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LEONARD PAIRS FROM THE EQUITABLE BASIS OF sl_2*

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Abstract. We construct Leonard pairs from finite-dimensional irreducible sl_2-modules, using the equitable basis for sl_2. We show that our construction yields all Leonard pairs of Racah, Hahn, dual Hahn, and Krawtchouk type, and no other types of Leonard pairs.

Key words. Lie algebra, Racah polynomials, Hahn polynomials, Dual Hahn polynomials, Krawtchouk polynomials.

AMS subject classifications. 17B10, 05E35, 33C45.

1. Introduction. In this paper, we construct Leonard pairs from each finite-dimensional irreducible sl_2-module. We show that this construction yields all Leonard pairs of Racah, Hahn, dual Hahn, and Krawtchouk type, and no other types of Leonard pairs.

Leonard pairs were introduced by P. Terwilliger [9] to abstract Bannai and Ito's [1] algebraic approach to a result of D. Leonard concerning the sequences of orthogonal polynomials with finite support for which the dual sequence of polynomials is also a sequence of orthogonal polynomials [7, 8]. These polynomials arise in connection with the finite-dimensional representations of certain Lie algebras and quantum groups, so one expects Leonard pairs to arise as well. Leonard pairs of Krawtchouk type have been constructed from finite-dimensional irreducible sl_2-modules [12]. In this paper, we give a more general construction based upon the equitable basis for sl_2 [2, 5]. The equitable basis of sl_2 arose in the study of the Tetrahedron algebra and the 3-point loop algebra of sl_2 [3]–[5]. These references consider the modules of these algebras and their connections with a generalization of Leonard pairs called tridiagonal. Here, we consider only Leonard pairs and sl_2, which has not been considered elsewhere.

2. Leonard pairs. We recall some facts concerning Leonard pairs; see [10]–[14] for more details. Fix an integer $d \geq 1$. Throughout this paper $\mathcal{F}$ shall denote a field whose characteristic is either zero or an odd prime greater than d. Also, V shall denote an $\mathcal{F}$-vector space of dimension $d+1$, and $\mathrm{End}(V)$ shall denote the $\mathcal{F}$-algebra

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of linear transformations from V to V. In addition, $\mathcal{F}^{d+1}$ shall denote the vector space over $\mathcal{F}$ consisting of column vectors of length $d+1$, and $\mathrm{Mat}_{d+1}(\mathcal{F})$ shall denote the $\mathcal{F}$-algebra of $(d+1)\times(d+1)$ matrices with entries in $\mathcal{F}$ having rows and columns indexed by $0, 1, \ldots, d$. Observe that $\mathrm{Mat}_{d+1}(\mathcal{F})$ acts on $\mathcal{F}^{d+1}$ by left multiplication.

A square matrix is said to be *tridiagonal* whenever every nonzero entry appears on, immediately above, or immediately below the main diagonal. A tridiagonal matrix is said to be *irreducible* whenever all entries immediately above and below the main diagonal are nonzero. A square matrix is said to be *upper* (resp., *lower*) *bidiagonal* whenever every nonzero entry appears on or immediately above (resp., below) the main diagonal.

Definition 2.1. By a *Leonard pair on V*, we mean an ordered pair A, A^* of elements from $\mathrm{End}(V)$ such that (i) there exists a basis of V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal; and (ii) there exists a basis of V with respect to which the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal.

An element of $\mathrm{End}(V)$ is *multiplicity-free* when it has $d+1$ mutually distinct eigenvalues in $\mathcal{F}$. Let $A \in \mathrm{End}(V)$ denote a multiplicity-free linear transformation. Let $\theta_0, \theta_1, \ldots, \theta_d$ denote an ordering of the eigenvalues of A, and for $0 \le i \le d$, set

$$E_i = \prod_{\substack{0\le j\le d \\ j\ne i}} \frac{A-\theta_j I}{\theta_i - \theta_j}, \tag{2.1}$$

where I denotes the identity map on V. By elementary linear algebra, $AE_i = E_iA = \theta_i E_i$ $(0 \le i \le d)$, $E_iE_j = \delta_{ij}E_i$ $(0 \le i,j \le d)$, and $\sum_{i=0}^{d} E_i = I$. It follows that $E_0, E_1, \ldots, E_d$ is a basis for the subalgebra of $\mathrm{End}(V)$ generated by A. We refer to E_i as the *primitive idempotent* of A *associated* with θ_i. Observe that $V = E_0V + E_1V + \cdots + E_dV$ (direct sum). For $0 \le i \le d$, E_iV is the (one-dimensional) eigenspace of A in V associated with the eigenvalue θ_i, and E_i acts on V as the projection onto this eigenspace.

Definition 2.2. [10] By a *Leonard system* on V, we mean a sequence of the form $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ of elements of $\mathrm{End}(V)$ that satisfies (i)–(v) below.

(i) A and A^* are multiplicity-free.
(ii) $E_0, E_1, \ldots, E_d$ is an ordering of the primitive idempotents of A.
(iii) $E_0^*, E_1^*, \ldots, E_d^*$ is an ordering of the primitive idempotents of A^*.
(iv) $E_iA^*E_j = \begin{cases} 0 & \text{if } |\, i-j \,| > 1, \\ \ne 0 & \text{if } |\, i-j \,| = 1 \end{cases} \quad (0 \le i,j \le d).$
(v) $E_i^*AE_j^* = \begin{cases} 0 & \text{if } |\, i-j \,| > 1, \\ \ne 0 & \text{if } |\, i-j \,| = 1 \end{cases} \quad (0 \le i,j \le d).$

We recall the relationship between Leonard systems and Leonard pairs. Suppose

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Volume 20, pp. 490-505, August 2010

$(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a Leonard system on V. For $0 \le i \le d$, pick any nonzero vectors $v_i \in E_iV$ and $v_i^* \in E_i^*V$. Then the sequence $\{v_i\}_{i=0}^d$ (resp., $\{v_i^*\}_{i=0}^d$) is a basis for V which satisfies condition (i) (resp., condition (ii)) of Definition 2.1. Thus, A, A^* is a Leonard pair. Conversely, suppose A, A^* is a Leonard pair on V. By [10, Lemma 1.3], each of A and A^* is multiplicity-free. Let $\{v_i\}_{i=0}^d$ (resp., $\{v_i^*\}_{i=0}^d$) be a basis of V which witnesses condition (i) (resp., condition (ii)) of Definition 2.1. For $\le i \le d$, v_i (resp., v_i^*) is an eigenvalue of A (resp., A^*); let E_i (resp., E_i^*) denote the corresponding primitive idempotent of A (resp., A^*). Then $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a Leonard system on V.

