2 x_i x_j}{\sum_{i \in V(G)} x_i^2} = \frac{\sum_{i=1}^{r} 2\,(x_u - x_v) x_{w_i}}{\sum_{i \in V(G)} x_i^2} \geq 0. \tag{2.1}$$

Now suppose G is connected, which implies the Perron vector $\mathbf{x}$ is a positive vector. Assume that $\theta_1 = \lambda_1$. Then all the inequalities must be equalities in equation (2.1). Requiring that the first inequality be equality implies that

$$\frac{\mathbf{x}^T(A(G_{u,v})\mathbf{x}}{\mathbf{x}^T\mathbf{x}}=\max_{\mathbf{z}\neq\mathbf{0}}\frac{\mathbf{z}^T(A(G_{u,v})\mathbf{z}}{\mathbf{z}^T\mathbf{z}}.$$

Thus, $\mathbf{x}$ is also a Perron vector for $G_{u,v}$. Then,

$$\sum_{y\sim_G u}x_y=(A(G)\mathbf{x})_u=\lambda_1 x_u=\theta_1 x_u=(A(G_{u,v})\mathbf{x})_u=\sum_{y\sim_G u}x_y+\sum_{i=1}^{r}x_{w_i},$$

where the equality $\theta_1 x_u=(A(G_{u,v})\mathbf{x})_u$ appears because $\mathbf{x}$ is a Perron vector of $A(G_{u,v})$. However, this is a contradiction since $\sum_{i=1}^{r}x_{w_i}>0$. Thus, $\theta_1>\lambda_1$ when G is connected.

Case 2. Let $N_G(v)\cap(N_G(u)\cup\{u\})=\emptyset$. If $x_u\geq x_v$, then we are done using the previous case. So we assume $x_v>x_u$. Consider the vector $\mathbf{z}$ defined as follows:

$$z_i=\begin{cases}x_i, \text{ for } i\neq u,v,\\ x_v, \text{ for } i=u,\\ 0, \text{ for } i=v.\end{cases}$$

Then

$$\theta_1-\lambda_1\geq\frac{\sum_{ij\in E(G_{u,v})}2z_iz_j}{(\sum_{i\in V(G)}x_i^2)-x_u^2}-\frac{\sum_{ij\in E(G)}2x_ix_j}{\sum_{i\in V(G)}x_i^2}\geq\frac{\sum_{i=1}^{r}2(x_v-x_v)x_{w_i}}{\sum_{i\in V(G)}x_i^2}=0.$$

Finally, we note that if G is connected, then $x_u>0$ and therefore the second inequality in the preceding equation is strict, so $\theta_1>\lambda_1$. □

Now we are ready to establish the next result, which gives a lower bound for $s^+(G_{u,v})$ in terms of $s^+(G)$ and $s^-(G_{u,v})$ in terms of $s^-(G)$.

THEOREM 2.10. *Let G be a graph, let $G_{u,v}$ be the graph obtained by moving the neighbors $\{w_1,w_2,\ldots,w_r\}$ from v to u, and let $\lambda_1\geq\cdots\geq\lambda_n$ and $\theta_1\geq\cdots\geq\theta_n$ denote the eigenvalues of $A(G)$ and $A(G_{u,v})$, respectively. Then $s^+(G_{u,v})\geq s^+(G)-\lambda_1^2$ and $s^-(G_{u,v})\geq s^-(G)-\lambda_n^2$.*

If 1) $x_u\geq x_v$ (where $\mathbf{x}$ is the Perron vector of $A(G)$) or 2) $N(u)\cap(N(v)\cup\{v\})=\emptyset$ and $\{w_1,w_2,\ldots,w_r\}=N_G(v)$, then the bound for $s^+(G)$ can be improved to $s^+(G_{u,v})\geq s^+(G)-\lambda_2^2$.

Proof. Let π_G and $\pi_{G_{u,v}}$ denote the number of positive eigenvalues of G and $G_{u,v}$, respectively. Theorem 2.8 implies that

$$\theta_i\geq\lambda_{i+1} \text{ for } i=1,\ldots,\pi_{G_{u,v}}, \qquad \text{and} \qquad \lambda_i\geq\theta_{i+1} \text{ for } i=\pi_{G_{u,v}}+1,\ldots,n-1.$$

Observe that $\lambda_{\pi_G-1}\geq\theta_{\pi_G}>0$ and $0\geq\theta_{\pi_{G_{u,v}}+1}\geq\lambda_{\pi_{G_{u,v}}+2}$, so $\pi_{G_{u,v}}-1\leq\pi_G\leq\pi_{G_{u,v}}+1$. We break the proof of $s^+(G_{u,v})\geq s^+(G)-\lambda_1^2$ into two parts. If $\pi_G=\pi_{G_{u,v}}+1$, then $\theta_i^2\geq\lambda_{i+1}^2$ for $i=1,\ldots,\pi_{G_{u,v}}$, and

$$s^+(G_{u,v})=\sum_{i=1}^{\pi_{G_{u,v}}}\theta_i^2\geq\sum_{i=2}^{\pi_G}\lambda_i^2=s^+(G)-\lambda_1^2.$$

If instead $\pi_G\leq\pi_{G_{u,v}}$, then $\theta_i^2\geq\lambda_{i+1}^2$ for $i=1,\ldots,\pi_G-1$ and

$$s^+(G_{u,v})=\left(\sum_{i=1}^{\pi_{G_{u,v}}-1}\theta_i^2\right)+\theta_{\pi_{G_{u,v}}}^2\geq\sum_{i=2}^{\pi_G}\lambda_i^2+\theta_{\pi_{G_{u,v}}}^2\geq s^+(G)-\lambda_1^2.$$

The proof that $s^-(G_{u,v}) \geq s^-(G) - \lambda_n^2$ is similar.

Now assume that G, u, and v satisfy the hypotheses of Lemma 2.9. Then $\sum_{i=1}^{p} \theta_i^2$ and $\sum_{i=2}^{\pi_G} \lambda_i^2$ (where p equals $\pi_{G_{u,v}}$ or $\pi_{G_{u,v}} - 1$ as needed) can be replaced by $\theta_1^2 + \left(\sum_{i=2}^{p} \theta_i^2\right)$ and $\lambda_1^2 + \left(\sum_{i=3}^{\pi_G} \lambda_i^2\right)$ to obtain $s^+(G_{u,v}) \geq s^+(G) - \lambda_2^2$. ☐

Remark 2.11. The process of moving neighbors from v to u is reversible. That is, $(G_{u,v})_{v,u} = G$ when the same set of vertices $\{w_1, \ldots, w_r\}$ is used for each move. Thus, from Theorem 2.10, we also obtain the bounds $s^+(G) \geq s^+(G_{u,v}) - \theta_1^2$ and $s^-(G) \geq s^-(G_{u,v}) - \theta_n^2$.

2.4. Induced subgraphs. We use interlacing to obtain lower bounds for the squared energies of G in terms of the squared energies of induced subgraphs of G.

