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REVERSE ORDER LAW AND FORWARD ORDER LAW FOR THE (b, c)-INVERSE[*]

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Abstract. The reverse order law and the forward order law have been studied for various types of generalized inverses. The (b, c)-inverse is a generalization of some well known generalized inverses, such as the Moore-Penrose inverse, the Drazin inverse, the core inverse, etc. In this paper, the reverse order law for the (b, c)-inverse, in a unital ring, is investigated and an equivalent condition for this law to hold for the (b, c)-inverse is derived. Also, some known results on this topic are generalized. Furthermore, the forward order law for the (b, c)-inverse in a ring with a unity is introduced, for different choices of b and c. Moreover, as corollaries of obtained results, equivalent conditions for the reverse order law and the forward order law for the inverse along an element are derived.

Key words. (b, c)-inverse, Inverse along an element, Reverse order law, Forward order law.

AMS subject classifications. 15A09, 16E50.

1. Introduction. The theory of generalized inverses has its beginning in the early years of the twentieth century [2]. Namely, in 1903, Fredholm defined a particular generalized inverse of an integral operator [12]. In 1920, an abstract of a talk, given by Moore at a meeting of the American Mathematical Society, had appeared in print [22]. In this abstract, Moore defined a unique generalized inverse for every finite matrix. In 1951, Bjerhammar [5, 6, 7] rediscovered Moore's inverse, and in 1955, Penrose [25] extended Bjerhammar's results. This inverse is now called the Moore–Penrose inverse, and it has been widely investigated by many authors. In 1958, Drazin [10] introduced a new generalized inverse, in associative rings, which was later called the Drazin inverse. Baksalary and Trenkler [1], in 2010, introduced a new generalized inverse – the core inverse, for complex matrices. Moreover, in 2011, Mary [18] defined a new generalized inverse using Green's preorders and named it the inverse along an element. In 2012, Drazin [11] introduced the concept of the (b, c)-inverse in semigroups and rings. The inverse along an element and the (b, c)-inverse both generalize some well-known generalized inverses, such as the Moore–Penrose, the Drazin inverse, the core and dual core inverse, and others. In 2021, it was proved that the inverse along an element and the (b, c)-inverse are equivalent concepts [19]. Many other generalized inverses were defined and studied.

Throughout this paper, we assume that R is a unital ring with the unity 1. It is well known that if $a, w \in R^{-1}$, then aw is also invertible and

$$(1.1) \qquad (aw)^{-1} = w^{-1}a^{-1}.$$

Equality (1.1) is known as the reverse order law. Since the reverse order law holds for the classical inverse, it is natural to investigate if this equality is valid for different generalized inverses. In 1966, Greville [13] was the first who studied this topic for the Moore-Penrose inverse for the product of two matrices and derived a necessary and sufficient condition for the reverse order law to hold for the Moore-Penrose inverse. Since then, many authors have been investigated under which conditions the reverse order law holds for various types of generalized inverses (e.g., see [4, 9, 19, 23, 24, 28, 33, 37]). In the second section of this paper, we will investigate the reverse order law for the (b,c)-inverse. Namely, we derive an equivalent condition for the reverse order law rule:

$$(1.2) \qquad (aw)^{(b,c)} = w^{(b,c_2)}a^{(b_1,c)}.$$

Note that $(aw)^{(b,c)} = w^{(b,c)}a^{(b,c)}$, $(aw)^{(b,c)} = w^{(b,c)}a^{(b_1,c)}$ and $(aw)^{(b,c)} = w^{(b,c_2)}a^{(b,c)}$ are all special cases of (1.2).

Let $b, c \in R$. An element $a \in R$ is (b,c)-invertible [11], if there exists $y \in R$ such that

$$y \in (bRy) \cap (yRc), \quad yab = b, \ cay = c.$$

If such y exists, it is unique and it is called the (b,c)-inverse of a, denoted by $a^{(b,c)}$. By $R^{(b,c)}$, we will denote the set of all (b,c)-invertible elements of a ring R. For more properties of the (b,c)-inverse, we refer the reader to see [3, 14, 15, 19, 23, 27, 29, 34].

Now, consider the following equality:

$$(1.3) \qquad (aw)^{-1} = a^{-1}w^{-1}.$$

Equality (1.3) is called the forward order law. Contrary to the reverse order law, even if a and w are both invertible, the forward order law is not valid in general. In 2003, Wang et al. [30] studied the forward order law for the outer inverse with prescribed range and null space in the matrix concept and derived necessary and sufficient conditions for the forward order law for the Moore-Penrose inverse, the weighted Moore-Penrose inverse, the Drazin inverse and the group inverse. After the mentioned publication, this topic has been investigated by some scholars, for different kinds of generalized inverses (see [8, 21, 35] for the Moore-Penrose inverse and [31, 32, 36] for {1}-, {1,2}-, {1,2,3}- and {1,2,4}-inverses), but not so widely as the reverse order law. Recently, Kumar and Mishra [16] and also Li, Mosić and Chen [17] studied the forward order law for the core inverse. However, in the present, there are no publications on the forward order for the (b,c)-inverse. In the third section of this paper, we will investigate the forward order law for the (b,c)-inverse in a unital ring. Actually, we obtain a necessary and sufficient condition for the forward order law rule:

$$(1.4) \qquad (aw)^{(b,c)} = a^{(b,c_1)}w^{(b_2,c)}.$$

Some special cases of (1.4) are $(aw)^{(b,c)} = a^{(b,c)}w^{(b,c)}$, $(aw)^{(b,c)} = a^{(b,c_1)}w^{(b,c)}$, and $(aw)^{(b,c)} = a^{(b,c)}w^{(b_2,c)}$. Note that if $(R, \cdot)$ is a ring, then $(R, *)$ is also a ring if we consider $a * b = b \cdot a$. Therefore, the obtained results of the manuscript can be dualized with no problem.

In what follows, we give some definitions, which we will use to obtain our results. An element $a \in R$ is regular if there exists $y \in R$ such that $aya = a$. Any inner inverse of a will be denoted by a^-. The set of all regular elements of a ring R will be denoted by R^-. For $a \in R$, we define image ideals aR and Ra by

$aR = \{ax : x \in R\}$ and $Ra = \{xa : x \in R\}$, respectively. Moreover, by a° and $^\circ a$, we denote kernel ideals (also known as annihilators) $a^\circ = \{x \in R : ax = 0\}$ and $^\circ a = \{x \in R : xa = 0\}$, respectively.

Let $d \in R$. An element $a \in R$ is said to be invertible along d [18], if there exists $y \in R$ such that

$$yad = d = day, \quad yR \subseteq dR, \quad Ry \subseteq Rd.$$

If such y exists, it is unique and it is called the inverse along an element d, denoted by $a^{\|d}$. We use notation $R^{\|d}$ to denote the set of all elements of R which are invertible along d. Obviously, the inverse along an element d is the (b,c)-inverse (for $b = c = d$, we have $a^{\|d} = a^{(d,d)}$). Recently, it was proved that the converse is also true [19, Theorem C.4, p.260]. More precisely, the (b,c)-inverse and the inverse along an element in a semigroup are actually genuine inverse when considered as morphisms in the Schutzenberger category of the semigroup [19]. For more properties of the inverse along an element, we suggest the reader to see [4, 19, 20, 37, 38].

Now, we will list some auxiliary lemmas, which we will use in our sequel development.

LEMMA 1.1. ([11, Theorem 2.2]) *For $a, b, c \in R$, the following statements are equivalent:*

(i) $a^{(b,c)}$ *exists.*
(ii) $b \in Rcab$ *and* $c \in cabR$.