Suppose A, A^* is a Leonard pair on V, and suppose $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is an associated Leonard system. Then the only other Leonard systems associated with A, A^* are $(A; \{E_i\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d)$, $(A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$, and $(A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d)$. Since $d \ge 1$, these four Leonard systems are distinct, so there is a one-to-four correspondence between Leonard pairs and Leonard systems here.

We recall the equivalence of Leonard systems and parameter arrays.

Definition 2.3. [10] By a *parameter array* over $\mathcal{F}$ of diameter d, we mean a sequence of scalars $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ taken from $\mathcal{F}$ that satisfy the following conditions:

$$\theta_i \ne \theta_k \quad (0 \le i < k \le d), \tag{2.2}$$

$$\theta_i^* \ne \theta_k^* \quad (0 \le i < k \le d), \tag{2.3}$$

$$\varphi_j \ne 0 \quad (1 \le j \le d), \tag{2.4}$$

$$\phi_j \ne 0 \quad (1 \le j \le d), \tag{2.5}$$

$$\varphi_j = \phi_1 \sum_{h=0}^{j-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_j^* - \theta_0^*)(\theta_{j-1} - \theta_d) \quad (1 \le j \le d), \tag{2.6}$$

$$\phi_j = \varphi_1 \sum_{h=0}^{j-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_j^* - \theta_0^*)(\theta_{d-j+1} - \theta_0) \quad (1 \le j \le d), \tag{2.7}$$

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = \frac{\theta_{k-2}^* - \theta_{k+1}^*}{\theta_{k-1}^* - \theta_k^*} \quad (2 \le i, k \le d-1). \tag{2.8}$$

Definition 2.4. Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ be a Leonard system on V. For each i $(0 \le i \le d)$, let θ_i be the eigenvalue of A associated with E_i. We refer to $\{\theta_i\}_{i=0}^d$ as an *eigenvalue sequence* of A. For each i $(0 \le i \le d)$, let θ_i^* be the eigenvalue of A^* associated with E_i^*. We refer to $\{\theta_i^*\}_{i=0}^d$ as an *eigenvalue sequence* of A^*.

Theorem 2.5. [11] *Let* $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ *be a Leonard system on*

Electronic Journal of Linear Algebra ISSN 1081-3810
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V. Let $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ denote the respective eigenvalue sequences for A and A^*. Fix a nonzero vector $v \in E_0^*V$.

(i) *For $0 \le i \le d$, define a vector $\omega_i = (A-\theta_{i-1}I)\cdots(A-\theta_1 I)(A-\theta_0 I)v$. Then $\{\omega_i\}_{i=0}^d$ is a basis for V with action*

$$A\omega_i = \theta_i\omega_i + \omega_{i+1} \quad (0 \le i \le d-1), \quad A\omega_d = \theta_d\omega_d$$
$$A^*\omega_0 = \theta_0^*, \quad A^*\omega_i = \varphi_i\omega_{i-1} + \theta_i^*\omega_i \quad (1 \le i \le d)$$

for some sequence of nonzero scalars $\{\varphi_j\}_{j=1}^d$ from $\mathcal{F}$, which we refer to as the first split sequence of Φ.

(ii) *For $0 \le i \le d$, define a vector $w_i = (A-\theta_{d-i+1}I)\cdots(A-\theta_{d-1}I)(A-\theta_d I)v$. Then $\{w_i\}_{i=0}^d$ is a basis for V with action*

$$Aw_i = \theta_{d-i}w_i + w_{i+1} \quad (0 \le i \le d-1), \quad Aw_d = \theta_0 w_d$$
$$A^*w_0 = \theta_0^*, \quad A^*w_i = \phi_i w_{i-1} + \theta_i^* w_i \quad (1 \le i \le d)$$

for some sequence of nonzero scalars $\{\phi_j\}_{j=1}^d$ from $\mathcal{F}$, which we refer to as the second split sequence of Φ.

(iii) *The sequence $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ is a parameter array, which we refer to as* the parameter array of Φ.

We say that a parameter array is *associated* with a Leonard pair whenever it is the parameter array of any associated Leonard system. Observe that with respect to the basis $\{\omega_i\}_{i=0}^d$ from Theorem 2.5, the matrices representing A and A^* are respectively lower bidiagonal and upper bidiagonal.

THEOREM 2.6. [10] *Let $B \in \mathrm{Mat}_{d+1}(\mathcal{F})$ be lower bidiagonal, and let $B^* \in \mathrm{Mat}_{d+1}(\mathcal{F})$ be upper bidiagonal. Then the following are equivalent:*

(i) *The pair B, B^* is a Leonard pair on $\mathcal{F}^{d+1}$.*

(ii) *There exists a parameter array $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ over $\mathcal{F}$ such that*

$$B(i,i) = \theta_i, \quad B^*(i,i) = \theta_i^* \quad (0 \le i \le d),$$
$$B(j,j-1)B^*(j-1,j) = \varphi_j \quad (1 \le j \le d).$$

When (i), (ii) *hold, B, B^* and $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ are associated.*

Theorem 2.5 constructs a parameter array from any Leonard system. Theorem 2.6 implies that certain matrices with entries taken from a parameter array form a Leonard pair on $\mathcal{F}^{d+1}$ associated with the parameter array. The first two subsequences of the parameter array are the eigenvalue sequences, so (2.1) yields the primitive idempotents of an associated Leonard system. Any Leonard systems with the same parameter array are isomorphic since they have the same action by Theorem 2.5.

