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POLYNOMIAL INEQUALITIES FOR NON-COMMUTING OPERATORS*

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Abstract. We prove an inequality for polynomials applied in a symmetric way to non-commuting operators.

Key words. Ando inequality, Non-commuting, Symmetric functional calculus.

AMS subject classifications. 15A60.

1. Introduction. J. von Neumann [9] proved an inequality about the norm of a polynomial applied to a contraction on a Hilbert space H. Let $\mathbb{D}$ be the unit disk and $\mathbb{T}$ the unit circle in $\mathbb{C}$, and for any polynomial p let $\|p\|_X$ be the supremum of the modulus of p on the set X. The result is that

$$T \in \mathcal{B}(H), \|T\| \leq 1 \Rightarrow \|p(T)\| \leq \|p\|_{\overline{\mathbb{D}}}. \tag{1.1}$$

For polynomials $p(z) = p(z_1, z_2, \ldots, z_n) = \sum_{|\alpha| \leq N} c_\alpha z^\alpha$ in n variables we use the standard multi-index notation (where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ has $0 \leq \alpha_j \in \mathbb{Z}$ for $1 \leq j \leq n$, $|\alpha| = \sum_{j=1}^n \alpha_j$, $z^\alpha = \prod_{j=1}^n z_j^{\alpha_j}$). There is an obvious way of applying p to an n-tuple $T = (T_1, T_2, \ldots, T_n)$ of *commuting* operators $T_j \in \mathcal{B}(H)$ $(1 \leq j \leq n)$, namely

$$p(T) = p(T_1, T_2, \ldots, T_n) = \sum_{|\alpha| \leq N} c_\alpha T^\alpha$$

(with $T^\alpha = \prod_{j=1}^n T_j^{\alpha_j}$ and $T_j^0 = I$).

T. Andô [2] proved an extension of von Neumann's inequality to pairs of commuting contractions.

THEOREM 1.1 (Andô). *If $T_1, T_2 \in \mathcal{B}(H)$, $\max(\|T_1\|, \|T_2\|) \leq 1$, $T_1T_2 = T_2T_1$ and $p(z) = p(z_1, z_2)$ is a polynomial, then*

$$\|p(T_1, T_2)\| \leq \|p\|_{\overline{\mathbb{D}}^2}.$$

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The purpose of this note is to look for analogues of Andô's inequality that are satisfied by *non-commuting* operators. For a polynomial p in n variables and an n-tuple of operators $T = (T_1, \dots, T_n)$ we define $p_{\text{sym}}(T)$ to be a symmetrized version of p applied to T (we make this precise in Section 2). We are looking for results of the form:

For all n-tuples T of operators in a certain set, there is a set K_1 in $\mathbb{C}^n$ such that

$$\|p_{\text{sym}}(T)\| \;\leq\; \|p\|_{K_1}. \tag{1.2}$$

and

For all n-tuples T of operators in a certain set, there is a set K_2 in $\mathbb{C}^n$ and a constant M such that

$$\|p_{\text{sym}}(T)\| \;\leq\; M\,\|p\|_{K_2}. \tag{1.3}$$

Our main result is:

Theorem 4.6 *There are positive constants M_n and R_n such that, whenever $T = (T_1, T_2, \dots, T_n) \in \mathcal{B}(H)^n$ satisfies*

$$\|\sum_{i=1}^{n} \zeta_i T_i\| \;\leq\; 1 \qquad \forall \zeta_i \in \overline{\mathbb{D}},$$

and p is a polynomial in n variables, then

$$\|p_{\text{sym}}(T)\| \;\leq\; \|p\|_{R_n \overline{\mathbb{D}}^n} \tag{1.4}$$

$$\|p_{\text{sym}}(T)\| \;\leq\; M_n \|p\|_{\overline{\mathbb{D}}^n}. \tag{1.5}$$

Moreover, one can choose $R_2 = 1.85$, $R_3 = 2.6$, $M_2 = 4.1$ and $M_3 = 16.6$.

