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THE ALLOW SEQUENCE OF DISTINCT EIGENVALUES FOR A SIGN PATTERN*

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Abstract. A sign pattern $\mathcal{A}$ is a matrix with entries in $\{+,-,0\}$. This article introduces the *allow sequence of distinct eigenvalues* for an $n \times n$ sign pattern $\mathcal{A}$, defined as $q_{\text{seq}}(\mathcal{A}) = \langle q_1, \ldots, q_n \rangle$, with $q_k = 1$ if there exists a real matrix with exactly k distinct eigenvalues having pattern $\mathcal{A}$, and $q_k = 0$ otherwise. For example, $q_{\text{seq}}(\mathcal{A}) = \langle 0, \ldots, 0, 1 \rangle$ is equivalent to $\mathcal{A}$ requiring all distinct eigenvalues, while $q_{\text{seq}}(\mathcal{A}) = \langle 1, 0, \ldots, 0 \rangle$ is equivalent to the digraph of $\mathcal{A}$ being acyclic. Relationships between the allow sequence for $\mathcal{A}$ and composite cycles of the digraph of $\mathcal{A}$ are explored to identify zeros in the sequence, while methods based on Jacobian matrices are developed to identify ones in the sequence. When $\mathcal{A}$ is an $n \times n$ irreducible sign pattern, the possible sequences for $q_{\text{seq}}(\mathcal{A})$ are completely determined when $n \leq 4$ and when the sequence has at least $n-4$ trailing zeros for $n \geq 5$.

Key words. Sign pattern, Eigenvalue, Digraph, Inverse eigenvalue problem, Distinct eigenvalues.

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1. Introduction and definitions. A *sign pattern* $\mathcal{A}$ is a symbolic matrix with entries in $\{+,-,0\}$. The *qualitative class* of an $n \times n$ sign pattern $\mathcal{A} = [\alpha_{ij}]$, denoted by $Q(\mathcal{A})$, is the set of all $n \times n$ real matrices $A = [a_{ij}]$ such that $\text{sgn}(a_{ij}) = \alpha_{ij}$ for $1 \leq i, j \leq n$ (here, for $c \in \mathbb{R}$, $\text{sgn}(c)$ is the sign of c, i.e., $+$, $-$, or 0). A matrix $A \in Q(\mathcal{A})$ is a *realization* of $\mathcal{A}$. For a matrix $A \in \mathbb{R}^{n \times n}$, let $q(A)$ be the number of distinct eigenvalues of A, and for a sign pattern $\mathcal{A}$, let $q(\mathcal{A}) = \min\{q(A) \mid A \in Q(\mathcal{A})\}$. The problem of determining $q(\mathcal{A})$ for a sign pattern $\mathcal{A}$ was initiated in [3]. Here we pose the problem of determining *all* possible values for $q(A)$ rather than focus on the minimum value for $q(A)$ for $A \in Q(\mathcal{A})$. To do this, we use a binary sequence to describe the number of distinct eigenvalues that a sign pattern allows. To be specific, for an $n \times n$ sign pattern $\mathcal{A}$ and $1 \leq k \leq n$, define

$$q_k(\mathcal{A}) = \begin{cases} 1, & \text{if } q(A) = k \text{ for some } A \in Q(\mathcal{A}), \\ 0, & \text{otherwise.} \end{cases}$$

For $n \geq 2$, the *allow sequence of distinct eigenvalues for $\mathcal{A}$* (abbreviated *allow sequence for $\mathcal{A}$*) is

$$q_{\text{seq}}(\mathcal{A}) = \langle q_1(\mathcal{A}), q_2(\mathcal{A}), \ldots, q_n(\mathcal{A}) \rangle.$$

A binary sequence s is *realizable* (as an allow sequence) if there is a sign pattern $\mathcal{A}$ with $q_{\text{seq}}(\mathcal{A}) = s$.

Some allow sequences for sign patterns have previously been studied in the literature. For example, determining the $n \times n$ sign patterns $\mathcal{A}$ requiring n distinct eigenvalues is equivalent to characterizing the

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sign patterns $\mathcal{A}$ for which $q_{\text{seq}}(\mathcal{A}) = \langle 0, \dots, 0, 1\rangle$. This problem was studied by Li and Harris [14] where their results give characterizations of all 2×2 (resp. 3×3) irreducible sign patterns with allow sequence $\langle 0, 1\rangle$ (resp. $\langle 0, 0, 1\rangle$). Determining 4×4 sign patterns with allow sequence $\langle 0, 0, 0, 1\rangle$ is addressed in [13] and [12].

For sign patterns $\mathcal{A}$ with $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, \dots, 1\rangle$, consider the class of spectrally arbitrary patterns: a sign pattern $\mathcal{A}$ is *spectrally arbitrary* if for every multiset Λ of n complex numbers closed under complex conjugation there exists $A \in Q(\mathcal{A})$ such that the spectrum of A is Λ (see, for example, [4, 5, 7]). Although every spectrally arbitrary sign pattern $\mathcal{A}$ has $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, \dots, 1\rangle$, there are also classes of non-spectrally arbitrary sign patterns $\mathcal{B}$ with $q_{\text{seq}}(\mathcal{B}) = \langle 1, 1, \dots, 1\rangle$ (several examples are provided within this paper). A sign pattern $\mathcal{A}$ is *potentially nilpotent* if there exists $A \in Q(\mathcal{A})$ such that A is nilpotent. Note that if $\mathcal{A}$ is potentially nilpotent, then $q_1(\mathcal{A}) = q(\mathcal{A}) = 1$; however, there also exist non-potentially nilpotent sign patterns $\mathcal{B}$ satisfying $q_1(\mathcal{B}) = q(\mathcal{B}) = 1$. We say that $\hat{\mathcal{A}}$ is a *superpattern* of $\mathcal{A}$ if $\hat{\mathcal{A}}$ can be obtained from $\mathcal{A}$ by replacing some, or none, of the zero entries with either $+$ or $-$.

An $n \times n$ sign pattern $\mathcal{A}$ can be represented with a *signed directed graph* $D(\mathcal{A}) = (V(D(\mathcal{A})), E(D(\mathcal{A})))$ that has vertices $v_1, \dots, v_n$, and an arc (v_i, v_j) if the (i, j) entry of the sign pattern is not zero. An arc of the form (v_i, v_i) is referred to as a *loop*. In the case that $D(\mathcal{A})$ has no loops, we call the sign pattern $\mathcal{A}$ *loopless*. For $v \in V(D(\mathcal{A}))$, the *indegree of* v and *outdegree of* v are defined as $\text{indeg}(v) = |\{(v_i, v) \in E(D(\mathcal{A})) : 1 \le i \le n\}|$ and $\text{outdeg}(v) = |\{(v, v_i) \in E(D(\mathcal{A})) : 1 \le i \le n\}|$, respectively. The arcs and loops of $D(\mathcal{A})$ are signed either positive or negative according to signs of the entries of $\mathcal{A}$; that is, the sign of (v_i, v_j) is the sign of the (i, j) entry of $\mathcal{A}$. The sign of a directed cycle is the product of the signs of the arcs in the cycle. A k-cycle is a directed cycle of order k (on k vertices). For convenience, we sometimes refer to cycles of $D(\mathcal{A})$ as cycles of $\mathcal{A}$.

For a directed graph D, a *composite cycle* of order k is a collection of vertex-disjoint directed cycles in D such that they cover precisely k vertices of D. The sign of a composite cycle is the product of the signs of the arcs in the composite cycle. If $U = \{\sigma_1, \dots, \sigma_r\}$ is a composite cycle with cycles σ_i, $1 \le i \le r$, then we define

$$V(U) = V(\sigma_1) \cup \dots \cup V(\sigma_r), \quad E(U) = E(\sigma_1) \cup \dots \cup E(\sigma_r),$$

and $|U| = r$, i.e., $|U|$ is the number of directed cycles in U (including loops). Let $\mathcal{U}_k$ denote the set of all composite cycles of D that cover precisely k vertices. If the characteristic polynomial of a real matrix $A = [a_{ij}] \in Q(\mathcal{A})$ is $p_A(z) = z^n + c_1 z^{n-1} + c_2 z^{n-2} + \dots + c_n$, then

$$c_k = (-1)^k E_k = \sum_{U \in \mathcal{U}_k} (-1)^{|U|} \prod_{(v_i, v_j) \in E(U)} a_{ij}, \tag{1.1}$$

where E_k is the sum of all $k \times k$ principal minors of A (see e.g., [10]).

For an $n \times n$ sign pattern $\mathcal{A}$, we define $\ell(\mathcal{A})$ to be the minimum length of a cycle in $D(\mathcal{A})$ and $c(\mathcal{A})$ to be the maximum order of a composite cycle in $D(\mathcal{A})$. For convenience, we say $\ell(\mathcal{A}) = c(\mathcal{A}) = 0$ if $D(\mathcal{A})$ has no cycles. We remark that $\ell(\mathcal{A})$ is also commonly known as the *girth* (or *digirth*) of $D(\mathcal{A})$. If $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, we let $\text{sgn}(A)$ denote the sign pattern whose (i, j) entry is $\text{sgn}(a_{ij})$ and we extend the notation $\ell(A)$ and $c(A)$ to real matrices in the obvious manner, i.e., $\ell(A) = \ell(\text{sgn}(A))$ and $c(A) = c(\text{sgn}(A))$. Since for sign patterns the allow sequence is preserved under negation, transposition, permutation similarity, and signature similarity, we say $\mathcal{A}$ is *equivalent* to $\mathcal{B}$ if $\mathcal{A}$ can be obtained from $\mathcal{B}$ by some combination of these four operations. In this case, we also say that $D(\mathcal{A})$ is *equivalent* to $D(\mathcal{B})$.

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The focus of this paper is on determining the sequences (or properties of sequences) that are the allow sequence for some $n \times n$ irreducible sign pattern. In Section 2, we develop techniques to identify zeros in $q_{\text{seq}}(\mathcal{A})$ for a sign pattern $\mathcal{A}$. We first prove that if $A \in Q(\mathcal{A})$ is not nilpotent, then $q(A) \geq \ell(A)$, and hence, $q_k(\mathcal{A}) = 0$ for $2 \leq k \leq \ell(\mathcal{A}) - 1$ when $\ell(\mathcal{A}) \geq 3$. We then give some conditions on $\mathcal{A}$ for improving these bounds. In Section 3, we develop techniques based on Jacobian matrices to both identify ones in $q_{\text{seq}}(\mathcal{A})$ and also to preserve some entries in the allow sequence for superpatterns of $\mathcal{A}$. Furthermore, for a matrix A with a repeated eigenvalue $\lambda \in \mathbb{R}$, we apply a Jacobian method to determine a condition that guarantees that superpatterns of $\text{sgn}(A)$ have realizations with repeated eigenvalues. In Section 4, we analyze some particular sequences, including $\langle 1, 1, \ldots, 1\rangle$, $\langle 0, \ldots, 0, 1\rangle$, cyclic sequences, and sequences that terminate in a string of zeros. For $t \geq 1$ and $t \geq n-4$, we determine each sequence of length n with exactly t trailing zeros that is the allow sequence for some $n \times n$ irreducible sign pattern. In the case that $t \geq 1$ and $t \geq n-3$, we give an allow sequence characterization of the $n \times n$ irreducible sign patterns $\mathcal{A}$ whose allow sequence has exactly t trailing zeros. In Section 5, we determine all sequences that can occur as the allow sequence for an $n \times n$ irreducible sign pattern when $n \leq 4$ and characterize the 2×2 and 3×3 irreducible sign patterns according to their allow sequence. Section 6 gives some concluding remarks.

2. Relationships between composite cycles and distinct eigenvalues.

Eschenbach and Johnson [9, Lemma (p. 172)] show that for an $n \times n$ sign pattern $\mathcal{A}$, the minimum algebraic multiplicity of the eigenvalue 0 occurring among matrices $A \in Q(\mathcal{A})$ is equal to $n - c(\mathcal{A})$, and furthermore, there is a matrix $A \in Q(\mathcal{A})$ with $c(\mathcal{A})$ distinct *nonzero* eigenvalues. In the case that $c(\mathcal{A}) = n$, this implies $q_n(\mathcal{A}) = 1$, otherwise $c(\mathcal{A}) \leq n-1$, and (including the zero eigenvalue) we get a matrix $A \in Q(\mathcal{A})$ with $c(\mathcal{A}) + 1$ distinct eigenvalues, and there is no realization with more distinct eigenvalues. We reformulate [9, Lemma (p. 172)] in terms of the allow sequence for $\mathcal{A}$ (see also [12]):

Lemma 2.1. *Let $\mathcal{A}$ be an $n \times n$ sign pattern and $c = c(\mathcal{A})$. If $c \leq n-2$, then $q_{c+1}(\mathcal{A}) = 1$ and $q_k(\mathcal{A}) = 0$ for $c+2 \leq k \leq n$. Further, $q_n(\mathcal{A}) = 1$ if and only if $c \geq n-1$.*

When $c(\mathcal{A}) \leq n-2$, Lemma 2.1 shows that the allow sequence for $\mathcal{A}$ ends in a string of zeros. In a similar fashion, when $\ell(\mathcal{A}) \geq 3$, we show in this section that the existence of zeros near the beginning of the allow sequence for $\mathcal{A}$ (except possibly $q_1(\mathcal{A})$) can be proved by analyzing the form of the characteristic polynomial of $A \in Q(\mathcal{A})$. In particular, we explore the effects of a matrix having a lacunary characteristic polynomial. Generally speaking, a polynomial is *lacunary* if there is a subsequence of zeros (a gap) in its sequence of coefficients. In Subsection 2.1 we focus on *lacunary polynomials with an initial gap,* that is, polynomials of the form $p(z) = z^n + \sum_{i=\ell}^{n} p_i z^{n-i}$ with $p_\ell \neq 0$, for some $\ell \geq 2$. In Subsection 2.2, we explore *lacunary polynomials with regular gaps*, that is, polynomials which have the form $p(z^m)$ for some $m \geq 2$. Then in Subsection 2.3, we explore some digraph structure that is necessary for $q_2(\mathcal{A}) = 1$

2.1. Sign patterns that allow lacunary characteristic polynomials with an initial gap.

It is known that if $p(z)$ is the lacunary polynomial $z^n + \sum_{i=\ell}^{n} p_i z^{n-i}$ with $p_\ell \neq 0$, for some $\ell > 1$, then $p(z)$ requires at least ℓ distinct nonzero roots, and no root of $p(z)$ can have multiplicity exceeding $n - \ell + 1$ (e.g., see [18] and [17, Theorem 3], respectively). For completeness, we provide a proof of the first statement based on the proof in [18].

Lemma 2.2. *Let ℓ, n be integers with $1 \leq \ell \leq n$ and $p(z) = z^n + \sum_{i=\ell}^{n} p_i z^{n-i}$ for some $p_\ell, \ldots, p_n \in \mathbb{R}$ with $p_\ell \neq 0$. Then, $p(z)$ has at least ℓ distinct nonzero roots (in $\mathbb{C}$).*

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Proof. First assume $p(0) \neq 0$. Let $d(z) = \gcd(p(z), p'(z))$ so that $p(z) = d(z)P(z)$ for some polynomial $P(z)$. The roots of $P(z)$ are simple and are exactly the distinct roots of $p(z)$ since if r is a root of $p(z)$ with multiplicity $m+1$, then r is a root of $p'(z)$ (and hence also of $d(z)$) of multiplicity m. Thus, $\deg(P(z))$ is equal to the number of distinct roots of $p(z)$. It suffices to show $\deg(P(z)) \geq \ell$ since $p(z)$ (and hence $P(z)$) has no zero root. Setting $f(z) = p(z) - z^n$, we observe that $d(z)$ divides

$$f'(z)p(z) - f(z)p'(z) = f'(z)(z^n + f(z)) - f(z)(nz^{n-1} + f'(z)) = z^{n-1}(zf'(z) - nf(z)).$$

Since $p(0) \neq 0$, z does not divide $p(z)$, and hence, z does not divide $d(z)$. Thus, $d(z)$ divides $zf'(z) - nf(z)$. Since the leading term of $zf'(z) - nf(z)$ is $-\ell p_\ell z^{n-\ell}$, it follows that

$$\deg(d(z)) \leq \deg(zf'(z) - nf(z)) = n - \ell.$$

Therefore, $\deg(P(z)) = n - \deg(d(z)) \geq \ell$ as required to show.

Now assume $p(0) = 0$. If 0 is a root of $p(z)$ with multiplicity equal to $n - N$, then letting

$$h(z) = z^N + \sum_{i=\ell}^{N} p_i z^{N-i},$$

we observe that $p(z) = z^{n-N}h(z)$ and $h(0) \neq 0$. Thus, the previous argument applies to $h(z)$. This implies that $h(z)$ (and hence also $p(z)$) has at least ℓ distinct nonzero roots. □

For an $n \times n$ matrix A, applying Lemma 2.2 to the characteristic polynomial of A gives a lower bound on $q(A)$ in terms of $\ell(A)$, i.e., the minimum length of a cycle in $D(\mathcal{A})$.

THEOREM 2.3. *Let A be an $n \times n$ matrix that is not nilpotent. Then, $q(A) \geq \ell(A)$. Furthermore, if $c(A) \leq n-1$, then $\ell(A) + 1 \leq q(A) \leq c(A) + 1$.*

Proof. Since A is not nilpotent, $p_A(z) = z^n + \sum_{i=\hat{\ell}}^{n} p_i z^{n-i}$ with $p_{\hat{\ell}} \neq 0$ for some $\hat{\ell} \geq \ell(A)$. By Lemma 2.2, $p_A(z)$ requires at least $\hat{\ell}$ distinct nonzero roots. Hence $q(A) \geq \hat{\ell} \geq \ell(A)$. If $c(A) \leq n-1$, then (including the zero eigenvalue) we get $q(A) \geq \hat{\ell} + 1 \geq \ell(A) + 1$. Lemma 2.1 gives the upper bound. □

EXAMPLE 2.4. Let $\mathcal{A}$ be a 7×7 sign pattern with the digraph given in Fig. 1. If $A \in Q(\mathcal{A})$ is not nilpotent, then $4 \leq q(A) \leq 5$ by Theorem 2.3 since $\ell(A) = 3$ and $c(A) = 4$. Further, by Lemma 2.1, $q_5(\mathcal{A}) = 1$. This implies $q_{\text{seq}}(\mathcal{A}) = \langle a, 0, 0, b, 1, 0, 0 \rangle$ for some $a, b \in \{0, 1\}$ depending on the signs of the cycles in $D(\mathcal{A})$. For example, if the two 3-cycles are oppositely signed and the two 4-cycles are oppositely signed then $a = 1$, otherwise, $a = 0$.