LEMMA 1.2. ([9, Lemma 2.1]) *Let $a, b, c \in R$. The following statements are equivalent:*

(i) $a^{(b,c)}$ *exists;*
(ii) $b, c \in R^-$ *and there exists $y \in R$ such that $yab = b$, $cay = c$, $y = bb^- y = yc^- c$.*

In that case, $y = a^{(b,c)}$.

LEMMA 1.3. ([14, Theorem 2.9]) *Let $a, b, c \in R$. The following statements are equivalent:*

(i) $a \in R^{(b,c)}$;
(ii) *there exists $y \in R$ such that $yay = y$, $yR = bR$ and $Ry = Rc$;*
(ii)′ $b \in R^-$ *and there exists $y \in R$ such that $yay = y$, $^\circ y = {}^\circ b$ and $Ry = Rc$;*
(ii)″ $c \in R^-$ *and there exists $y \in R$ such that $yay = y$, $yR = bR$ and $y^\circ = c^\circ$;*
(ii)‴ $b, c \in R^-$ *and there exists $y \in R$ such that $yay = y$, $^\circ y = {}^\circ b$ and $y^\circ = c^\circ$;*
(iii) $b \in R^-$, $^\circ a \cap Rc = \{0\}$ *and $R = Rca \oplus {}^\circ b$;*
(iii)′ $c \in R^-$, $a^\circ \cap bR = \{0\}$ *and $R = abR \oplus c^\circ$.*

In this case, $y = a^{(b,c)}$.

LEMMA 1.4. ([34, Lemma 3.11]) *Let $a, b_1, b_2, c_1, c_2 \in R$. If $b_1 R = b_2 R$ and $Rc_1 = Rc_2$, then a is (b_1, c_1)-invertible if and only if a is (b_2, c_2)-invertible. In this case, we have $a^{(b_1, c_1)} = a^{(b_2, c_2)}$.*

LEMMA 1.5. ([29, Lemma 4.1]) *Let $b, c \in R$ and $a, w \in R^{(b,c)}$. Then, $w^{(b,c)} = w^{(b,c)} a a^{(b,c)} = a^{(b,c)} a w^{(b,c)}$.*

LEMMA 1.6. ([14, Lemma 2.8]) *Let $b_1, b_2 \in R^-$. Then, the following hold:*

(i) *the condition $b_1 R = b_2 R$ is equivalent to $^\circ b_1 = {}^\circ b_2$;*
(ii) *the condition $Rb_1 = Rb_2$ is equivalent to $b_1^\circ = b_2^\circ$.*

LEMMA 1.7. ([26, Theorem 3.3]) *Let $b, c, w \in R$ and $a \in R^{(b,c)}$. Then, $cab = cwb$ if and only if $w \in R^{(b,c)}$ and $a^{(b,c)} = w^{(b,c)}$.*

Proof. Let $cab = cwb$ and let $y = a^{(b,c)}$. Hence, $b \in Rcab = Rcwb$ and $c \in cabR = cwbR$. Thus, by Lemma 1.1, we have that $w \in R^{(b,c)}$. Furthermore, we have $ca - cw \in {}^\circ b$. By Lemma 1.3 (ii)$'$, we have ${}^\circ b = {}^\circ y$, and therefore, $cay - cwy = 0$, that is $c = cwy$. Further, we have $ab - wb \in c^\circ$. By Lemma 1.3 (ii)$''$, $c^\circ = y^\circ$ holds, so we get $yab - ywb = 0$ and thereby $ywb = b$. Since $y = a^{(b,c)}$, we have that $y \in (bRy) \cap (yRc)$. Therefore, by the definition of the (b, c)-inverse, we have that $w^{(b,c)} = y$.

Now, let us prove that the converse also holds. Let $a, w \in R^{(b,c)}$ and $y = a^{(b,c)} = w^{(b,c)}$. Then, $b = yab = ywb$, so we have $y(ab - wb) = 0$. Hence, $ab - wb \in y^\circ$. By Lemma 1.3 (ii)$''$, we have $y^\circ = c^\circ$. Thus, $cab - cwb = 0$. □

LEMMA 1.8. ([26, Theorem 4.4]) *Let $b, c \in R$ and $a, 1 \in R^{(b,c)}$. Then, $a^{(b,c)} = a^{(b,c)}1^{(b,c)} = 1^{(b,c)}a^{(b,c)}$.*

Proof. By Lemma 1.5, for every $a, w \in R^{(b,c)}$, $a^{(b,c)} = a^{(b,c)}ww^{(b,c)} = w^{(b,c)}wa^{(b,c)}$ holds. Hence, taking $w = 1$, we get that the statement of this lemma is valid. □

LEMMA 1.9. ([27, Theorem 3.2]) *Let $b, c \in R$ and $a \in R^{(b,c)}$. If $aa^{(b,c)} = a^{(b,c)}a$, then $1 \in R^{(b,c)}$ and $1^{(b,c)} = aa^{(b,c)} = a^{(b,c)}a$.*

LEMMA 1.10. ([27, Theorem 3.1]) *Let $b, c \in R$ and $a \in R^{(b,c)}$. Then, $a^{(b,c)} \in R^{(b,c)}$ if and only if $1 \in R^{(b,c)}$. In that case, $(a^{(b,c)})^{(b,c)} = 1^{(b,c)}a1^{(b,c)}$.*

2. Reverse order law. We begin this section with the following example, where we show that the reverse order law (and also the forward order law) is not valid for the (b, c)-inverse in general.

EXAMPLE 2.1. Let M_2 stands for the algebra of 2×2 complex matrices and let $b, c, \alpha \in M_2$ be such that:

$$b = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \alpha = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}, \quad \text{where } \alpha_3 \neq 0.$$

We will use Lemma 1.2 to prove that α is (b, c)-invertible. Namely, we should prove that there exists $y \in M_2$, $y = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}$, such that $y\alpha b = b$, $c\alpha y = c$ and $y = bb^- y = yc^- c$ is satisfied. We have that:

$$b^- = \begin{bmatrix} 1 & b_2 \\ b_3 & b_4 \end{bmatrix}, \quad c^- = \begin{bmatrix} c_1 & c_2 \\ c_3 & 1 \end{bmatrix}, \quad \text{for arbitrary } b_2, b_3, b_4, c_1, c_2, c_3 \in \mathbb{C}.$$

Now, we get:

$$bb^- y = \begin{bmatrix} y_1 + b_2 y_3 & y_2 + b_2 y_4 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad yc^- c = \begin{bmatrix} 0 & y_1 c_2 + y_2 \\ 0 & y_3 c_2 + y_4 \end{bmatrix}.$$

From the conditions $bb^- y = y$ and $yc^- c = y$, we get that $y_1 = y_3 = y_4 = 0$. Further, we obtain:

$$y\alpha b = \begin{bmatrix} y_2 \alpha_3 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad c\alpha y = \begin{bmatrix} 0 & 0 \\ 0 & \alpha_3 y_2 \end{bmatrix}.$$

Now, from the conditions $y\alpha b = b$ and $c\alpha y = c$, we obtain $y_2 = \alpha_3^{-1}$. Therefore, we get that $y\alpha b = b$, $c\alpha y = c$ and $y = bb^- y = yc^- c$ is satisfied for

$$y = \begin{bmatrix} 0 & \alpha_3^{-1} \\ 0 & 0 \end{bmatrix}.$$

Hence, by Lemma 1.2, α is (b, c)-invertible and $\alpha^{(b,c)} = y$. Now, consider matrices

$$a = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}, \quad w = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

We have that a, w, and $aw = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$ are all (b, c)-invertible and:

$$a^{(b,c)} = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, \quad w^{(b,c)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad (aw)^{(b,c)} = \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & 0 \end{bmatrix}.$$

However, we have that

$$w^{(b,c)}a^{(b,c)} = a^{(b,c)}w^{(b,c)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $(aw)^{(b,c)} \neq w^{(b,c)}a^{(b,c)}$ and $(aw)^{(b,c)} \neq a^{(b,c)}w^{(b,c)}$.