Lemma 2.12. *Suppose a connected graph G has an induced subgraph H. Then $\pi_G \geq \pi_H$, $\nu_G \geq \nu_H$, $s^+(G) \geq s^+(H)$, and $s^-(G) \geq s^-(H)$.*

Proof. Let n_G be the number of vertices of G and let n_H be the number of vertices of H. Here, we denote the ith largest eigenvalues of G and H by $\lambda_i(G)$ and $\lambda_i(H)$, so $\lambda_i(G) \geq \lambda_i(H) \geq \lambda_{n_G - n_H + i}(G)$ by the Interlacing Theorem. This implies that $\pi_G \geq \pi_H$ and $\nu_G \geq \nu_H$. Furthermore,

$$\lambda_i(G)^2 \geq \lambda_i(H)^2, \quad i = 1, \ldots, \pi_H \quad \text{and} \quad \lambda_{n_G - n_H + i}(G)^2 \geq \lambda_i(H)^2, \quad i = n_H - \nu_H + 1, \ldots, n_H.$$

This implies $s^+(G) \geq s^+(H)$ and $s^-(G) \geq s^-(H)$. ☐

We obtain the next result as an immediate corollary.

Corollary 2.13. *If a graph G on n vertices has an induced subgraph H with $s^+(H) \geq n-1$ (respectively, $s^-(H) \geq n-1$), then $s^+(G) \geq n-1$ (respectively, $s^-(G) \geq n-1$).*

Since we may not know information about the squared energies of the induced subgraphs, the next result may be more useful (it is immediate from the fact that a bipartite graph H with ℓ edges has $s^+(H) = s^-(H) = \ell$).

Corollary 2.14. *If a graph G on n vertices has an induced bipartite subgraph with at least $n-1$ edges, then $s^+(G) \geq n-1$ and $s^-(G) \geq n-1$.*

Corollary 2.14 is applied to cactus graphs in Section 3.5.

2.5. Quotient matrices. Let $M = [m_{ij}]$ be an $n \times n$ matrix and $X = (X_1, \ldots, X_p)$ be a partition of $\{1, \ldots, n\}$. Then the partition X defines a $p \times p$ matrix $[M_{ij}]$ where $M_{ij} = M[X_i|X_j]$ is the submatrix with row indices in X_i and column indices in X_j The *characteristic matrix* $S = [s_{ij}]$ of X is the $n \times p$ matrix defined by $s_{ij} = 1$ if $i \in X_j$ and $s_{ij} = 0$ if $i \notin X_j$.

The *quotient matrix* $B = [b_{ij}]$ of M for this partition is the $p \times p$ matrix with entry b_{ij} equal to the average row sum of the submatrix M_{ij}. More precisely, $b_{ij} = \frac{1}{|X_i|} \mathbf{1}_n^T M_{ij} \mathbf{1}_p = \frac{1}{|X_i|} (S^T M S)_{ij}$ where $\mathbf{1}_k$ is a k-vector with every entry equal to one.

Lemma 2.15. [9] *If B is the quotient matrix of a symmetric matrix M with respect to a partition, then the eigenvalues of B interlace the eigenvalues of M.*

The proof of the next result is analogous to the proof of Lemma 2.12 using the previous result.

Proposition 2.16. *If* G *has a partition of the vertices with quotient matrix* B *of* $A(G)$*, then* $s^+(G) \geq s^+(B)$ *and* $s^-(G) \geq s^-(B)$*.*

We can give a simple application of Proposition 2.16 for a graph with an edge cut.

Lemma 2.17. *Let* G *be a graph on* n *vertices and let* S *be a subset of* $V(G)$ *of order* s*. Let* c *be the number of edges that are incident to exactly one vertex of* S*. Let* d_1 *be the average degree of the subgraph of* G *induced by* S *and* d_2 *be the average degree of the subgraph of* G *induced by* $V(G) \setminus S$*. If* $d_1 d_2 - \frac{c^2}{(n-s)s} \geq 0$*, then*

$$s^+(G) \geq d_1^2 + d_2^2 + \frac{2c^2}{s(n-s)}.$$

If $d_1 d_2 - \frac{c^2}{(n-s)s} < 0$*, then*

$$s^+(G) \geq \lambda_1(G)^2 \geq \frac{1}{4}\left(d_1 + d_2 + \sqrt{(d_1-d_2)^2 + \frac{4c^2}{s(n-s)}}\right)^2 \text{ and } s^-(G) \geq \lambda_n(G)^2 \geq \frac{1}{4}\left(d_1 + d_2 - \sqrt{(d_1-d_2)^2 + \frac{4c^2}{s(n-s)}}\right)^2.$$

Proof. We consider the partition of $V(G)$ into S and $V \setminus S$. The quotient matrix of $A(G)$ with respect to this partition is

$$B = \begin{bmatrix} d_1 & \frac{c}{s} \\ \frac{c}{n-s} & d_2 \end{bmatrix}.$$

We can find the characteristic polynomial of B in variable t as follows:

$$|tI_2 - B| = \begin{vmatrix} t - d_1 & -\frac{c}{s} \\ -\frac{c}{n-s} & t - d_2 \end{vmatrix} = (t-d_1)(t-d_2) - \frac{c^2}{s(n-s)} = t^2 - (d_1+d_2)t + d_1 d_2 - \frac{c^2}{s(n-s)}.$$

Thus, the eigenvalues of B are

$$\lambda_\pm = \frac{d_1 + d_2 \pm \sqrt{(d_1+d_2)^2 - 4d_1 d_2 + \frac{4c^2}{s(n-s)}}}{2} = \frac{d_1 + d_2 \pm \sqrt{(d_1-d_2)^2 + \frac{4c^2}{s(n-s)}}}{2}.$$

If the determinant $d_1 d_2 - \frac{c^2}{(n-s)s} \geq 0$ then $\lambda_- > 0$ and $s^+(G) \geq \lambda_+^2 + \lambda_-^2$ and thus

$$\begin{aligned} s^+(G) &\geq \frac{1}{4}\left(d_1 + d_2 + \sqrt{(d_1-d_2)^2 + \frac{4c^2}{s(n-s)}}\right)^2 + \frac{1}{4}\left(d_1 + d_2 - \sqrt{(d_1-d_2)^2 + \frac{4c^2}{s(n-s)}}\right)^2 \\ &= \frac{1}{2}\left(d_1+d_2\right)^2 + \frac{1}{2}(d_1-d_2)^2 + \frac{2c^2}{s(n-s)} \\ &= d_1^2 + d_2^2 + \frac{2c^2}{s(n-s).} \end{aligned}$$

Otherwise, $\lambda_- < 0$ and we obtain $s^+(G) \geq \lambda_1(G)^2 \geq \lambda_+^2$ and $s^-(G) \geq \lambda_n(G)^2 \geq \lambda_-^2$ from Proposition 2.16. □

Lemma 2.17 will be applied in Section 3.2 to establish the conjecture for a family of unicyclic graphs. Here, we apply Lemma 2.17 to conclude that a join of a graph H with itself satisfies the conjecture provided H does not have too high density.