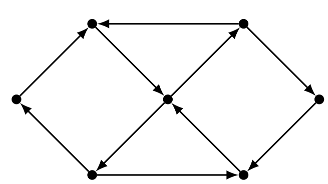

FIGURE 1. *A digraph with cycles of length 3 and 4.*

In Theorem 2.3, if $\mathcal{A} = \text{sgn}(A)$ is not potentially nilpotent then every $B \in Q(\mathcal{A})$ satisfies $q(B) \geq \ell(\mathcal{A})$. This implies the bound also holds for $q(\mathcal{A})$.

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Corollary 2.5. *Let $\mathcal{A}$ be an $n \times n$ sign pattern that is not potentially nilpotent. Then, $q(\mathcal{A}) \geq \ell(\mathcal{A})$. Furthermore, if $c(\mathcal{A}) \leq n-1$, then $\ell(\mathcal{A})+1 \leq q(\mathcal{A}) \leq c(\mathcal{A})+1$.*

We next consider the allow sequence when an $n \times n$ sign pattern $\mathcal{A}$ satisfies $\ell(\mathcal{A}) = c(\mathcal{A})$. When $\ell(\mathcal{A}) = c(\mathcal{A}) = n$, the digraph of $\mathcal{A}$ consists of a single n-cycle and it follows that $q(\mathcal{A}) = n$ (i.e., $q_{\text{seq}}(\mathcal{A}) = \langle 0, \ldots, 0, 1\rangle$) by Corollary 2.5. When $\ell(\mathcal{A}) = c(\mathcal{A}) \leq n-1$, we have the following result.

Corollary 2.6. *Let $\mathcal{A}$ be an $n \times n$ sign pattern and suppose $\ell = \ell(\mathcal{A}) = c(\mathcal{A}) \leq n-1$. Then, $q_{\ell+1}(\mathcal{A}) = 1$ and $q_k(\mathcal{A}) = 0$ for $k \neq \ell+1$ and $k \neq 1$. Further, $q_1(\mathcal{A}) = 1$ if and only if $\mathcal{A}$ is potentially nilpotent (i.e., $D(\mathcal{A})$ has at least two oppositely signed ℓ-cycles).*

Example 2.7. Let $n \geq 3$ and suppose $\mathcal{A}$ and $\mathcal{B}$ are the $n \times n$ loopless sign patterns

$$\mathcal{A} = \begin{bmatrix} 0 & + & \cdots & + \\ + & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ + & 0 & \cdots & 0 \\ + & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} 0 & + & \cdots & + \\ + & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ + & 0 & \cdots & 0 \\ - & 0 & \cdots & 0 \end{bmatrix}.$$

Since $\ell(\mathcal{A}) = \ell(\mathcal{B}) = c(\mathcal{A}) = c(\mathcal{B}) = 2$ and $\mathcal{B}$ is potentially nilpotent but $\mathcal{A}$ is not, it follows by Corollary 2.6 that $q_{\text{seq}}(\mathcal{A}) = \langle 0, 0, 1, 0, 0, \ldots, 0\rangle$ and $q_{\text{seq}}(\mathcal{B}) = \langle 1, 0, 1, 0, 0, \ldots, 0\rangle$.

For a sign pattern $\mathcal{A}$, Theorem 2.3 can give zeros in $q_{\text{seq}}(\mathcal{A})$, e.g., if $\ell(\mathcal{A}) = 3$ then $q_2(\mathcal{A}) = 0$. More generally, we have the following result.

Corollary 2.8. *Let $\mathcal{A}$ be an $n \times n$ sign pattern with $\ell(\mathcal{A}) \geq 3$. Then, $q_k(\mathcal{A}) = 0$ for $2 \leq k \leq \ell(\mathcal{A})-1$, and further, $q_1(\mathcal{A}) = 1$ if and only if $\mathcal{A}$ is potentially nilpotent.*

Corollary 2.8 can be improved in the case that $c(\mathcal{A}) \leq n-1$.

Corollary 2.9. *Let $\mathcal{A}$ be an $n \times n$ sign pattern with $\ell(\mathcal{A}) \geq 2$ and $c(\mathcal{A}) \leq n-1$. Then, $q_k(\mathcal{A}) = 0$ for $2 \leq k \leq \ell(\mathcal{A})$, and further, $q_1(\mathcal{A}) = 1$ if and only if $\mathcal{A}$ is potentially nilpotent.*

Example 2.10. Let $n \geq 4$ and consider the $n \times n$ sign pattern

$$\mathcal{A} = \begin{bmatrix} 0 & + & 0 & \cdots & 0 & 0 \\ 0 & 0 & + & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \vdots & & 0 & + & + \\ + & 0 & & 0 & 0 & + \\ - & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

whose digraph consists of two oppositely signed cycles of order $n-1$ and one cycle of order n. Then, $\ell(\mathcal{A}) = n-1$ and $c(\mathcal{A}) = n$. Since $\mathcal{A}$ is not potentially nilpotent, Corollary 2.8 implies $q_1(\mathcal{A}) = \cdots = q_{n-2}(\mathcal{A}) = 0$ and Lemma 2.1 implies $q_n(\mathcal{A}) = 1$. We can deduce that $q_{n-1}(\mathcal{A}) = 1$ by considering $A = [a_{ij}] \in Q(\mathcal{A})$ with every nonzero entry having magnitude equal to 1 except $a_{n,1} = 1-n < 0$ and $a_{n-1,1} = 2n-1 > 0$. Therefore, $q_{\text{seq}}(\mathcal{A}) = \langle 0, \ldots, 0, 1, 1\rangle$.

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2.2. Sign patterns that allow lacunary characteristic polynomials with regular gaps. When $p(z)$ is a non-constant polynomial and m is a positive integer, the number of distinct *nonzero* roots of $p(z^m)$ is divisible by m. For completeness, we provide a proof of this result.

LEMMA 2.11. *Let m be a positive integer and $p(z)$ be a polynomial with $t \geq 1$ distinct nonzero roots (in $\mathbb{C}$). Then, $p(z^m)$ has mt distinct nonzero roots.*

Proof. We may write $p(z) = a\prod_{i=1}^{n}(z - z_i)$ for some $a, z_1, \ldots, z_n \in \mathbb{C}$ with $a \neq 0$. Note if r is a root of both $z^m - z_i$ and $z^m - z_j$ where $i \neq j$, then it must be that $z_i = z_j$; thus, there are no common roots of $z^m - z_i$ and $z^m - z_j$ whenever $z_i \neq z_j$. Since $z^m - z_i$ has m distinct nonzero roots whenever $z_i \neq 0$, the polynomial $p(z^m)$ must have mt distinct nonzero roots for some integer t. □

For an $n \times n$ sign pattern $\mathcal{A}$, if there is a constant m such that the order of every composite cycle in $D(\mathcal{A})$ is divisible by m, then Lemma 2.11 gives restrictions on the values for $q(\mathcal{A})$. In turn, this gives $q_k(\mathcal{A}) = 0$ for values of k outside of this restriction. A sign pattern $\mathcal{A}$ is *sign nonsingular* if every $A \in Q(\mathcal{A})$ is nonsingular.

THEOREM 2.12. *Suppose $\mathcal{A}$ is an $n \times n$ sign pattern and the order of every composite cycle in $D(\mathcal{A})$ is divisible by m for some positive integer m. Then, the following statements hold:*

(i) *If $c(\mathcal{A}) \leq n - 1$, then $q_k(\mathcal{A}) = 0$ for $k \not\equiv 1 \pmod{m}$.*
(ii) *If $c(\mathcal{A}) = n$, then $q_k(\mathcal{A}) = 0$ for $k \not\equiv 0, 1 \pmod{m}$.*
(iii) *If $\mathcal{A}$ is sign nonsingular, then $q_k(\mathcal{A}) = 0$ for $k \not\equiv 0 \pmod{m}$.*

Proof. In case (i), any $A \in Q(\mathcal{A})$ has zero as an eigenvalue, and the number of distinct nonzero roots of the characteristic polynomial is a multiple of m by Lemma 2.11. In case (ii), either zero is an eigenvalue as in the previous case, or the total number of distinct roots is a multiple of m. In case (iii), zero is not an eigenvalue of any $A \in Q(\mathcal{A})$. □

A *bipartite* sign pattern $\mathcal{A}$ is a pattern for which the underlying graph of $D(\mathcal{A})$ is bipartite. Theorem 2.12 gives an immediate result for bipartite sign patterns since bipartite graphs have no odd cycles:

COROLLARY 2.13. *Let $\mathcal{A}$ be an $n \times n$ bipartite sign pattern.*

(i) *If $c(\mathcal{A}) \leq n - 1$, then $q_{2k}(\mathcal{A}) = 0$ for $k \leq \lfloor n/2 \rfloor$.*
(ii) *If $\mathcal{A}$ is sign nonsingular, then $q_{2k-1}(\mathcal{A}) = 0$ for $k \leq \lfloor n/2 \rfloor$.*

Note that the hypothesis of Corollary 2.13(ii) implies that n is even, since in this case $D(\mathcal{A})$ would have a composite n-cycle for the bipartite pattern $\mathcal{A}$.

EXAMPLE 2.14. Let $\mathcal{B}$ be the bipartite sign pattern:

$$\mathcal{B} = \begin{bmatrix} 0 & 0 & + & 0 \\ 0 & 0 & 0 & + \\ + & + & 0 & 0 \\ - & 0 & 0 & 0 \end{bmatrix}.$$

Then, $q_1(\mathcal{B}) = q_3(\mathcal{B}) = 0$ by Corollary 2.13(ii) since $\mathcal{B}$ is sign nonsingular, $q_4(\mathcal{B}) = 1$ by Lemma 2.1 and $q_2(\mathcal{B}) = 1$ by considering $B = [b_{ij}] \in Q(\mathcal{B})$ with every nonzero entry having magnitude 1 except $b_{31} = 2$ (in this case, $p_B(z) = (z-1)^2(z+1)^2$). Therefore, $q_{\text{seq}}(\mathcal{B}) = \langle 0, 1, 0, 1 \rangle$.

The following example demonstrates that the assumptions in Corollary 2.13 are necessary for the conclusion to hold for a bipartite sign pattern.

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Example 2.15. For $a, b > 0$, consider the 4×4 bipartite sign pattern $\mathcal{A}$ and a realization A:

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & + & + \\ 0 & 0 & + & + \\ + & + & 0 & 0 \\ - & - & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 0 & 1 & a \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & -b & 0 & 0 \end{bmatrix}.$$

Then, $c(\mathcal{A}) = 4$ and $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, 1, 1\rangle$ since $q(A) = 1$ when $(a, b) = (1, 1)$, $q(A) = 2$ when $(a, b) = (2, 2)$, $q(A) = 3$ when $(a, b) = (1, 2)$ and $q(A) = 4$ when $(a, b) = (2, 3)$.

2.3. Signed directed graphs and $q_2(\mathcal{A}) = 1$. For an $n \times n$ sign pattern $\mathcal{A}$, there are some known structural properties that $D(\mathcal{A})$ must satisfy when $q(\mathcal{A}) = 1$ (see [3, Lemma 3.2, Theorems 3.3 and 3.5]). Analogous properties hold in the case that $c(\mathcal{A}) \leq n - 1$ and $q_2(\mathcal{A}) = 1$. These properties are used later in the paper to show that certain sequences cannot be an allow sequence for any $n \times n$ irreducible sign pattern (in particular, see Theorem 4.13 and Theorem 4.14). We first give a comparable result to [3, Lemma 3.2]. A proof is provided since the proof of [3, Lemma 3.2] relies on observations concerning potentially nilpotent sign patterns (i.e., [3, Observation 3.1]).

Theorem 2.16. *For $n \geq 2$, let $\mathcal{A}$ be an $n \times n$ sign pattern with $c(\mathcal{A}) \leq n - 1$. If there exists $A \in Q(\mathcal{A})$ with $q(A) = 2$ and the coefficient of $z^{n-c(\mathcal{A})}$ in $p_A(z)$ is nonzero, then either $D(\mathcal{A})$ has a negative 2-cycle or $D(\mathcal{A})$ has exactly $c(\mathcal{A})$ loops of the same sign. In the latter case, $D(\mathcal{A})$ either has no 2-cycles, or has both positive and negative 2-cycles.*

Proof. Suppose $A = [a_{ij}] \in Q(\mathcal{A})$ with $q(A) = 2$ and $p_A(z) = z^n - E_1 z^{n-1} + E_2 z^{n-2} + \cdots + (-1)^n E_n$. By (1.1), $E_i = 0$ for $i > c(\mathcal{A})$, and since the coefficient of $z^{n-c(\mathcal{A})}$ in $p_A(z)$ is nonzero, $p_A(z) = z^{n-N}(z^N - E_1 z^{N-1} + E_2 z^{N-2} + \cdots + (-1)^N E_N)$ for $N = c(\mathcal{A}) \leq n - 1$ and $E_N \neq 0$. Without loss of generality, we may assume that A (or $-A$) has eigenvalues 0 with multiplicity $n - N$ and 1 with multiplicity N. We consider the case that A has these eigenvalues and note that $D(\mathcal{A})$ and $D(-\mathcal{A})$ have the same number of loops and the same number of negative 2-cycles. Then, $E_1 = N$ and $E_2 = N(N-1)/2$ implying that $E_1^2 - 2E_2 = N$. By (1.1),

$$E_2 = \sum_{i<j} a_{ii}a_{jj} - \sum_{i<j} a_{ij}a_{ji} = \frac{1}{2}\left[\left(\sum_{i=1}^{n} a_{ii}\right)^2 - \sum_{i=1}^{n} a_{ii}^2\right] - \sum_{i<j} a_{ij}a_{ji},$$

and hence,

$$2\sum_{i<j} a_{ij}a_{ji} = E_1^2 - 2E_2 - \sum_{i=1}^{n} a_{ii}^2 = N - \sum_{i=1}^{n} a_{ii}^2.$$

Suppose $D(\mathcal{A})$ has t loops. Then, $t \geq 1$ since $\sum_{i=1}^n a_{ii} = \sum_{i=1}^n \lambda_i = N$. The Cauchy-Schwarz inequality gives

$$\sum_{i=1}^{n} a_{ii}^2 = \sum_{i \in L} a_{ii}^2 \geq \frac{1}{t}\left(\sum_{i \in L} a_{ii}\right)^2 = \frac{N^2}{t},$$

where $L = \{i : a_{ii} \neq 0,\ 1 \leq i \leq n\}$ (note $|L| = t$), with equality if and only if $a_{ii} = N/t$ for every $i \in L$. Since $t \leq c(\mathcal{A}) = N$,

$$2\sum_{i<j} a_{ij}a_{ji} \leq N(1 - N/t) \leq 0,$$

with equality if and only if $t = N$ and $a_{ii} = 1$ for every $i \in L$. If $2\sum_{i<j} a_{ij}a_{ji} < 0$, then $D(\mathcal{A})$ has a negative 2-cycle. Otherwise $2\sum_{i<j} a_{ij}a_{ji} = 0$, and hence, $a_{ii} = 1$ for every $i \in L$. In the latter case, $D(\mathcal{A})$

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has exactly $c(\mathcal{A})$ loops of the same sign, and furthermore, $D(\mathcal{A})$ either has no 2-cycles or both positive and negative 2-cycles. ☐

Theorem 2.16 immediately gives the following analogue to [3, Theorem 3.3].

COROLLARY 2.17. *For $n \geq 2$, let $\mathcal{A}$ be an $n\times n$ sign pattern with $c(\mathcal{A}) \leq n-1$. If there exists $A \in Q(\mathcal{A})$ with $q(A) = 2$ and the coefficient of $z^{n-c(\mathcal{A})}$ in $p_A(z)$ is nonzero, and any one of the following conditions hold:*

(i) $D(\mathcal{A})$ has a 2-cycle;
(ii) $D(\mathcal{A})$ has between 1 and $c(\mathcal{A}) - 1$ loops; or
(iii) $D(\mathcal{A})$ has $c(\mathcal{A})$ loops, and two oppositely signed loops,

then $D(\mathcal{A})$ has a negative 2-cycle.

The next theorem is an analogue to [3, Theorem 3.5] with an almost identical proof, but for completeness, we provide a proof.

THEOREM 2.18. *For $n \geq 4$, let $\mathcal{A}$ be an $n \times n$ sign pattern with $3 \leq c(\mathcal{A}) \leq n-1$ and suppose $D(\mathcal{A})$ has at least one negative loop and no positive loops. If there exists $A \in Q(\mathcal{A})$ with $q(A) = 2$ and such that the coefficient of $z^{n-c(\mathcal{A})}$ in $p_A(z)$ is nonzero, then either $D(\mathcal{A})$ has exactly $c(\mathcal{A})$ loops of the same sign or all of the following conditions must hold:*

(i) for every $2 \leq k \leq c(\mathcal{A})$, $D(\mathcal{A})$ has a composite cycle U of order k and sign $(-1)^{|U|}$ that is not the product of loops.
(ii) $D(\mathcal{A})$ has a negative 3-cycle, or a loop and a negative 2-cycle that are vertex disjoint.
(iii) $D(\mathcal{A})$ has a positive 3-cycle, or a loop incident to a negative 2-cycle.