2. Tuples of noncommuting contractions. There are several natural ways one might apply a polynomial $p(z_1, z_2)$ in two variables to pairs $T = (T_1, T_2) \in \mathcal{B}(H)^2$ of operators. A simple case is for polynomials of the form $p(z_1, z_2) = p_1(z_1) + p_2(z_2)$ where we could naturally consider $p(T_1, T_2)$ to mean $p_1(T_1) + p_2(T_2)$.

A recent result of Drury [4] is that if $p(z_1, z_2) = p_1(z_1) + p_2(z_2)$, $T_1, T_2 \in \mathcal{B}(H)$ (no longer necessarily commuting), $\max(\|T_1\|, \|T_2\|) \leq 1$, then

$$\|p(T_1, T_2)\| \leq \sqrt{2}\|p\|_{\overline{\mathbb{D}}^2}. \tag{2.1}$$

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Moreover, Drury [4] shows that the constant $\sqrt{2}$ is best possible.

One way to apply a polynomial $p(z_1, z_2) = \sum_{j,k=0}^{n} a_{j,k} z_1^j z_2^k$ to two noncommuting operators T_1 and T_2 is by mapping each monomial $z_1^j z_2^k$ to the average over all possible products of j number of T_1 and k number of T_2, and then extend this map by linearity to all polynomials. We use the notation $p_{\rm sym}(T_1, T_2)$ and the formula

$$p_{\rm sym}(T_1, T_2) = \sum_{j,k=0}^{n} \frac{a_{j,k}}{\binom{j+k}{j}} \sum_{S \in \mathcal{P}(j+k,j)} \prod_{i=1}^{j+k} T_{2-\chi_S(i)}$$

where $\mathcal{P}(j+k, j)$ denotes the subsets of $\{1, 2, \ldots, j+k\}$ of cardinality j. The empty product, which arises for $j = k = 0$, should be taken as the identity operator. The notation $\prod_{i=1}^{j+k} T_{2-\chi_S(i)}$ is intended to mean the ordered product

$$T_{2-\chi_S(1)} T_{2-\chi_S(2)} \cdots T_{2-\chi_S(j+k)},$$

and $\chi_S(\cdot)$ denotes the indicator function of S.

Remarks 2.1. The operation $p \mapsto p_{\rm sym}(T_1, T_2)$ is not an algebra homomorphism (from polynomials to operators). It is a linear operation and does not respect squares in general.

For example, if $p(z_1, z_2) = z_1^2 + z_2^2$, then

$$p_{\rm sym}(T_1, T_2) = T_1^2 + T_2^2$$

but for $q(z_1, z_2) = (p(z_1, z_2))^2 = z_1^4 + z_2^4 + 2z_1^2 z_2^2$ we have

$$(p_{\rm sym}(T_1, T_2))^2 = T_1^4 + T_2^4 + T_1^2 T_2^2 + T_2^2 T_1^2 \neq q_{\rm sym}(T_1, T_2)$$

in general.

Similarly for $p(z_1, z_2) = 2z_1 z_2$ and

$$q(z_1, z_2) = (p(z_1, z_2))^2 = 4z_1^2 z_2^2,$$

$p_{\rm sym}(T_1, T_2) = T_1 T_2 + T_2 T_1$,

$$(p_{\rm sym}(T_1, T_2))^2 = T_1 T_2 T_1 T_2 + T_1 T_2^2 T_1 + T_2 T_1^2 T_2 + T_2 T_1 T_2 T_1 \neq q_{\rm sym}(T_1, T_2)$$

in general.

However in the very restricted situation that $p(z_1, z_2) = \alpha + \beta z_1 + \gamma z_2$ and $q = p^m$, then we do have $q_{\rm sym}(T_1, T_2) = (p_{\rm sym}(T_1, T_2))^m$.

The symmetrizing idea generalizes in the obvious way to $n > 2$ variables. We will use the notation $p_{\rm sym}(T)$ for n-tuples $T \in \mathcal{B}(H)^n$ for $n \geq 2$.

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3. Example. The analogue of Andô's inequality for $n \geq 3$ commuting Hilbert space contractions and polynomials norms on $\mathbb{D}^n$ is known to fail (see Varopoulos [8], Crabb & Davie [3], Lotto & Steger [6], Holbrook [5]).