Proposition 2.18. *Suppose* H *is a graph of order* $r \geq 8$ *with average degree* $d \leq \frac{r}{2}$*. Then* $G = H \vee H$ *satisfies Conjecture 1.1.*

Proof. Note that the order n of G is $2r$. Since G has a $K_{r,r}$ subgraph, $s^+(G) \geq n-1$. Partition the vertices of G into the vertices of the two copies of H. Then $d_1 = d_2 = d, s = n-s = r$, and $c = r^2$. So $d_1d_2 - \frac{c^2}{(n-s)s} \leq \frac{r^2}{4} - r^2 < 0$ and by Lemma 2.17,

$$s^-(G) \geq \frac{1}{4}\left(2d - \sqrt{4r^2}\right)^2 \geq \left(\frac{r}{2} - r\right)^2 = \frac{r^2}{4} \geq 2r-1 = n-1,$$

where the last inequality holds because $r \geq 8$. ∎

Note that $d_1d_2 - \frac{c^2}{(n-s)s} < 0$ without the assumption that $d \leq \frac{r}{2}$, since $d \leq r-1$ for any graph of order r.

2.6. Equitable partitions and twins. Let $M = [m_{ij}]$ be an $n \times n$ matrix. The partition $X = (X_1, \ldots, X_p)$ of $\{1, \ldots, n\}$ is *equitable* for M if for every pair $i, j \in \{1, \ldots, p\}$, the row sums of M_{ij} are constant. The material in this section will be applied in Section 3.1. It is adapted from [10], where analogous results for distance matrices are presented; the results there could also be adapted to show similar results for the Laplacian, signless Laplacian, and normalized Laplacian matrices of a graph.

Lemma 2.19. *Let M be a symmetric $n \times n$ matrix, let X be an equitable partition of $\{1, \ldots, n\}$, let B be the quotient matrix of M for X, and let $\mathbf{w}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^p$.*

1. [3, p. 24] *$MS = SB$.*
2. [10] *If $i \in X_j$, then $(S\mathbf{w})_i = w_j$ where $(S\mathbf{w})_i$ denotes the ith coordinate of $S\mathbf{w}$ and w_j denotes the jth coordinate of $\mathbf{w}$.*
3. [10] *If $S\mathbf{w} = S\mathbf{y}$, then $\mathbf{w} = \mathbf{y}$.*
4. [10] *If $S\mathbf{z}$ is an eigenvector of M, then $\mathbf{z}$ is an eigenvector of B for the same eigenvalue.*
5. [3, Lemmas 2.3.1] *If $\mathbf{z}$ is an eigenvector of B, then $S\mathbf{z}$ is an eigenvector of M for the same eigenvalue.*

Let v_1, v_2 be vertices of a graph G of order at least three that have the same neighbors other than v_1 and v_2. If $N(v_1) = N(v_2)$ (so v_1 and v_2 are not adjacent), then v_1 and v_2 are called *independent twins*. If $N[v_1] = N[v_2]$ (so v_1 and v_2 are adjacent), then v_1 and v_2 are called *adjacent twins*. Both cases are referred to as *twins*. Note that twins have the same degree. Observe that if v_k and v_{k+i} are twins for $i = 1, \ldots, r-1$, then for $i \neq j \in \{1, \ldots, r-1\}$, v_{k+i} and v_{k+j} are twins of the same type as v_k and v_{k+i}, because $N(v_{k+i}) = N(v_k) = N(v_{k+j})$ for independent twins and $N[v_{k+i}] = N[v_k] = N[v_{k+j}]$ for adjacent twins.

It is useful to partition the vertices with one or more partition sets consisting of twins and to use the partition to create block matrices, as in the proofs of Proposition 2.20 and Theorem 2.21. We make no claim that the next proposition is new but include the brief proof for completeness.

Proposition 2.20. *Let G be a graph of order at least three and suppose that v_k and v_{k+i} are twins for $i = 1, \ldots, r-1$. For $i = 1, \ldots, r-1$, let $\mathbf{z}_i = [0, \ldots, 1, 0, \ldots, 0, -1, 0, \ldots, 0]^T$ be the vector where the kth coordinate is 1 and the $k+i$th coordinate is -1. Then for $i = 1, \ldots, r-1$, $\mathbf{z}_i$ is an eigenvector for $A(G)$ for eigenvalue $\alpha = 0$ if v_k and v_{k+i} are independent or $\alpha = -1$ if v_k and v_{k+i} are adjacent. Thus eigenvalue α has multiplicity at least $r-1$.*

Proof. We show that $\mathbf{w} = \mathbf{z}_1$ is an eigenvector for eigenvalue $\alpha = 0$ or $\alpha = -1$ of $A = A(G)$ where v_1 and v_2 are independent or adjacent twins (the argument is the same for v_k and v_{k+i} but the notation is messier).

Suppose v_1 and v_2 are independent twins. Apply the partition $\{1, 2\}, \{3, \ldots, n\}$ to A and $\mathbf{w}$ to define block matrices and multiply:

$$A\mathbf{w} = \begin{bmatrix} A_{1,1} & A_{2,1}^T \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} = \begin{bmatrix} A_{1,1}\mathbf{w}_1 + A_{2,1}^T\mathbf{w}_2 \\ A_{2,1}\mathbf{w}_1 + A_{2,2}\mathbf{w}_2 \end{bmatrix}$$

Since $A_{1,1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $A_{2,1} = \begin{bmatrix} \mathbf{v} & \mathbf{v} \end{bmatrix}$ for some vector $\mathbf{v}$, $\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\mathbf{w}_2 = \mathbf{0}_{n-2}$,

$$A_{1,1}\mathbf{w}_1 + A_{2,1}^T\mathbf{w}_2 = \mathbf{0}_2 + \mathbf{0}_2 = 0\mathbf{w}_1 \text{ and } A_{2,1}\mathbf{w}_1 + A_{2,2}\mathbf{w}_2 = \mathbf{0}_{n-2} + \mathbf{0}_{n-2} = 0\mathbf{w}_2.$$

Thus, $A\mathbf{w} = 0\mathbf{w}$.

Suppose v_1 and v_2 are adjacent twins. Apply the partition $\{1,2\},\{3,\dots,n\}$ to $A = A(G)$ and $\mathbf{w}$ to define block matrices and multiply:

$$A\mathbf{w} = \begin{bmatrix} A_{1,1} & A_{2,1}^T \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} = \begin{bmatrix} A_{1,1}\mathbf{w}_1 + A_{2,1}^T\mathbf{w}_2 \\ A_{2,1}\mathbf{w}_1 + A_{2,2}\mathbf{w}_2 \end{bmatrix}$$

Since $A_{1,1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_{2,1} = \begin{bmatrix} \mathbf{v} & \mathbf{v} \end{bmatrix}$ for some vector $\mathbf{v}$, $\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\mathbf{w}_2 = \mathbf{0}_{n-2}$,

$$A_{1,1}\mathbf{w}_1 + A_{2,1}^T\mathbf{w}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \mathbf{0}_2 = (-1)\mathbf{w}_1 \text{ and} A_{2,1}\mathbf{w}_1 + A_{2,2}\mathbf{w}_2 = \mathbf{0}_{n-2} + \mathbf{0}_{n-2} = (-1)\mathbf{w}_2.$$

Thus, $A\mathbf{w} = (-1)\mathbf{w}$. ◻

Sets of twins in a graph naturally provide an equitable partition of the adjacency matrix. Proposition 2.20 and Lemma 2.19 can be combined to determine the spectrum.