Electronic Journal of Linear Algebra ISSN 1081-3810
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That is to say, there is a one-to-one correspondence between parameter arrays and isomorphism classes of associated Leonard systems. In light of the discussion following Definition 2.2, there is a one-to-four correspondence between associated Leonard pairs and parameter arrays.

3. Parameter arrays of classical type. In [14], parameter arrays are classified into 13 families, each named for certain associated sequences of orthogonal polynomials. The four families which arise in this paper share a common property. Given a parameter array, let β be the common value of (2.8) minus one if $d \geq 3$, and let β be any scalar in $\mathcal{F}$ if $d \leq 2$.

Definition 3.1. A parameter array is of *classical type* whenever $\beta = 2$.

We shall show that only the four classical families arise from sl_2 via the construction of this paper. The following results characterize these types.

Theorem 3.2. [14, Example 5.10] *Fix nonzero h, $h^* \in \mathcal{F}$ and s, s^*, r_1, r_2, θ_0, $\theta_0^* \in \mathcal{F}$ such that $r_1 + r_2 = s + s^* + d + 1$ and none of r_1, r_2, $s^* - r_1$, $s^* - r_2$ is equal to $-j$ for $1 \leq j \leq d$ and that neither of s, s^* is equal to $-i$ for $2 \leq i \leq 2d$. Let*

$$\begin{aligned}
\theta_i &= \theta_0 + hi(i+1+s) \quad (0 \leq i \leq d),\\
\theta_i^* &= \theta_0^* + h^*i(i+1+s^*) \quad (0 \leq i \leq d),\\
\varphi_j &= hh^*j(j-d-1)(j+r_1)(j+r_2) \quad (1 \leq j \leq d),\\
\phi_j &= hh^*j(j-d-1)(j+s^*-r_1)(j+s^*-r_2) \quad (1 \leq j \leq d).
\end{aligned}$$

Then $\Phi = (\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ is a parameter array; we say that it is of Racah type. *We refer to the scalars r_1, r_2, s, s^*, h, h^*, θ_0, θ_0^* as* hypergeometric parameters *for Φ.*

Theorem 3.3. [14, Example 5.11] *Fix nonzero s, $h^* \in \mathcal{F}$ and s^*, r, θ_0, $\theta_0^* \in \mathcal{F}$ such that neither of r, $s^* - r$ is equal to $-j$ for $1 \leq j \leq d$ and that s^* is not equal $-i$ for $2 \leq i \leq 2d$. Let*

$$\begin{aligned}
\theta_i &= \theta_0 + si \quad (0 \leq i \leq d),\\
\theta_i^* &= \theta_0^* + h^*i(i+1+s^*) \quad (0 \leq i \leq d),\\
\varphi_j &= h^*sj(j-d-1)(j+r) \quad (1 \leq j \leq d),\\
\phi_j &= -h^*sj(j-d-1)(j+s^*-r) \quad (1 \leq j \leq d).
\end{aligned}$$

Then $\Phi = (\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ is a parameter array; we say that it is of Hahn type. *We refer to the scalars r, s, s^*, h^*, θ_0, θ_0^* as* hypergeometric parameters *for Φ.*

Theorem 3.4. [14, Example 5.12] *Fix nonzero h, $s^* \in \mathcal{F}$ and s, r, θ_0, $\theta_0^* \in \mathcal{F}$ such that neither of r, $s - r$ is equal to $-j$ for $1 \leq j \leq d$, and that s is not equal $-i$*

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 20, pp. 490-505, August 2010

for $2 \leq i \leq 2d$. Let

$$\begin{aligned}
\theta_i &= \theta_0 + hi(i+1+s) \quad (0 \leq i \leq d),\\
\theta^*_i &= \theta^*_0 + s^* i \quad (0 \leq i \leq d),\\
\varphi_j &= hs^* j(j-d-1)(j+r) \quad (1 \leq j \leq d),\\
\phi_j &= hs^* j(j-d-1)(j+r-s-d-1) \quad (1 \leq j \leq d).
\end{aligned}$$

Then $\Phi = (\{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ is a parameter array; we say that it is of dual Hahn type. *We refer to the scalars r, s, s^*, h, θ_0, θ^*_0 as* hypergeometric parameters *of Φ.*

Theorem 3.5. [14, Example 5.13] *Fix nonzero r, s, $s^* \in \mathcal{F}$ and θ_0, $\theta^*_0 \in \mathcal{F}$ such that $r \neq ss^*$. Let*

$$\begin{aligned}
\theta_i &= \theta_0 + si \quad (0 \leq i \leq d),\\
\theta^*_i &= \theta^*_0 + s^* i \quad (0 \leq i \leq d),\\
\varphi_j &= rj(j-d-1) \quad (1 \leq j \leq d),\\
\phi_j &= (r-ss^*)j(j-d-1) \quad (1 \leq j \leq d).
\end{aligned}$$

Then $\Phi = (\{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ is a parameter array; we say that it is of Krawtchouk type. *We refer to the scalars r, s, s^*, θ_0, θ^*_0 as* hypergeometric parameters Φ.

Theorem 3.6. [14] *A parameter array is of classical type if and only if it is of Racah, Hahn, dual Hahn, or Krawtchouk type.*

The parameter arrays of classical type are not distinct when $d = 1$; it is customary to define the type to be Krawtchouk in this case. If $d \geq 2$, then the parameter arrays of classical type are distinguished by their eigenvalue sequences. Indeed, one need only determine which eigenvalue sequences are linear and which are quadratic in their subscript.

Given a parameter array, all associated Leonard pairs and Leonard systems are said to be of the same type as the parameter array. Assume $d \geq 3$. Then β is the same in all four parameter arrays associated with a given Leonard pair; in particular, the type of a Leonard pair is well-defined.