The explicit counterexamples of Kaijser & Varopoulos [8], and Crabb & Davie [3]) have $p(T)$ nilpotent (and so of spectral radius 0). While the examples of Lotto & Steger [6] and Holbrook [5]) do not have this property, they are obtained by perturbing examples where $p(T)$ is nilpotent (and so $p(T)$ has relatively small spectral radius).

It is not known whether there is a constant C_n so that the multi-variable inequality

$$\|p(T)\| = \|p(T_1, T_2, \ldots, T_n)\| \leq C_n \|p\|_{\overline{\mathbb{D}}^n} \tag{3.1}$$

holds for all polynomials $p(z)$ in n variables and for all n-tuples T of commuting Hilbert space contractions. However, it is well-known that a spectral radius version of Andô's inequality is true — indeed, it holds in any Banach algebra.

Proposition 3.1. *If p is a polynomial in n variables and $T = (T_1, T_2, \ldots, T_n)$ is an n-tuple of commuting elements in a Banach algebra, each with norm at most one, then*

$$\rho(p(T)) = \lim_{m\to\infty} \|(p(T))^m\|^{1/m} \leq \|p\|_{\overline{\mathbb{D}}^n} \tag{3.2}$$

(for $\rho(\cdot)$ the spectral radius).

Proof. We consider a fixed n. It follows from the Cauchy integral formula, that if $\max_{1\leq j\leq n} \|T_j\| \leq r < 1$, then

$$\|p(T)\| = \|p(T_1, T_2, \ldots, T_n)\| \leq C_r \|p\|_{\overline{\mathbb{D}}^n} \tag{3.3}$$

for a constant C_r depending on r (and n).

To see this write

$$p(T) = \frac{1}{(2\pi i)^n} \int_{\zeta\in\mathbb{T}^n} \prod_{j=1}^{n} p(\zeta) \prod_{j=1}^{n} (\zeta_j - T_j)^{-1}\, d\zeta_1\, d\zeta_2\, \ldots\, d\zeta_n$$

and estimate with the triangle inequality. This shows that $C_r = (1-r)^{-n}$ will work.

Applying (3.3) to powers of p and using the spectral radius formula, we get

$$\rho(p(T)) \leq \|p\|_{\overline{\mathbb{D}}^n},$$

(provided $\max_{1\leq j\leq n} \|T_j\| \leq r < 1$). However, for the general case $\max_{1\leq j\leq n} \|T_j\| = 1$, we can apply this to rT to get

$$\rho(p(T)) = \lim_{r\to 1^-} \rho(p(rT)) \leq \|p\|_\infty. \qquad \square$$

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Example 3.2.

Let $p(z,w) = (z-w)^2 + 2(z+w) + 1 = z^2 + w^2 - 2zw + 2(z+w) + 1$,

$$T_1 = \begin{pmatrix} \cos(\pi/3) & \sin(\pi/3) \\ \sin(\pi/3) & -\cos(\pi/3) \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ -\sin(\pi/3) & -\cos(\pi/3) \end{pmatrix}.$$

Note that $\|p\|_{\mathbb{D}^2} \geq p(1,-1) = 5$. To show that $\|p\|_{\mathbb{D}^2} \leq 5$, consider the homogeneous polynomial

$$q(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_1z_3 - 2z_2z_3$$

and observe first that $p(z,w) = q(z,w,-1)$. Moreover

$$\|p\|_{\mathbb{D}^2} = \|p\|_{\mathbb{T}^2} = \|q\|_{\mathbb{T}^3} = \|q\|_{\mathbb{D}^3},$$

by homogeneity of q and the maximum principle. Holbrook [5, Proposition 2] gives a proof that $\|q\|_{\mathbb{D}^3} = 5$.

We have

$$\begin{aligned} p_{\mathrm{sym}}(T_1, T_2) &= (T_1 - T_2)^2 + 2(T_1 + T_2) + I \\ &= \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}^2 + 2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + I \\ &= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \end{aligned}$$

So $\|p_{\mathrm{sym}}(T_1, T_2)\| = 6 > 5 = \|p\|_{\mathbb{D}^2}$.