THEOREM 2.21. *Let G be a graph with $V(G) = \{1,\dots,n\}$, let $X = (X_1,\dots,X_p)$ be a partition of the vertices of G with $|X_1| \le \dots \le |X_p|$, and let k be the least index such that $|X_k| \ge 2$. Suppose that $v,u \in X_j$ implies $v = u$ or v and u are twins. For $j = k,\dots,p$, let α_j denote the eigenvalue α specified in Proposition 2.20 for the type of twin in X_j. Let B denote the quotient matrix of $A(G)$ for X. Then*

$$\operatorname{spec}(A(G)) = \{\alpha_k^{(n_k-1)},\dots,\alpha_p^{(n_p-1)}\} \cup \operatorname{spec}(B),$$

(as multisets).

Proof. Let $n_i = |X_i|$ for $i = 1,\dots,p$. Apply Proposition 2.20 to construct $n_j - 1$ eigenvectors for α_j, $j = k,\dots,p$ and denote this entire collection of eigenvectors by $\mathbf{w}_1,\dots,\mathbf{w}_{n-p}$; let W_j denote the span of the subset of these vectors that are associated with α_j. It is immediate that $\{\alpha_k^{(n_k-1)},\dots,\alpha_p^{(n_p-1)}\} \subset \operatorname{spec}(A(G))$ (as multisets). By Lemma 2.19, every eigenvector $\mathbf{z}$ of B for eigenvalue α yields an eigenvector $S\mathbf{z}$ of $A(G)$ for α. Furthermore, $S\mathbf{z}$ is orthogonal to (and thus independent of) $\mathbf{w}_1,\dots,\mathbf{w}_{n-m}$. Hence it suffices to show that B has a basis of eigenvectors.

Extend $\{\mathbf{w}_1,\dots,\mathbf{w}_{n-m}\}$ to a basis of eigenvectors

$$\{\mathbf{w}_1,\dots,\mathbf{w}_{n-m},\mathbf{w}_{n-m+1},\dots,\mathbf{w}_n\},$$

of $A(G)$ (a basis of eigenvectors exists because $A(G)$ is symmetric). Consider $\mathbf{w}_h$ with $h > n-m$. If the associated eigenvalue α_h of M is distinct from α_j, then $\mathbf{w}_h$ is orthogonal to the eigenvectors for α_j. If $\alpha_h = \alpha_j$, then let $\mathbf{w}_h' = \mathbf{w}_h - \operatorname{proj}_{W_j}(\mathbf{w}_h)$ (this step can be applied more than once if needed). Then $\mathbf{w}_h'$ is an eigenvector for α_h and is orthogonal to $\mathbf{w}_\ell$ for $\ell = 1,\dots,n-p$. This implies $\mathbf{w}_h'$ is constant on the coordinates in X_j for $j = 1,\dots,m$, so $\mathbf{w}_h' = S\mathbf{z}_h$ for some p-vector $\mathbf{z}$. By Lemma 2.19, $\mathbf{z}_h$ is an eigenvector for B for α_h. Thus, B has a basis of eigenvectors and $\operatorname{spec}(A(G)) = \{\alpha_k^{(n_k-1)},\dots,\ \alpha_p^{(n_p-1)}\} \cup \operatorname{spec}(B)$ (as multisets). ◻

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 307-326, May 2023.

3. Results on graph classes. In this section, we use the tools obtained in the preceding section and other known results to establish Conjecture 1.1 for several graph classes.

3.1. Extended barbell graphs. A *barbell graph* is a graph composed of two cliques of the same size connected by an edge. It was shown in [4] that Conjecture 1.1 is true for barbell graphs. Here, we apply the results of Section 2.6 to establish Conjecture 1.1 for extended barbell graphs. An *extended barbell graph* is a graph composed of two cliques of the same size, say of size k, connected by a path P of length 2. +We label the vertices of the cliques by $v_1, \dots, v_k$ and $v_{k+1}, \dots, v_{n-1}$, and the degree-two vertex of the path by v_n, so that $V(P) = \{v_k, v_n, v_{k+1}\}$.

PROPOSITION 3.1. *Let G be an extended barbell graph on $n = 2k+1$ vertices for $k \geq 3$. The eigenvalues of G are $k-1$, -1 with multiplicity $n-4$, and the three roots of $f(x) = x^3 - (k-2)x^2 - (1+k)x + 2(k-2)$.*

Proof. Let G be an extended barbell graph and let $X = (X_1, X_2, X_3, X_4, X_5)$ be the partition of $V(G)$ defined by $X_1 = \{v_k\}, X_2 = \{v_{k+1}\}, X_3 = \{v_n\}, X_4 = \{v_1, \dots, v_{k-1}\}$, and $X_5 = \{v_{k+2}, \dots, v_{n-1}\}$. The partition X is equitable with quotient matrix

$$B = \begin{bmatrix} 0 & 0 & 1 & k-1 & 0 \\ 0 & 0 & 1 & 0 & k-1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & k-2 & 0 \\ 0 & 1 & 0 & 0 & k-2 \end{bmatrix},$$

and its characteristic polynomial is given by $p_B(x) = f(x)(x-k+1)(x+1)$ where $f(x) = x^3 - (k-2)x^2 - (1+k)x + 2(k-2)$. Since the twins are adjacent, $\operatorname{spec}(G) = \{(-1)^{(2k-4)}\} \cup \operatorname{spec}(B)$ by Theorem 2.21. This establishes that the eigenvalues of G are -1 with multiplicity $2k-3 = n-4$, $k-1$, and the three roots of $f(x) = x^3 - (k-2)x^2 - (1+k)x + 2(k-2)$. □

THEOREM 3.2. *Let G be an extended barbell graph on $n = 2k+1$ vertices for $k \geq 3$. Then $\lambda_2 = k-1$, $\lambda_4 = \cdots = \lambda_{n-1} = -1$, $s^+(G) \geq n-1$, and $s^-(G) \geq n-1$.*

Proof. By Proposition 3.1, the eigenvalues of G are -1 with multiplicity $2k-3 = n-4$, $k-1$, and the three roots $\mu_1 \geq \mu_2 \geq \mu_3$ of $f(x) = x^3 - (k-2)x^2 - (1+k)x + 2(k-2)$. Since $f(k-1) = -2 < 0$ and $f(-1) = 2k-2 > 0$, $\mu_1 > k-1 > \mu_2 > -1 > \mu_3$. Thus $\lambda_1(G) = \mu_1$, $\lambda_2(G) = k-1$, $\lambda_3(G) = \mu_2$, $\lambda_4(G) = \cdots = \lambda_{n-1}(G) = -1$ and $\lambda_n(G) = \mu_3$.

For $s^+(G)$, we have that

$$s^+(G) \geq \lambda_1^2(G) + \lambda_2^2(G) > 2(k-1)^2 = 2k^2 - 4k + 2.$$

This implies that $s^+(G) > n-1 = 2k$ since $k \geq 3$.