Each set of hypergeometric parameters uniquely determines a parameter array. Suppose $d \geq 2$. Then each parameter array of Hahn, dual Hahn, and Krawtchouk type has a unique set of hypergeometric parameters. Swapping hypergeometric parameters r_1 and r_2 in Theorem 3.2 (Racah type) gives a sequence of hypergeometric parameters for the same parameter array (it might be the case that $r_1 = r_2$).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 490-505, August 2010

4. The Lie algebra sl_2. In this section, we recall some facts concerning the Lie algebra sl_2.

DEFINITION 4.1. [6] The Lie algebra sl_2 is the Lie algebra over $\mathcal{F}$ that has a basis e, f, h satisfying the following conditions:

$$[h,e]=2e, \quad [h,f]=-2f, \quad [e,f]=h,$$

where $[-,-]$ denotes the Lie bracket.

LEMMA 4.2. [5] *With reference to Definition 4.1, let*

$$x=2e-h, \quad y=-2f-h, \quad z=h.$$

Then x, y, z is a basis for sl_2, and

$$[x,y]=2x+2y, \quad [y,z]=2y+2z, \quad [z,x]=2z+2x.$$

We call x, y, z the *equitable basis* for the Lie algebra sl_2.

Observe that the map $x \mapsto y \mapsto z \mapsto x$ defines an automorphism of sl_2. Thus, for simplicity, we shall state all results for x, y, z with the understanding that they are readily extended by applying any cyclic shift to the equitable basis.

LEMMA 4.3. [6] *There is a finite-dimensional irreducible sl_2-module V_d with basis $v_0, v_1, \ldots, v_d$ and action $hv_i=(d-2i)v_i$ $(0 \le i \le d)$, $fv_i=(i+1)v_{i+1}$ $(0 \le i \le d-1)$, $fv_d=0$, $ev_0=0$, $ev_i=(d-i+1)v_{i-1}$ $(1 \le i \le d)$. Moreover, up to isomorphism, V_d is the unique irreducible sl_2-module of dimension $d+1$.*

LEMMA 4.4. [5] *With reference to Lemmas 4.2 and 4.3,*

$$\begin{gathered}(x+dI)v_0=0, \quad (x+(d-2i)I)v_i=2(d-i+1)v_{i-1} \ \ (1 \le i \le d),\\ (y+(d-2i)I)v_i=-2(i+1)v_{i+1} \ \ (0 \le i \le d-1), \quad (y-dI)v_d=0,\\ (z-(d-2i)I)v_i=0 \ \ (0 \le i \le d).\end{gathered}$$

5. A pair of linear operators. Let $U(sl_2)$ denote the universal enveloping algebra of sl_2, that is, the associative $\mathcal{F}$-algebra with generators e, f, h and relations $[h,e]=2e$, $[h,f]=-2f$, $[e,f]=h$, where $[a,b]=ab-ba$ is commutator of a and b.

DEFINITION 5.1. Let $A \in U(sl_2)$ denote an arbitrary linear combination of 1, y, z, and yz, and let $A^* \in U(sl_2)$ denote an arbitrary linear combination of 1, z, x, and zx. Write

$$A=\kappa 1+\lambda y+\mu z+\nu yz, \quad A^*=\kappa^* 1+\lambda^* z+\mu^* x+\nu^* zx.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 20, pp. 490-505, August 2010

Our goal is to characterize when A and A^* act on V_d as a Leonard pair. In this section, we show that this is the case if and only if the following sequences of scalars form a parameter array.

Definition 5.2. With reference to Definition 5.1, define

$$\begin{aligned}
&\theta_i = \kappa - (\lambda - \mu)(d-2i) - (d-2i)^2\nu \quad (0 \le i \le d),\\
&\theta^*_i = \kappa^* + (\lambda^* - \mu^*)(d-2i) - (d-2i)^2\nu^* \quad (0 \le i \le d),\\
&\varphi_j = -4j(d-j+1)(\lambda + (d-2(j-1))\nu)(\mu^* + (d-2(j-1))\nu^*) \quad (1 \le j \le d),\\
&\phi_j = 4j(d-j+1)\left((\lambda + d\nu)(\mu^* + d\nu^*)\right.\\
&\qquad \left.+(\lambda - \mu + 2(j-1)\nu)(\lambda^* - \mu^* - 2(d-j)\nu^*)\right) \quad (1 \le j \le d).
\end{aligned}$$

Lemma 5.3. *The pair A, A^* of Definition 5.1 act on the sl_2-module V_d as follows. Referring to the basis $\{v_i\}_{i=0}^d$ of Lemma 4.3,*

$$\begin{gathered}
Av_i = \theta_i v_i + \sigma_i v_{i+1} \quad (0 \le i \le d-1), \quad Av_d = \theta_d v_d,\\
A^* v_0 = \theta^*_0 v_0, \quad A^* v_i = \tau^*_i v_{i-1} + \theta^*_i v_i \quad (1 \le i \le d),
\end{gathered}$$

where $\{\theta_i\}_{i=0}^d$ and $\{\theta^*_i\}_{i=0}^d$ are as in Definition 5.2 and where

$$\begin{aligned}
\sigma_i &= -2(i+1)(\lambda + (d-2i)\nu) \quad (0 \le i \le d-1),\\
\tau^*_i &= 2(d-i+1)(\mu^* + (d-2(i-1))\nu^*) \quad (1 \le i \le d).
\end{aligned}$$

Proof. Straightforward from Lemma 4.4. □

Lemma 5.4. *With reference to Definition 5.2 and Lemma 5.3,*

$$\varphi_j = \sigma_{j-1}\tau^*_j \quad (1 \le j \le d). \tag{5.1}$$

Proof. Straightforward. □

Theorem 5.5. *The pair A, A^* of Definition 5.1 acts on the sl_2-module V_d as a Leonard pair if and only if the sequence of scalars $(\{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ from Definition 5.2 forms a parameter array, in which case they are associated.*