Remark 3.3. The example has hermitian T_1 and T_2 and a polynomial with real coefficients and yet $\rho(p_{\mathrm{sym}}(T_1, T_2)) > \|p\|_{\mathbb{D}^2}$. Thus even Proposition 3.1 does not hold for non-commuting pairs.

The referee has provided an argument to show that for the polynomial p of Example 3.2, one has the inequality $\|p_{\mathrm{sym}}(T_1, T_2)\| \leq 6$ for all contractions T_1 and T_2 (and thus the example is optimal for that p). This inequality is a substantial improvement over using the sum of the absolute values of the coefficients of p, so one is led to ask how well can one bound $\|p_{\mathrm{sym}}(T)\|$ for general p?

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4. $\|\sum \zeta_i T_i\| \leq 1$**.** In this section, we shall consider n-tuples $T = (T_1, \ldots, T_n)$ of operators, not assumed to be commuting, and we shall make the standing assumption:

$$\|\sum_{i=1}^{n} \zeta_i T_i\| \leq 1 \qquad \forall\, \zeta_i \in \overline{\mathbb{D}}. \tag{4.1}$$

This will hold, for example, if the condition

$$\sum_{i=1}^{n} \|T_i\| \leq 1 \tag{4.2}$$

holds. We wish to derive bounds on $\|p_{\mathrm{sym}}(T)\|$. We start with the following lemma:

Lemma 4.1. *If $S \in \mathcal{B}(H)$ and $\|S\| < 1$ then*

$$\Re((I+S)(I-S)^{-1}) \geq 0.$$

Proof.

$$\begin{aligned}
&2\Re((I+S)(I-S)^{-1})\\
&= (I-S^*)^{-1}(I+S^*) + (I+S)(I-S)^{-1}\\
&= (I-S^*)^{-1}\big[(I+S^*)(I-S) + (I-S^*)(I+S)\big](I-S)^{-1}\\
&= 2(I-S^*)^{-1}[I - S^*S](I-S)^{-1}\\
&\geq 0. \qquad \square
\end{aligned}$$

If $p(z) = \sum c_\alpha z^\alpha$, define

$$\Gamma p(z) = \sum c_\alpha \frac{\alpha!}{|\alpha|!} z^\alpha \tag{4.3}$$

(as usual, $\alpha!$ means $\alpha_1! \cdots \alpha_n!$). We let Λ denote the inverse of Γ:

$$\Lambda \sum d_\alpha z^\alpha = \sum d_\alpha \frac{|\alpha|!}{\alpha!} z^\alpha.$$

Proposition 4.2. *Let $T = (T_1, T_2, \ldots, T_n) \in \mathcal{B}(H)^n$ satisfy (4.1) and $p(z)$ be a polynomial in n variables. Then*

$$\|p_{\mathrm{sym}}(T)\| \leq \|\Gamma p\|_{\overline{\mathbb{D}}^n}. \tag{4.4}$$

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Proof. We first restrict to the case

$$\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{T}^n \Rightarrow \|\zeta \cdot T\| = \left\| \sum_{j=1}^{n} \zeta_j T_j \right\| < 1$$

and hence by Lemma 4.1 the operator

$$(I + \zeta \cdot T)(I - \zeta \cdot T)^{-1} = (I + \zeta \cdot T) \sum_{j=0}^{\infty} (\zeta \cdot T)^j = I + 2 \sum_{j=1}^{\infty} (\zeta \cdot T)^j$$

has positive real part

$$\begin{aligned} K(\zeta, T) &= \Re\left((I + \zeta \cdot T)(I - \zeta \cdot T)^{-1}\right) \\ &= I + \sum_{j=1}^{\infty} (\zeta \cdot T)^j + \sum_{j=1}^{\infty} (\bar{\zeta} \cdot T^*)^j \\ &= 2\Re\left[\sum_{\alpha_1, \ldots, \alpha_n = 0}^{\infty} \frac{|\alpha|!}{\alpha!} \zeta^\alpha (z^\alpha)_{\mathrm{sym}}(T) \right] - I. \end{aligned}$$

We can compute that for polynomials $p(z) = p(z_1, z_2, \ldots, z_n)$,

$$p_{\mathrm{sym}}(T) = \int_{\mathbb{T}^n} \Gamma p(\zeta) K(\bar{\zeta}, T)\, d\sigma(\zeta)$$

with $d\sigma$ indicating normalised Haar measure on the torus $\mathbb{T}^n$ (and $\bar{\zeta} = (\bar{\zeta}_1, \bar{\zeta}_2, \ldots, \bar{\zeta}_n)$).