For $s^-(G)$ we have that

$$s^-(G) \geq \sum_{j=1}^{n-4} (-1)^2 + \lambda_n^2(G) = n - 4 + \lambda_n^2(G).$$

Since $f(-\frac{9}{5}) = \frac{14}{25}k - \frac{194}{125}$, for $k \geq 3$ we have $f(-\frac{9}{5}) \geq \frac{16}{125} > 0$. This implies $\lambda_n(G) < -\frac{9}{5}$ and thus

$$s^-(G) > n - 4 + \left(-\frac{9}{5}\right)^2 = n - 4 + 3.24 > n - 1.$$

□

3.2. Unicyclic graphs. A *unicyclic graph* is a connected graph that has exactly one cycle. In this section, we apply results from Section 2.5 and results established in other papers to unicyclic graphs. We begin by applying Lemma 2.17 to the family of unicyclic graphs $U_{n,3}$ obtained by adding an edge between two leaves of the star $K_{1,n-1}$ (a *leaf* is a vertex of degree one); see Figure 3.

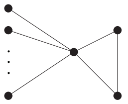

FIGURE 3. *The graph* $U_{n,3}$.

PROPOSITION 3.3. *For* $n \geq 3$, $s^+(U_{n,3}) \geq n-1$ *and* $s^-(U_{n,3}) \geq n-1$.

Proof. Observe first that $U_{n,3}$ contains a $K_{1,n-1}$ subgraph, so $s^+(U_{n,3}) \geq n-1$. Let S be the set consisting of only the $n-1$-degree vertex of $U_{n,3}$. Then in the notation of Lemma 2.17, $s=1$, $n-s=n-1$, $c=n-1$, $d_1=0$ and $d_2=\frac{1}{n-1}$. Since $d_1d_2-\frac{c^2}{(n-s)s}=-\frac{(n-1)^2}{n-1}<0$, by Lemma 2.17,

$$\begin{aligned}\lambda_n(U_{n,3})^2 &\geq \frac{1}{4}\left(\frac{1}{n-1}-\sqrt{\left(\frac{1}{n-1}\right)^2+4\frac{(n-1)^2}{n-1}}\right)^2\\ &=\frac{1}{4}\left(\left(\frac{1}{n-1}\right)^2-\frac{2}{n-1}\sqrt{\left(\frac{1}{n-1}\right)^2+4(n-1)}+\left(\frac{1}{n-1}\right)^2+4(n-1)\right).\\ &\geq (n-1)-1.\end{aligned}$$

Since the two vertices of degree two are adjacent twins, $-1 \in \operatorname{spec}(U_{n,3})$ by Proposition 2.20 and

$$s^-(U_{n,3}) \geq \lambda_n(U_{n,3})^2+1 \geq n-1. \qquad \square$$

Next we apply results of Guo and Spiro in [6] to unicyclic graphs. A *homomorphism* from a graph G to a graph H is a map $\varphi : V(G) \to V(H)$ such that $\varphi(u)\varphi(v) \in E(H)$ whenever $uv \in E(G)$. Observe that if G is bipartite with vertex partition $V(G) = X \cup Y$ and $V(K_2)=\{1,2\}$, then $\varphi : V(G) \to V(K_2)$ defined by $\varphi(x)=1$ for $x \in X$ and $\varphi(y)=2$ for $y \in Y$ is a homomorphism. The *Kneser graph* $\mathrm{Kn}(a,k)$ is the graph whose vertices are the k-subsets of an a-element set, and two k-subsets are adjacent whenever they are disjoint. The *fractional chromatic number* of a graph G is given by

$$\chi_f(G)=\inf_{(a,k)}\frac{a}{k},$$

where the infimum runs over all pairs (a,k) such that there exist a homomorphism from G to $\mathrm{Kn}(a,k)$. For more background on the fractional chromatic number, see [12]. Guo and Spiro recently extended a bound of Ando and Lin [2] to the fractional chromatic number:

THEOREM 3.4. [6] *For any graph* G,

$$\chi_f(G) \geq 1+\max\left\{\frac{s^+(G)}{s^-(G)},\frac{s^-(G)}{s^+(G)}\right\}.$$

COROLLARY 3.5. *For* $m \geq 1$, $s^+(C_{2m+1}), s^-(C_{2m+1}) \geq 2m = |V(C_{2m+1})|-1$.

Proof. It is well-known that the fractional chromatic number of the odd cycle C_{2m+1} is $2+\frac{1}{m}$. Thus,

$$2+\frac{1}{m} \geq 1+\max\left\{\frac{s^+(G)}{s^-(G)}, \frac{s^-(G)}{s^+(G)}\right\}, \quad \text{which implies} \quad \frac{m+1}{m} \geq \frac{s^+(G)}{s^-(G)} \quad \text{and} \quad \frac{m+1}{m} \geq \frac{s^-(G)}{s^+(G)}.$$

Since $s^+(G)+s^-(G)=2|E(G)|=2(2m+1)$, we can substitute $s^+(G)=2(2m+1)-s^-(G)$ into the first expression and obtain

$$\begin{aligned}(m+1)s^-(G) &\geq ms^+(G) = m(2(2m+1)-s^-)\\ (m+1+m)s^-(G) &\geq 2m(2m+1)\\ s^-(G) &\geq 2m.\end{aligned}$$

By a similar argument, we also obtain $s^+(G) \geq 2m$. ∎

We note that the conjecture was shown to be true for all regular graphs except for odd cycles in Theorem 8 in [4]. It was claimed there that the conjecture was also true for odd cycles but no proof was presented. Thus, Corollary 3.5 resolves the last regular graph case.

Using a stronger theorem from [6], we can show that unicyclic graphs containing a long odd cycle also satisfy the conjecture.

Theorem 3.6. [6] *If G has a homomorphism to an edge-transitive graph H, then*

$$\frac{\lambda_{\max}(H)}{|\lambda_{\min}(H)|} \geq \max\left\{\frac{s^+(G)}{s^-(G)}, \frac{s^-(G)}{s^+(G)}\right\},$$

where $\lambda_{\max}(H), \lambda_{\min}(H)$ denote the greatest and least eigenvalue of H, respectively.

Theorem 3.7. *Let G be a unicylic graph of order n with an odd cycle of length $2m+1$ such that*

$$m \geq \frac{\pi}{2\arccos\left(\frac{n-1}{n+1}\right)} - \frac{1}{2}.$$

Then $s^+(G) \geq n-1$ and $s^-(G) \geq n-1$.

Proof. We first show that G has a homomorphism to C_{2m+1}: Since G is unicyclic, $G = C_{2m+1} \cup \left(\cup_i^k T_i\right)$ where T_i is a tree and $|V(C) \cap V(T_i)| = 1$; we denote the unique vertex in $V(C_{2m+1}) \cap V(T_i)$ by v_i. Since each T_i is bipartite, we have a homomorphism from T_i to C_{2m+1} by mapping the partite class containing v_i to v_i and the other partite class to a neighbor of v_i on the cycle. Together these maps define a homomorphsim from G to C_{2m+1}.

Since C_{2m+1} is edge-transitive and there exists a homomorphism from G to C_{2m+1}, by Theorem 3.6, we have that

$$\frac{\lambda_{\max}(C_{2m+1})}{|\lambda_{\min}(C_{2m+1})|} \geq \max\left\{\frac{s^+(G)}{s^-(G)}, \frac{s^-(G)}{s^+(G)}\right\}.$$

The eigenvalues of C_{2m+1} are $2\cos\left(\frac{2\pi j}{2m+1}\right)$ for $j=0,\ldots,2m$. The largest eigenvalue is equal to 2 and the least eigenvalue is

$$2\cos\left(\frac{2\pi m}{2m+1}\right) = -2\cos\frac{\pi}{2m+1}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 307-326, May 2023.