Proof. Let B and B^* denote the respective matrices representing A and A^* with respect to the basis $\{v_i\}_{i=0}^d$ of Lemma 4.3. This defines an $\mathcal{F}$-algebra isomorphism from $\mathrm{End}(V)$ to $\mathrm{Mat}_{d+1}(\mathcal{F})$, so A, A^* act on V_d as a Leonard pair if and only if B,

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 20, pp. 490-505, August 2010

B^* is a Leonard pair on $\mathcal{F}^{d+1}$. By Lemma 5.3, B is lower bidiagonal and B^* is upper bidiagonal with $B(i,i) = \theta_i$, $B^*(i,i) = \theta^*_i$ $(0 \le i \le d)$ and $B(j,j-1)B^*(j-1,j) = \sigma_{j-1}\tau^*_j$ $(1 \le j \le d)$. Recall that $\varphi_j = \sigma_{j-1}\tau^*_j$ $(1 \le j \le d)$ by (5.1).

Suppose $(\{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ is a parameter array. Then B, B^* is a Leonard pair by Theorem 2.6. The same theorem also implies that this parameter array is associated with B, B^*.

Now suppose B, B^* is a Leonard pair. Then by Theorem 2.6, there is an associated parameter array $(\{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi'_j\}_{j=1}^d)$ for some scalars $\{\phi'_j\}_{j=1}^d$. It remains to verify that $\phi_j = \phi'_j$ $(1 \le j \le d)$. Because it is part of a parameter array, ϕ'_j is given by the right-hand side of (2.7) (which is well-defined since the θ_i are distinct). Simplifying ϕ'_j verifies that $\phi_j = \phi'_j$ $(1 \le j \le d)$. This calculation will appear with more detail in the next section. □

6. The associated parameter array. In this section, we characterize when the scalars $(\{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ of Definition 5.2 form a parameter array. The first two conditions of Definition 5.2 (equations (2.2) and (2.3)) require that the eigenvalue sequences consist of distinct elements, so we make a preliminary calculation.

LEMMA 6.1. *With reference to Definition 5.2,*

$$\theta_i - \theta_k = 2(i-k)(\lambda - \mu + 2(d-i-k)\nu) \quad (0 \le i,\, k \le d), \tag{6.1}$$
$$\theta^*_i - \theta^*_k = 2(k-i)(\lambda^* - \mu^* - 2(d-i-k)\nu^*) \quad (0 \le i,\, k \le d). \tag{6.2}$$

Proof. Clear from the definition of the θ_i and θ^*_i. □

LEMMA 6.2. *With reference to Definition 5.2, the following hold:*

(i) *Equation* (2.2) *holds if and only if*

$$\lambda - \mu + 2(d-\ell)\nu \ne 0 \quad (1 \le \ell \le 2d-1). \tag{6.3}$$

(ii) *Equation* (2.3) *holds if and only if*

$$\lambda^* - \mu^* + 2(d-\ell)\nu^* \ne 0 \quad (1 \le \ell \le 2d-1). \tag{6.4}$$

Proof. Here $i \ne k$, so $2(i-k) \ne 0$. Also, $\ell = i+k$ is $1, 2, \ldots,$ or $2d-1$. The result follows from (6.1) and (6.2). □

The second pair of conditions of Definition 5.2 (equations (2.4) and (2.5)) require that the split sequences be nonzero.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 490-505, August 2010

Lemma 6.3. *With reference to Definition 5.2, equation* (2.4) *holds if and only if both*

$$\lambda - (d-2j)\nu \neq 0 \quad (1 \leq j \leq d) \tag{6.5}$$

and

$$\mu^* - (d-2j)\nu^* \neq 0 \quad (1 \leq j \leq d). \tag{6.6}$$

Proof. Recall that $\varphi_j = \sigma_{j-1}\tau_j^*$ $(1 \leq j \leq d)$ by (5.1). Discard the nonzero factors in the expressions for σ_{j-1} and τ_j^* of Lemma 5.3, and then reverse and shift the indices. This gives that $\sigma_{j-1} \neq 0$ $(1 \leq j \leq d)$ if and only if (6.5) holds and that $\tau_j^* \neq 0$ $(1 \leq j \leq d)$ if and only if (6.6) holds. The result follows. □

Lemma 6.4. *With reference to Definition 5.2, equation* (2.5) *holds if and only if*

$$(\lambda + d\nu)(\mu^* + d\nu^*) \neq -(\lambda - \mu + 2(j-1)\nu)(\lambda^* - \mu^* - 2(d-j)\nu^*) \quad (1 \leq j \leq d). \tag{6.7}$$

Proof. Clear from the definition of ϕ_j. □

The next pair of conditions of Definition 5.2 (equations (2.6) and (2.7)) relate the eigenvalue and split sequences.

Lemma 6.5. *With reference to Definition 5.2, suppose $\theta_0 \neq \theta_d$. Then*

$$\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} = \frac{i(d-i+1)}{d} \quad (1 \leq i \leq d). \tag{6.8}$$

Lemma 6.6. *With reference to Definition 5.2, suppose $\theta_0 \neq \theta_d$.*

(i) *If $d = 1$, then equation* (2.6) *holds.*
(ii) *Suppose that $d \geq 2$. Then equation* (2.6) *holds if and only if*

$$\lambda^*\nu + \mu\nu^* + 2\nu\nu^* = 0. \tag{6.9}$$

Proof. For $1 \leq j \leq d$, let φ'_j denote the right-hand side of equation (2.6). Simplifying φ'_j with (6.8) and expanding with Definition 5.2 gives

$$\begin{aligned}\varphi'_j = -4j(d-j+1)&((\lambda - \mu)(\lambda^* - \mu^* - 2(d-1)\nu^*)\\ &+ (\lambda + d\nu)(\mu^* + d\nu^*) - (\lambda - \mu - 2(j-1)\nu)(\lambda^* - \mu^* - 2(d-j)\nu^*)).\end{aligned}$$

Now $\varphi_j - \varphi'_j = -8j(d-j+1)(\lambda^*\nu + \mu\nu^* + 2\nu\nu^*)$ $(1 \leq j \leq d)$. If $d = 1$, the term $(d-j+1)$ is zero for $1 \leq j \leq d = 1$. If $d \geq 2$, $\varphi_j = \varphi'_j$ for $1 \leq j \leq d$ if and only if (6.9) holds. □

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 490-505, August 2010

Lemma 6.7. *With reference to Definition 5.2, suppose $\theta_0 \neq \theta_d$. Then equation* (2.7) *holds.*

Proof. Simplify the right-hand side of (2.7) with (6.8) to verify the equality. □

The final condition of Definition 5.2 (equation (2.8)) requires that a certain expression involving the eigenvalue sequences be equal and independent of the subscript.