As

$$K(\bar{\zeta}, T)\, d\sigma(\zeta)$$

is a positive operator valued measure on $\mathbb{T}^n$, we then have a positive unital linear map $C(\mathbb{T}^n) \to \mathcal{B}(H)$ given by $f \mapsto \int_{\mathbb{T}^n} f(\zeta) K(\bar{\zeta}, T)\, d\sigma(\zeta)$. As this map is then of norm 1, we can conclude

$$\|p_{\mathrm{sym}}(T)\| \leq \|\Gamma p\|_{\overline{\mathbb{D}}^n}.$$

For the remaining case $\sup_{\zeta \in \mathbb{T}^n} \|\zeta \cdot T\| = 1$, we have

$$\|p_{\mathrm{sym}}(T)\| = \lim_{r \to 1^-} \|p_{\mathrm{sym}}(rT)\| \leq \|\Gamma p\|_{\overline{\mathbb{D}}^n}. \qquad \square$$

Remark 4.3. The technique of the above proof is derived from methods of [7].

Now we want to estimate $\|\Gamma p\|_{\overline{\mathbb{D}}^N}$.

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PROPOSITION 4.4. *For each $n \geq 2$ there is a constant M_n so that*

$$\|\Gamma p\|_{\overline{\mathbb{D}}^n} \;\leq\; M_n \, \|p\|_{\overline{\mathbb{D}}^n}.$$

Moreover,

$$\begin{aligned} M_2 &\leq 4.07 \\ M_3 &\leq 16.6 \end{aligned}$$

Proof. Define

$$J(\eta) \;=\; \sum_{\alpha_1=0,\ldots,\alpha_n=0}^{\infty} \frac{\alpha!}{|\alpha|!}\eta^{\alpha}. \tag{4.5}$$

Then

$$\Gamma p(z) \;=\; \int_{\mathbb{T}^n} p(\zeta)[J(z_1\bar{\zeta}_1,\ldots,z_n\bar{\zeta}_n)]d\sigma(\zeta). \tag{4.6}$$

To use (4.6), we break J into two parts — the sum J_0 where the minimum of the α_i is 0, and the remaining terms J_1.

$$J_1(\eta) \;=\; \sum_{\alpha_1=1,\ldots,\alpha_n=1}^{\infty} \frac{\alpha!}{|\alpha|!}\eta^{\alpha}.$$

Case: $n = 2$. Here,

$$\int_{\mathbb{T}^2} p(\zeta)J_0(z_1\bar{\zeta}_1, z_2\bar{\zeta}_2)d\sigma(\zeta) \;=\; p(z_1,0)+p(0,z_2)-p(0,0). \tag{4.7}$$

So the norm of the left-hand side of (4.7) is dominated by $3\|p\|_{\overline{\mathbb{D}}^2}$.

For J_1, we will use the estimate

$$\left|\int_{\mathbb{T}^2} p(\zeta)J_1(z_1\bar{\zeta}_1, z_2\bar{\zeta}_2)d\sigma(\zeta)\right| \;\leq\; \|p\|_\infty\|J_1\|_{L^1} \;\leq\; \|p\|_\infty\|J_1\|_{L^2}.$$