Now, we give the next result, which will be one of our main tools in investigating the reverse order law for the (b, c)-inverse. Namely, in [29, Lemma 4.1], Wang, Castro–González, and Chen proved that for arbitrary $a, w \in R^{(b,c)}$, $a^{(b,c)} = a^{(b,c)}ww^{(b,c)} = w^{(b,c)}wa^{(b,c)}$ is valid. In the following two theorems, we generalize this result.

THEOREM 2.2. *Let* $a, b_1, b_2, c_1, c_2 \in R$, $w \in R^{(b_2, c_2)}$ *and* $b_1 R = b_2 R$. *The following conditions are equivalent:*

(i) $a \in R^{(b_1, c_1)}$;
(ii) $a \in R^{(b_2, c_1)}$.

Moreover, if any of the above conditions is satisfied, then

$$(2.5) \qquad\qquad a^{(b_1, c_1)} = a^{(b_2, c_1)} = w^{(b_2, c_2)}wa^{(b_1, c_1)}.$$

Proof. Since $b_1 R = b_2 R$, applying Lemma 1.4, we get that $a \in R^{(b_1, c_1)}$ if and only if $a \in R^{(b_2, c_1)}$. Further, in that case:

$$(2.6) \qquad\qquad a^{(b_1, c_1)} = a^{(b_2, c_1)}.$$

Now, let $a \in R^{(b_1, c_1)}$, $w \in R^{(b_2, c_2)}$ and let us prove that $w^{(b_2, c_2)}wa^{(b_1, c_1)}$ is the (b_2, c_1)-inverse of a. By the hypothesis and by Lemma 1.2, we have $b_1, b_2, c_1, c_2 \in R^-$. Denote by $y_1 = a^{(b_1, c_1)}$, $y_2 = w^{(b_2, c_2)}$, and $y = y_2 w y_1$. Since $b_1 R = b_2 R$, we have $b_1 = b_2 x$ and $b_2 = b_1 z$, for some $x, z \in R$. Therefore,

$$(2.7) \qquad yab_2 = y_2 wy_1 ab_2 = y_2 w(y_1 ab_1)z = y_2 w(b_1 z) = y_2 wb_2 = b_2.$$

Further, since $y_1 = a^{(b_1, c_1)}$, we have that $y_1 \in b_1 Ry_1$. Thereby, there exists $u \in R$ such that $y_1 = b_1 uy_1$. Thus, we have:

$$(2.8) \qquad \begin{aligned} c_1 ay &= c_1 ay_2 wy_1 = c_1 ay_2 wb_1 uy_1 = c_1 a(y_2 wb_2)xuy_1 \\ &= c_1 a(b_2 x)uy_1 = c_1 a(b_1 uy_1) = c_1 ay_1 = c_1. \end{aligned}$$

Moreover, we have:

$$(2.9) \qquad\qquad b_2 b_2^- y = (b_2 b_2^- w^{(b_2, c_2)})wa^{(b_1, c_1)} = y,$$

$$(2.10) \qquad\qquad yc_1^- c_1 = w^{(b_2, c_2)}w(a^{(b_1, c_1)}c_1^- c_1) = y.$$

Now, from (2.7), (2.8), (2.9), and (2.10), using Lemma 1.2, we get that $y = a^{(b_2, c_1)}$. Further, from (2.6), we get that (2.5) is valid. $\qquad\square$

As a direct corollary of Theorem 2.2, we get the following result.

COROLLARY 2.3. *Let $a, b, c_1, c_2 \in R$ and $a \in R^{(b,c_1)}$. Then for arbitrary, but fixed $c_2 \in R$ and for every $w \in R^{(b,c_2)}$, the following is valid:*

$$a^{(b,c_1)} = w^{(b,c_2)} w a^{(b,c_1)}.$$

Next theorem is dual to Theorem 2.2, and therefore, its proof is omitted.

THEOREM 2.4. *Let $a, b_1, b_2, c_1, c_2 \in R$, $w \in R^{(b_2,c_2)}$ and $Rc_1 = Rc_2$. The following conditions are equivalent:*

(i) $a \in R^{(b_1,c_1)}$;
(ii) $a \in R^{(b_1,c_2)}$.

Moreover, if any of the above conditions is satisfied, then

$$a^{(b_1,c_1)} = a^{(b_1,c_2)} = a^{(b_1,c_1)} w w^{(b_2,c_2)}.$$

Now, we give the following corollary of Theorem 2.4.

COROLLARY 2.5. *Let $b_1, c \in R$ and $a \in R^{(b_1,c)}$. Then for arbitrary, but fixed $b_2 \in R$ and for every $w \in R^{(b_2,c)}$, the following holds:*

$$a^{(b_1,c)} = a^{(b_1,c)} w w^{(b_2,c)}.$$

REMARK 2.6. Note that, by Lemma 1.6, the condition $b_1 R = b_2 R$ from Theorem 2.2 can be replaced by conditions $b_1 \in R^-$ and $^\circ b_1 = {}^\circ b_2$. Similarly, by Lemma 1.6, the condition $Rc_1 = Rc_2$ from Theorem 2.4 can be replaced by conditions $c_1 \in R^-$ and $c_1^\circ = c_2^\circ$.

In the following corollary, we give some special cases of Corollary 2.3 and Corollary 2.5, which will be useful in our sequel development.

COROLLARY 2.7. *Let $b_1, b_2, c_1, c_2 \in R$.*

(i) *If $a \in R^{(b,c_1)}$ and $1 \in R^{(b,c_2)}$, then:*
(a) $1^{(b,c_2)} a^{(b,c_1)} = a^{(b,c_1)}$;
(b) $a^{(b,c_1)} a 1^{(b,c_2)} = 1^{(b,c_2)}$.
(ii) *If $a \in R^{(b_1,c)}$ and $1 \in R^{(b_2,c)}$, then:*
(c) $a^{(b_1,c)} 1^{(b_2,c)} = a^{(b_1,c)}$;
(d) $1^{(b_2,c)} a a^{(b_1,c)} = 1^{(b_2,c)}$.

In the next theorem, we show that (b, c)-invertibility of the unity, for specific b and c, is a necessary condition for the reverse order law to hold for the (b, c)-inverse in a ring. We remark that the condition $1 \in R^{(b,c)}$ coincides with the condition cb is a trace product ($b \in Rcb$ and $c \in cbR$) of [19]. Therefore, Theorem 2.8 and Theorem 2.9 follow from the results of [19, Theorem C.2, p.264] and [19, Theorem C.7, p.262]. In the present paper, these theorems are proved in the ring setting, using direct sum decompositions.