Lemma 6.8. *With reference to Definition 5.2, assume that both sets of equivalent conditions in Lemma 6.2 hold. Then $\beta = 2$. That is to say, equation* (2.8) *holds with*

$$\frac{\theta_{i-2}-\theta_{i+1}}{\theta_{i-1}-\theta_i} = \frac{\theta^*_{i-2}-\theta^*_{i+1}}{\theta^*_{i-1}-\theta^*_i} = 3 \quad (2 \leq i \leq d-1). \tag{6.10}$$

Proof. Straightforward. □

The results of this section give the following.

Theorem 6.9. *With reference to Definition 5.2, assume $d \geq 2$. Then $(\{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ is a parameter array if and only if equations* (6.3), (6.4), (6.5), (6.6), (6.7), *and* (6.9) *hold.*

Proof. By Lemma 6.2, (2.2) holds if and only if (6.3) holds, and (2.3) holds if and only if (6.4) holds. By Lemma 6.3, (2.4) holds if and only if (6.5) and (6.6) hold. By Lemma 6.4, (2.5) holds if and only if (6.7) holds. Assume that the equivalent conditions of Lemma 6.2(i) hold. Then by Lemma 6.7, (2.6) holds. By Lemma 6.6, (2.7) holds if and only if (6.9) holds. Finally, (2.8) holds by (6.10). The result follows by Definition 2.3. □

Together, Theorems 5.5 and 6.9 give our main result concerning A, A^*.

Theorem 6.10. *With reference to Definitions 5.1 and 5.2, assume $d \geq 2$. Then A, A^* act on V_d as a Leonard pair if and only if equations* (6.3), (6.4), (6.5), (6.6), (6.7), *and* (6.9) *hold.*

Proof. The result follows from Theorems 5.5 and 6.9. □

In Theorems 6.9 and 6.10, if the assumption $d \geq 2$ is replaced with $d = 1$, then (6.9) must be removed from the list of conditions.

7. Recognizing the types of Leonard pairs. We consider which types of Leonard pairs/parameter arrays arise from our construction.

Theorem 7.1. *With reference to Definition 5.1, suppose A, A^* act on V_d as a Leonard pair. Then this Leonard pair is of Racah, Hahn, dual Hahn, or Krawtchouk type.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 490-505, August 2010

Proof. Clear from Theorem 3.6 and Lemma 6.8. ◻

The type of the Leonard pair arising in Theorem 6.10 is determined by ν and ν^*.

THEOREM 7.2. *Assume $d \geq 2$. With reference to Definition 5.1, suppose A, A^* act on V_d as a Leonard pair. Then type of this Leonard pair is determined by ν and ν^* as follows.*

ν, ν^*:	$\nu \neq 0$, $\nu^* \neq 0$	$\nu = 0$, $\nu^* \neq 0$	$\nu \neq 0$, $\nu^* = 0$	$\nu = 0$, $\nu^* = 0$
Type:	*Racah*	*Hahn*	*dual Hahn*	*Krawtchouk*

Proof. As an abuse of notation, in this proof take A, A^* to mean the Leonard pair on V_d arising from their action. Let $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ be as in Definition 5.2. Observe that $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ are eigenvalue sequences for A and A^*, respectively, since they are part of an associated parameter array by Theorem 6.9.

For $0 \leq i \leq d$, $\theta_i = \kappa - (\lambda - \mu)(d-2i) - \nu(d-2i)^2$. If $\nu \neq 0$, then the θ_i are quadratic functions of their subscripts. If $\nu = 0$, then the θ_i are linear functions of their subscripts. Similarly, ν^* determines the form of the θ_i^*. By Theorem 7.1, the type of A, A^* is Racah, Hahn, dual Hahn, or Krawtchouk. Now compare the eigenvalue sequences in Theorems 3.2–3.5 to those of A, A^* to complete the proof. ◻

When one or both of ν, ν^* vanish, the conditions of Theorem 6.10 become simpler.

LEMMA 7.3. *Assume $d \geq 2$. With reference to Definition 5.1, A, A^* acts on V_d as a Leonard pair of Hahn type if and only if*

$$\begin{gathered} \lambda \neq 0, \quad \mu = 0, \quad \nu = 0, \quad \nu^* \neq 0, \\ \mu^* - (d-2i)\nu^* \neq 0, \quad \lambda^* - (d-2i)\nu^* \neq 0 \quad (1 \leq i \leq d), \\ \lambda^* - \mu^* + 2\nu^*(d-i) \neq 0 \quad (1 \leq i \leq 2d-1). \end{gathered}$$

Proof. Set $\nu = 0$ and assume $\nu^* \neq 0$ in the lines referred to in Theorem 6.10 and simplify. Here, (6.9) implies that $\mu = 0$. ◻

LEMMA 7.4. *Assume $d \geq 2$. With reference to Definition 5.1, A, A^* acts on V_d as a Leonard pair of dual Hahn type if and only if*

$$\begin{gathered} \lambda^* = 0, \quad \nu \neq 0, \quad \mu^* \neq 0, \quad \nu^* = 0, \\ \lambda - (d-2i)\nu \neq 0, \quad \mu - (d-2i)\nu \neq 0 \quad (1 \leq i \leq d), \\ \lambda - \mu + 2\nu(d-i) \neq 0 \quad (1 \leq i \leq 2d-1). \end{gathered}$$