We have

$$\begin{aligned} \|J_1\|^2_{L^2} &= \sum_{\alpha_1,\alpha_2=1}^{\infty}\left(\frac{\alpha_1!\alpha_2!}{(\alpha_1+\alpha_2)!}\right)^2 \\ &= \sum_{\alpha_1=1}^{\infty}\frac{1}{(\alpha_1+1)^2} + \sum_{\alpha_2=2}^{\infty}\frac{1}{(\alpha_2+1)^2} + \sum_{\alpha_1,\alpha_2=2}^{\infty}\left(\frac{\alpha_1!\alpha_2!}{(\alpha_1+\alpha_2)!}\right)^2 \\ &\leq \left(\frac{\pi^2}{3}-\frac{9}{4}\right) + \sum_{k=4}^{\infty}(k-3)\left(\frac{2}{k(k-1)}\right)^2 \\ &\leq (1.069)^2. \end{aligned}$$

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(In the penultimate line, we let $k = \alpha_1 + \alpha_2$; there are $k-3$ terms with this sum, and the largest they can be is when either α_1 or α_2 is 2.) Adding the two estimates, we get $M_2 \leq 4.07$.

Case: $n = 3$. Again, we estimate the contributions of J_0 and J_1 separately. We have

$$\begin{aligned}
&\int p(\zeta) J_0(z_1\bar{\zeta}_1, z_2\bar{\zeta}_2, z_3\bar{\zeta}_3) d\sigma(\zeta) \\
&= \Gamma p(0, z_2, z_3) + [\Gamma p(z_1, 0, z_3) - p(0, 0, z_3)] \\
&\quad + [\Gamma p(z_1, z_2, 0) - p(z_1, 0, 0) - p(0, z_2, 0) + p(0, 0, 0)]
\end{aligned}$$

where we have had to subtract some terms to avoid double-counting. Thus the contribution of J_0 is at most $3M_2 + 4$.

To calculate the contribution of J_1, we make the following estimate on $\|J_1\|_{L^2}$, which is valid for all $n \geq 3$:

We want to bound

$$\sum_{\alpha_1=1,\dots,\alpha_n=1}^{\infty} \left(\frac{\alpha!}{|\alpha|!} \right)^2 \tag{4.8}$$

Let $k = |\alpha|$ in (4.8). Note first that the number of terms for each k is the number of ways of writing k as a sum of n distinct positive integers (order matters), and this is exactly $\binom{k-1}{n-1}$. Moreover, as each α_i is at least 1, we have

$$\frac{\alpha!}{|\alpha|!} \leq \frac{1}{k(k-1)\cdots(k-n+2)}.$$

Therefore (4.8) is bounded by

$$\begin{aligned}
&\sum_{k=n}^{\infty} \binom{k-1}{n-1} \left(\frac{1}{k(k-1)\cdots(k-n+2)} \right)^2 \\
&= \sum_{k=n}^{\infty} \frac{k-n+1}{(n-1)!k} \frac{1}{k(k-1)\cdots(k-n+2)}.
\end{aligned}$$

The terms on the right-hand side of (4.9) decay like $1/k^{n-1}$, so the series converges for all $n \geq 3$. When $n = 3$, the series is

$$\sum_{k=3}^{\infty} \frac{k-2}{2k^2(k-1)} \leq (0.381)^2.$$

Therefore $M_3 \leq 3M_2 + 4.381 < 16.59$.

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We now proceed by induction on n. The contribution from J_0 is dominated by applying Γ to the restriction of p to the slices with one or more coordinates equal to 0, and these are bounded by the inductive hypothesis. The contribution from J_1 is bounded by (4.8). ☐

We have proved that the polydisk is an M-spectral set for T; we can make the constant one by enlarging the domain.

PROPOSITION 4.5. *There is a constant R_n so that*

$$\|\Gamma p\|_{\overline{\mathbb{D}}^n} \;\leq\; \|p\|_{R_n \overline{\mathbb{D}}^n}. \tag{4.9}$$

Moreover,

$$\begin{aligned} R_2 &\leq 1.85 \\ R_3 &\leq 2.6 \end{aligned}$$

Proof. Let $L(\eta) = 2\Re J(\eta) - 1$. Adding terms that are not conjugate analytic powers of ζ inside the bracket in (4.6) will not change the value of the integral, so, writing $z\bar{\zeta}$ for the n-tuple $(z_1\bar{\zeta}_1, \ldots, z_n\bar{\zeta}_n)$, we get

$$\Gamma p(z) \;=\; \int_{\mathbb{T}^n} p(\zeta)[L(z\bar{\zeta})] d\sigma(\zeta). \tag{4.10}$$