THEOREM 2.8. *Let $b, b_1, c, c_2 \in R$, $a \in R^{(b_1,c)}$, $w \in R^{(b,c_2)}$ and $aw \in R^{(b,c)}$. If $(aw)^{(b,c)} = w^{(b,c_2)} a^{(b_1,c)}$, then $1 \in R^{(b_1,c_2)}$.*

Proof. Let the assumptions of the theorem hold. Using Lemma 1.3 (ii), we have:

$$Rc = R(aw)^{(b,c)} = Rw^{(b,c_2)} a^{(b_1,c)} = Rc_2 a^{(b_1,c)}.$$

Since $a \in R^{(b_1,c)}$, by Lemma 1.3 (iii), we have $R = Rca \oplus {}^\circ b_1$. Thus,

$$R = Rc_2 a^{(b_1,c)} a \oplus {}^\circ b_1.$$

Therefore, there exist some $u,v \in R$, such that $1 = uc_2 a^{(b_1,c)} a + v$ and $vb_1 = 0$. Hence, $b_1 = uc_2 a^{(b_1,c)} ab_1 = uc_2 b_1$ and thereby:

$$(2.11) \qquad\qquad\qquad\qquad b_1 \in Rc_2 b_1.$$

Moreover, using Lemma 1.3 again, we get:

$$bR = (aw)^{(b,c)} R = w^{(b,c_2)} a^{(b_1,c)} R = w^{(b,c_2)} b_1 R.$$

Since $w \in R^{(b,c_2)}$, by Lemma 1.3 (iii'), we get $R = wbR \oplus c_2^\circ$. Hence,

$$R = ww^{(b,c_2)} b_1 R \oplus c_2^\circ.$$

Thus, there are some $s,t \in R$, such that $1 = ww^{(b,c_2)} b_1 s + t$ and $c_2 t = 0$. Therefore, $c_2 = c_2 ww^{(b,c_2)} b_1 s = c_2 b_1 s$. Hence,

$$(2.12) \qquad\qquad\qquad\qquad c_2 \in c_2 b_1 R.$$

Using Lemma 1.1, by (2.11) and (2.12), we get that $1 \in R^{(b_1,c_2)}$. $\qquad\square$

The following result will be our key tool for obtaining equivalent conditions for the reverse order law and the forward order law to hold, for the (b,c)-inverse. Namely, in [27, Theorem 3.1], authors proved that in the case when $a,w,1 \in R^{(b,c)}$, then $a1^{(b,c)}w \in R^{(b,c)}$ and $(a1^{(b,c)}w)^{(b,c)} = w^{(b,c)} a^{(b,c)}$. In the next theorem, we generalize this result.

THEOREM 2.9. *Let* $b,b_1,c,c_2 \in R$, $a \in R^{(b_1,c)}$, $w \in R^{(b,c_2)}$ *and* $1 \in R^{(b_1,c_2)}$. *Then* $a1^{(b_1,c_2)}w \in R^{(b,c)}$ *and*

$$(a1^{(b_1,c_2)}w)^{(b,c)} = w^{(b,c_2)} a^{(b_1,c)}.$$

Proof. Let the assumptions of the theorem hold. We will prove that $w^{(b,c_2)} a^{(b_1,c)}$ is the (b,c)-inverse of $a1^{(b_1,c_2)}w$. By Corollary 2.7 (a) and (d), we have $1^{(b_1,c_2)} a^{(b_1,c)} = a^{(b_1,c)}$ and $1^{(b_1,c_2)} = 1^{(b_1,c_2)} ww^{(b,c_2)}$. Hence, we have:

$$(2.13) \qquad ca1^{(b_1,c_2)} ww^{(b,c_2)} a^{(b_1,c)} = ca1^{(b_1,c_2)} a^{(b_1,c)} = caa^{(b_1,c)} = c.$$

Using Corollary 2.7 (b) and (c), we get that $a^{(b_1,c)} a1^{(b_1,c_2)} = 1^{(b_1,c_2)}$ and $w^{(b,c_2)} 1^{(b_1,c_2)} = w^{(b,c_2)}$. Therefore:

$$(2.14) \qquad w^{(b,c_2)} a^{(b_1,c)} a1^{(b_1,c_2)} wb = w^{(b,c_2)} 1^{(b_1,c_2)} wb = w^{(b,c_2)} wb = b.$$

Moreover, we have that:

$$(2.15) \qquad\qquad\qquad bb^- w^{(b,c_2)} a^{(b_1,c)} = w^{(b,c_2)} a^{(b_1,c)},$$

$$(2.16) \qquad\qquad\qquad w^{(b,c_2)} a^{(b_1,c)} c^- c = w^{(b,c_2)} a^{(b_1,c)}.$$

By (2.13), (2.14), (2.15) and (2.16), using Lemma 1.2, we get that $a1^{(b_1,c_2)}w \in R^{(b,c)}$ and

$$(a1^{(b_1,c_2)}w)^{(b,c)} = w^{(b,c_2)} a^{(b_1,c)}. \qquad\square$$

Now we can derive an equivalent condition for the reverse order law rule $(aw)^{(b,c)} = w^{(b,c_2)}a^{(b_1,c)}$ to hold.

THEOREM 2.10. *Let* $b, b_1, c, c_2 \in R$, $a \in R^{(b_1,c)}$ *and* $w \in R^{(b,c_2)}$. *The following conditions are equivalent:*

(i) $aw \in R^{(b,c)}$ *and* $(aw)^{(b,c)} = w^{(b,c_2)}a^{(b_1,c)}$;

(ii) $1 \in R^{(b_1,c_2)}$ *and* $1^{(b_1,c_2)} = a^{(b_1,c)}aww^{(b,c_2)}$.

Proof. Let $a \in R^{(b_1,c)}$ and $w \in R^{(b,c_2)}$.

(i) $\Rightarrow$ (ii). Let $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = w^{(b,c_2)}a^{(b_1,c)}$. By Theorem 2.8, it follows that $1 \in R^{(b_1,c_2)}$. Further, by Theorem 2.9, we have that $a1^{(b_1,c_2)}w \in R^{(b,c)}$ and $(a1^{(b_1,c_2)}w)^{(b,c)} = w^{(b,c_2)}a^{(b_1,c)}$. Hence, $(aw)^{(b,c)} = (a1^{(b_1,c_2)}w)^{(b,c)}$. By Lemma 1.7, it follows that:

$$cawb = ca1^{(b_1,c_2)}wb.$$

Therefore, $awb - a1^{(b_1,c_2)}w \in c^\circ$. By Lemma 1.3, we have $c^\circ = (a^{(b_1,c)})^\circ$ and thereby:

$$a^{(b_1,c)}awb - a^{(b_1,c)}a1^{(b_1,c_2)}wb = 0.$$

Using Corollary 2.7 (b), we get $a^{(b_1,c)}a1^{(b_1,c_2)} = 1^{(b_1,c_2)}$. Hence,

$$a^{(b_1,c)}awb - 1^{(b_1,c_2)}wb = 0.$$

Thus, $a^{(b_1,c)}aw - 1^{(b_1,c_2)}w \in^\circ b$. By Lemma 1.3, we have $^\circ b =^\circ (w^{(b,c_2)})$, so:

$$a^{(b_1,c)}aww^{(b,c_2)} - 1^{(b_1,c_2)}ww^{(b,c_2)} = 0.$$

By Corollary 2.7 (d), we have $1^{(b_1,c_2)}ww^{(b,c_2)} = 1^{(b_1,c_2)}$. Therefore, $a^{(b_1,c)}aww^{(b,c_2)} - 1^{(b_1,c_2)} = 0$.