Thus, we have that

$$\frac{2}{2\cos\frac{\pi}{2m+1}} = \frac{1}{\cos\frac{\pi}{2m+1}} \geq \max\left\{\frac{s^+(G)}{s^-(G)}, \frac{s^-(G)}{s^+(G)}\right\}.$$

Since $s^+(G)+s^-(G) = 2|E(G)| = 2n$, we can rearrange to obtain that

$$s^-(G) \geq \frac{2n\cos\frac{\pi}{2m+1}}{1+\cos\frac{\pi}{2m+1}}, \quad s^+(G) \geq \frac{2n\cos\frac{\pi}{2m+1}}{1+\cos\frac{\pi}{2m+1}}.$$

Let $m_0 = \frac{\pi}{2\arccos\frac{n-1}{n+1}} - \frac{1}{2}$. Then, we have that $\arccos\frac{n-1}{n+1} = \frac{\pi}{2m_0+1}$ and so $\cos\frac{\pi}{2m_0+1} = \frac{n-1}{n+1}$. Thus

$$\frac{2n\cos\frac{\pi}{2m_0+1}}{1+\cos\frac{\pi}{2m_0+1}} = \frac{2n\frac{n-1}{n+1}}{\frac{n+1}{n+1}+\frac{n-1}{n+1}} = \frac{2n(n-1)}{2n} = n-1.$$

Since $\cos\frac{\pi}{2x+1}$ increases as x increases, we obtain that

$$\cos\frac{\pi}{2m+1} \geq \frac{n-1}{n+1}.$$

for $m \geq m_0$. □

To give an idea of this bound, for a unicyclic graph on 100 vertices, we need $m \geq 7.38$ for the theorem to apply. Figure 4 shows a plot of (n, m) where $m = \frac{\pi}{2\arccos\frac{n-1}{n+1}} - \frac{1}{2}$, as in Theorem 3.7.

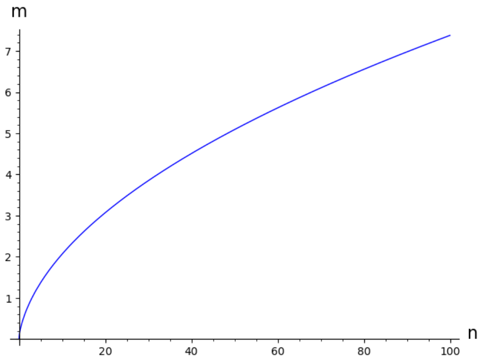

FIGURE 4. *Plot of n and $m(n) = \frac{\pi}{2\arccos\frac{n-1}{n+1}} - \frac{1}{2}$, as in Theorem 3.7.*

3.3. Graphs with two positive eigenvalues.

In this section, we show that if G is a graph with exactly two positive eigenvalues, then $s^+(G) \geq s^-(G)$ and thus $s^+(G) \geq n-1$. Conjecture 1.1 was established in [4] for every graph that has exactly one positive eigenvalue, one negative eigenvalue, or (two negative eigenvalues and minimum degree at least two). Since $\lambda_1 \geq \lambda_n$ for every graph, $\nu_G = 1$ implies $\pi_G = 1$ and thus $\pi_G = 2$ implies $\nu_G \geq 2$.

PROPOSITION 3.8. *Let G be a connected graph of order $n \geq 4$ with $\pi_G = 2$ positive eigenvalues. Define $\mu_i = \lambda_i, i = 1, 2$, $\mu_i = 0, i = 3, \ldots, \nu_G$, and $\theta_i = |\lambda_{n+1-i}|, i = 1, \ldots, \nu_G$. Then the positive eigenvalues majorize the (reordered) absolute values of the negative eigenvalues, i.e., $\sum_{i=1}^k \mu_i \geq \sum_{i=1}^k \theta_i$ for all $k \leq \nu_G - 1$ and $\sum_{i=1}^{\nu_G} \mu_i = \sum_{i=1}^{\nu_G} \theta_i$. Whenever the positive eigenvalues majorize the (reordered) absolute values of the negative eigenvalues, $s^+(G) \geq s^-(G)$ and $s^+(G) \geq n-1$.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 307-326, May 2023.

Proof. Note that $\sum_{i=1}^{k}\mu_i=\sum_{i=1}^{k}\theta_i$ since $\operatorname{tr}A(G)=0$. Since $\mu_1\geq\theta_1$, the positive eigenvalues majorize the reordered absolute values of the negative eigenvalues. This implies $s^+(G)\geq s^-(G)$ by Karamata's inequality, and so $s^+(G)\geq n-1$. ☐

We can apply Proposition 3.8 to show that $s^+(H_n^3)\geq s^-(H_n^3)$ and thus $s^+(H_n^3)\geq n-1$ for the family of unicyclic graphs H_n^3 defined as follows: For $n\geq k+2$, define H_n^k to be the graph obtained from $K_{1,n-k}$ and C_k by identifying a degree 1 vertex of $K_{1,n-k}$ with a vertex of C_k; H_n^k has n vertices. The graph in Figure 5 is H_9^3. The graphs H_n^3 appear to minimize $s^-(H_n^3)$ over graphs of order n, as discussed in Section 4.

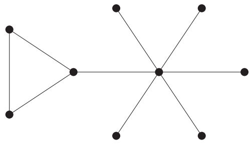

FIGURE 5. *The unicyclic graph* H_9^3

PROPOSITION 3.9. *The graph* H_n^3 *has exactly two positive eigenvalues and three negative eigenvalues. Thus* $s^+(H_n^3)\geq s^-(H_n^3)$ *and* $s^+(H_n^3)\geq n-1$.

Proof. The statements about $s^+(H_n^3)$ follow from Proposition 3.8 once it is established that H_n^3 has exactly two positive eigenvalues. It is verified computationally in [1] that H_5^3 has exactly two positive eigenvalues and three negative eigenvalues; specifically, $\operatorname{spec}(H_5^3)=\{2.214320,1,-0.539189,-1,-1.675131\}$ (to six decimal places). Thus, $\pi_{H_n^3}\geq 2$ and $\nu_{H_n^3}\geq 3$ by Lemma 2.12. Since H_n^3 has a set of $n-4$ independent twins, 0 is an eigenvalue of H_n^3 with multiplicity $n-5$ by Proposition 2.20. Thus $\pi_{H_n^3}=2$ and $\nu_{H_n^3}=3$. ☐

3.4. Graphs with a certain fraction of positive (or negative) eigenvalues.

In this section, we utilize graph energy to show that if a sufficiently small percentage of the nonzero eigenvalues of a graph G of order n are positive (respectively, negative) then $s^+(G)\geq n-1$ (respectively, $s^-(G)\geq n-1$).