As L is real and has integral 1, if we can choose r_n so that if $|z_i| \leq r_n$ for each i then $L(z\bar{\zeta})$ is non-negative for all ζ, then its L^1 norm would equal its integral, and so we would get from (4.10) that

$$|\Gamma p(z)| \;\leq\; \|p\|_{\overline{\mathbb{D}}^n}.$$

Letting $R_n = 1/r_n$ gives (4.9). As the series (4.5) converges absolutely for all $\eta \in \mathbb{D}^n$, and $L(0) = 1$, the existence of some r_n now follows by continuity.

Let us turn now to obtaining quantitative estimates.

Case: $n = 2$. Adding terms to J that are not analytic will not affect the integral (4.10), so let us consider

$$L'(\eta) \;=\; \Re\left[\frac{1+\eta_1}{1-\eta_1}\right] \cdot \Re\left[\frac{1+\eta_2}{1-\eta_2}\right] \;-\; \sum_{\alpha_1=1,\alpha_2=1}^{\infty} (1 - \frac{\alpha!}{|\alpha|!})(\eta_1^{\alpha_1} - \bar{\eta}_1^{\alpha_1})(\eta_2^{\alpha_2} - \bar{\eta}_2^{\alpha_2}).$$

Then L' has integral 1 and (4.10) is unchanged if L is replaced by L'. So we wish to find the largest r so that L' is positive on $r\mathbb{D}^2$.

It can be checked numerically that $r = 0.5406$ works, so the best R_2 is smaller than the reciprocal of 0.5406, which is less than 1.85.

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Case: $n = 3$. As in the case $n = 2$, we consider the kernel

$$\begin{aligned} L'(\eta) \;&=\; \Re\left[\frac{1+\eta_1}{1-\eta_1}\right] \cdot \Re\left[\frac{1+\eta_2}{1-\eta_2}\right] \cdot \Re\left[\frac{1+\eta_3}{1-\eta_3}\right] \\ &\quad - \sum_{\alpha_1=1,\alpha_2=1,\alpha_3=0}^{\infty} (1 - \frac{\alpha!}{|\alpha|!})(\eta_1^{\alpha_1} - \bar\eta_1^{\alpha_1})(\eta_2^{\alpha_2} - \bar\eta_2^{\alpha_2})(\eta_3^{\alpha_3} + \bar\eta_3^{\alpha_3}). \end{aligned}$$

(Note that there is a plus in the last factor to keep L' real.) Again, a computer search can find r so that L' is positive on $r\mathbb{D}^3$, and $r = .39$ works, so $R_3 < 2.6$. $\square$

Combining Propositions 4.2, 4.4 and 4.5, we get the main result of this section.

Theorem 4.6. *There are positive constants M_n and R_n such that whenever $T = (T_1, T_2, \ldots, T_n) \in \mathcal{B}(H)^n$ satisfies (4.1) and $p(z)$ is a polynomial in n variables, then*

$$\|p_{\rm sym}(T)\| \;\le\; \|p\|_{R_n\overline{\mathbb{D}}^n} \tag{4.11}$$
$$\|p_{\rm sym}(T)\| \;\le\; M_n\|p\|_{\overline{\mathbb{D}}^n}. \tag{4.12}$$

Moreover, one can choose $R_2 = 1.85$, $R_3 = 2.6$, $M_2 = 4.1$ and $M_3 = 16.6$.