Proof. Suppose $A = [a_{ij}] \in Q(\mathcal{A})$ satisfies $q(A) = 2$ and such that the coefficient of $z^{n-c(\mathcal{A})}$ in $p_A(z)$ is nonzero. Then, zero is an eigenvalue of A with multiplicity equal to $n - c(\mathcal{A})$. Thus, we may scale A so that it has characteristic polynomial

$$p_A(z) = z^{n-N}(z+1)^N = z^{n-N}\left(z^N + \binom{N}{1}z^{N-1} + \binom{N}{2}z^{N-2} + \cdots + \binom{N}{N}\right), \tag{2.2}$$

where $N = c(\mathcal{A}) \geq 3$. Let $\mathcal{U}_k$ denote the set of all composite cycles of $D(\mathcal{A})$ that cover precisely k vertices and $\mathcal{V}_k \subseteq \mathcal{U}_k$ be those that consist entirely of loops. For every $2 \leq k \leq N$, by Eq. (1.1) and (2.2),

$$\binom{N}{k} = \sum_{U\in \mathcal{V}_k} (-1)^k \prod_{(v_i,v_i)\in E(U)} a_{ii} + \sum_{U\in \mathcal{U}_k\setminus\mathcal{V}_k} (-1)^{|U|} \prod_{(v_i,v_j)\in E(U)} a_{ij}.$$

Note that $a_{ii} \leq 0$ for $1 \leq i \leq n$. Without loss of generality (otherwise consider a permutation similarity), we may assume that the nonzero diagonal entries of A occur in positions (i, i) with $1 \leq i \leq t$, where t is the number of loops in $D(\mathcal{A})$ (note $t \leq N$). Let $x_i = |a_{ii}|$ for $1 \leq i \leq N$ and observe that $\sum_{i=1}^{N} x_i = N$ by Eq. (2.2). By [3, Proposition 3.4(iii)],

$$\sum_{U\in \mathcal{V}_k} (-1)^k \prod_{(v_i,v_i)\in E(U)} a_{ii} = \sum_{1\leq i_1<\cdots<i_k\leq N} x_{i_1}x_{i_2}\cdots x_{i_k} \leq \binom{N}{k},$$

with equality if and only if $t = N$ and $x_1 = \cdots = x_N = 1$. In the equality case, $D(\mathcal{A})$ has exactly $N = c(\mathcal{A})$ loops of the same sign; otherwise, (i) follows. For (ii), since $N \geq 3$, we substitute $k = 3$ in (i) and recall that

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$D(\mathcal{A})$ only has negative loops. For (iii), Eq. (2.2) gives

$$\binom{N}{1} = \sum_{i=1}^{N} x_i, \tag{2.3}$$

$$\binom{N}{2} = \sum_{1\le i<j\le N} x_i x_j - \sum_{1\le i<j\le n} a_{ij}a_{ji}, \tag{2.4}$$

$$\binom{N}{3} = \sum_{1\le i<j<k\le N} x_i x_j x_k - \sum_{k=1}^{N} x_k \sum_{\substack{1\le i<j\le n\\ i\ne k, j\ne k}} a_{ij}a_{ji} - \sum_{1\le i<j<k\le n} a_{ij}a_{jk}a_{ki}. \tag{2.5}$$

Observe that

$$\sum_{k=1}^{N} x_k \sum_{\substack{1\le i<j\le n\\ i\ne k, j\ne k}} a_{ij}a_{ji} = \sum_{k=1}^{N} x_k \sum_{1\le i<j\le n} a_{ij}a_{ji} - \sum_{k=1}^{N} x_k \sum_{\substack{1\le i\le n\\ i\ne k}} a_{ik}a_{ki}. \tag{2.6}$$

Define $X = \sum_{1\le i<j<k\le n} a_{ij}a_{jk}a_{ki} - \sum_{k=1}^{N} x_k \sum_{\substack{1\le i\le n\\ i\ne k}} a_{ik}a_{ki}$. From an identical argument to [3, Theorem 3.5], it follows that $X \ge \frac{N(N-t)(N+t)}{3t^2}$ with equality if and only if $t = N$ and $x_1 = \cdots = x_N = 1$. Thus, one of three scenarios occur: $t = N$ and $D(\mathcal{A})$ has exactly $N = c(\mathcal{A})$ loops of the same sign, or $D(\mathcal{A})$ has a positive simple 3-cycle, or $D(\mathcal{A})$ has a loop incident to a negative 2-cycle. □

If an $n \times n$ sign pattern $\mathcal{A}$ has $c(\mathcal{A})$ loops, then every cycle of $D(\mathcal{A})$ of length $r \ge 2$ must be incident to r loops (otherwise there exists a composite cycle of order greater than $c(\mathcal{A})$). Thus, in the case that $c(\mathcal{A}) \le n-1$ and $D(\mathcal{A})$ has $c(\mathcal{A})$ loops, we have the following remark.

Remark 2.19. Let $\mathcal{A}$ be an $n \times n$ sign pattern. If $c(\mathcal{A}) \le n-1$ and $D(\mathcal{A})$ has exactly $c(\mathcal{A})$ loops, then $\mathcal{A}$ is reducible.

3. Techniques utilizing Jacobian matrices and vertex duplication.

In this section, we first develop methods, based on Jacobian matrices, that preserve allow sequences for superpatterns. We also explore the effect that vertex duplication has on the allow sequence in some special cases.

3.1. Jacobian methods.

In general, taking a superpattern of a pattern $\mathcal{A}$ will not preserve the allow sequence for $\mathcal{A}$, nor $q_k(\mathcal{A})$ for $1 \le k \le n$. For example, if

$$\mathcal{Z}_1 = \begin{bmatrix} 0 & + & 0\\ - & 0 & +\\ 0 & + & 0 \end{bmatrix} \text{ and } \hat{\mathcal{Z}}_1 = \begin{bmatrix} + & + & 0\\ - & 0 & +\\ 0 & + & 0 \end{bmatrix},$$

then $q_{\text{seq}}(\mathcal{Z}_1) = \langle 1,0,1\rangle$ but $q_{\text{seq}}(\hat{\mathcal{Z}}_1) = \langle 0,1,1\rangle$ (see Theorem 5.2(iii)). However, for some sign patterns, some entries of the allow sequence are preserved for its superpatterns when certain Jacobian constraints hold. In this section, we develop methods to demonstrate this idea.

As defined in [3], we say that a real $n \times n$ matrix $A = [a_{ij}]$ *allows a Jacobian with rank* r if A satisfies the following conditions:

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(i) A has at least r nonzero entries.
(ii) Among the nonzero entries of A, there are r entries, say $a_{i_1j_1}, \ldots, a_{i_rj_r}$, such that if X is the matrix obtained from A by replacing the entries $a_{i_1j_1}, \ldots, a_{i_rj_r}$ by real variables $x_1, \ldots, x_r$ and the characteristic polynomial of X is

$$p_X(z) = z^n + p_1 z^{n-1} + p_2 z^{n-2} + \cdots + p_{n-1}z + p_n,$$

then the $n \times r$ Jacobian matrix J with (i, j) entry equal to $\frac{\partial p_i}{\partial x_j}(x_1, \ldots, x_r)$ has rank r evaluated at $(x_1, \ldots, x_r) = (a_{i_1j_1}, \ldots, a_{i_rj_r})$.

We first start with an auxiliary lemma. This lemma simply notes that for any multiset of complex numbers closed under complex conjugation having k distinct elements, there are multisets of complex numbers closed under complex conjugation sufficiently close to the original set having k' distinct elements for any $k' \geq k$ with $k' - k$ even (taking into account complex conjugates) while maintaining the same sum.

Lemma 3.1. *Let $S = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a multiset of n complex numbers, closed under complex conjugation, containing k distinct elements. For k' satisfying $k \leq k' \leq n$ with $k' - k$ even, there exists a multiset $S' = \{\mu_1, \mu_2, \ldots, \mu_n\}$ of n complex numbers, closed under complex conjugation, with the following properties:*

(i) the number of distinct elements in S' is k'.
(ii) μ_i is sufficiently close to λ_i for $1 \leq i \leq n$.
(iii) $\displaystyle\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} \lambda_i$.

Furthermore, if S contains a real number of multiplicity at least two, then the condition that $k' - k$ is even may be removed in the statement above.

Now, [6, Corollary 3.3] and Lemma 3.1 give the following theorem.

Theorem 3.2. *Let $\mathcal{A}$ be an $n \times n$ sign pattern and $\hat{\mathcal{A}}$ a superpattern of $\mathcal{A}$. Suppose $A \in Q(\mathcal{A})$ allows a Jacobian with rank n. If A has a real eigenvalue with algebraic multiplicity at least two, then $q_k(\hat{\mathcal{A}}) = 1$ for $q(A) \leq k \leq n$, otherwise, $q_k(\hat{\mathcal{A}}) = 1$ for $q(A) \leq k \leq n$ with $k - q(A)$ even.*

When $q(A) = 1$ or $q(A) = 2$ in Theorem 3.2, we get the following corollary.

Corollary 3.3. *Let $\mathcal{A}$ be an $n \times n$ sign pattern and $\hat{\mathcal{A}}$ be a superpattern of $\mathcal{A}$. Suppose $A \in Q(\mathcal{A})$ allows a Jacobian with rank n.*

(i) If $q(A) = 1$, then $q_{\text{seq}}(\hat{\mathcal{A}}) = \langle 1, 1, \ldots, 1\rangle$.
(ii) If $q(A) = 2$, then $q_k(\hat{\mathcal{A}}) = 1$ for all even k.

In the next example, we illustrate how Corollary 3.3 may be applied to a stable matrix, that is, a matrix whose eigenvalues all have strictly negative real parts.

Example 3.4. The stable realization C of the companion pattern $\mathcal{C}$ in [3, Example 2.12] with $q(C) = 1$ allows a Jacobian with rank n and hence by Corollary 3.3, each superpattern $\hat{\mathcal{C}}$ of $\mathcal{C}$ satisfies $q_{\text{seq}}(\hat{\mathcal{C}}) = \langle 1, 1, \ldots, 1\rangle$. Note that $\mathcal{C}$ is not potentially nilpotent, and hence not spectrally arbitrary, but $\mathcal{C}$ can obtain any number of distinct eigenvalues.

Let $\mathcal{A}$ be an $n \times n$ sign pattern such that $D(\mathcal{A})$ has no loops and suppose $A \in Q(\mathcal{A})$ allows a Jacobian with rank $n - 1$. By using Lemma 3.1 and extending the proof of [3, Theorem 2.16], we can obtain a conclusion identical to that of Theorem 3.2 without having a Jacobian with rank n.

Theorem 3.5. *Let $\mathcal{A}$ be an $n \times n$ loopless sign pattern and $\hat{\mathcal{A}}$ be a superpattern of $\mathcal{A}$. Suppose $A \in Q(\mathcal{A})$ allows a Jacobian with rank $n-1$. If A has a real eigenvalue with algebraic multiplicity at least two, then $q_k(\hat{\mathcal{A}}) = 1$ for $q(A) \leq k \leq n$, otherwise, $q_k(\hat{\mathcal{A}}) = 1$ for $q(A) \leq k \leq n$ with $k - q(A)$ even.*

Proof. Let $\mathcal{A}$ be an $n \times n$ sign pattern such that $D(\mathcal{A})$ has no loops. If $\hat{\mathcal{A}}$ is a superpattern of $\mathcal{A}$ and $A \in Q(\mathcal{A})$ allows a Jacobian with rank $n-1$, then by [3, Theorem 2.16], there exists $B \in Q(\hat{\mathcal{A}})$ such that $q(B) = q(A)$. In fact, it was also shown in the proof of [3, Theorem 2.16] that B allows a Jacobian with rank n when $\hat{\mathcal{A}}$ is not loopless.

Suppose that the spectrum of A is $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and consider k satisfying $q(A) \leq k \leq n$ (and with $k - q(A)$ an even integer in the case that A does not have a real eigenvalue with algebraic multiplicity at least two). Note that $\sum_{i=1}^n \lambda_i = 0$ since A has a zero diagonal.

First consider a superpattern $\hat{\mathcal{A}}$ of $\mathcal{A}$ that has a zero main diagonal. By Lemma 3.1, there is a multiset $\{\mu_1, \mu_2, \ldots, \mu_n\}$ of n complex numbers with k distinct elements, closed under complex conjugation, with each μ_i sufficiently close to λ_i satisfying $\sum_{i=1}^n \mu_i = \sum_{i=1}^n \lambda_i = 0$. Note that by [3, Theorem 2.16], the pattern $\hat{\mathcal{B}}$, derived from $\hat{\mathcal{A}}$ by allowing the diagonal entry (t,t) to be free, realizes every characteristic polynomial sufficiently close to that of A. Now, since the coefficients of the characteristic polynomial of a matrix are continuous functions of its eigenvalues, and since the coefficient of z^{n-1} in $\prod_{i=1}^n (z - \mu_i)$ is equal to 0, the sign pattern $\hat{\mathcal{A}}$ has a realization B whose spectrum is $\{\mu_1, \mu_2, \ldots, \mu_n\}$. Thus, $q_k(\hat{\mathcal{A}}) = 1$.

Next consider a superpattern $\hat{\mathcal{A}}$ of $\mathcal{A}$ where the zero (t,t) entry is replaced by $+$ (resp. $-$) and every other entry of $\hat{\mathcal{A}}$ equals that of $\mathcal{A}$. By Lemma 3.1, there is a multiset $\{\mu_1', \mu_2', \ldots, \mu_n'\}$ of n complex numbers with k distinct elements, closed under complex conjugation, with each μ_k' sufficiently close to λ_k satisfying $\sum_{i=1}^n \mu_i' = \sum_{i=1}^n \lambda_i = 0$.

Let $\epsilon > 0$ (resp. $\epsilon < 0$) be sufficiently close to zero and construct $S = \{\mu_1 \ldots, \mu_n\}$ with $\mu_i = \mu_i' + \epsilon/n$. Then, S satisfies (i)–(ii) of Lemma 3.1 but $\sum_{i=1}^n \mu_i = \epsilon + \sum_{i=1}^n \lambda_i$. Since the coefficient of x^{n-1} in $\prod_{i=1}^n (x - \mu_i)$ is equal to $-\epsilon$, it has the same sign as $\operatorname{sgn}(\hat{\mathcal{A}}_{tt})$, i.e., $+$ (resp. $-$). Thus, the sign pattern $\hat{\mathcal{A}}$ has a realization B having spectrum $\sigma(B) = \{\mu_1, \mu_2, \ldots, \mu_n\}$ implying that $q_k(\hat{\mathcal{A}}) = 1$. Furthermore, B allows a Jacobian with rank n since $B_{tt} \neq 0$. Thus, by Theorem 3.2, the result also follows for all superpatterns of $\mathcal{A}$. Since the argument applies to every $1 \leq t \leq n$, if $\hat{\mathcal{A}}$ is a superpattern of $\mathcal{A}$ that has at least one nonzero diagonal entry, $q_k(\hat{\mathcal{A}}) = 1$. ∎

In the next example, we illustrate how Theorem 3.5 may be applied to a matrix having two distinct eigenvalues.

Example 3.6. The realization A of the $n \times n$ loopless companion pattern $\mathcal{A}$ defined in [3, Example 2.17] with $q(A) = 2$ allows a Jacobian with rank $n-1$ and has a real eigenvalue with algebraic multiplicity at least two. Thus, by Theorem 3.5, each superpattern $\hat{\mathcal{A}}$ of $\mathcal{A}$ satisfies $q_k(\hat{\mathcal{A}}) = 1$ for $k \geq 2$. In particular, we can conclude that either $q_{\text{seq}}(\hat{\mathcal{A}}) = \langle 0, 1, 1, \ldots, 1\rangle$ or $q_{\text{seq}}(\hat{\mathcal{A}}) = \langle 1, 1, \ldots, 1\rangle$. Note that whether $q_1(\hat{\mathcal{A}}) = 0$ or $q_1(\hat{\mathcal{A}}) = 1$ depends on the particular superpattern $\hat{\mathcal{A}}$. For example, if $\hat{\mathcal{A}} = \mathcal{A}$ then $q_{\text{seq}}(\hat{\mathcal{A}}) = \langle 0, 1, 1, \ldots, 1\rangle$ since it was shown that $q(\mathcal{A}) = 2$ in [3, Example 2.17].

In the next theorem, the implicit function theorem is applied to a matrix with a repeated real eigenvalue.

Theorem 3.7. *Let $\mathcal{A}$ be an $n \times n$ sign pattern and $\hat{\mathcal{A}}$ a superpattern of $\mathcal{A}$. Suppose $A \in Q(\mathcal{A})$ has the following properties:*

(i) $\lambda \in \mathbb{R}$ is an eigenvalue of A with multiplicity equal to $m \geq 1$,

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(ii) *A has at least m nonzero entries,*

(iii) *among the nonzero entries of A, there are m entries, say $a_{i_1j_1}, \ldots, a_{i_mj_m}$, such that if X is the matrix obtained from A by replacing the entries $a_{i_1j_1}, \ldots, a_{i_mj_m}$ by real variables $x_1, \ldots, x_m$ and the characteristic polynomial of X is $p_X(z)$, then the $m \times m$ Jacobian matrix J with*

$$J_{ij} = \frac{\partial f_{i-1}(\lambda)}{\partial x_j}(x_1, \ldots, x_m),$$

has rank m at $(x_1, \ldots, x_m) = (a_{i_1j_1}, \ldots, a_{i_mj_m})$, where $f_i(\lambda)$ is the ith derivative of $p_X(z)$ evaluated at $z = \lambda$.

Then, there exists $\hat{A} \in Q(\hat{\mathcal{A}})$ having λ as an eigenvalue with multiplicity equal to m and whose remaining eigenvalues are some numbers sufficiently close to the remaining eigenvalues of A. Furthermore, if A has $n - m + 1$ distinct eigenvalues, then $q_k(\hat{\mathcal{A}}) = 1$ for $k \geq n - m + 1$.

Proof. Consider the $n \times n$ matrix Y with (i, j) entry equal to

$$Y_{ij} = \begin{cases} X_{ij} & \text{if } \mathcal{A}_{ij} \neq 0, \\ \epsilon_{ij} & \text{if } \mathcal{A}_{ij} = 0 \text{ and } \hat{\mathcal{A}}_{ij} > 0, \\ -\epsilon_{ij} & \text{if } \mathcal{A}_{ij} = 0 \text{ and } \hat{\mathcal{A}}_{ij} < 0. \end{cases}$$

Let $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{c}$ be any vector that contains all the ϵ_{ij}'s. Denote the characteristic polynomial of Y by $p_Y(z; \mathbf{x}, \mathbf{c})$. Observe that when $\mathbf{x} = (a_{i_1j_1}, \ldots, a_{i_mj_m})$ and $\mathbf{c} = (0, \ldots, 0)$, then $p_Y(z; \mathbf{x}, \mathbf{c}) = p_A(z)$. By taking $\epsilon_{ij} > 0$ sufficiently close to 0 and $\mathbf{x}$ sufficiently close to $(a_{i_1j_1}, \ldots, a_{i_mj_m})$, we can guarantee that the roots of $p_Y(z; \mathbf{x}, \mathbf{c})$ are sufficiently close to the roots of $p_A(z)$. Thus, it suffices to find nonzero values of $x_1, \ldots, x_m$ that satisfy $\operatorname{sgn}(x_r) = \operatorname{sgn}(a_{i_rj_r})$, $1 \leq r \leq m$, and

$$\begin{gathered} p_Y(\lambda; \mathbf{x}, \mathbf{c}) = 0 \\ p'_Y(\lambda; \mathbf{x}, \mathbf{c}) = 0 \\ \vdots \\ p_Y^{(m-1)}(\lambda; \mathbf{x}, \mathbf{c}) = 0, \end{gathered}$$

or equivalently,

$$p_Y(\lambda; \mathbf{x}, \mathbf{0}) + h_0(\mathbf{x}, \mathbf{c}) = 0, \quad p'_Y(\lambda; \mathbf{x}, \mathbf{0}) + h_1(\mathbf{x}, \mathbf{c}) = 0, \quad \ldots, \quad p_Y^{(m-1)}(\lambda; \mathbf{x}, \mathbf{0}) + h_{m-1}(\mathbf{x}, \mathbf{c}) = 0,$$

in which $h_0, \ldots, h_{m-1}$ are polynomial functions each of whose terms contain at least one ϵ_{ij} as a factor. Define the functions $g_0, \ldots, g_{m-1}$ of $\mathbf{x}$ and $\mathbf{c}$ to be the left sides of the equations in order. Each g_i has continuous partial derivatives with respect to all variables and

$$g_i(a_{i_1j_1}, \ldots, a_{i_mj_m}, 0, \ldots, 0) = 0.$$

Let $\tilde{J}$ be $\partial(g_0, \ldots, g_{m-1})/\partial(x_1, \ldots, x_m)$ evaluated at $(a_{i_1j_1}, \ldots, a_{i_mj_m}, 0, \ldots, 0)$. Calculating $\tilde{J}$ can be simplified by setting $\epsilon_{ij} = 0$ before taking any partial derivatives since terms involving one or more ϵ_{ij} do not influence the value of the Jacobian. Then, $\tilde{J} = J$, and thus, $|\tilde{J}| \neq 0$ by the assumption in the statement of the theorem. By the implicit function theorem, for $\mathbf{c}$ sufficiently close to $\mathbf{0}$, there are unique continuous functions $x_1, \ldots, x_m$ of ϵ_{ij} that maintain $g_0 = \cdots = g_{m-1} = 0$. Taking any $\mathbf{c}$ positive sufficiently close to $\mathbf{0}$ guarantees that $\operatorname{sgn}(x_r) = \operatorname{sgn}(a_{i_rj_r})$, $1 \leq r \leq m$ (as $\mathbf{x}$ can be made arbitrarily close to $(a_{i_1j_1}, \ldots, a_{i_mj_m})$).