LEMMA 3.10. [4] *Let* G *be a graph with* π_G *positive eigenvalues and* ν_G *negative eigenvalues. Then*

$$s^+(G)\geq\frac{\mathcal{E}(G)^2}{4\pi_G}\quad\text{and}\quad s^-(G)\geq\frac{\mathcal{E}(G)^2}{4\nu_G}.$$

THEOREM 3.11. *Let* G *be a connected graph with* $n\geq 3$ *vertices. If* $\pi_G\leq\frac{(\operatorname{rank}A(G))^2}{4(n-1)}$, *then* $s^+(G)\geq n-1$. *If* $\nu_G\leq\frac{(\operatorname{rank}A(G))^2}{4(n-1)}$, *then* $s^-(G)\geq n-1$.

Proof. Let $r=\pi_G+\nu_G=\operatorname{rank}A(G)$. Let a_r denote the product of the nonzero eigenvalues. Since $a_r=S_r(\lambda_1,\dots,\lambda_n)$ is the rth symmetric function of all the eigenvalues, $(-1)^r a_r$ is the coefficient of $xn-r$ in the characteristic polynomial $p(x)$ of $A(G)$. Since all the entries of $A(G)$ are integers, every coefficient in $p(x)$ is an integer. Since $a_r\neq 0$, this implies that $|a_r|\geq 1$. By Lemma 3.10 and the arithmetic mean-geometric mean inequality,

$$s^-(G)\geq\frac{\mathcal{E}(G)^2}{4\nu_G}=\frac{(|\lambda_1|+\cdots+|\lambda_{\pi_G}|+|\lambda_{n-\nu_G+1}|+\cdots+|\lambda_n|)^2}{4\nu_G}\geq\frac{\left[r\Big(\prod_{\lambda_i\neq 0}|\lambda_i|\Big)^{\frac{1}{r}}\right]^2}{4\nu_G}\geq\frac{r^2}{4\nu_G}.$$

If $\nu_G\leq\frac{r^2}{4(n-1)}$, then $s^-(G)\geq n-1$.

An analogous argument shows the same statement for $s^+(G)$. ☐

3.5. Cactus graphs. A *cactus graph* is a connected graph in which any two cycles have at most one vertex in common. For a cactus graph G that has sufficiently many even cycles relative to the number of odd cycles and its maximum degree, we can delete vertices to obtain an induced bipartite graph and apply results from Section 2.4 to conclude G satisfies Conjecture 1.1. While the strategy of deleting vertices to obtain a bipartite graph (see Lemma 3.12 below) can be applied to any graph, it is particularly easy to use on cactus graphs. The *maximum degree* of a graph G is $\Delta(G) = \max\{\deg v : v \in V(G)\}$.

Lemma 3.12. *Let G be a graph on n vertices and let $S \subset V(G)$ be such that the subgraph of G obtained by deleting the vertices in S is bipartite. Then*

$$s^+(G) \geq |E(G)| - |S|\,\Delta(G) \text{ and } s^-(G) \geq |E(G)| - |S|\,\Delta(G).$$

Proof. Let $H = G - S$, the graph induced by $V(G) \setminus S$. Observe that deleting a single vertex can delete at most $\Delta(G)$ edges, so $E(H) \geq |E(G)| - |S|\Delta(G)$. Thus by Corollary 2.14, $s^+(G) \geq |E(G)| - |S|\,\Delta(G)$ and $s^-(G) \geq |E(G)| - |S|\,\Delta(G)$. ☐

The next corollary applies Lemma 3.12 to cactus graphs.

Corollary 3.13. *Let G be a cactus graph on n vertices with k odd cycles and ℓ even cycles. If $\ell \geq k(\Delta(G) - 1)$, then G satisfies Conjecture 1.1*

Proof. It is easy to see that $|E(G)| = n - 1 + k + \ell$. The deletion of k vertices, one from each odd cycle, will result in a bipartite graph. So if $\ell \geq k(\Delta(G) - 1)$, then $|E(G)| = n - 1 + k + \ell \geq n - 1 + k\Delta(G)$ and the result follows from Lemma 3.12. ☐

However, one can often break multiple odd cycles by deleting a single vertex, in which case applying Lemma 3.12 or Corollary 2.14 directly is preferred, as in the next example.

Example 3.14. Let G be the cactus graph shown in Figure 6. Deleting the one vertex incident to the two 3-cycles results in a bipartite graph with $13 = |V(G)| - 1$ edges, so G satisfies Conjecture 1.1 by Corollary 2.14.

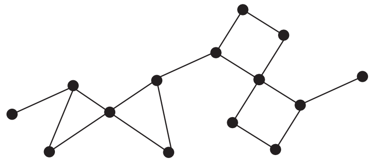

Figure 6. *A cactus to which Lemma 3.12 applies.*

4. Concluding remarks and open problems. In this section, we discuss some questions whose solutions may shed light on the conjecture and present related computational data. In particular, we look at the relative magnitude of $s^+(G)$ and $s^-(G)$ and at some graph families that seem difficult for the conjecture.

From computations on small graphs, as summarized in Table 1, we see that it is much more common that $s^+(G)$ is larger.

Question 4.1. *Does the percentage of graphs G with $s^-(G) > s^+(G)$ tend to zero as n goes to infinity?*

We also note that for $n \leq 8$, there were no nonbipartite graphs on n vertices that had $s^+(G) = s^-(G)$. This naturally raises the question, “Do there exist non-biparite graphs for which $s^+(G) = s^-(G)$?” This

n	# graphs	# $s^+ > s^-$	# $s^- > s^+$	% $s^- > s^+$	# bipartite	# $s^+ = s^-$
2	1	0	0	0.000000	1	1
3	2	1	0	0.000000	1	1
4	6	3	0	0.000000	3	3
5	21	15	1	4.76190	5	5
6	112	93	2	1.78571	17	17
7	853	795	14	1.64127	44	44
8	11117	10848	87	0.782585	182	182

TABLE 1
The number of graphs where $s^+(G) > s^-(G)$, $s^+(G) = s^-(G)$, and $s^-(G) > s^+(G)$ for connected graphs on up to 8 vertices.

question appeared in an earlier draft, and a positive answer is given in the next example, which was provided by Clive Elphick and others.

EXAMPLE 4.2. The strongly regular graph X with parameters $(n, k, a, c) = (15, 6, 1, 3)$, which is the complement of the line graph of K_6 and also the point graph of the generalized quadrangle $GQ(2, 2)$, has $\text{spec}(X) = \{6, 1^{(9)}, (-3)^{(5)}\}$ [5, pp. 227, 235]. Thus, $s^+(X) = s^-(X) = 45$ and X is not bipartite because it has 3 cycles.

The class of nonbipartite unicyclic graphs seems to be a particularly difficult case for Conjecture 1.1. In particular, the conjecture is still open for unicyclic graphs G of order $n \geq 10$ such that G has a 3-cycle and is not isomorphic to $U_{n,3}$. It is not surprising that unicyclic graphs challenge the conjecture, since they are close to graphs that achieve equality in the bound: A connected unicyclic graph of order n has n edges and a connected graph T of order n with $n-1$ edges is a tree and has $s^+(T) = s^-(T) = n-1$ (however, the graph with the maximum number of edges, K_n, also has $s^-(K_n) = n-1$).