Proof. Set $\nu^* = 0$ and assume $\nu \neq 0$ in the lines referred to in Theorem 6.10 and simplify. Here, (6.9) implies that $\lambda^* = 0$. ◻

LEMMA 7.5. *Assume $d \geq 2$. With reference to Definition 5.1, A, A^* acts on V_d*

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 20, pp. 490-505, August 2010

as a Leonard pair of Krawtchouk type if and only if

$$\mu \neq \lambda, \quad \mu^* \neq \lambda^*, \quad \lambda \neq 0, \quad \mu^* \neq 0, \quad \nu = 0, \quad \nu^* = 0, \quad \lambda\lambda^* - \mu\lambda^* + \mu\mu^* \neq 0.$$

Proof. Set $\nu = \nu^* = 0$ in the lines referred to in Theorem 6.10, and simplify. ∎

8. Hypergeometric parameters. We now consider the hypergeometric parameters of the parameter arrays arising from sl_2.

LEMMA 8.1. *Assume $d \geq 2$. With reference to Definition 5.1, suppose A, A^* acts on V_d as a Leonard pair of Racah type. Then this Leonard pair has hypergeometric parameters*

$$\begin{aligned}
&\theta_0 = \kappa - d(\lambda - \mu + d\nu), && \theta_0^* = \kappa^* + d(\lambda^* - \mu^* - d\nu^*),\\
&h = -4\nu, && h^* = -4\nu^*,\\
&s = -\frac{\lambda - \mu}{2\nu} - d - 1, && s^* = \frac{\lambda^* - \mu^*}{2\nu^*} - d - 1,\\
&\{r_1, r_2\} = \left\{ -\frac{\lambda}{2\nu} - \frac{d}{2} - 1, -\frac{\mu^*}{2\nu^*} - \frac{d}{2} - 1 \right\}.
\end{aligned}$$

Proof. This choice of parameters in Theorem 3.2 gives the same sequences $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$, $\{\varphi_j\}_{j=1}^d\}_{j=1}^d$, and $\{\phi_j\}_{j=1}^d\}_{j=1}^d$ as in Theorem 6.9. Thus, A, A^* act as on V_d as a Leonard pair of Racah type with the given hypergeometric parameters by Theorem 6.10.

(The inequalities and equalities in both Theorem 6.9 and Theorem 3.2 derive from those on a general parameter array in Definition 2.3. One may also verify directly that those of Theorem 3.2 are a consequence of those of Theorem 6.9.) ∎

We omit proofs for the other three types as the above argument proceeds virtually identically in each case.

LEMMA 8.2. *Assume $d \geq 2$. With reference to Definition 5.1, suppose A, A^* acts on V_d as a Leonard pair of Hahn type. Then this Leonard pair has hypergeometric parameters*

$$\begin{aligned}
&\theta_0 = \kappa - d\lambda, && \theta_0^* = \kappa^* + d(\lambda^* - \mu^* - d\nu^*),\\
&s = 2\lambda, && s^* = \frac{\lambda^* - \mu^*}{2\nu^*} - d - 1,\\
&h^* = -4\nu^*, && r = -\frac{\mu^*}{2\nu^*} - \frac{d}{2} - 1.
\end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 490-505, August 2010

LEMMA 8.3. *Assume $d \geq 2$. With reference to Definition 5.1, suppose A, A^* acts on V_d as a Leonard pair of dual Hahn type. Then this Leonard pair has hypergeometric parameters*

$$\begin{aligned} &\theta_0 = \kappa - d(\lambda - \mu + d\nu), && \theta_0^* = \kappa^* - d\mu^*, \\ &s = -\frac{\lambda - \mu}{2\nu} - d - 1, && s^* = 2\mu^*, \\ &h = -4\nu, && r = -\frac{\lambda}{2\nu} - \frac{d}{2} - 1. \end{aligned}$$

LEMMA 8.4. *Assume $d \geq 2$. With reference to Definition 5.1, suppose A, A^* acts on V_d as a Leonard pair of Krawtchouk type. Then this Leonard pair has hypergeometric parameters*

$$\begin{aligned} &\theta_0 = \kappa - d(\lambda - \mu), && \theta_0^* = \kappa^* + d(\lambda^* - \mu^*), \\ &s = 2(\lambda - \mu), && s^* = -2(\lambda^* - \mu^*), \\ &r = 4\lambda\mu^*. \end{aligned}$$

9. Leonard pairs of classical type. We prove a converse to Theorem 6.10. We treat each type individually.

LEMMA 9.1. *Assume $d \geq 2$. Let A, A^* be a Leonard pair on V of Racah type. Let h, h^* s, s^*, r_1, r_2, θ_0, and θ_0^* be hypergeometric parameters of A, A^*. Then for each ℓ and m such that $\{\ell, m\} = \{1, 2\}$, there exists an irreducible sl_2-module structure on V in which A and A^* act respectively as $\kappa 1 + \lambda y + \mu z + \nu yz$ and $\kappa^* 1 + \lambda^* z + \mu^* x + \nu^* zx$, where*

$$\begin{aligned} &\kappa = \theta_0 + \frac{dh(2s + d + 2)}{4}, && \kappa^* = \theta_0^* + \frac{dh^*(2s^* + d + 2)}{4}, \\ &\lambda = \frac{h(2r_m + d + 2)}{4}, && \lambda^* = -\frac{h^*(2s^* - 2r_\ell + d)}{4}, \\ &\mu = -\frac{h(2s - 2r_m + d)}{4}, && \mu^* = \frac{h^*(2r_\ell + d + 2)}{4}, \\ &\nu = -\frac{h}{4}, && \nu^* = -\frac{h^*}{4}. \end{aligned}$$

Proof. For $0 \leq i \leq d$, write $S_j = 2(j+1)(\lambda + (d-2j)\nu) = h(j+1)(r_m + j + 1)$ and $P_i = \prod_{j=0}^{i-1} S_j$. By Theorem 3.2, $S_j \neq 0$ $(0 \leq j \leq d-1)$.