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This produces a realization $\hat{A} \in Q(\hat{\mathcal{A}})$ having λ as an eigenvalue with multiplicity equal to m and whose remaining eigenvalues are sufficiently close to those of A.

Next assume that every eigenvalue of A not equal to λ is simple and let $\mu_1, \dots, \mu_{n-m}$ denote the eigenvalues of A that are not equal to λ. Note that $q_{n-m+1}(\hat{\mathcal{A}}) = 1$. Let $k = n - m + 2$. We prove that there exists $\hat{A} \in Q(\hat{\mathcal{A}})$ having k distinct eigenvalues and satisfying (i)-(iii) with m replaced by $m-1$. Let ϵ be a new variable and follow the previous argument except let $\mathbf{c}$ be a vector that contains both ϵ and the ϵ_{ij}'s. Define the functions $g_0, \dots, g_{m-2}$ as before and $g_{m-1} = p_Y^{(m-1)}(\lambda; \mathbf{x}, \mathbf{c}) - \epsilon$. The Jacobian $\partial(g_0, \dots, g_{m-1})/\partial(x_1, \dots, x_m)$ evaluated at $(a_{i_1 j_1}, \dots, a_{i_m j_m}, 0, \dots, 0)$ is equal to J. By the implicit function theorem, for $\mathbf{c}$ sufficiently close to $\mathbf{0}$, there are unique continuous functions $x_1, \dots, x_m$ of ϵ_{ij} and ϵ that maintain $g_0 = \cdots = g_{m-1} = 0$. Now taking any $\mathbf{c}$ positive sufficiently close to $\mathbf{0}$ also guarantees that $g_{m-1} = \epsilon \neq 0$. This produces a realization $\hat{A} \in Q(\hat{\mathcal{A}})$ having λ as an eigenvalue with multiplicity equal to $m-1$ and whose remaining eigenvalues are sufficiently close to $\{\lambda, \mu_1, \dots, \mu_{n-m}\}$, and hence, every eigenvalue of $\hat{A}$ not equal to λ can be guaranteed to be simple since $\mu_i \neq \lambda$. Let J' be equal to J with the last row deleted (as this row corresponds to $p_Y^{(m-1)}(\lambda; \mathbf{x}, \mathbf{c})$). To see that (iii) holds for $\hat{A}$, we observe that at $(x_1, \dots, x_m) = (a_{i_1 j_1}, \dots, a_{i_m j_m})$, J' has rank $m-1$ since J has rank m. Thus, there is a subset of $m-1$ columns of J' that are linearly independent, and we may choose the corresponding entries in $\hat{A}$ to replace by variables so that (iii) holds.

Repeatedly applying the above argument implies $q_k(\hat{\mathcal{A}}) = 1$ for $k \geq n - m + 1$. □

We demonstrate the method described in Theorem 3.7 with a simple example.

Example 3.8. For $n \geq 3$, let $\mathcal{A}$ be an $n \times n$ sign pattern whose digraph $D(\mathcal{A})$ is a positive n-cycle with two positive loops. Consider $A \in Q(\mathcal{A})$ with nonzero diagonal entries equal to $\frac{2n-3}{2n-5}$ and 3, and nonzero off-diagonal entries equal to 1 with the exception of one entry equal to $\frac{4}{2n-5}$. Replace the entry $\frac{2n-3}{2n-5}$ by x_1 and the entry $\frac{4}{2n-5}$ by x_2 to form the matrix X. Then,

$$p_X(z) = z^n - (x_1 + 3)z^{n-1} + 3x_1 z^{n-2} - x_2,$$

and hence,

$$p_X(1) = 1 - (x_1 + 3) + 3x_1 - x_2 \quad \text{and} \quad p'_X(1) = n - (n-1)(x_1 + 3) + 3(n-2)x_1.$$

Since $p_A(1) = p'_A(1) = 0$ (and a verification shows that $p''_A(1) \neq 0$), $\lambda = 1$ is an eigenvalue of A with multiplicity $m = 2$. The Jacobian matrix defined in Theorem 3.7 is

$$J = \begin{bmatrix} 2 & -1 \\ (2n-5) & 0 \end{bmatrix},$$

and has rank $m = 2$ at $(x_1, x_2) = \left(\frac{2n-3}{2n-5}, \frac{4}{2n-5}\right)$. By Theorem 3.7, if $\hat{\mathcal{A}}$ is any superpattern of $\mathcal{A}$, then there exists $\hat{A} \in Q(\hat{\mathcal{A}})$ having 1 as an eigenvalue with multiplicity equal to 2.

The next lemma shows the utility of Theorem 3.7 and sparse examples such as Example 3.8; in particular, when applied to Example 3.8 it implies that if $\mathcal{A}$ is a sign pattern whose digraph contains a positive r-cycle (with $r \geq 3$) incident to two positive loops, then $\mathcal{A}$ has a realization having $\lambda = 1$ as an eigenvalue with multiplicity 2. First, we state some definitions. If $\hat{\mathcal{A}}$ is a superpattern of $\mathcal{A}$ then $\mathcal{A}$ is a *subpattern* of $\hat{\mathcal{A}}$. For an $n \times n$ sign pattern $\mathcal{A}$ and index set $\eta = \{i_1, \dots, i_m\} \subset [n] = \{1, 2, \dots n\}$, we let $\mathcal{A}_\eta$ denote the *principal submatrix* of $\mathcal{A}$ formed by the rows and columns indexed by η. Note that a principal submatrix of $\mathcal{A}$ is

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different from a subpattern since subpatterns of $\mathcal{A}$ are required to be the same order as $\mathcal{A}$. Further, if $\mathcal{B}$ is a principal submatrix of a subpattern of $\mathcal{A}$, then we call $D(\mathcal{B})$ a *subdigraph* of $D(\mathcal{A})$. We also let O_r denote the $r \times r$ zero matrix.

LEMMA 3.9. *Let $\mathcal{A}$ be an $n \times n$ sign pattern, $\eta \subset [n]$ and $\mathcal{B}$ a subpattern of $\mathcal{A}_\eta$. If there exists $B \in Q(\mathcal{B})$ having $\lambda \in \mathbb{R}$ as an eigenvalue with multiplicity equal to $m \geq 1$ and satisfying (i)-(iii) of Theorem 3.7, then there exists $A \in Q(\mathcal{A})$ having λ as an eigenvalue with multiplicity equal to m and whose remaining eigenvalues are some numbers sufficiently close to the remaining eigenvalues of $B \oplus O_{n-|\eta|}$.*

Proof. Suppose $B \in Q(\mathcal{B})$ satisfies (i)-(iii) of Theorem 3.7 for entries $b_{i_1j_1}, \ldots, b_{i_mj_m}$, variables $x_1, \ldots, x_m$ and matrix X. Let $M = B \oplus O_{n-|\eta|}$ and $Y = X \oplus O_{n-|\eta|}$. Let $f_i(z)$ be the ith derivative of $p_X(z)$ and $g_i(z)$ be the ith derivative of $p_Y(z)$. Since $p_Y(z) = z^{n-|\eta|}p_X(z)$, the $m \times m$ Jacobian matrices

$$J_B = \frac{\partial f_{i-1}(\lambda)}{\partial x_j}(x_1, \ldots, x_m) \quad \text{and} \quad J_M = \frac{\partial g_{i-1}(\lambda)}{\partial x_j}(x_1, \ldots, x_m),$$

have the same rank at $(x_1, \ldots, x_m) = (a_{i_1j_1}, \ldots, a_{i_mj_m})$, and hence, M also satisfies (i)-(iii) of Theorem 3.7. Since $\mathcal{A}$ is equivalent to a superpattern of $\operatorname{sgn}(M)$, the result follows. □

Additional examples where Theorem 3.7 is applicable are given in Appendix B and it is straight-forward to verify that each of $H_1, \ldots, H_{17}$ (as defined in Figs. 7, 8 and 9) satisfy (i)-(iii) of Theorem 3.7 (each has 1 as an eigenvalue with multiplicity equal to 2 and replacing entries indicated by a box by variables determines a Jacobian matrix with rank 2). Furthermore, $c(H_i) = 3$ for $1 \leq i \leq 10$ and $c(H_i) = 4$ for $11 \leq i \leq 17$. Combined with Lemma 3.9, this gives the following lemma.

LEMMA 3.10. *Let $\mathcal{A}$ be an $n \times n$ sign pattern and suppose $D(\mathcal{A})$ has a subdigraph equivalent to $D(\mathcal{H}_i)$ (defined in Appendix B) for some $1 \leq i \leq 17$. Then, the following statements hold.*

(i) If $n = 4$, then $q_3(\mathcal{A}) = 1$.
(ii) If $n \geq 5$, then $q_k(\mathcal{A}) = 1$ for some value of k satisfying $c(\mathcal{H}_i) \leq k \leq c(\mathcal{A})$. In particular, if $1 \leq i \leq 10$ and $c(\mathcal{A}) = 3$ then $q_3(\mathcal{A}) = 1$, and if $11 \leq i \leq 17$ and $c(\mathcal{A}) = 4$ then $q_4(\mathcal{A}) = 1$.

Proof. If $1 \leq i \leq 10$ (resp. $11 \leq i \leq 17$), by Theorem 3.7 and Lemma 3.9, there exists $A \in Q(\mathcal{A})$ having 1 as an eigenvalue with multiplicity equal to two and whose eigenvalues are sufficiently close to those of $H_i \oplus O$ where O is the $(n-3) \times (n-3)$ (resp. $(n-4) \times (n-4)$) zero matrix; let the spectrum of H_i be denoted as in Figs. 7, 8 and 9.

For $n = 4$, this implies the eigenvalues of A are sufficiently close to $\{1, 1, \lambda_i, 0\}$ with $\lambda_i \notin \{0, 1\}$ (resp. $\{1, 1, \lambda_i, \mu_i\}$ with $\lambda_i, \mu_i \notin \{0, 1\}$, $\lambda_i \neq \mu_i$), thus, $q_3(\mathcal{A}) = 1$. Otherwise $n \geq 5$ and since $\lambda_i \neq 0$ (resp. $\lambda_i, \mu_i \neq 0$) we may assume A has at least two (resp. three) distinct nonzero eigenvalues. But A also has 0 as an eigenvalue with multiplicity at least $n - c(\mathcal{A})$. Thus, if $c(\mathcal{A}) \leq n - 1$, then $\mathcal{A}$ has at least 3 (resp. 4) distinct eigenvalues and at most $c(\mathcal{A})$ distinct eigenvalues (and if $c(\mathcal{A}) = n$ then $q_n(\mathcal{A}) = 1$). □

EXAMPLE 3.11. Let

$$\mathcal{A} = \begin{bmatrix} 0 & + & + & + \\ - & 0 & + & 0 \\ + & 0 & 0 & 0 \\ 0 & 0 & - & 0 \end{bmatrix}.$$

Observe that $D(\mathcal{A})$ has a subdigraph equivalent to $D(\mathcal{H}_6)$. Thus, $q_3(\mathcal{A}) = 1$ by Lemma 3.10. Furthermore, $q_1(\mathcal{A}) = 1$ since the matrix $A \in Q(\mathcal{A})$ with every nonzero entry having magnitude 1 is nilpotent and $q_2(\mathcal{A}) = 0$ by Corollary 2.9 since $c(\mathcal{A}) = 3$. Thus, $q_{\text{seq}}(\mathcal{A}) = \langle 1, 0, 1, 1 \rangle$.

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We next describe a family of sparse sign patterns whose digraphs consist of cycles with alternating signs that may be used with Lemma 3.9. Under certain assumptions, this class of sign patterns satisfies the conditions in Theorem 3.7 but to demonstrate this fact, we employ the following combinatorial statement (which we prove in Appendix C).

LEMMA 3.12. *Let $n \geq m \geq 2$ and $1 \leq \ell_1 < \ell_2 < \cdots < \ell_m \leq n$ be integers. Then for $0 \leq i \leq m-1$,*

$$(n)_i = \sum_{r=1}^{m} (n-\ell_r)_i \prod_{j \neq r} \frac{\ell_j}{\ell_j - \ell_r}, \quad \textit{but} \quad (n)_m \neq \sum_{r=1}^{m} (n-\ell_r)_m \prod_{j \neq r} \frac{\ell_j}{\ell_j - \ell_r},$$

where $(n)_i = \frac{n!}{(n-i)!}$ is a falling factorial.

A digraph D is *intercyclic* if D does not contain two vertex-disjoint cycles.

THEOREM 3.13. *Let $m \geq 2$ and $1 \leq \ell_1 < \ell_2 < \cdots < \ell_m \leq n$. Suppose $\mathcal{A} = [\alpha_{ij}]$ is an $n \times n$ sign pattern whose digraph is intercyclic and is composed of exactly m cycles $C_1, C_2, \ldots, C_m$ and such that for $1 \leq r \leq m$, cycle C_r has sign $(-1)^{r+1}$, length ℓ_r and an arc (i_r, j_r) that does not belong to any other cycle. Then every superpattern of $\mathcal{A}$ has a realization having 1 as an eigenvalue with multiplicity equal to m.*

Proof. We show there exists $A \in Q(\mathcal{A})$ satisfying conditions (i)-(iii) in Theorem 3.7. For $1 \leq r \leq m$, let

$$a_r = (-1)^{r+1} \prod_{j \neq r} \frac{\ell_j}{\ell_j - \ell_r},$$

and observe that $a_r > 0$. Form the matrix $A \in Q(\mathcal{A})$ so that for $1 \leq r \leq m$, the (i_r, j_r) entry of A has magnitude a_r, and every other nonzero entry has magnitude 1. By (1.1),

$$p_A(z) = z^n + \sum_{r=1}^{m} (-1)^r a_r z^{n-\ell_r}.$$

Next construct X by replacing the m entries $a_{i_1 j_1}, \ldots, a_{i_m j_m}$ by real variables $x_1, \ldots, x_m$. Note that $a_{i_r j_r} = \operatorname{sgn}(\alpha_{i_r j_r}) a_r$. Now

$$p_X(z) = z^n + \sum_{r=1}^{m} (-1)^r \operatorname{sgn}(\alpha_{i_r j_r}) x_r z^{n-\ell_r},$$

and for $0 \leq i \leq m$,

$$p_X^{(i)}(1) = (n)_i + \sum_{r=1}^{m} (-1)^r \operatorname{sgn}(\alpha_{i_r j_r}) x_r (n-\ell_r)_i.$$

By Lemma 3.12, it follows that $p_A^{(i)}(1) = 0$ for $0 \leq i \leq m-1$ and $p_A^{(m)}(1) \neq 0$. Thus, 1 is an eigenvalue of A with multiplicity equal to m. Let J be the Jacobian matrix defined in Theorem 3.7 and note that J is independent of the x_r's. For $0 \leq i \leq m-1$ and $1 \leq r \leq m$, the $(i+1, r)$ entry of J is $(-1)^r \operatorname{sgn}(\alpha_{i_r j_r})(n - \ell_r)_{i-1}$. Define the matrix K with $(i+1, r)$ entry equal to $(L_r)_{i-1}$ where $L_r = n - \ell_r$, and observe that $|\det(J)| = |\det(K)|$. Note that the $(i+1, r)$ entry of K is a monic polynomial in L_r with degree i, thus, we may apply suitable row operations in the obvious manner to transform K into a Vandermonde matrix V whose $(i+1, r)$ entry is equal to L_r^i (e.g., start by adding the second row to the third row). Since $\ell_i \neq \ell_r$ for $i \neq r$, it follows that $L_i \neq L_r$ for $i \neq r$ and hence, $|\det(V)| \neq 0$ implying that $|\det(J)| \neq 0$. Thus, $A \in Q(\mathcal{A})$ satisfies conditions (i)-(iii) in Theorem 3.7, and hence, every superpattern $\hat{\mathcal{A}}$ of $\mathcal{A}$ has a realization having 1 as an eigenvalue with multiplicity equal to m. ∎

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The matrix realization in the proof of Theorem 3.13 could have eigenvalues other than 1 that are non-simple. For example, the digraph $D(\mathcal{B})$ where $\mathcal{B}$ is defined in Example 2.14 belongs to the family of patterns in Theorem 3.13 and any realization having 1 as an eigenvalue with multiplicity 2 will also have -1 as an eigenvalue with multiplicity 2 since $\mathcal{B}$ is a bipartite sign pattern.

Theorem 3.13 and Lemma 3.9 can be used to show that if $\mathcal{A}$ is defined as in Example 2.4, then $q_4(\mathcal{A}) = 1$ when $D(\mathcal{A})$ has a negative 4-cycle (if both 4-cycles are positive then an analysis of the characteristic polynomial shows that $q_4(\mathcal{A}) = 0$).

3.2. The effect of vertex duplication on the allow sequence. Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ sign pattern and $v_k \in V(D(\mathcal{A}))$ for some $1 \leq k \leq n$. Define the $(n+1) \times (n+1)$ sign pattern $\mathcal{B} = [\beta_{ij}]$ whose entries are $\beta_{n+1,i} = \alpha_{k,i}$, $\beta_{i,n+1} = \alpha_{i,k}$ for $1 \leq i \leq n$, $\beta_{n+1,n+1} = \alpha_{k,k}$ and $\beta_{ij} = \alpha_{ij}$ otherwise. In this case, we say that the digraph $D(\mathcal{B})$ is obtained from $D(\mathcal{A})$ by *duplicating* v_k.