(ii) $\Rightarrow$ (i). Let $1 \in R^{(b_1,c_2)}$ and $a^{(b_1,c)}aww^{(b,c_2)} = 1^{(b_1,c_2)}$. Denote by $y = w^{(b,c_2)}a^{(b_1,c)}$. We will prove that y is the (b,c)-inverse of aw. By Corollary 2.7 (a), we have that $1^{(b_1,c_2)}a^{(b_1,c)} = a^{(b_1,c)}$. Thus,

$$cawy = caww^{(b,c_2)}a^{(b_1,c)} = caa^{(b_1,c)}aww^{(b,c_2)}a^{(b_1,c)} = ca1^{(b_1,c_2)}a^{(b_1,c)} = caa^{(b_1,c)} = c.$$

Further, by Corollary 2.7 (c), $w^{(b,c_2)}1^{(b_1,c_2)} = w^{(b,c_2)}$. Hence,

$$yawb = w^{(b,c_2)}a^{(b_1,c)}awb = w^{(b,c_2)}a^{(b_1,c)}aww^{(b,c_2)}wb = w^{(b,c_2)}1^{(b_1,c_2)}wb = b.$$

Moreover,

$$bb^-y = (bb^-w^{(b,c_2)})a^{(b_1,c)} = y \quad \text{and} \quad yc^-c = w^{(b,c_2)}(a^{(b_1,c)}c^-c) = y.$$

Now, by Lemma 1.2, we have that $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = y$. □

REMARK 2.11. Using Lemma 1.4 and Theorem 2.10, one can get an equivalent condition for the reverse order law rule $(aw)^{(b,c)} = w^{(b_2,c_2)}a^{(b_1,c_1)}$ to hold, under assumptions $bR = b_2R$ and $Rc = Rc_1$. Namely, let $b, b_1, b_2, c, c_1, c_2 \in R$, $a \in R^{(b_1,c_1)}$ and $w \in R^{(b_2,c_2)}$. If $bR = b_2R$ and $Rc = Rc_1$, then the following conditions are equivalent:

(i) $aw \in R^{(b,c)}$ *and* $(aw)^{(b,c)} = w^{(b_2,c_2)}a^{(b_1,c_1)}$;

(ii) $1 \in R^{(b_1,c_2)}$ *and* $1^{(b_1,c_2)} = a^{(b_1,c_1)}aww^{(b_2,c_2)}$.

REMARK 2.12. Note that the reverse order law rules $(aw)^{(b,c)} = w^{(b,c_2)}a^{(b,c)}$, $(aw)^{(b,c)} = w^{(b,c)}a^{(b_1,c)}$, and $(aw)^{(b,c)} = w^{(b,c)}a^{(b,c)}$ are all special cases of the reverse order law rule investigated in Theorem 2.10.

Recently, in [27, Theorem 3.8], Višnjić et al. proved that if $ww^{(b,c)} = w^{(b,c)}w$, then $(aw)^{(b,c)} = w^{(b,c)}a^{(b,c)}$. And in [28, Corollary 3.4], Wang proved that if $aa^{(b,c)} = a^{(b,c)}a$, then the reverse order law rule $(aw)^{(b,c)} = w^{(b,c)}a^{(b,c)}$ holds. In the following theorems, we generalize these results.

THEOREM 2.13. *Let $b, c, c_2 \in R$, $a \in R^{(b,c)}$ and $w \in R^{(b,c_2)}$. If $ww^{(b,c_2)} = w^{(b,c_2)}w$, then $aw \in R^{(b,c)}$ and*

$$(aw)^{(b,c)} = w^{(b,c_2)}a^{(b,c)}.$$

Proof. Let the assumptions of the theorem hold and let $ww^{(b,c_2)} = w^{(b,c_2)}w$. By Lemma 1.9, it follows that $1 \in R^{(b,c_2)}$ and $ww^{(b,c_2)} = w^{(b,c_2)}w = 1^{(b,c_2)}$. Moreover, by Corollary 2.7 (b), we have $a^{(b,c)}a1^{(b,c_2)} = 1^{(b,c_2)}$. Hence,

$$a^{(b,c)}aww^{(b,c_2)} = a^{(b,c)}a1^{(b,c_2)} = 1^{(b,c_2)}.$$

Now, using Theorem 2.10 for $b_1 = b$, we get that $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = w^{(b,c_2)}a^{(b,c)}$.   □

THEOREM 2.14. *Let $b, b_1, c \in R$, $a \in R^{(b_1,c)}$ and $w \in R^{(b,c)}$. If $aa^{(b_1,c)} = a^{(b_1,c)}a$, then $aw \in R^{(b,c)}$ and*

$$(aw)^{(b,c)} = w^{(b,c)}a^{(b_1,c)}.$$

EXAMPLE 2.15. Let $b, b_1, c, \alpha, \beta \in M_2$ be such that:

$$b = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \alpha = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix},$$

where $\alpha_1 \neq 0$ and $\beta_2 \neq 0$. By Lemma 1.2, we get that α is (b_1,c)-invertible, β is (b,c)-invertible and:

$$\alpha^{(b_1,c)} = \begin{bmatrix} \alpha_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \beta^{(b,c)} = \begin{bmatrix} 0 & 0 \\ \beta_2^{-1} & 0 \end{bmatrix}.$$

Now, let $a, w \in M_2$ be such that $a = \begin{bmatrix} a_1 & 0 \\ 0 & a_4 \end{bmatrix}$, $w = \begin{bmatrix} 0 & w_2 \\ w_3 & w_4 \end{bmatrix}$, where $a_1 \neq 0$ and $w_2 \neq 0$. We have that w and aw are both (b,c)-invertible and

$$w^{(b,c)} = \begin{bmatrix} 0 & 0 \\ w_2^{-1} & 0 \end{bmatrix} \quad \text{and} \quad (aw)^{(b,c)} = \begin{bmatrix} 0 & 0 \\ a_1^{-1}w_2^{-1} & 0 \end{bmatrix}.$$

Since $cab = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, using Lemma 1.1, we get that a is not (b,c)-invertible. Thereby, the reverse order law rule $(aw)^{(b,c)} = w^{(b,c)}a^{(b,c)}$ does not hold. However, we have that a is (b_1,c)-invertible and

$$a^{(b_1,c)} = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Moreover, one can easily check that $aa^{(b_1,c)} = a^{(b_1,c)}a$, and therefore, we can apply Theorem 2.14. Thus, the reverse order law rule $(aw)^{(b,c)} = w^{(b,c)}a^{(b_1,c)}$ holds.

As a direct corollary of Theorem 2.10, in the case when $b_1 = b = c_2 = c$, we get a necessary and sufficient condition for the reverse order law to hold for the inverse along an element.

COROLLARY 2.16. *Let $d \in R$ and $a, w \in R^{\|d}$. The following are equivalent:*

(i) $aw \in R^{\|d}$ *and* $(aw)^{\|d} = w^{\|d}a^{\|d}$;
(ii) $1 \in R^{\|d}$ *and* $1^{\|d} = a^{\|d}aww^{\|d}$.

For more results on the reverse order law for the inverse along an element, we refer the reader to see [4, 19, 37].

3. Forward order law. As for the reverse order law (Theorem 2.8), we first prove that the forward order law for the (b,c)-inverse implies (b,c)-invertibility of the unity 1. Our results on the forward order law will follow.