Let $\{\omega_i\}_{i=0}^d$ be the basis of V from Theorem 2.5, and let $\{v_i\}_{i=0}^d$ be the basis for V_d from Lemma 4.3. Define a linear transformation $\psi : V \to V_d$ by $\psi(\omega_i) = P_i v_i$. Now ψ is a bijection since V and V_d both have dimension $d+1$ and the P_i are nonzero. The map Ψ : $\mathrm{End}(V_d) \to \mathrm{End}(V_d)$ defined by $\Psi(X) = \psi X \psi^{-1}$ is an $\mathcal{F}$-algebra isomorphism.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 490-505, August 2010

Let A and A^* act on V_d as $\Psi(A)$ and $\Psi(A^*)$. Using Theorem 2.5, compute the action of A, A^*. For $0 \le i \le d-1$, $Av_i = \psi(A\psi^{-1}(v_i)) = \psi(AP_i^{-1}\omega_i) = \psi(\theta_i P_i^{-1}\omega_i + P_i^{-1}\omega_{i+1}) = \theta_i\psi(P_i^{-1}\omega_i) + S_i\psi(P_{i+1}^{-1}\omega_{i+1}) = \theta_i v_i + S_i v_{i+1}$. Similarly, $Av_d = \theta_d v_d$. For $1 \le i \le d$, $A^*v_i = \psi(A^*\psi^{-1}(v_i)) = \psi(A^*P_i^{-1}\omega_i) = \psi(\theta_i^* P_i^{-1}\omega_i + P_i^{-1}\varphi_i\omega_{i-1}) = \theta_i^*\psi(P_i^{-1}\omega_i) + S_i^{-1}\varphi_i\psi(P_{i-1}^{-1}\omega_{i-1}) = \theta_i v_i + S_i^{-1}\varphi_i v_{i+1}$. Also, $A^*v_0 = \theta_0^* v_0$.

Compare the respective actions of A and A^* to those of $\kappa 1 + \lambda y + \mu z + \nu yz$ and $\kappa^*1 + \lambda^* z + \mu^* x + \nu^* zx$ in Lemma 5.3. With the given coefficients, the formulas for $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ from Theorem 3.2 and Lemma 5.3 coincide. Also $\sigma_i = S_i$ $(0 \le i \le d-1)$ and $\tau_i^* = S_i^{-1}\varphi_i$ $(1 \le i \le d)$. The actions coincide as required, so the result follows. □

We omit proofs for the other three types as the above argument is modified only by choosing S_j so that it equals σ_j in each case.

Lemma 9.2. *Assume $d \ge 2$. Let A, A^* be a Leonard pair on V of Hahn type. Let h, s, s^*, r, θ_0, and θ_0^* be hypergeometric parameters of A, A^*. Then there exists an irreducible sl_2-module structure on V in which A and A^* act respectively as $\kappa 1 + \lambda y$ and $\kappa^*1 + \lambda^* z + \mu^* x + \nu^* zx$, where*

$$\begin{aligned} \kappa &= \theta_0 + \frac{ds}{2}, & \kappa^* &= \theta_0^* + \frac{dh^*(2s^*+d+2)}{4}, \\ \lambda &= \frac{s}{2}, & \lambda^* &= -\frac{h^*(2s^*-2r+d)}{4}, \\ & & \mu^* &= \frac{h^*(2r+d+2)}{4}, \\ & & \nu^* &= -\frac{h^*}{4}. \end{aligned}$$

Lemma 9.3. *Assume $d \ge 2$. Let A, A^* be a Leonard pair on V of dual Hahn type. Let h, s, s^*, r, θ_0, and θ_0^* be hypergeometric parameters of A, A^*. Then there exists an irreducible sl_2-module structure on V in which A and A^* act respectively as $\kappa 1 + \lambda y + \mu z + \nu yz$ and $\kappa^*1 + \mu^* x$, where*

$$\begin{aligned} \kappa &= \theta_0 + \frac{dh(2s+d+2)}{4}, & \kappa^* &= \theta_0^* + \frac{ds^*}{2}, \\ \lambda &= \frac{h(2r+d+2)}{4}, & & \\ \mu &= -\frac{h(2s-2r+d)}{4}, & \mu^* &= \frac{s^*}{2}, \\ \nu &= -\frac{h}{4}. & & \end{aligned}$$

Lemma 9.4. *Assume $d \ge 2$. Let A, A^* be a Leonard pair on V of Krawtchouk type. Let s^*, s, r, θ_0, and θ_0^* be hypergeometric parameters of A, A^*. Then there*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 490-505, August 2010

exists an irreducible sl_2*-module structure on* V *in which* A *and* A^* *act respectively as* $\kappa 1 + \lambda y + \mu z$ *and* $\kappa^* 1 + \lambda^* z + \mu^* x$*, where for any nonzero* $t \in \mathcal{F}$*,*

$$\begin{aligned} \kappa &= \theta_0 + \frac{ds}{2}, & \kappa^* &= \theta_0^* + \frac{ds^*}{2}, \\ \lambda &= \frac{r}{4t}, & \lambda^* &= t - \frac{s^*}{2}, \\ \mu &= \frac{r}{4t} - \frac{s}{2}, & \mu^* &= t. \end{aligned}$$

Combining Lemmas 9.1–9.4 gives the following.

THEOREM 9.5. *Assume* $d \geq 2$*. Let* A*,* A^* *be a Leonard pair on* V *of Racah, Hahn, dual Hahn, or Krawtchouk type. Then there exists an irreducible* sl_2*-module structure on* V *in which* A *and* A^* *act respectively as linear combinations of* 1*,* y*,* z*,* yz *and of* 1*,* z*,* x*,* zx*.*

One may apply a cyclic shift to the equitable basis to get two other actions.

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