Theorem 3.14. *Let $\mathcal{A}$ be an $n \times n$ sign pattern and suppose $v \in V(D(\mathcal{A}))$ is not incident to a loop. Let D' be obtained from $D(\mathcal{A})$ by duplicating v and let $\mathcal{B}$ be the sign pattern whose digraph is D'.*

(i) If $c(\mathcal{A}) \leq n-1$, then $q_k(\mathcal{B}) \geq q_k(\mathcal{A})$ for $k = 1, 2, \ldots, n$.
(ii) If $\mathcal{A}$ is sign nonsingular, then $q_{k+1}(\mathcal{B}) \geq q_k(\mathcal{A})$ for $k = 1, 2, \ldots, n$.

Proof. Let $A \in Q(\mathcal{A})$ with $q(A) = k$ and suppose the last row and column of A corresponds to v. Then, $A = \begin{bmatrix} \tilde{A} & 2\mathbf{x} \\ \mathbf{y}^T & 0 \end{bmatrix}$ for some $\tilde{A} \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{(n-1)\times 1}$. For (i) (resp. (ii)), we show there exists $B \in Q(\mathcal{B})$ with $q(B) = k$ (resp. $q(B) = k+1$). Consider $B = \begin{bmatrix} \tilde{A} & \mathbf{x} & \mathbf{x} \\ \mathbf{y}^T & 0 & 0 \\ \mathbf{y}^T & 0 & 0 \end{bmatrix}$ and observe that $B \in Q(\mathcal{B})$ since $D(\mathcal{B})$ is obtained from $D(\mathcal{A})$ by duplicating v. It is straight-forward to verify that $p_B(z) = zp_A(z)$ holds by (1.1); hence, p_A and p_B have the same number of distinct *nonzero* roots. For (i), since $c(\mathcal{A}) \leq n-1$, zero is a root of $p_A(z)$, thus p_A and p_B have the same number of distinct roots. Hence, $q(B) = q(A) = k$. For (ii), since $\mathcal{A}$ is sign nonsingular, zero is not a root of $p_A(z)$ for every $A \in Q(\mathcal{A})$, thus $q(B) = q(A) + 1 = k + 1$. □

In Theorem 3.14, if v has $\operatorname{outdeg}(v) \leq 1$ or $\operatorname{indeg}(v) \leq 1$, then the allow sequence for $\mathcal{B}$ is completely determined by $q_{\mathrm{seq}}(\mathcal{A})$ in both statements (i) and (ii).

Theorem 3.15. *Let $\mathcal{A}$ be an $n \times n$ sign pattern with $q_{\mathrm{seq}}(\mathcal{A}) = \langle s_1, s_2, \ldots, s_n \rangle$. Suppose $v \in V(D(\mathcal{A}))$ is not incident to a loop and* $\operatorname{outdeg}(v) \leq 1$ *or* $\operatorname{indeg}(v) \leq 1$. *Let D' be obtained from $D(\mathcal{A})$ by duplicating v and $\mathcal{B}$ be the sign pattern whose digraph is D'.*

(i) If $c(\mathcal{A}) \leq n-1$, then $q_{\mathrm{seq}}(\mathcal{B}) = \langle s_1, s_2, \ldots, s_n, 0 \rangle$.
(ii) If $\mathcal{A}$ is sign nonsingular, then $q_{\mathrm{seq}}(\mathcal{B}) = \langle 0, s_1, s_2, \ldots, s_n \rangle$.

Proof. Let $B \in Q(\mathcal{B})$ with $q(B) = k$ and suppose the last two rows and columns of B correspond to v and v', respectively, where v' is the duplication of v. For (i) (resp. (ii)), we show there exists $A \in Q(\mathcal{A})$ with $q(A) = k$ (resp. $q(A) = k-1$). We may assume $\operatorname{outdeg}(v) \leq 1$ (otherwise, consider the transpose of $\mathcal{A}$), and thus, the last two rows of B have at least $n-1$ zeros. By applying a diagonal similarity if necessary, we may assume the last two rows of B are equal. Thus, without loss of generality, $B = \begin{bmatrix} \tilde{A} & \mathbf{x}_1 & \mathbf{x}_2 \\ \mathbf{y}^T & 0 & 0 \\ \mathbf{y}^T & 0 & 0 \end{bmatrix}$ for some $\tilde{A} \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y} \in \mathbb{R}^{(n-1)\times 1}$. Consider $A = \begin{bmatrix} \tilde{A} & \mathbf{x}_1 + \mathbf{x}_2 \\ \mathbf{y}^T & 0 \end{bmatrix}$ and observe that $A \in Q(\mathcal{A})$

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since $D(\mathcal{B})$ is obtained from $D(\mathcal{A})$ by duplicating v. By (1.1), it follows that $p_B(z) = zp_A(z)$. For (i), zero is a root of $p_A(z)$ since $c(\mathcal{A}) \le n-1$, and hence, p_A and p_B have the same number of distinct roots. Thus, $q(A) = q(B) = k$. For (ii), since $\mathcal{A}$ is sign nonsingular, zero is not a root of $p_A(z)$ for every $A \in Q(\mathcal{A})$, thus $q(A) = q(B) - 1 = k - 1$ (note that this implies $k \ge 2$, and hence, $q_1(\mathcal{B}) = 0$). Finally, for (i), first observe that $c(\mathcal{B}) = c(\mathcal{A})$ due to the degree condition on vertex v. Thus, $c(\mathcal{B}) \le n-1$ and so $q_{n+1}(\mathcal{B}) = 0$ by Lemma 2.1, noting that $\mathcal{B}$ is an $(n+1)\times(n+1)$ sign pattern. The result now follows by Theorem 3.14. □

4. Realizable and nonrealizable order n sequences. We turn our attention to determining which binary sequences of length n can be the allow sequence for an $n \times n$ sign pattern. We begin with some observations on sign patterns having allow sequence $\langle 1, 1, \dots, 1\rangle$ or $\langle 0, \dots, 0, 1\rangle$. We then look at two examples demonstrating some realizable cyclic sequences, for example, $\langle 0, 1, 0, 1, \dots, 0, 1\rangle$ and $\langle 1, 0, 1, 0, \dots, 1, 0, 1\rangle$. We end this section with an analysis of sequences that terminate in a string of zeros, i.e., $\langle s_1, \dots, s_r, 0, \dots, 0\rangle$ where $s_1, \dots, s_r \in \{0, 1\}$.

4.1. Sign patterns $\mathcal{A}$ with $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, \dots, 1\rangle$. When Corollary 3.3 is applied to an $n \times n$ nilpotent matrix that allows a Jacobian with rank n, the corresponding sign pattern $\mathcal{A}$ is spectrally arbitrary (see the nilpotent-Jacobian method [7]) and every superpattern $\hat{\mathcal{A}}$ of $\mathcal{A}$ (including $\mathcal{A}$ itself) has allow sequence $q_{\text{seq}}(\hat{\mathcal{A}}) = \langle 1, 1, \dots, 1\rangle$. One can also construct non-spectrally-arbitrary sign patterns that have the allow sequence $\langle 1, 1, \dots, 1\rangle$. For example, the companion pattern $\mathcal{C}$ in Example 3.4 can be used to demonstrate how Corollary 3.3 may be applied to an $n \times n$ stable matrix A that allows a Jacobian with rank n and satisfies $q(A) = 1$. In particular, consider superpatterns of the companion pattern $\mathcal{C}$ with exactly one nonzero diagonal entry. Such patterns require a nonzero eigenvalue and hence can not be spectrally arbitrary.

In general, $q(\mathcal{A}) = 1$ does not imply that $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, \dots, 1\rangle$, for example, see the sign pattern $\mathcal{B}$ defined in Example 2.7. However, when $\mathcal{A}$ is a full sign pattern (that is, $\mathcal{A}$ has no zero entries), then $q(\mathcal{A}) = 1$ is sufficient to guarantee that $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, \dots, 1\rangle$.

Theorem 4.1. *If $\mathcal{A}$ is a full sign pattern with $q(\mathcal{A}) = 1$, then $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, \dots, 1\rangle$.*

Proof. The proof is analogous to that of [15, Theorem 1.2] and [5, Lemma 2.10]: For an $n \times n$ full sign pattern $\mathcal{A}$, suppose $A \in Q(\mathcal{A})$ satisfies $q(A) = 1$. Let $J = S^{-1}AS$ be its Jordan canonical form. Since $q(A) = 1$, the matrix J is triangular with equal diagonal entries. Now small perturbations of J will be able to obtain full matrices with k distinct eigenvalues. For example, assume $\epsilon > 0$ and fix k with $2 \le k \le n$. Define the $n \times n$ diagonal matrix $D_k = [d_{ij}]$ with entries $d_{jj} = j\epsilon$ if $1 \le j \le k-1$ and $d_{ij} = 0$ otherwise. Note that $q(J + D_k) = k$. For $\epsilon > 0$ sufficiently small, $B = S(J + D_k)S^{-1} \in Q(\mathcal{A})$ and $q(B) = k$, thus, $q_k(\mathcal{A}) = 1$. □

Corollary 2.8 shows that if $\mathcal{A}$ is a sign pattern with $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, \dots, 1\rangle$, then $\ell(\mathcal{A}) \le 2$. Both $\ell(\mathcal{A}) = 1$ and $\ell(\mathcal{A}) = 2$ are possible: any spectrally arbitrary sign pattern $\mathcal{A}$ has $\ell(\mathcal{A}) = 1$ and $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, \dots, 1\rangle$, whereas the sign pattern $\mathcal{A}$ in Example 2.15 has $\ell(\mathcal{A}) = 2$ and $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, \dots, 1\rangle$.

In Section 5, the 2×2 (resp. 3×3) irreducible sign patterns with $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1\rangle$ (resp. $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, 1\rangle$) are characterized (see Theorems 5.1 and 5.2, respectively). The digraphs of each of these sign patterns have a negative 2-cycle; however, this is not necessarily the case when $n \ge 4$ as illustrated in the following example.

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EXAMPLE 4.2. Consider the 4×4 sign pattern and realization

$$\mathcal{A} = \begin{bmatrix} + & + & 0 & 0 \\ 0 & + & + & 0 \\ - & 0 & + & + \\ 0 & + & 0 & + \end{bmatrix}, \qquad A = \begin{bmatrix} \boxed{1} & \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ -1 & 0 & \boxed{a} & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

with $a > 0$ and observe that $D(\mathcal{A})$ has no 2-cycles. We show that $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, 1, 1 \rangle$. Observe that $p_A(z) = (z-1)^3(z-a)$. When $a = 2$, it is straight-forward to verify that A allows a Jacobian with rank 4 (putting variables in the positions indicated by boxes), hence, by Theorem 3.2, $q_k(\mathcal{A}) = 1$ for $2 \leq k \leq 4$. When $a = 1$, $q(A) = 1$ (however, A does not allow a Jacobian with rank 4 in this case). Thus, $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, 1, 1 \rangle$. We remark that by Theorem 3.2, superpatterns $\hat{\mathcal{A}}$ of $\mathcal{A}$ must have either $q_{\text{seq}}(\hat{\mathcal{A}}) = \langle 0, 1, 1, 1 \rangle$ or $q_{\text{seq}}(\hat{\mathcal{A}}) = \langle 1, 1, 1, 1 \rangle$. Whether $q_1(\hat{\mathcal{A}}) = 0$ or $q_1(\hat{\mathcal{A}}) = 1$ depends on the particular superpattern. For example, if $D(\hat{\mathcal{A}})$ has a 2-cycle but has no negative 2-cycle, then $q_1(\hat{\mathcal{A}}) = 0$ by [3, Theorem 3.3], and if $\hat{\mathcal{A}} = \mathcal{A}$ then $q_1(\hat{\mathcal{A}}) = 1$.

4.2. Sign patterns $\mathcal{A}$ that require all distinct eigenvalues, i.e., $q_{\text{seq}}(\mathcal{A}) = \langle 0, \ldots, 0, 1 \rangle$. Li and Harris [14] defined $\mathcal{DE}$ to be the set of all sign patterns that require the property of all distinct eigenvalues and characterized the 2×2 and 3×3 irreducible sign patterns $\mathcal{A} \in \mathcal{DE}$ (summarized in Theorem 5.1(ii) and Theorem 5.2(iv) respectively). They also proved for an $n \times n$ sign pattern $\mathcal{A}$, if $D(\mathcal{A})$ has at least $n-1$ loops and a (simple) cycle of length $2r+1$ for some $r \geq 1$, then $\mathcal{A} \notin \mathcal{DE}$ (see [14, Theorem 3.3]). The 4×4 irreducible patterns in $\mathcal{DE}$ are addressed in [13, 12]. The techniques we present are useful for the problem of characterizing the $n \times n$ sign patterns $\mathcal{A}$ that require all distinct eigenvalues since $\mathcal{A} \in \mathcal{DE}$ if and only if $q(\mathcal{A}) = n$ if and only if $q_{\text{seq}}(\mathcal{A}) = \langle 0, \ldots, 0, 1 \rangle$. For example, a stronger version of [14, Theorem 3.3] holds.

REMARK 4.3. Let $\mathcal{A}$ be an $n \times n$ sign pattern whose digraph has a (simple) cycle of length $r \geq 3$ and at least three loops incident to the r-cycle. Then $\mathcal{A} \notin \mathcal{DE}$ (i.e., $q(\mathcal{A}) \leq n-1$).

Proof. Up to equivalence, we may assume $D(\mathcal{A})$ has two positive loops that are incident to an r-cycle C_r with $r \geq 3$ (otherwise consider $-\mathcal{A}$). If C_r is positive, the result follows by Example 3.8 and Lemma 3.9, otherwise the result follows by Theorem 3.13 and Lemma 3.9. ∎

In Remark 4.3, having an r-cycle ($r \geq 3$) incident to two loops is not sufficient to guarantee $\mathcal{A} \notin \mathcal{DE}$ since if $\mathcal{A}$ is a 4×4 sign pattern whose digraph is a positive 4-cycle having exactly one positive loop and one negative loop, then it is straight-forward to verify that $\mathcal{A} \in \mathcal{DE}$ by [14, Theorem 2.5] or [12, Theorem 4.12] (the resultant of $p_A(z)$ and $p'_A(z)$ is negative for every $A \in Q(\mathcal{A})$). Results analogous to Remark 4.3 can be derived by considering the examples in this paper where Theorem 3.7 is applied to show the existence of a repeated eigenvalue for every superpattern.

REMARK 4.4. Any pattern $\mathcal{A}$ with a subdigraph of $D(\mathcal{A})$ that satisfies the conditions of Theorem 3.13 will have $\mathcal{A} \notin \mathcal{DE}$ by Lemma 3.9. For example, if $\mathcal{A}$ is an $n \times n$ sign pattern and if $D(\mathcal{A})$ (or $D(-\mathcal{A})$) has a positive ℓ_1-cycle and a negative ℓ_2-cycle that share exactly one vertex for some $\ell_2 > \ell_1 \geq 1$, then $\mathcal{A} \notin \mathcal{DE}$.

For $\mathcal{A} \in \mathcal{DE}$, Appendix B gives a list of some forbidden (signed) subdigraphs for $D(\mathcal{A})$.

4.3. Cyclic sequences. For a positive integer m, let $s = \langle 0, \ldots, 0, 1 \rangle$ have length $m \geq 2$. In the following examples, we show that $\langle s, s, \ldots, s \rangle$, $\langle 0, s, s, \ldots, s \rangle$, and $\langle 1, s, s, \ldots, s \rangle$ are realizable.

EXAMPLE 4.5. Let $n \geq m \geq 2$ be integers and suppose that $n = mt$ for a positive integer t. Consider the $n \times n$ proper Hessenberg sign pattern $\mathcal{A} = [\alpha_{ij}]$ with a positive superdiagonal and some negative entries

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in the last row, namely, $\alpha_{i,i+1} = +$ for $1 \le i \le n-1$, $\alpha_{n,n-mj+1} = -$ for $1 \le j \le t$, and $\alpha_{ij} = 0$ otherwise. For example, if $m = 3$ the last row of the pattern is $[-, 0, 0, -, 0, 0, \dots, -, 0, 0]$. Let $A = [a_{ij}] \in Q(\mathcal{A})$. By applying a diagonal similarity, if necessary, we may assume without loss of generality that $a_{i,i+1} = 1$ for $1 \le i \le n-1$ and $a_{n,n-mj+1} = -b_j$, $1 \le j \le t$, for some $b_1, b_2, \dots, b_t > 0$. Since A is a companion matrix, its characteristic polynomial is

$$p_A(z) = z^n + \sum_{j=1}^{t} b_j z^{n-mj} = P(z^m)$$

where $P(z) = z^t + \sum_{j=1}^{t} b_j z^{t-j}$. Since $\mathcal{A}$ is sign nonsingular, by Theorem 2.12(iii), it follows that $q_k(\mathcal{A}) = 0$ for $k \not\equiv 0 \pmod{m}$. Now let $k = \hat{k}m$ for some positive integer $1 \le \hat{k} \le t$. Let $r_1, r_2, \dots, r_t > 0$ so that there are exactly $\hat{k}$ distinct numbers among $r_1, r_2, \dots, r_t$. Then, the number of distinct roots of the polynomial $p(z) = (z^m + r_1)(z^m + r_2)\cdots(z^m + r_t)$ is equal to k. But $p(z) = z^n + \sum_{j=1}^{t} c_j z^{n-mj}$ for some $c_1, c_2, \dots, c_t > 0$, thus, taking $b_j = c_j > 0$ for $1 \le j \le t$ gives $A \in Q(\mathcal{A})$ with $q(A) = k$. Therefore, $q_{\text{seq}}(\mathcal{A}) = \langle s, s, \dots, s\rangle$ where $s = \langle 0, \dots, 0, 1\rangle$ has length m.

Example 4.6. If $\mathcal{A}$ is defined as in Example 4.5 and $\mathcal{B}$ is the sign pattern whose digraph is D', where D' is obtained from $D(\mathcal{A})$ by duplicating v_n, then Theorem 3.15(ii) gives $q_{\text{seq}}(\mathcal{A}) = \langle 0, s, s, \dots, s\rangle$ where $s = \langle 0, \dots, 0, 1\rangle$ has length m.