THEOREM 3.1. *Let $b, b_2, c, c_1 \in R$, $a \in R^{(b,c_1)}$, $w \in R^{(b_2,c)}$ and $aw \in R^{(b,c)}$. If $(aw)^{(b,c)} = a^{(b,c_1)}w^{(b_2,c)}$, then $1 \in R^{(b_2,c_1)}$.*

Proof. The proof is close to that of Theorem 2.8. By Lemma 1.3 (ii), we have:

$$Rc = R(aw)^{(b,c)} = Ra^{(b,c_1)}w^{(b_2,c)} = Rc_1 w^{(b_2,c)},$$

$$bR = (aw)^{(b,c)}R = a^{(b,c_1)}w^{(b_2,c)}R = a^{(b,c_1)}b_2 R.$$

Moreover, since $w \in R^{(b_2,c)}$ and $a \in R^{(b,c_1)}$, by Lemma 1.3 (iii) and (iii)', we have $R = Rcw \oplus {}^\circ b_2$ and $R = abR \oplus c_1^\circ$. Hence,

$$R = Rc_1 w^{(b_2,c)}w \oplus {}^\circ b_2 \quad \text{and} \quad R = aa^{(b,c_1)}b_2 R \oplus c_1^\circ.$$

Therefore, there exist some $u, s \in R$, $v \in {}^\circ b_2$ and $t \in c_1^\circ$, such that $1 = uc_1 w^{(b_2,c)}w + v$ and $1 = aa^{(b,c_1)}b_2 s + t$. Thus,

$$b_2 = uc_1 w^{(b_2,c)}wb_2 = uc_1 b_2 \quad \text{and} \quad c_1 = c_1 aa^{(b,c_1)}b_2 s = c_1 b_2 s.$$

Hence, $b_2 \in Rc_1 b_2$ and $c_1 \in c_1 b_2 R$. Using Lemma 1.1 completes the proof. □

In the next theorem, we derive necessary and sufficient conditions for the forward order law rule $(aw)^{(b,c)} = a^{(b,c_1)}w^{(b_2,c)}$ to hold.

THEOREM 3.2. *Let $b, b_2, c, c_1 \in R$, $a \in R^{(b,c_1)}$ and $w \in R^{(b_2,c)}$. The following conditions are equivalent:*

(i) $aw \in R^{(b,c)}$ *and* $(aw)^{(b,c)} = a^{(b,c_1)}w^{(b_2,c)}$;
(ii) $1 \in R^{(b_2,c_1)}$ *and* $1^{(b_2,c_1)} = w^{(b_2,c)}awa^{(b,c_1)}$.

Proof. Using the similar method as in the proof of Theorem 2.10, one can get that the statement of this theorem is true. Here we give just key steps.

(i) $\Rightarrow$ (ii). If $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = a^{(b,c_1)}w^{(b_2,c)}$, then by Theorem 3.1 $1 \in R^{(b_2,c_1)}$. Moreover, by Theorem 2.9 $w1^{(b_2,c_1)}a \in R^{(b,c)}$ and $(w1^{(b_2,c_1)}a)^{(b,c)} = a^{(b,c_1)}w^{(b_2,c)}$. Now, by Lemma 1.7, we have $cawb = cw1^{(b_2,c_1)}ab$. Using Lemma 1.3 (ii)''', we get $w^{(b_2,c)}awa^{(b,c_1)} = 1^{(b_2,c_1)}$.

(ii) $\Rightarrow$ (i). If $1 \in R^{(b_2,c_1)}$ and $w^{(b_2,c)}awa^{(b,c_1)} = 1^{(b_2,c_1)}$, then $a^{(b,c_1)}w^{(b_2,c)}$ is the (b,c)-inverse of aw. Indeed, denote by $y = a^{(b,c_1)}w^{(b_2,c)}$. Using Corollary 2.7 (a) and (c), one can get that $cawy = c$ and $yawb = b$. Moreover, $y = bb^-y = yc^-c$ holds and thereby, by Lemma 1.2, $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = a^{(b,c_1)}w^{(b_2,c)}$. □

REMARK 3.3. By Lemma 1.4 and Theorem 3.2, we can obtain a necessary and sufficient condition for the forward order law rule $(aw)^{(b,c)} = a^{(b_1,c_1)}w^{(b_2,c_2)}$ to hold, under assumptions $bR = b_1 R$ and $Rc = Rc_2$. Actually, we have that the following is valid. Let $b, b_1, b_2, c, c_1, c_2 \in R$, $a \in R^{(b_1,c_1)}$ and $w \in R^{(b_2,c_2)}$. If $bR = b_1 R$ and $Rc = Rc_2$, then the following conditions are equivalent:

(i) $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = a^{(b_1,c_1)}w^{(b_2,c_2)}$;
(ii) $1 \in R^{(b_2,c_1)}$ and $1^{(b_2,c_1)} = w^{(b_2,c_2)}awa^{(b_1,c_1)}$.

REMARK 3.4. Note that the forward order law rules $(aw)^{(b,c)} = a^{(b,c_1)}w^{(b,c)}$, $(aw)^{(b,c)} = a^{(b,c)}w^{(b_2,c)}$ and $(aw)^{(b,c)} = a^{(b,c)}w^{(b,c)}$ are all special cases of the forward order law rule studied in Theorem 3.2.

It is well known that, for two square and nonsingular matrices a and w, the forward order law $(aw)^{-1} = a^{-1}w^{-1}$ holds if and only if $aw = wa$. In the following example, we consider two square matrices a and w, for which $aw \neq wa$ and for which the forward order law rule $(aw)^{(b,c)} = a^{(b,c)}w^{(b_2,c)}$ holds.

EXAMPLE 3.5. Let $b, b_2, c, a, w \in M_2$ be such that:

$$b = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad a = \begin{bmatrix} 0 & a_2 \\ a_3 & a_4 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_1 \end{bmatrix},$$

where $a_2 \neq 0$, $a_4 \neq 0$, $w_1 \neq 0$ and $w_2 \neq 0$. It can be checked easily that $aw \neq wa$. Moreover, using Example 2.15, we get that a is (b, c)-invertible, w is (b_2, c)-invertible and

$$a^{(b,c)} = \begin{bmatrix} 0 & 0 \\ a_2^{-1} & 0 \end{bmatrix}, \quad w^{(b_2,c)} = \begin{bmatrix} w_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Furthermore, we have that the identity matrix $u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is (b_2, c)-invertible and $u^{(b_2,c)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. By straightforward computation, we get $u^{(b_2,c)} = w^{(b_2,c)}awa^{(b,c)}$. Hence, by Theorem 3.2, aw is (b, c)-invertible and the forward order law rule $(aw)^{(b,c)} = a^{(b,c)}w^{(b_2,c)}$ holds.

In the sequel, we study the forward order law for the (b, c)-inverse, for different choices of b and c.

THEOREM 3.6. Let $b, c, c_1 \in R$, $a \in R^{(b,c_1)}$ and $w \in R^{(b,c)}$. If $awa^{(b,c_1)} = wa^{(b,c_1)}a$ and $w^{(b,c)}aw = aw^{(b,c)}w$, then $aw \in R^{(b,c)}$ and

$$(aw)^{(b,c)} = a^{(b,c_1)}w^{(b,c)}.$$

Proof. Let the hypothesis of the theorem hold. We will prove that $1 \in R^{(b,c_1)}$ and $1^{(b,c_1)} = w^{(b,c)}awa^{(b,c_1)}$. Indeed,

$$w^{(b,c)}(awa^{(b,c_1)}) \cdot 1 \cdot b = w^{(b,c)}wa^{(b,c_1)}ab = w^{(b,c)}wb = b.$$

Moreover, by Corollary 2.3, we have $w^{(b,c)}wa^{(b,c_1)} = a^{(b,c_1)}$ and thereby:

$$c_1 \cdot 1 \cdot (w^{(b,c)}aw)a^{(b,c_1)} = c_1 aw^{(b,c)}wa^{(b,c_1)} = c_1 aa^{(b,c_1)} = c_1.$$

Further, $bb^- w^{(b,c)}awa^{(b,c_1)} = w^{(b,c)}awa^{(b,c_1)} = w^{(b,c)}awa^{(b,c_1)}c_1^- c_1$. Using Lemma 1.2, we get that $1 \in R^{(b,c_1)}$ and $1^{(b,c_1)} = w^{(b,c)}awa^{(b,c_1)}$. Hence, using Theorem 3.2 for $b_2 = b$, we get that the statement of the theorem is true.   □

Dually, we have the following result.