Remark 4.7. Another way to estimate $\|p_{\rm sym}(T)\|$, under the assumption (4.2), would be to crash through with absolute values. Let $\Delta_n = \{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j| \le 1\}$ and let r_n denote the Bohr radius of Δ_n, *i.e.* the largest r such that whenever $p(z) = \sum c_\alpha z^\alpha$ has modulus less than or equal to one on Δ_n, then $q(z) = \sum |c_\alpha| z^\alpha$ has modulus bounded by one on $r\Delta_n$. One then has the estimate that, under the hypothesis (4.2), and writing $C_n = 1/r_n$,

$$\|p_{\rm sym}(T)\| \;\le\; \|q\|_{\Delta_n} \;\le\; \|p\|_{C_n\,\Delta_n}. \tag{4.13}$$

It was shown by L. Aizenberg [1, Thm. 9] that

$$\frac{1}{3e^{1/3}} \;<\; r_n \;\le\; \frac{1}{3}.$$

So the estimate in (4.11) for pairs satisfying (4.2) does not follow from (4.13).

5. n-tuples of contractions. In an attempt to use the above technique for tuples $T \in \mathcal{B}(H)^n$ such that $\max_{1\le j\le n} \|T_j\| \le 1$, we consider restricting ζ to belong to Δ_n, and we replace σ by some probability measure μ supported on Δ_n.

Suppose we can find some function q such that

$$\Lambda_\mu(q)(z) \;:=\; \int_{\Delta_n} q(\zeta)\Re\frac{1+\bar\zeta\cdot z}{1-\bar\zeta\cdot z}\,d\mu(\zeta) \tag{5.1}$$

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equals $p(z)$. We do not actually need q to be a polynomial; having an absolutely convergent power series on Δ_n (in ζ and $\bar{\zeta}$) is enough.

LEMMA 5.1. *With notation as above, assume $\Lambda_\mu(q) = p$ and that $T \in \mathcal{B}(H)^n$ is an n-tuple of contractions. Then*

$$\|(p)_{\mathrm{sym}}(T)\| \leq \|q\|_{\mathrm{suppt}(\mu)} \leq \sup\{|q(z)| : z \in \Delta_n\}.$$

Proof. We assume first that $\max_{1\leq j\leq n} \|T_j\| < 1$ and use the notation $K(\zeta, T)$ from the proof of Proposition 4.2 (which is permissible as $\|\zeta \cdot T\| < 1$ for $\zeta \in \Delta_n$). We have

$$(\Lambda_\mu q)_{\mathrm{sym}}(T) = \int_{\Delta_n} q(\zeta) K(\bar{\zeta}, T)\, d\sigma(\zeta)$$

and hence the inequality $\|(p)_{\mathrm{sym}}(T)\| \leq \|q\|_{\mathrm{suppt}(\mu)}$ follows as in the previous proof.

If $\max_{1\leq j\leq n} \|T_j\| = 1$, we deduce the result from $\|(p)_{\mathrm{sym}}(rT)\| \leq \|q\|_{\Delta_n}$ for $0 < r < 1$. ◻

REMARK 5.2. For an arbitrary measure μ, there might be no q such that $\Lambda_\mu(q) = p$. If μ is chosen to be circularly symmetric, though, one gets

$$\Lambda_\mu(z^\alpha) \;=\; \left[\frac{|\alpha|!}{\alpha_1! \ldots \alpha_n!} \int |\zeta^\alpha|^2 d\mu(\zeta)\right] z^\alpha. \tag{5.2}$$

As long as none of the moments on the right of (5.2) vanish, inverting Λ_μ is now straightforward.

To make use of the lemma to bound $p_{\mathrm{sym}}(T)$ we need to find a way to choose another polynomial q and a μ on Δ_n so that $p = \Lambda_\mu q$ and $\|q\|_{\Delta_n}$ is small. We do not know a good way to do this.

QUESTION 1. *What is the smallest constant R_n such that, for every n-tuple T of contractions and every polynomial p, one has*

$$\|p_{\mathrm{sym}}(T)\| \;\leq \|p\|_{R_n \overline{\mathbb{D}}^n} ? \tag{5.3}$$

We do not know if one can choose R_n smaller than the reciprocal of the Bohr radius of the polydisk, even when $n = 2$.

QUESTION 2. *Is there a constant M_n such that, for every n-tuple T of contractions and every polynomial p, one has*

$$\|p_{\mathrm{sym}}(T)\| \;\leq M_n \|p\|_{\overline{\mathbb{D}}^n} ? \tag{5.4}$$

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