Example 4.7. Let $n \ge m \ge 2$ be integers and suppose that $n = mt + 1$ for a positive integer t. Consider the $n \times n$ proper Hessenberg sign pattern $\mathcal{A} = [\alpha_{ij}]$ with $\alpha_{i,i+1} = +$ for $1 \le i \le n-1$, $\alpha_{n,n-mj+1} = -$ for $1 \le j \le t$, $\alpha_{m,1} = +$ and $\alpha_{ij} = 0$ otherwise. For example, if $m = 2$ the pattern is

$$\mathcal{A} = \begin{bmatrix} 0 & + & 0 & \cdots & \cdots & \cdots & 0 \\ + & 0 & + & \ddots & & & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & \ddots & + \\ 0 & - & 0 & - & \cdots & - & 0 \end{bmatrix}.$$

Let $A = [a_{ij}] \in Q(\mathcal{A})$. By applying a diagonal similarity, if necessary, we may assume without loss of generality that $a_{i,i+1} = 1$ for $1 \le i \le n-1$, $a_{n,n-mj+1} = -b_j$ for $1 \le j \le t$ for some $b_1, b_2, \dots, b_t > 0$ and $a_{m,1} = d$ for some $d > 0$. Defining $b_0 = 1$, it is straight-forward to verify that the characteristic polynomial is

$$p_A(z) = z\left(z^n + \sum_{j=1}^{t} (b_j - db_{j-1}) z^{n-mj}\right) = zP(z^m),$$

where $P(z) = z^t + \sum_{j=1}^{t} (b_j - db_{j-1}) z^{t-j}$. Since $c(\mathcal{A}) \le n-1$, by Theorem 2.12(i), it follows that $q_k(\mathcal{A}) = 0$ for $k \not\equiv 1 \pmod{m}$. Now let $1 \le \hat{k} \le t$ be a positive integer and $r_1, r_2, \dots, r_t > 0$ so that there are exactly $\hat{k}$ distinct numbers among $r_1, r_2, \dots, r_t$. Then, the number of distinct roots (including the zero root) of the polynomial $p(z) = z(z^m + r_1)(z^m + r_2)\cdots(z^m + r_t)$ is equal to $\hat{k}m + 1$. But $p(z) = z\left(z^n + \sum_{j=1}^{t} f_j z^{n-mj}\right)$ for some $f_1, f_2, \dots, f_t > 0$, thus, taking $d = 1$ and $b_j = 1 + \sum_{i=1}^{j} f_j > 0$ for $1 \le j \le t$ gives $A \in Q(\mathcal{A})$ with $q(A) = \hat{k}m + 1$. Since the realization with $d = b_1 = \cdots = b_t = 1$ gives a nilpotent matrix, it follows that $q_k(\mathcal{A}) = 1$ for $k \equiv 1 \pmod{m}$. Therefore, $q_{\text{seq}}(\mathcal{A}) = \langle 1, s, s, \dots, s\rangle$ where $s = \langle 0, \dots, 0, 1\rangle$ has length m.

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4.4. Allow sequences with trailing zeros. In this section, for each positive integer $r \leq 4$, we characterize possible allow sequences for $n \times n$ irreducible sign patterns that end in a string of $n - r$ zeros whenever $n \geq r + 1$. Lemma 2.1 immediately gives the following remark.

REMARK 4.8. Let $\mathcal{A}$ be an $n \times n$ sign pattern and $1 \leq t \leq n - 1$. Then, $q_{\text{seq}}(\mathcal{A})$ has exactly t trailing zeros if and only if $c(\mathcal{A}) = n - t - 1$.

Since $\langle 0, 0, \ldots, 0\rangle$ is never the allow sequence for a sign pattern, we start with sequences of length n having exactly $n - 1$ or $n - 2$ trailing zeros and do not assume the sign patterns are irreducible in the next result (the irreducible case when $n = 2$ is given in Theorem 5.1 and is omitted here). Case (i) of Theorem 4.9 is also true for $n = 2$.

THEOREM 4.9. *Let $n \geq 3$ and $\mathcal{A}$ be an $n \times n$ sign pattern. Then*

(i) $q_{\text{seq}}(\mathcal{A}) = \langle 1, 0, 0, \ldots, 0\rangle$ *if and only if $D(\mathcal{A})$ is acyclic.*
(ii) $q_{\text{seq}}(\mathcal{A}) = \langle 0, 1, 0, 0, \ldots, 0\rangle$ *if and only if $D(\mathcal{A})$ has precisely one cycle whose length is* 1.
(iii) $q_{\text{seq}}(\mathcal{A}) \neq \langle 1, 1, 0, 0, \ldots, 0\rangle$.

Proof. (i) By Remark 4.8, $q_{\text{seq}}(\mathcal{A})$ has exactly $t = n - 1 \geq 1$ trailing zeros if and only if $c(\mathcal{A}) = 0$. The result now follows since $c(\mathcal{A}) = 0$ if and only if $D(\mathcal{A})$ is acyclic.

(ii) and (iii) By Remark 4.8, $q_{\text{seq}}(\mathcal{A})$ has exactly $t = n - 2 \geq 1$ trailing zeros if and only if $c(\mathcal{A}) = 1$. It is straight-forward to verify that $c(\mathcal{A}) = 1$ if and only if $D(\mathcal{A})$ has precisely one cycle whose length is 1. But if $D(\mathcal{A})$ has precisely one cycle whose length is 1, then every $A \in Q(\mathcal{A})$ requires a nonzero eigenvalue with multiplicity 1 and a zero eigenvalue with multiplicity $n - 1$, hence, $q_{\text{seq}}(\mathcal{A}) = \langle 0, 1, 0, 0, \ldots, 0\rangle$. □

The sign patterns with digraphs described in Theorem 4.9(i) and (ii) are reducible; thus, the following corollary holds for irreducible sign patterns.

COROLLARY 4.10. *Given $n \geq 3$, sequences $\langle s_1, s_2, 0, \ldots, 0\rangle$ with $s_1, s_2 \in \{0, 1\}$ are not realizable by any $n \times n$ irreducible sign pattern.*

We next consider sequences of length n with exactly $n - 3$ trailing zeros. There are four possible sequences: $\langle s_1, s_2, 1, 0, \ldots, 0\rangle$ with $s_1, s_2 \in \{0, 1\}$. It can be shown that each of these sequences is realizable by some sign pattern. In particular, the sequence $\langle 1, 1, 1, 0, 0, \ldots, 0\rangle$ is realizable by $\mathcal{B} \oplus \mathcal{O}$ where $\mathcal{B}$ is any 2×2 spectrally arbitrary sign pattern and $\mathcal{O}$ is the $(n-2) \times (n-2)$ zero pattern; examples for the remaining three sequences are given in Theorem 4.11. When $\mathcal{A}$ is an $n \times n$ *irreducible* sign pattern with $n \geq 4$, the sequence $\langle 1, 1, 1, 0, 0, \ldots 0\rangle$ is the only sequence of these four that is not realizable (the case when $n = 3$ is given in Theorem 5.2 and is omitted here). A *star digraph* on n vertices is a strongly connected digraph whose underlying graph is a star. Example 2.7 describes allow sequences for some loopless star digraphs. The next theorem describes allow sequences for some further star digraphs.

THEOREM 4.11. *Let $n \geq 4$ and $s = \langle s_1, s_2, 1, 0, \ldots, 0\rangle$ have length n with $s_1, s_2 \in \{0, 1\}$. Then, $\langle 1, 1, 1, 0, 0, \ldots, 0\rangle$ is the only sequence s not realizable by an $n \times n$ irreducible sign pattern. Furthermore, if $\mathcal{A}$ be an $n \times n$ irreducible sign pattern, then*

(i) $q_{\text{seq}}(\mathcal{A}) = \langle 1, 0, 1, 0, 0, \ldots, 0\rangle$ *if and only if $D(\mathcal{A})$ is a star digraph with no loops and both positive and negative 2-cycles.*
(ii) $q_{\text{seq}}(\mathcal{A}) = \langle 0, 1, 1, 0, 0, \ldots, 0\rangle$ *if and only if $D(\mathcal{A})$ is a star digraph with at least one negative 2-cycle and exactly one loop that is located on the central vertex.*

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(iii) $q_{\text{seq}}(\mathcal{A}) = \langle 0, 0, 1, 0, 0, \ldots, 0\rangle$ *if and only if* $D(\mathcal{A})$ *is a star digraph with either*
   (a) no loops and all 2*-cycles of the same sign, or*
   (b) no negative 2*-cycles and exactly one loop that is located on the central vertex.*

Proof. Suppose $q_{\text{seq}}(\mathcal{A}) = \langle s_1, s_2, 1, 0, \ldots, 0\rangle$ for some $s_1, s_2 \in \{0, 1\}$. Then, $c(\mathcal{A}) = 2$ by Remark 4.8. Since $\mathcal{A}$ is irreducible, $D(\mathcal{A})$ is a star digraph with either no loops or exactly one loop on the central vertex. To complete the proof, it suffices to analyze all star digraphs of this form and in each case determine their allow sequence.

If $D(\mathcal{A})$ has no loops, then $q_2(\mathcal{A}) = 0$ by Corollary 2.9 and $q_1(\mathcal{A}) = 1$ precisely when $\mathcal{A}$ is potentially nilpotent. Thus, either $D(\mathcal{A})$ has no loops and both positive and negative 2-cycles in which case $q_{\text{seq}}(\mathcal{A}) = \langle 1, 0, 1, 0, ..., 0\rangle$ or $D(\mathcal{A})$ has no loops and all 2-cycles of the same sign in which case $q_{\text{seq}}(\mathcal{A}) = \langle 0, 0, 1, 0, ..., 0\rangle$.

If $D(\mathcal{A})$ has exactly one loop on the central vertex, then $q_1(\mathcal{A}) = 0$ since every $A \in Q(\mathcal{A})$ has $\det(A) = 0$ and $\operatorname{tr}(A) \neq 0$. We show that $q_2(\mathcal{A}) = 1$ precisely when $D(\mathcal{A})$ has a negative 2-cycle. Let $A \in Q(\mathcal{A})$. Then, $p_A(z) = z^{n-2}(z^2 + a_1 z + a_2)$ for some $a_1 \neq 0$ and $a_2 \in \mathbb{R}$. Observe that $p_A(z)$ has two distinct roots when $a_2 = a_1^2/4$ and three distinct roots otherwise. If $D(\mathcal{A})$ has only positive 2-cycles, then we require $a_2 < 0$, and thus, $q_{\text{seq}}(\mathcal{A}) = \langle 0, 0, 1, 0, ..., 0\rangle$ in this case. Otherwise, $D(\mathcal{A})$ has a negative 2-cycle, and it is straight-forward to construct a realization A so that $a_2 = a_1^2/4$ holds, and hence, $q_2(\mathcal{A}) = 1$. In this case, $q_{\text{seq}}(\mathcal{A}) = \langle 0, 1, 1, 0, ..., 0\rangle$.

It now also follows that $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, 1, 0, ..., 0\rangle$ is not possible for irreducible patterns. □

A *star sign pattern* $\mathcal{A}$ is a sign pattern such that $D(\mathcal{A})$ is a star digraph. Remark 4.8 and Theorem 4.11 give a characterization of the realizable sequences for $n \times n$ star sign patterns $\mathcal{A}$ with $n \geq 4$ satisfying $c(\mathcal{A}) = 2$. Before we consider sequences of length n with exactly $n - 4$ trailing zeros, we first give a characterization of the realizable sequences for star sign patterns $\mathcal{A}$ satisfying $c(\mathcal{A}) = 3$. Given $n \geq m$, we say that an $n \times n$ pattern $\mathcal{A}$ is a (k, ℓ)*-duplication* of an $m \times m$ pattern $\mathcal{S}$ if $D(\mathcal{A})$ is obtained from $D(\mathcal{S})$ by $n - m$ vertex duplications, with each vertex duplication being a duplication of either vertex v_k or v_ℓ.

THEOREM 4.12. *Let* $n \geq 4$ *and* $\mathcal{A}$ *be an* $n \times n$ *star sign pattern with* $c(\mathcal{A}) = 3$. *Then,* $q_{\text{seq}}(\mathcal{A}) \in \{\langle 1, 1, 1, 1, 0, \ldots, 0\rangle, \langle 0, 1, 1, 1, 0, \ldots, 0\rangle, \langle 0, 0, 1, 1, 0, \ldots, 0\rangle, \langle 0, 0, 0, 1, 0, \ldots, 0\rangle\}$. *Furthermore, taking*

$$\mathcal{S} = \begin{bmatrix} \alpha_0 & + & + & + \\ \alpha_1 & - & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \\ \alpha_3 & 0 & 0 & 0 \end{bmatrix},$$

and $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$*, then*

(i) $q_{\text{seq}}(\mathcal{A}) = \langle 0, 0, 0, 1, 0, \ldots, 0\rangle$ *if and only if* $\mathcal{A}$ *is equivalent to a (3,4)-duplication of* $\mathcal{S}$ *with* $\alpha \in \{(0, +, +, +), (+, +, +, +), (-, +, +, +)\}$.
(ii) $q_{\text{seq}}(\mathcal{A}) = \langle 0, 0, 1, 1, 0, \ldots, 0\rangle$ *if and only if* $\mathcal{A}$ *is equivalent to a (3,4)-duplication of* $\mathcal{S}$ *with* $\alpha \in \{(0, +, -, +), (+, +, -, +), (0, +, -, -), (+, +, -, -), (0, -, +, +), (-, -, +, +)\}$.
(iii) $q_{\text{seq}}(\mathcal{A}) = \langle 0, 1, 1, 1, 0, \ldots, 0\rangle$ *if and only if* $\mathcal{A}$ *is equivalent to a (3,4)-duplication of* $\mathcal{S}$ *with* $\alpha \in \{(-, +, -, +), (-, +, -, -), (+, -, +, +), (0, -, -, -), (+, -, -, -), (-, -, -, -), (0, -, +, -), (-, -, +, -)\}$.
(iv) $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, 1, 1, 0, \ldots, 0\rangle$ *if and only if* $\mathcal{A}$ *is equivalent to a (3,4)-duplication of* $\mathcal{S}$ *with* $\alpha = (+, -, +, -)$.

Proof. In this proof, we reference some of the digraphs $D(\mathcal{H}_i)$ listed in Fig. 11 of Appendix B.

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Suppose $n = 4$. Since $c(\mathcal{A}) = 3$, $D(\mathcal{A})$ has exactly one non-central loop (and possibly a loop on the center vertex). Thus, up to equivalence, we may assume $\mathcal{A}_{12} = \mathcal{A}_{13} = \mathcal{A}_{14} = +$, $\mathcal{A}_{22} = -$ and $\mathcal{A}_{33} = \mathcal{A}_{44} = 0$. Hence $\mathcal{A}$ is equivalent to $\mathcal{S}$ for some $\alpha_i \in \{+,-\}$, $i = 1,2,3$ and $\alpha_0 \in \{0,+,-\}$. There are 18 non-equivalent sign patterns to consider since sign patterns with $(\alpha_2,\alpha_3) = (+,-)$ are equivalent to those with $(\alpha_2,\alpha_3) = (-,+)$, and it can be verified that all 18 are listed in (i) to (iv) in the statement of the theorem. Thus, it suffices to determine the allow sequence of $\mathcal{S}$ for each α listed in the statement of the theorem. Since $c(\mathcal{S}) = 3$, one eigenvalue of each realization of $\mathcal{S}$ is zero and $q_4(\mathcal{S}) = 1$ by Lemma 2.1.

For any $\alpha_i \in \{+,-\}$ for $i = 1,2,3$ and $\alpha_0 \in \{0,+,-\}$, without loss of generality, $S \in Q(\mathcal{S})$ can be scaled to have the form

$$S = \begin{bmatrix} a & 1 & 1 & 1 \\ b_1 & -1 & 0 & 0 \\ b_2 & 0 & 0 & 0 \\ b_3 & 0 & 0 & 0 \end{bmatrix}, \tag{4.7}$$

where $\operatorname{sgn}(a) = \alpha_0$ and $\operatorname{sgn}(b_i) = \alpha_i$, $i = 1,2,3$. Note $p_S(z) = z^4 + (1-a)z^3 - (a+b_1+b_2+b_3)z^2 - (b_2+b_3)z$. Thus, $q_1(\mathcal{S}) = 1$ if and only if there is an $S \in Q(\mathcal{S})$ that is nilpotent if and only if $\alpha = (+,-,+,-)$ or $\alpha = (+,-,-,+)$. Note that these two α sequences give equivalent patterns.

(i) Suppose $\alpha \in \{(0,+,+,+),(+,+,+,+),(-,+,+,+)\}$. Then, $\mathcal{S}$ requires four distinct eigenvalues by [12, Lemma 2.5]. Thus $q_{\text{seq}}(\mathcal{S}) = \langle 0,0,0,1\rangle$.

(ii) Suppose $\alpha \in \{(0,+,-,+),(+,+,-,+),(0,+,-,-),(+,+,-,-),(0,-,+,+),(-,-,+,+)\}$. By the nilpotent analysis above, $q_1(\mathcal{S}) = 0$. Further, $D(\mathcal{S})$ has a subdigraph equivalent to either $D(\mathcal{H}_3)$ when $\alpha_1 = -$, or $D(\mathcal{H}_4)$ when $\alpha_1 = +$. Thus, $q_3(\mathcal{S}) = 1$ by Lemma 3.10.

In the case that $\alpha \in \{(0,-,+,+),(-,-,+,+)\}$, the coefficient of z in $p_S(z)$ is nonzero for every $S \in Q(\mathcal{S})$, and furthermore, $D(\mathcal{S})$ does not have a loop and a negative 2-cycle that are vertex disjoint. Hence, $q_2(\mathcal{S}) = 0$ by Theorem 2.18, and thus, $q_{\text{seq}}(\mathcal{S}) = \langle 0,0,1,1\rangle$.

Otherwise, $\alpha \in \{(0,+,-,-),(0,+,-,+),(+,+,-,+),(+,+,-,-)\}$. We show that $q_2(\mathcal{S}) = 0$. Assume there is an $S \in Q(\mathcal{S})$ of the form in (4.7) with $q(S) = 2$. Note $a \geq 0$ and $b_1 > 0$. If $b_2 + b_3 = 0$, then 0 is an eigenvalue of S with multiplicity two, and since the coefficient of z^2 is negative, S also requires a positive and a negative eigenvalue contradicting that $q(S) = 2$. Thus, $\lambda = (a-1)/3$ is an eigenvalue of S with multiplicity three and $p_S(z) = z(z-\lambda)^3 = z^4 + (1-a)z^3 + \frac{1}{3}(1-a)^2z^2 + \frac{1}{27}(1-a)^3z$. Then, $b_2 + b_3 = -\frac{1}{27}(1-a)^3$ and analyzing the coefficient of z^2 in $p_S(z)$ gives $b_1 = -\frac{1}{27}a^3 - \frac{2}{9}a^2 - \frac{4}{9}a - \frac{8}{27} < 0$, contradicting that $b_1 > 0$. It follows that $q(S) \neq 2$ for every $S \in Q(\mathcal{S})$, and thus, $q_{\text{seq}}(\mathcal{S}) = \langle 0,0,1,1\rangle$.