We performed computations on nonbipartite, connected unicyclic graphs (equivalently, connected graphs on n vertices with n edges) for $n = 3, \ldots, 18$ [1]. We summarize the minimum value of $s^+(G)$ and $s^-(G)$ among these graphs, as well as the number of isomorphism classes of such graphs, in Table 2. Recall that H_n^k is the graph of order n obtained from $K_{1,n-k}$ and C_k by identifying a degree 1 vertex of $K_{1,n-k}$ with a vertex of C_k. It was shown in Proposition 3.9 that $s^+(H_n^3) \geq n-1$. By Theorem 3.7, Conjecture 1.1 is true for H_n^5 for $n \leq 48$. For each case $n = 3, \ldots, 18$ and each of $s^+(G)$ and $s^-(G)$, the minimum value is attained by only one isomorphism class of graphs [1] (since $s^+(G) + s^-(G) = 2|E(G)| = 2n$ for a unicyclic graph, a minimizer for $s^+(G)$ is a maximizer for $s^-(G)$ and vice versa). In particular, the minimizer of $s^+(G)$ among non-bipartite unicyclic graphs of order $n = 7, \ldots, 18$ is H_n^5 and the minimizer of $s^-(G)$ among nonbipartite unicyclic graphs of order $n = 5, \ldots, 18$ is H_n^3. This leads us to ask whether this is true in general.

QUESTION 4.3. *Let G be a nonbipartite unicyclic graph on $n \geq 19$ vertices. Is $s^+(G) \geq s^+(H_n^5)$? Is $s^-(G) \geq s^-(H_n^3)$?*

Next we describe some preliminary efforts to show that $s^-(H_n^3) \geq n-1$. By Proposition 2.20, $A(H_n^3)$ has eigenvalues -1 and 0 with multiplicity $n-5$. From Proposition 3.9, we know that $\pi_{H_n^3} = 2$ and $\nu_{H_n^3} = 3$. In H_n^3, label the vertices as follows: The degree-2 vertices are 1 and 2, the degree-3 vertex is 3, the degree-$(n-3)$ vertex adjacent to one or more leaves is 4, and the leaves are $5, \ldots, n$. The partition $X_1 = \{1, 2\}$, $X_2 = \{3\}$, $X_3 = \{4\}$, $X_4 = \{5, \ldots, n\}$ is equitable. The quotient matrix is

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
total graphs	1	1	4	8	23	55	155	403	1116	3029	8417	23285	65137	182211	512625	1444444
$\min s^+(G)$	4.0	4.806063	4.763932	5.8548	6.797054	7.786641	8.78153	9.778404	10.776269	11.774708	12.773512	13.772564	14.771792	15.771151	16.77061	17.770146
$\min s^-(G)$	2.0	3.193937	4.096788	5.073208	6.060343	7.051905	8.045829	9.041196	10.037521	11.034519	12.032012	13.029882	14.028045	15.026442	16.025029	17.023774

TABLE 2
The minimum values of $s^+(G)$ and $s^-(G)$, rounded to 6 decimal places, among non-bipartite unicyclic graphs of order n.

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & n-4 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of B is $p_B(x) = x^4 - x^3 - (n-1)x^2 + (n-3)x + 2(n-4)$. Denote the eigenvalues of B by $\mu_1 > \mu_2 > \mu_3 > \mu_4$. Thus, $\operatorname{spec}(A(H_n^3)) = \{\mu_1, \mu_2, \mu_3, \mu_4, -1, 0^{(n-5)}\}$. Since $A(H_n^3)$ has exactly two positive eigevalues, $\mu_2 > 0 > \mu_3$. To prove the conjecture for H_n^3, it is suffices to show that $\mu_3^2 + \mu_4^2 \geq n-2$. Similar methods can be applied to H_n^5 (a 5×5 quotient matrix can be obtained by using an equitable partition that groups the two cycle neighbors of the degree-3 cycle vertex together and groups the other two degree-2 cycle vertices together).

Acknowledgment. This project started and was made possible by *Spectral Graph and Hypergraph Theory: Connections & Applications*, December 6-10, 2021, a workshop of the American Institute of Mathematics with support from the US National Science Foundation. The authors thank AIM and also thank Sam Spiro for many fruitful discussions. We thank Clive Elphick for bringing Conjecture 1.1 to Aida Abiad's attention and also for suggesting Example 4.2. We thank the reviewer for helpful comments and examples that have improved the paper. Aida Abiad is partially supported by the Dutch Research Council through the grant VI.Vidi.213.085 and by the Research Foundation Flanders through the grant 1285921N. Leonardo de Lima is partially supported by CNPq grant 315739/2021-5.

REFERENCES

[1] A. Abiad, K. Guo, and L. Hogben. Sage code for square energies of graphs with examples and computations. Code available at https://sage.math.iastate.edu/home/pub/153/. PDF available at https://aimath.org/~hogben/PositiveNegativeSquareEnergiesGraphs--Sage.pdf.

[2] T. Ando and M. Lin. Proof of a conjectured lower bound on the chromatic number of a graph. *Linear Algebra Appl.*, 485:480–484, 2015.

[3] A.E. Brouwer and W.H. Haemers. *Spectra of Graphs.* Springer, New York, NY, 2011.

[4] C. Elphick, M. Farber, F. Goldberg, and P. Wocjan. Conjectured bounds for the sum of squares of positive eigenvalues of a graph. *Disc. Math.*, 339:2215–2223, 2016.

[5] C.D. Godsil and G.F. Royle. *Algebraic Graph Theory.* Springer-Verlag, New York, 2001.

[6] K. Guo and S. Spiro. New Eigenvalue Bound for the Fractional Chromatic Number. arXiv:2211.04499.

[7] I. Gutman. The energy of a graph: Old and new results. In *Algebraic Combinatorics and Applications* (A. Betten, A. Kohner, R. Laue and A. Wassermann, eds.), 196–211, Springer, Berlin, Germany, 2001.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 307-326, May 2023.

[8] F.J. Hall, K. Patel, and M. Stewart. Interlacing results on matrices associated with graphs. *Comb. Math. Comb. Comput.* 68:113–127, 2009.

[9] W.H. Haemers. Interlacing eigenvalues and graphs. *Linear Algebra Appl.,* 226-228:593-616, 1995.

[10] L. Hogben and C. Reinhart. Spectra of variants of distance matrices of graphs and digraphs: A survey. *La Matematica* 1:186–224, 2022.

[11] V. Nikiforov. The energy of graphs and matrices. *Journal of Mathematical Analysis and Applications* 326:1472–1475, 2007.

[12] E. Scheinerman and D. Ullman. *Fractional Graph Theory: A Rational Approach to the Theory of Graphs*. Dover Publications, Mineola, NY, 2011.

[13] P. Wocjan and C. Elphick. New spectral bounds on the chromatic number encompassing all eigenvalues of the adjacency matrix. *Electron. J. Comb.,* 20: #P39, 2013.

[14] B. Wu and C. Elphick. Upper bounds for the achromatic and coloring numbers of a graph. *Disc. Appl. Math.,* 217 part 2:375–380, 2017.

[15] B.-F. Wu, J.-Y. Shao, and Y. Liu. Interlacing eigenvalues on some operations of graphs, *Linear Algebra Appl.* 430:1140–1150, 2009.