THEOREM 3.7. Let $b, b_2, c \in R$, $a \in R^{(b,c)}$ and $w \in R^{(b_2,c)}$. If $awa^{(b,c)} = aa^{(b,c)}w$ and $w^{(b_2,c)}aw = ww^{(b_2,c)}a$, then $aw \in R^{(b,c)}$ and

$$(aw)^{(b,c)} = a^{(b,c)}w^{(b_2,c)}.$$

As we have proved in Theorem 3.1, (b, c)-invertibility of 1 (for suitable choices of b and c) is a necessary condition for the forward order law to hold for the (b, c)-inverse. In other words, if 1 is not (b, c)-invertible for suitable choices of b and c, then the forward order law for the (b, c)-inverse does not hold. Hence, it is natural to assume that 1 is (b, c)-invertible, for appropriate b and c, in investigations concerning the forward order law for the (b, c)-inverse. In the next theorem, we suppose that $1 \in R^{(b,c_1)}$, and we obtain that the forward order law rule $(aw)^{(b,c)} = a^{(b,c_1)}w^{(b,c)}$ holds if only one of two conditions ($awa^{(b,c_1)} = wa^{(b,c_1)}a$ and $w^{(b,c)}aw = aw^{(b,c)}w$) of Theorem 3.6 is valid.

THEOREM 3.8. *Let* $b, c, c_1 \in R$, $a \in R^{(b,c_1)}$, $w \in R^{(b,c)}$ *and* $1 \in R^{(b,c_1)}$. *Consider the conditions:*

(i) $awa^{(b,c_1)} = wa^{(b,c_1)}a$;
(ii) $w^{(b,c)}aw = aw^{(b,c)}w$.

If any of the above conditions is satisfied, then $aw \in R^{(b,c)}$ *and*

$$(aw)^{(b,c)} = a^{(b,c_1)}w^{(b,c)}.$$

Proof. Let the assumptions of the theorem hold.

(i). Let $awa^{(b,c_1)} = wa^{(b,c_1)}a$. We will prove that, for $b_2 = b$, the condition (ii) of Theorem 3.2 holds. By Lemma 1.5, we have $a^{(b,c_1)}a1^{(b,c_1)} = 1^{(b,c_1)}$. Also, by Corollary 2.7 (b), we have $w^{(b,c)}w1^{(b,c_1)} = 1^{(b,c_1)}$. Therefore,

$$w^{(b,c)}awa^{(b,c_1)} = w^{(b,c)}(awa^{(b,c_1)})1^{(b,c_1)} = w^{(b,c)}w(a^{(b,c_1)}a1^{(b,c_1)})$$
$$= w^{(b,c)}w1^{(b,c_1)} = 1^{(b,c_1)}.$$

Hence, by Theorem 3.2, $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = a^{(b,c_1)}w^{(b,c)}$.

(ii). Let $w^{(b,c)}aw = aw^{(b,c)}w$. By Corollary 2.7 (a), we have $1^{(b,c_1)}w^{(b,c)} = w^{(b,c)}$. Moreover, by Corollary 2.3, we have $w^{(b,c)}wa^{(b,c_1)} = a^{(b,c_1)}$. Thus,

$$w^{(b,c)}awa^{(b,c_1)} = 1^{(b,c_1)}(w^{(b,c)}aw)a^{(b,c_1)} = 1^{(b,c_1)}a(w^{(b,c)}wa^{(b,c_1)})$$
$$= 1^{(b,c_1)}aa^{(b,c_1)} = 1^{(b,c_1)}.$$

Analogously as in the previous theorem, in the next theorem we assume that $1 \in R^{(b_2,c)}$ and we derive that the forward order law rule $(aw)^{(b,c)} = a^{(b,c)}w^{(b_2,c)}$ is valid if only one of two conditions ($awa^{(b,c)} = aa^{(b,c)}w$ and $w^{(b_2,c)}aw = ww^{(b_2,c)}a$) of Theorem 3.7 holds.

THEOREM 3.9. *Let* $b, b_2, c \in R$, $a \in R^{(b,c)}$, $w \in R^{(b_2,c)}$ *and* $1 \in R^{(b_2,c)}$. *Consider the conditions:*

(i) $awa^{(b,c)} = aa^{(b,c)}w$;
(ii) $w^{(b_2,c)}aw = ww^{(b_2,c)}a$.

If any of the above conditions is satisfied, then $aw \in R^{(b,c)}$ *and*

$$(aw)^{(b,c)} = a^{(b,c)}w^{(b_2,c)}.$$

Now, we consider the forward order law rule $(aw)^{(b,c)} = a^{(b,c)}w^{(b,c)}$. Actually, in the similar manner as in [29, Theorem 4.4], where authors offered equivalent conditions for the reverse order law to hold for the (b,c)-inverse, we derive equivalent conditions for the forward order law rule $(aw)^{(b,c)} = a^{(b,c)}w^{(b,c)}$ to hold.

THEOREM 3.10. *Let* $b, c \in R$ *and* $a, w \in R^{(b,c)}$. *The following conditions are equivalent:*

(i) $aw \in R^{(b,c)}$ *and* $(aw)^{(b,c)} = a^{(b,c)}w^{(b,c)}$;
(ii) $w^{(b,c)} = w^{(b,c)}awa^{(b,c)}w^{(b,c)} = a^{(b,c)}w^{(b,c)}aww^{(b,c)}$;
(iii) $a^{(b,c)} = a^{(b,c)}awa^{(b,c)}w^{(b,c)} = a^{(b,c)}w^{(b,c)}awa^{(b,c)}$.

Proof. Let the assumptions of the theorem hold.

(i) $\Rightarrow$ (ii). Let $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = a^{(b,c)}w^{(b,c)}$. By Lemma 1.5, we get:

$$w^{(b,c)} = w^{(b,c)}aw(aw)^{(b,c)} = w^{(b,c)}awa^{(b,c)}w^{(b,c)},$$

$$w^{(b,c)} = (aw)^{(b,c)}aww^{(b,c)} = a^{(b,c)}w^{(b,c)}aww^{(b,c)}.$$

(ii) $\Rightarrow$ (iii). Assume that (ii) holds, that is:

(3.17) $$w^{(b,c)} = w^{(b,c)}awa^{(b,c)}w^{(b,c)},$$

(3.18) $$w^{(b,c)} = a^{(b,c)}w^{(b,c)}aww^{(b,c)}.$$

If we multiply (3.17) by $a^{(b,c)}w$ on the left side and applying Lemma 1.5, we get:

$$a^{(b,c)} = (a^{(b,c)}w)w^{(b,c)} = a^{(b,c)}ww^{(b,c)}awa^{(b,c)}w^{(b,c)} = a^{(b,c)}awa^{(b,c)}w^{(b,c)}.$$

Similarly, if we multiply (3.18) by $wa^{(b,c)}$ on the right side and using Lemma 1.5, we get:

$$a^{(b,c)} = w^{(b,c)}(wa^{(b,c)}) = a^{(b,c)}w^{(b,c)}aw(w^{(b,c)}wa^{(b,c)}) = a^{(b,c)}w^{(b,c)}awa^{(b,c)}.$$

(iii) $\Rightarrow$ (i). Let $a^{(b,c)} = a^{(b,c)}awa^{(b,c)}w^{(b,c)} = a^{(b,c)}w^{(b,c)}awa^{(b,c)}$. Denote by $y = a^{(b,c)}w^{(b,c)}$. We will prove that y is the (b, c)-inverse of aw. Since $b = a^{(b,c)}ab$, we have:

$$yawb = a^{(b,c)}w^{(b,c)}awb = (a^{(b,c)}w^{(b,c)}awa^{(b,c)})ab = a^{(b,c)}ab = b.$$

Further, since $c = caa^{(b,c)}$, we have:

$$cawy = cawa^{(b,c)}w^{(b,c)} = ca(a^{(b,c)}awa^{(b,c)}w^{(b,c)}) = caa^{(b,c)} = c.$$

Moreover, $bb^-y = (bb^-a^{(b,c)})w^{(b,c)} = a^{(b,c)}w^{(b,c)} = y$ and $yc^-c = a^{(b,c)}(w^{(b,c)}c^-c) = a^{(b,c)}w^{(b,c)} = y$. Thus, by Lemma 1.2, we get that $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = a^{(b,c)}w^{(b,c)}$. □

As a special case of Theorem 3.2 and Theorem 3.10, we get equivalent conditions for the forward order law for the inverse along and element to hold.