(iii) Suppose $\alpha \in \{(-,+,-,+),(-,+,-,-),(+,-,+,+),(0,-,-,-),(+,-,-,-),(-,-,-,-),$ $(0,-,+,-),(-,-,+,-)\}$. By the nilpotent analysis above, $q_1(\mathcal{S}) = 0$. If $\alpha_1 = \alpha_2 = \alpha_3 = -$, then $D(\mathcal{S})$ has a subdigraph equivalent to $D(\mathcal{H}_8)$, otherwise, $D(\mathcal{S})$ has a subdigraph equivalent to either $D(\mathcal{H}_3)$ when $\alpha_1 = -$, or $D(\mathcal{H}_4)$ when $\alpha_1 = +$. Thus, $q_3(\mathcal{S}) = 1$ by Lemma 3.10.

When (a,b_1,b_2,b_3) is one of $(-1,1,-1,1)$, $(-5,1,-4,-4)$, $(4,-8,\frac{1}{2},\frac{1}{2})$, $(0,-\frac{8}{27},-\frac{1}{54},-\frac{1}{54})$, $(\frac{2}{3},-\frac{512}{729},-\frac{1}{1458},-\frac{1}{1458})$, $(-1,-\frac{1}{27},-\frac{1}{6},-\frac{7}{54})$, $(0,-\frac{1}{4},\frac{1}{4},-\frac{1}{4})$ or $(-3,-1,1,-1)$, then $q(S) = 2$. Thus, $q_{\text{seq}}(\mathcal{S}) = \langle 0,1,1,1\rangle$.

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(iv) Suppose $\alpha \in \{(+,-,+,-)\}$. By the nilpotent analysis above, $q_1(\mathcal{S}) = 1$. Further, $D(\mathcal{S})$ has a subdigraph equivalent to $D(\mathcal{H}_3)$; thus, $q_3(\mathcal{S}) = 1$ by Lemma 3.10. When $(a, b_1, b_2, b_3) = (2, -2, 1, -1)$, then $q(S) = 2$. Thus, $q_{\text{seq}}(\mathcal{S}) = \langle 1, 1, 1, 1\rangle$.

Observe that for $n \geq 5$, any $n \times n$ star sign pattern $\mathcal{A}$ with $c(\mathcal{A}) = 3$ is equivalent to a $(3, 4)$-duplication of $\mathcal{S}$ for some $\alpha_i \in \{+,-\}$ for $i = 1, 2, 3$ and $\alpha_0 \in \{0,+,-\}$. The result now follows since if $q_{\text{seq}}(\mathcal{S}) = \langle s_1, s_2, s_3, 1\rangle$, then any $(3, 4)$-duplication of $\mathcal{S}$ has allow sequence $\langle s_1, s_2, s_3, 1, 0, 0, \ldots, 0\rangle$ by Theorem 3.15(i). ◻

We next give sufficient conditions that imply an irreducible sign pattern allows three distinct eigenvalues.

THEOREM 4.13. *Let $n \geq 4$ and $\mathcal{A}$ be an $n \times n$ irreducible sign pattern. If $c(\mathcal{A}) = 3$ and $q_2(\mathcal{A}) = 1$, then $q_3(\mathcal{A}) = 1$.*

Proof. Let $\mathcal{A}$ be an irreducible $n \times n$ sign pattern with $c(\mathcal{A}) = 3$ and $q_2(\mathcal{A}) = 1$. By Corollary 2.9, it follows that $\ell(\mathcal{A}) = 1$; thus, $D(\mathcal{A})$ has a loop. If $D(\mathcal{A})$ has no 3-cycle, then $D(\mathcal{A})$ is a star digraph and $q_3(\mathcal{A}) = 1$ by Theorem 4.12. Thus, assume $D(\mathcal{A})$ has a 3-cycle. We show that there exists $1 \leq i \leq 10$ such that $D(\mathcal{A})$ has a subdigraph equivalent to $D(\mathcal{H}_i)$ (as defined in Appendix B), and from this, it then follows that $q_3(\mathcal{A}) = 1$ by Lemma 3.10. Since $c(\mathcal{A}) = 3$, every 3-cycle is incident to every loop (and also incident to every 2-cycle, if any exist). If $D(\mathcal{A})$ has a 3-cycle and a loop that are oppositely signed, then $D(\mathcal{A})$ has a subdigraph equivalent to $D(\mathcal{H}_5)$. Now assume that every loop and every 3-cycle have the same sign, say negative. If $D(\mathcal{A})$ has a positive 2-cycle, then $D(\mathcal{A})$ has a subdigraph equivalent to $D(\mathcal{H}_6)$ or $D(\mathcal{H}_9)$. Thus, assume all 2-cycles (if any) are negative. By (1.1), the coefficient of z^{n-3} in $p_A(z)$ is nonzero for every $A \in Q(\mathcal{A})$ since every 3×3 principal minor is positive. But $q_2(\mathcal{A}) = 1$, thus, there is $A \in Q(\mathcal{A})$ with $q(A) = 2$ and such that the coefficient of z^{n-3} in $p_A(z)$ is nonzero. Since $\mathcal{A}$ is irreducible, Remark 2.19 implies that $D(\mathcal{A})$ has at most two negative loops. Thus, by Theorem 2.18(iii), $D(\mathcal{A})$ has a loop incident to a negative 2-cycle. Hence, $D(\mathcal{A})$ has a subdigraph equivalent to $D(\mathcal{H}_7)$ or $D(\mathcal{H}_{10})$. ◻

We now give the characterization of possible allow sequences of length n with exactly $n-4$ trailing zeros for $n \times n$ irreducible sign patterns with $n \geq 5$ (the case when $n = 4$ is given in Theorem 5.4 and is omitted here).

THEOREM 4.14. *Let $n \geq 5$ and $s = \langle s_1, s_2, s_3, 1, 0, 0, \ldots, 0\rangle$ have length n with $s_1, s_2, s_3 \in \{0, 1\}$. Then, there exists an $n \times n$ irreducible sign pattern $\mathcal{A}$ with $q_{\text{seq}}(\mathcal{A}) = s$ if and only if $s \neq \langle s_1, 1, 0, 1, 0, \ldots, 0\rangle$.*

Proof. By Remark 4.8 and Theorem 4.13, the sequences $\langle s_1, 1, 0, 1, 0, 0, \ldots, 0\rangle$ with $s_1 \in \{0, 1\}$ are unattainable by an $n \times n$ irreducible sign pattern for $n \geq 5$.

If $\mathcal{A}$ is a 4×4 sign pattern whose digraph is two oppositely signed 3-cycles that have one arc in common, then $q_{\text{seq}}(\mathcal{A}) = \langle 1, 0, 0, 1\rangle$ by Corollary 2.6. The sign pattern $\mathcal{A}$ listed in Example 3.11 has $q_{\text{seq}}(\mathcal{A}) = \langle 1, 0, 1, 1\rangle$. Finally, by Theorem 4.12, there is a 4×4 sign pattern $\mathcal{A}$ with allow sequence $\langle 1, 1, 1, 1\rangle$ (resp. $\langle 0, 1, 1, 1\rangle$, $\langle 0, 0, 1, 1\rangle$ and $\langle 0, 0, 0, 1\rangle$). For each of these six 4×4 sign patterns, $c(\mathcal{A}) = 3$. By Theorem 3.15(i), vertex duplications can be applied to these six 4×4 patterns to generate sign patterns of every order $n \geq 5$ with the appropriate sequence. ◻

The conclusion of Theorem 4.14 does not hold when $n = 4$ since $\langle 0, 1, 0, 1\rangle$ is realizable by the irreducible sign pattern $\mathcal{B}$ given in Example 2.14. Note that Theorem 3.15(i) cannot be applied to $\mathcal{B}$ to generate sign patterns that realize $\langle 0, 1, 0, 1, 0, 0, \ldots, 0\rangle$ since $c(\mathcal{B}) = 4$. Instead, since $\mathcal{B}$ is sign nonsingular, we can apply Theorem 3.15(ii) to $\mathcal{B}$ (where v corresponds to the last row and column of $\mathcal{B}$) to give a sign pattern $\mathcal{B}_1$ with $q_{\text{seq}}(\mathcal{B}_1) = \langle 0, 0, 1, 0, 1\rangle$. Since $\mathcal{B}_1$ is a 5×5 sign pattern with $c(\mathcal{B}_1) = 4$, for $n \geq 5$, we can apply

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Theorem 3.15(i) to $\mathcal{B}_1$ repeatedly by duplicating v a total of $n-5$ times to produce an $n \times n$ sign pattern with allow sequence $\langle 0,0,1,0,1,0,0,\dots,0\rangle$.

5. Allow sequences of $n \times n$ irreducible sign patterns with $n \leq 4$.

This section characterizes the sequences that are realizable as an allow sequence for 2×2, 3×3 and 4×4 *irreducible* sign patterns. For 2×2 and 3×3 sign patterns, this is accomplished by using the characterizations according to the value of $q(\mathcal{A})$ in [3]. For 4×4 irreducible sign patterns, techniques and examples introduced earlier in this paper are applied to give the result. We state our characterizations using relaxations of patterns (as introduced in [5]).

The definition of a sign pattern was relaxed in [5] and later used in [3] to help state the characterizations of 3×3 irreducible sign patterns $\mathcal{A}$ according to the value of $q(\mathcal{A})$. Let $\mathbb{S} = \{+,-,0,*,\oplus,\ominus,\circledast\}$, where $+$ (resp. $-$, $\oplus$ and $\ominus$) denotes a positive (resp. negative, nonnegative and nonpositive) real number, and $*$ (resp. $\circledast$) denotes a nonzero (resp. arbitrary) real number. An $\mathbb{S}$-*pattern* is a matrix with entries in $\mathbb{S}$. The definition of $Q(\mathcal{A})$ for sign patterns $\mathcal{A}$ extends to $\mathbb{S}$-patterns in the obvious manner. An $\mathbb{S}$-pattern $\mathcal{B}$ is a *relaxation* of an $\mathbb{S}$-pattern $\mathcal{A}$ if $Q(\mathcal{A}) \subseteq Q(\mathcal{B})$. If $\mathcal{B}$ is an $\mathbb{S}$-pattern, then $\mathcal{A}$ is a *fixed signing* of $\mathcal{B}$ if $\mathcal{A}$ is a sign pattern and $\mathcal{B}$ is a relaxation of $\mathcal{A}$.

THEOREM 5.1. *Let $\mathcal{A}$ be a 2×2 irreducible sign pattern. Then, $q_{\text{seq}}(\mathcal{A}) \in \{\langle 0,1\rangle, \langle 1,1\rangle\}$ and*

(i) $q_{\text{seq}}(\mathcal{A}) = \langle 1,1\rangle$ if and only if $\mathcal{A}$ is equivalent to a superpattern of $\begin{bmatrix} + & + \\ - & 0 \end{bmatrix}$.

(ii) $q_{\text{seq}}(\mathcal{A}) = \langle 0,1\rangle$ if and only if $\mathcal{A}$ is equivalent to $\begin{bmatrix} 0 & + \\ - & 0 \end{bmatrix}$ or a superpattern of $\begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$.

Proof. Since $\mathcal{A}$ is irreducible, $D(\mathcal{A})$ has a 2-cycle, and hence $q_2(\mathcal{A}) = 1$ by Lemma 2.1. Thus $q_{\text{seq}}(\mathcal{A}) \in \{\langle 0,1\rangle, \langle 1,1\rangle\}$. By [3, Theorem 4.1], $q_1(\mathcal{A}) = 1$ if and only if $\mathcal{A}$ is equivalent to a superpattern of $\begin{bmatrix} + & + \\ - & 0 \end{bmatrix}$. Therefore, (i) holds and (ii) is the complementary event. □

For the sake of convenience in stating the next result, we reproduce the catalogue of patterns in [3] but list them in Appendix A according to their allow sequence. Note that the characterization of 3×3 irreducible sign patterns with $q_{\text{seq}}(\mathcal{A}) = \langle 0,0,1\rangle$ was proved in [14, Theorem 4.12].

THEOREM 5.2. *Let $\mathcal{A}$ be a 3×3 irreducible sign pattern. Then $q_{\text{seq}}(\mathcal{A}) \in \{\langle 1,1,1\rangle, \langle 1,0,1\rangle, \langle 0,1,1\rangle, \langle 0,0,1\rangle\}$ and*

(i) $q_{\text{seq}}(\mathcal{A}) = \langle 1,1,1\rangle$ if and only if $\mathcal{A}$ is equivalent to $\mathcal{Z}_2$ or a superpattern of $\mathcal{Y}_i$ for some $1 \leq i \leq 6$.
(ii) $q_{\text{seq}}(\mathcal{A}) = \langle 1,0,1\rangle$ if and only if $\mathcal{A}$ is equivalent to $\mathcal{Z}_1$.
(iii) $q_{\text{seq}}(\mathcal{A}) = \langle 0,1,1\rangle$ if and only if $\mathcal{A}$ is equivalent to a fixed signing of $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{W}$ or $\mathcal{S}_{i,j}$ for some $(i,j) \in \{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2)\}$.
(iv) $q_{\text{seq}}(\mathcal{A}) = \langle 0,0,1\rangle$ if and only if $\mathcal{A}$ is equivalent to a fixed signing of $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ or $\mathcal{F}_4$.

Proof. By [3, Theorem 6.1], it suffices to determine the allow sequence for each pattern in Appendix A. If $\mathcal{A}$ is one of these patterns, then $c(\mathcal{A}) \geq 2$ since $\mathcal{A}$ is irreducible, thus, $q_3(\mathcal{A}) = 1$ by Lemma 2.1. If $\mathcal{A}$ is equivalent to a superpattern of $\mathcal{Y}_i$ in Fig. 4 for some $1 \leq i \leq 6$, then the realizations Y_i in the proof of [3, Lemma 5.4] have $q(Y_i) = 1$ and allow a Jacobian of rank 3; thus by Corollary 3.3, $q_{\text{seq}}(\mathcal{A}) = \langle 1,1,1\rangle$. The star sign pattern $\mathcal{Z}_1$ in Fig. 3 is equivalent to the sign pattern $\mathcal{B}$ (with $n=3$) in Example 2.7; thus, $q_{\text{seq}}(\mathcal{Z}_1) = \langle 1,0,1\rangle$. The digraph $D(\mathcal{Z}_2)$ has a subdigraph of a positive 2-cycle and a negative 3-cycle that

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share a vertex, and hence, by Theorem 3.13, $\mathcal{Z}_2$ in Fig. 3 has a realization A having 1 as an eigenvalue of multiplicity 2 and hence $q(A) = 2$. It follows by [3, Theorem 6.1] that $q_{\text{seq}}(\mathcal{Z}_2) = \langle 1, 1, 1\rangle$. If $\mathcal{A}$ is equivalent to a fixed signing of $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{W}$ or $\mathcal{S}_{i,j}$ in Fig. 5, then $q(\mathcal{A}) = 2$ by [3, Theorem 6.1], and hence, $q_{\text{seq}}(\mathcal{A}) = \langle 0, 1, 1\rangle$. If $\mathcal{A}$ is equivalent to a fixed signing of $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ or $\mathcal{F}_4$ in Fig. 6, then $q(\mathcal{A}) = 3$ by [14, Theorem 4.12], and hence, $q_{\text{seq}}(\mathcal{A}) = \langle 0, 0, 1\rangle$. ☐

For 4×4 irreducible sign patterns, we determine which sequences are realizable by some 4×4 irreducible sign pattern. We begin with a preliminary result on 4×4 bipartite sign patterns, which is used in the proof of Theorem 5.4.

Lemma 5.3. *Suppose $\mathcal{A}$ is a 4×4 bipartite sign pattern such that $c(\mathcal{A}) = 2$ or $c(\mathcal{A}) = 4$ and $\mathcal{A}$ allows singularity. Then, $q_3(\mathcal{A}) = 1$.*

Proof. If $c(\mathcal{A}) = 2$, then $q_3(\mathcal{A}) = 1$ by Lemma 2.1. Otherwise $c(\mathcal{A}) = 4$, and since $\mathcal{A}$ allows singularity, $D(\mathcal{A})$ requires a 2-cycle. Thus, without loss of generality, $\mathcal{A}$ is a fixed signing of

$$\begin{bmatrix} 0 & 0 & * & \circledast \\ 0 & 0 & \circledast & \circledast \\ * & \circledast & 0 & 0 \\ \circledast & \circledast & 0 & 0 \end{bmatrix}.$$

Let $A = [a_{ij}] \in Q(\mathcal{A})$. Then

$$p_A(z) = z^4 - (a_{13}a_{31} + a_{14}a_{41} + a_{23}a_{32} + a_{24}a_{42})z^2 + (a_{31}a_{42} - a_{32}a_{41})(a_{13}a_{24} - a_{14}a_{23})$$

for appropriately signed a_{ij} (possibly zero) with $a_{13}, a_{31} \neq 0$. Furthermore, since $c(\mathcal{A}) = 4$ and $\mathcal{A}$ allows singularity, either

(i) $a_{42} \neq 0$ and both $a_{31}a_{42}$ and $a_{32}a_{41}$ have the same nonzero sign, or
(ii) $a_{24} \neq 0$ and both $a_{13}a_{24}$ and $a_{14}a_{23}$ have the same nonzero sign.

In case (i), consider $A \in Q(\mathcal{A})$ where each of a_{31}, a_{32}, $a_{41,}$ and a_{42} have magnitude 1 and each of a_{14}, $a_{23,}$ and a_{24} are sufficiently small (possibly zero). When a_{13} is sufficiently large, $p_A(z) = z^4 - \alpha z^2$ for some $\alpha \neq 0$, and thus, $q(A) = 3$ implying $q_3(\mathcal{A}) = 1$. In case (ii), consider A^T and apply the argument in (i). ☐

Example 2.14 demonstrates that the singularity hypothesis of Lemma 5.3 is needed.

Theorem 5.4. *Let $s = \langle s_1, s_2, s_3, s_4\rangle$ with $s_i \in \{0, 1\}$. There exists a 4×4 irreducible sign pattern $\mathcal{A}$ with $q_{\text{seq}}(\mathcal{A}) = s$ if and only if $s \notin \{\langle 0, 0, 0, 0\rangle, \langle 1, 0, 0, 0\rangle, \langle 0, 1, 0, 0\rangle, \langle 1, 1, 0, 0\rangle, \langle 1, 1, 1, 0\rangle, \langle 1, 1, 0, 1\rangle\}$.*

Proof. Corollary 4.10 gives the result for sequences $\langle s_1, s_2, 0, 0\rangle$ with $s_1, s_2 \in \{0, 1\}$. Theorem 4.11 gives the result for sequences $\langle s_1, s_2, 1, 0\rangle$ with $s_1, s_2 \in \{0, 1\}$. With the exception of $\langle 1, 1, 0, 1\rangle$, the sequences $\langle s_1, s_2, s_3, 1\rangle$ where $s_1, s_2, s_3 \in \{0, 1\}$ are realizable by Example 2.14 ($\langle 0, 1, 0, 1\rangle$), Corollary 2.6 ($\langle 1, 0, 0, 1\rangle$), Example 3.11 ($\langle 1, 0, 1, 1\rangle$), and Theorem 4.12 ($\langle 1, 1, 1, 1\rangle$, $\langle 0, 1, 1, 1\rangle$, $\langle 0, 0, 1, 1\rangle$, $\langle 0, 0, 0, 1\rangle$).