COROLLARY 3.11. Let $d \in R$ and $a, w \in R^{\|d}$. The following conditions are equivalent:

(i) $aw \in R^{\|d}$ and $(aw)^{\|d} = a^{\|d}w^{\|d}$;
(ii) $1 \in R^{\|d}$ and $1^{\|d} = w^{\|d}awa^{\|d}$;
(iii) $w^{\|d} = w^{\|d}awa^{\|d}w^{\|d} = a^{\|d}w^{\|d}aww^{\|d}$;
(iv) $a^{\|d} = a^{\|d}awa^{\|d}w^{\|d} = a^{\|d}w^{\|d}awa^{\|d}$.

Moreover, as a direct consequence of Theorem 3.6, 3.7, 3.8, and 3.9, we get the following result concerning the forward order law for the inverse along an element.

COROLLARY 3.12. Let $d \in R$ and $a, w \in R^{\|d}$. Consider the conditions:

(i) $awa^{\|d} = wa^{\|d}a$;
(ii) $w^{\|d}aw = aw^{\|d}w$;
(iii) $awa^{\|d} = aa^{\|d}w$;

(iv) $w^{\|d}aw = ww^{\|d}a$.

The following statements hold:

(a) *If conditions* (i) *and* (ii) *are satisfied, then* $aw \in R^{\|d}$ *and* $(aw)^{\|d} = a^{\|d}w^{\|d}$;
(b) *If conditions* (iii) *and* (iv) *are valid, then* $aw \in R^{\|d}$ *and* $(aw)^{\|d} = a^{\|d}w^{\|d}$;
(c) *If* $1 \in R^{\|d}$ *and any of the conditions* (i)–(iv) *holds, then* $aw \in R^{\|d}$ *and* $(aw)^{\|d} = a^{\|d}w^{\|d}$.

In order to prove our next result, we obtain the following proposition.

PROPOSITION 3.13. *Let* $b, c \in R$ *and* $a, w \in R^{(b,c)}$. *If* $aw^{(b,c)} = w^{(b,c)}a$, *then* $a^{(b,c)}w^{(b,c)} = w^{(b,c)}a^{(b,c)}$.

Proof. Let the hypothesis of the proposition hold and let

$$aw^{(b,c)} = w^{(b,c)}a.$$

If we multiply the above equality by $a^{(b,c)}$ from the left side, we get

$$a^{(b,c)}aw^{(b,c)} = a^{(b,c)}w^{(b,c)}a.$$

By Lemma 1.5, we have that $a^{(b,c)}aw^{(b,c)} = w^{(b,c)}$ and therefore

$$w^{(b,c)} = a^{(b,c)}w^{(b,c)}a.$$

Now, if we multiply the last equality by $a^{(b,c)}$ from the right side, we get

$$w^{(b,c)}a^{(b,c)} = a^{(b,c)}w^{(b,c)}aa^{(b,c)}.$$

Using Lemma 1.5 again, we have $w^{(b,c)}aa^{(b,c)} = w^{(b,c)}$ and therefore

$$w^{(b,c)}a^{(b,c)} = a^{(b,c)}w^{(b,c)}. \qquad \square$$

Now, we consider under which condition the both reverse order law and the forward order law hold for the (b, c)-inverse. Namely, in the following corollary, we give sufficient conditions for $(aw)^{(b,c)} = a^{(b,c)}w^{(b,c)} = w^{(b,c)}a^{(b,c)}$ to hold.

THEOREM 3.14. *Let* $b, c \in R$ *and* $a, w \in R^{(b,c)}$. *Consider the conditions:*

(i) $aw^{(b,c)} = w^{(b,c)}a$;
(ii) $wa^{(b,c)} = a^{(b,c)}w$.

If any of the above conditions is satisfied, then $aw \in R^{(b,c)}$ *and*

$$(aw)^{(b,c)} = a^{(b,c)}w^{(b,c)} = w^{(b,c)}a^{(b,c)}.$$

Proof. Let the assumptions of the theorem hold.

(i) Let $aw^{(b,c)} = w^{(b,c)}a$. By Proposition 3.13, we have that

(3.19) $$w^{(b,c)}a^{(b,c)} = a^{(b,c)}w^{(b,c)}.$$

Now, we will prove that $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = a^{(b,c)}w^{(b,c)}$, by using Theorem 3.10 (ii). Applying Lemma 1.5 and (3.19), we get

$$w^{(b,c)}aw(a^{(b,c)}w^{(b,c)}) = (w^{(b,c)}a)ww^{(b,c)}a^{(b,c)} = a(w^{(b,c)}ww^{(b,c)})a^{(b,c)}$$
$$= (aw^{(b,c)})a^{(b,c)} = w^{(b,c)}aa^{(b,c)} = w^{(b,c)}.$$

Also, we have

$$a^{(b,c)}(w^{(b,c)}a)ww^{(b,c)} = (a^{(b,c)}aw^{(b,c)})ww^{(b,c)} = w^{(b,c)}ww^{(b,c)} = w^{(b,c)}.$$

Hence, we have that condition (ii) of Theorem 3.10 is satisfied. Thereby, $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = a^{(b,c)}w^{(b,c)}$.

(ii) Similarly as in the proof of the part (i), from $wa^{(b,c)} = a^{(b,c)}w$ we get that (3.19) holds. Also, it can be checked that the condition (iii) of Theorem 3.10 is satisfied. Hence, $aw \in R^{(b,c)}$ and $(aw)^{(b,c)} = a^{(b,c)}w^{(b,c)} = w^{(b,c)}a^{(b,c)}$. □

EXAMPLE 3.15. Let $b, c, a, w \in M_2$ be such that:

$$b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad a = \begin{bmatrix} a_1 & 0 \\ 0 & a_4 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix},$$

where $a_1 \neq 0$, $a_4 \neq a_1$, $w_1 \neq 0$ and $w_2 \neq 0$. Using Example 2.15, we get that a and w are both (b, c)-invertible and

$$a^{(b,c)} = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad w^{(b,c)} = \begin{bmatrix} w_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Moreover, we have that $aw^{(b,c)} = w^{(b,c)}a$. Therefore, by Theorem 3.14, aw is also (b, c)-invertible and $(aw)^{(b,c)} = a^{(b,c)}w^{(b,c)} = w^{(b,c)}a^{(b,c)}$. We remark that $aw \neq wa$.

REMARK 3.16. Note that for $b = c$, as a direct consequence of Theorem 3.14, we have that if $aw^{\|d} = w^{\|d}a$ or $wa^{\|d} = a^{\|d}w$ holds, then $(aw)^{\|d} = a^{\|d}w^{\|d} = w^{\|d}a^{\|d}$ is valid.

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