It remains to prove there is no 4×4 irreducible sign pattern $\mathcal{A}$ with $q_{\text{seq}}(\mathcal{A}) = \langle 1, 1, 0, 1\rangle$. Let $\mathcal{A}$ be a 4×4 irreducible sign pattern with $q_1(\mathcal{A}) = q_2(\mathcal{A}) = q_4(\mathcal{A}) = 1$. It suffices to show that $q_3(\mathcal{A}) = 1$. Since the result follows by Lemma 2.1 and Theorem 4.13 if $c(\mathcal{A}) \leq 3$, we assume $c(\mathcal{A}) = 4$. In the rest of this proof, we reference some of the digraphs $D(\mathcal{H}_i)$ listed in Fig. 11 of Appendix B.

We first consider the case that $D(\mathcal{A})$ has no 2-cycles. Then, $D(\mathcal{A})$ has at least one loop by Corollary 2.8 since $q_2(\mathcal{A}) = 1$. Since $q_1(\mathcal{A}) = 1$, by [3, Lemma 3.2] it follows that $D(\mathcal{A})$ has four loops of the same sign,

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and furthermore, the subpattern $\mathcal{B}$ obtained from $\mathcal{A}$ by replacing every diagonal entry by 0 is potentially nilpotent. To determine the structure of $D(\mathcal{B})$, we note there are four strongly connected digraphs G_1, G_2, G_3, and G_4 with no 2-cycles as depicted in Fig. 2. The digraphs G_1, G_2, and G_4 each have exactly one

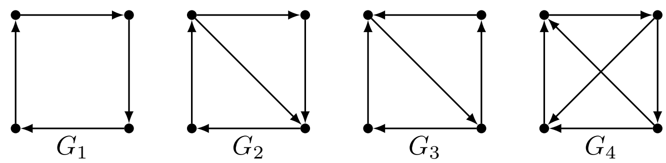

FIGURE 2. *The four strongly connected digraphs of order 4 with no 2-cycles.*

4-cycle and no other composite cycles of order 4, thus, cannot be the digraph of a potentially nilpotent sign pattern. Hence, $D(\mathcal{B})$ is isomorphic to G_3. Since $\mathcal{B}$ is potentially nilpotent, $D(\mathcal{B})$ (and hence $D(\mathcal{A})$) has oppositely signed 3-cycles. But $D(\mathcal{A})$ has four loops, thus, it has a subdigraph equivalent to $D(\mathcal{H}_5)$ and it follows that $q_3(\mathcal{A}) = 1$ by Lemma 3.10(i).

We now consider the case that $D(\mathcal{A})$ has a 2-cycle. By [3, Theorem 3.3], $D(\mathcal{A})$ has a negative 2-cycle. Since $q_1(\mathcal{A}) = 1$ and $\mathcal{A}$ is a 4×4 sign pattern, if $A \in Q(\mathcal{A})$ has $q(A) = 1$ then $\det(A) \geq 0$. Recalling that $c(\mathcal{A}) = 4$, it now follows from Equation (1.1) that $D(\mathcal{A})$ has a composite cycle U of order 4 with sign $(-1)^{|U|}$. There are four cases to consider depending on the number of cycles in U. In each case, we show that there exists $1 \leq i \leq 17$ such that $D(\mathcal{A})$ has a subdigraph equivalent to $D(\mathcal{H}_i)$. It then follows that $q_3(\mathcal{A}) = 1$ by Lemma 3.10(i).

Case 1. $|U| = 4$. In this case, $D(\mathcal{A})$ has four loops and a negative 2-cycle, hence $D(\mathcal{A})$ has a subdigraph equivalent to $D(\mathcal{H}_1)$ or $D(\mathcal{H}_2)$.

Case 2. $|U| = 3$. In this case, U has one 2-cycle and two loops (that are not incident to the 2-cycle).

Subcase 2.1. If the 2-cycle in U is positive, then the two loops in U are oppositely signed since U has sign $(-1)^3$. But $D(\mathcal{A})$ has a negative 2-cycle and thus has a subdigraph equivalent to $D(\mathcal{H}_3)$ or $D(\mathcal{H}_{11})$.

Subcase 2.2. If the 2-cycle in U is negative, then the two loops in U are the same sign since U has sign $(-1)^3$. Without loss of generality, we may assume both loops are negative (otherwise consider $-\mathcal{A}$). If $D(\mathcal{A})$ has three or four loops, then $D(\mathcal{A})$ has a subdigraph equivalent to $D(\mathcal{H}_1)$ or $D(\mathcal{H}_2)$. If $D(\mathcal{A})$ has exactly two loops, by [3, Theorem 3.5], $D(\mathcal{A})$ has either a positive 3-cycle implying it has a subdigraph equivalent to $D(-\mathcal{H}_5)$, or $D(\mathcal{A})$ has a negative 2-cycle that is incident to a loop implying it has a subdigraph equivalent to $D(\mathcal{H}_1)$.

Case 3. $|U| = 2$. There are two distinct structures to consider that have sign $(-1)^2$.

Subcase 3.1. U has two disjoint 2-cycles of the same sign. If $\mathcal{A}$ is bipartite (and hence, $\mathcal{A}$ allows singularity since $q_1(\mathcal{A}) = 1$), then $q_3(\mathcal{A}) = 1$ by Lemma 5.3. Thus, assume $D(\mathcal{A})$ has either a loop or a 3-cycle. First suppose both 2-cycles in U are negative. If $D(\mathcal{A})$ has a loop, then it has a subdigraph equivalent to $D(\mathcal{H}_{12})$, and if $D(\mathcal{A})$ has a 3-cycle, then it has a subdigraph equivalent to $D(\mathcal{H}_{15})$. Now suppose both 2-cycles in U are positive and note that $D(\mathcal{A})$ also has a negative 2-cycle. If $D(\mathcal{A})$ has a loop, then it has a subdigraph equivalent to $D(\mathcal{H}_4)$ or $D(\mathcal{H}_{13})$, and if $D(\mathcal{A})$ has a 3-cycle, then it has a subdigraph equivalent to $D(\mathcal{H}_9)$.

Subcase 3.2. U has a 3-cycle and a loop (that is not incident to the 3-cycle) of the same sign. Without loss of generality, we may assume the 3-cycle and loop are both negative (otherwise consider $-\mathcal{A}$). If $D(\mathcal{A})$

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has a positive loop, then it has a subdigraph equivalent to $D(\mathcal{H}_5)$. If $D(\mathcal{A})$ has a positive 2-cycle, then $D(\mathcal{A})$ has a subdigraph equivalent to $D(\mathcal{H}_6)$ or $D(\mathcal{H}_9)$.

Suppose all loops and 2-cycles of $D(\mathcal{A})$ are negative. By [3, Theorem 3.5], $D(\mathcal{A})$ has either: (a) 4 negative loops, (b) a negative 2-cycle incident to a negative loop, or (c) a positive 3-cycle. In (a), since $D(\mathcal{A})$ has a negative 2-cycle, it has a subdigraph equivalent to $D(-\mathcal{H}_1)$. In (b), $D(\mathcal{A})$ has a subdigraph equivalent to $D(-\mathcal{H}_1)$ or $D(-\mathcal{H}_{14})$.

Now suppose $D(\mathcal{A})$ has a positive 3-cycle as in (c). If this positive 3-cycle is incident to the negative loop in U, then $D(\mathcal{A})$ has a subdigraph equivalent to $D(-\mathcal{H}_5)$. This implies that $\mathcal{A}$ is equivalent to a superpattern of

$$\begin{bmatrix} 0 & + & + & 0 \\ - & 0 & + & 0 \\ - & - & 0 & 0 \\ 0 & 0 & 0 & - \end{bmatrix}.$$

If $D(\mathcal{A})$ has at least two loops (recall that all loops must be negative), then $D(\mathcal{A})$ has a subdigraph equivalent to $D(-\mathcal{H}_1)$. Thus, suppose $D(\mathcal{A})$ has exactly one loop. If $D(\mathcal{A})$ has a 2-cycle incident to the loop (recall that all 2-cycles must be negative), then $D(\mathcal{A})$ has a subdigraph equivalent to $D(-\mathcal{H}_{14})$. Thus, suppose the loop of $D(\mathcal{A})$ is not incident to a 2-cycle. As $\mathcal{A}$ is irreducible, it follows that $\mathcal{A}$ is equivalent to a fixed signing of one of

$$\mathcal{A}_1 = \begin{bmatrix} 0 & + & + & + \\ - & 0 & + & 0 \\ - & - & 0 & \circledast \\ 0 & + & \circledast & - \end{bmatrix}, \ \mathcal{A}_2 = \begin{bmatrix} 0 & + & + & + \\ - & 0 & + & 0 \\ - & - & 0 & \circledast \\ 0 & - & \circledast & - \end{bmatrix}, \ \mathcal{A}_3 = \begin{bmatrix} 0 & + & + & + \\ - & 0 & + & 0 \\ - & - & 0 & 0 \\ 0 & 0 & + & - \end{bmatrix} \ \text{ or } \ \mathcal{A}_4 = \begin{bmatrix} 0 & + & + & + \\ - & 0 & + & 0 \\ - & - & 0 & 0 \\ 0 & 0 & - & - \end{bmatrix}.$$

Since $\mathcal{A}_3$ requires four distinct eigenvalues by [12, Theorem 4.12], $\mathcal{A}$ cannot be equivalent to $\mathcal{A}_3$. If $\mathcal{A}$ is equivalent to a fixed signing of $\mathcal{A}_1$ or $\mathcal{A}_4$, then $D(\mathcal{A})$ has a subdigraph equivalent to $D(-\mathcal{H}_{16})$. If $\mathcal{A}$ is equivalent to a fixed signing of $\mathcal{A}_2$, then $D(\mathcal{A})$ has a subdigraph equivalent to $D(-\mathcal{H}_5)$.

Case 4. $|U| = 1$. In this case, U is a negative 4-cycle. Suppose $\mathcal{A}$ is bipartite. If $\mathcal{A}$ is sign nonsingular, then $q_1(\mathcal{A}) = 0$ by Corollary 2.13(ii), whereas if $\mathcal{A}$ allows singularity, then $q_3(\mathcal{A}) = 1$ by Lemma 5.3. Thus, suppose $\mathcal{A}$ is not bipartite. Then, $D(\mathcal{A})$ has either a loop or a 3-cycle. If $D(\mathcal{A})$ has a loop, then it has a subdigraph equivalent to $D(\mathcal{H}_{16})$, and if $D(\mathcal{A})$ has a 3-cycle, then it has a subdigraph equivalent to $D(\mathcal{H}_{17})$. □

6. Concluding comments. In this paper, we introduced the allow sequence of distinct eigenvalues for a sign pattern. We also developed combinatorial and analytical techniques for obtaining information about an allow sequence. A summary of the results on realizable sequences from Sections 4 and 5 is given in the following theorem that characterizes sequences of the form $\langle s_1, s_2, s_3, s_4, 0, \ldots, 0\rangle$.

THEOREM 6.1. *Let $n \geq 2$, $s = \langle s_1, s_2, \ldots, s_n\rangle$ with $s_i \in \{0, 1\}$ for $1 \leq i \leq 4$ and $s_i = 0$ for every $i \geq 5$. Then there exists an $n \times n$ irreducible sign pattern $\mathcal{A}$ with $q_{\text{seq}}(\mathcal{A}) = s$ if and only if s is not equal to any of*

(i) $\langle 0, 0, \ldots, 0\rangle$,
(ii) $\langle 1, 0, 0, \ldots, 0\rangle$,
(iii) $\langle 0, 1, 0, 0, \ldots, 0\rangle$ *if* $n \geq 3$,
(iv) $\langle 1, 1, 0, 0, \ldots, 0\rangle$ *if* $n \geq 3$,
(v) $\langle 1, 1, 1, 0, 0, \ldots, 0\rangle$ *if* $n \geq 4$,

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(vi) $\langle 1,1,0,1,0,0,\ldots,0\rangle$ *if* $n \geq 4$, *or*
(vii) $\langle 0,1,0,1,0,0,\ldots,0\rangle$ *if* $n \geq 5$.

Proof. If $s_3 = s_4 = 0$ then Corollary 4.10 and Theorem 5.1 give (i)-(iv). If $s_3 = 1$ and $s_4 = 0$, then Theorem 4.11 gives (v). If $s_4 = 1$, then Theorems 4.14 and 5.4 give (vi) and (vii). ☐

In the case that either $n \leq 3$ or $c(\mathcal{A}) \leq 2$, we provide a complete characterization of all $n \times n$ irreducible sign patterns $\mathcal{A}$ classified by their allow sequence (see Corollary 4.10 and Theorems 4.11, 5.1 and 5.2). A natural next step is to characterize the $n \times n$ irreducible sign patterns $\mathcal{A}$ according to their allow sequence when either $n = 4$ or $c(\mathcal{A}) = 3$ holds. The case $n = 4$ will be aided by the very recent work [12] on 4×4 sign patterns $\mathcal{A}$ with $q_{\text{seq}}(\mathcal{A}) = \langle 0,0,0,1\rangle$. We characterized star sign patterns $\mathcal{A}$ with $c(\mathcal{A}) = 3$ in Theorem 4.12. It would be interesting to classify realizable sequences for star sign patterns with $c(\mathcal{A}) \geq 4$. It would also be interesting to determine other sequences (not already listed in Theorem 6.1) that are not the allow sequence for any $n \times n$ irreducible sign pattern.

In Section 4, we explored conditions for when $q_3(\mathcal{A}) = 1$. We wonder what other conditions on $\mathcal{A}$ or $D(\mathcal{A})$ can guarantee that $q_3(\mathcal{A}) = 1$. For $n \geq 3$, Theorem 4.9(iii) shows that there is no $n \times n$ sign pattern $\mathcal{A}$ with allow sequence $\langle 1,1,0,0,\ldots,0\rangle$ and Theorem 4.14 shows that for $n \geq 5$ there is no $n \times n$ irreducible sign pattern $\mathcal{A}$ with allow sequence $\langle 1,1,0,1,0,\ldots,0\rangle$. Note that Theorem 5.4 shows that there is no 4×4 irreducible sign pattern with allow sequence $\langle 1,1,0,1\rangle$. Hence, we ask the following question for $n \geq 3$ and $\mathcal{A}$ an $n \times n$ irreducible sign pattern: does $q_1(\mathcal{A}) = q_2(\mathcal{A}) = 1$ imply that $q_3(\mathcal{A}) = 1$?

We note that we can compare one of our lower bounds on $q(\mathcal{A})$ to an analogous bound for graphs (see, for example, [1]). For G an undirected graph, $q(G)$ is defined to be the minimum number of distinct eigenvalues taken over all real symmetric matrices A respecting the graph G (and having arbitrary diagonal entries). It is known ([1],[8]) that (i) if T is a tree then $q(T) \geq \operatorname{diam}(T) + 1$, and (ii) for any connected graph G, if A is nonnegative and is compatible with G, then $q(A) \geq \operatorname{diam}(G) + 1$. To compare this to the problem for sign patterns $\mathcal{A}$, since $\ell(\mathcal{A})$ is the girth of $D(\mathcal{A})$, Corollary 2.5 shows that the girth of $D(\mathcal{A})$ is a lower bound on $q(\mathcal{A})$ (i.e., $q(\mathcal{A}) \geq \ell(\mathcal{A})$) when $\mathcal{A}$ is not potentially nilpotent.

For the graph problem, some strong properties were introduced [2] and applied to the problem of determining $q(G)$. In particular, the strong spectral property (SSP) and strong multiplicity property (SMP) are important tools used in the inverse eigenvalue problem for graphs. For example, a matrix A has the SSP if $X = O$ is the only symmetric matrix satisfying $A \circ X = O$, $I \circ X = O$, and $[A, X] = O$, where $A \circ X$ is the entrywise product of A and X, $[A, X] = AX - XA$ and O denotes the zero matrix. It is known that if a symmetric $n \times n$ matrix A with graph G has the SSP, then for every q' with $q(A) \leq q' \leq n$ there exists a matrix A' with graph G and $q(A') = q'$ (see [11, Theorem 4.1]). Thus, if $q(A) = q(G)$ and A has the SSP then the allow sequence for G is $\langle 0,0,\ldots,0,1,1,\ldots,1\rangle$ where the first 1 appears in the $q(A)$ position. It is an open problem whether the SSP assumption can be dropped.

The notion of the non-symmetric strong spectral property (nSSP) was introduced in [11]. In particular, if A is a real $n \times n$ matrix, then A has the nSSP if $X = O$ is the only matrix satisfying $A \circ X = O$ and $[A, X^T] = O$. If A has the nSSP, then Theorems 5.3 and 5.4 of [11] can be used to prove that the conclusion of Theorems 3.2 and 3.7 hold for any superpattern $\mathcal{A}'$ of $\mathcal{A} = \operatorname{sgn}(A)$. We remark that the Jacobian method described in Theorem 3.7 has utility in cases where the matrix does not have the nSSP. For example, H_6 (defined in Appendix B) does not have the nSSP (take X to be the 3×3 identity matrix) but does satisfy the conditions of Theorem 3.7. Some of the matrices in this paper do have the nSSP. For example, the realization considered in Example 3.6 has the nSSP as do H_1, H_2, H_3, H_4, H_7 and H_{13} in Appendix B.

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It was noted in [11] that there is a connection between Jacobian matrices and strong properties. In Section 3, Jacobian matrices were used to develop techniques to help determine $q_{\text{seq}}(\mathcal{A})$ for an $n \times n$ sign pattern $\mathcal{A}$. In the future, it would be interesting to explore how relevant strong properties could be defined to study allow sequences for sign patterns.

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Appendix A. 3×3 sign patterns $\mathcal{A}$ and $q_{\text{seq}}(\mathcal{A})$. In this appendix, we provide a catalogue of all the irreducible 3×3 sign patterns $\mathcal{A}$ according to $q_{\text{seq}}(\mathcal{A})$.

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$$\mathcal{Z}_1 = \begin{bmatrix} 0 & + & 0 \\ - & 0 & + \\ 0 & + & 0 \end{bmatrix} \quad \mathcal{Z}_2 = \begin{bmatrix} 0 & + & + \\ - & 0 & + \\ + & + & 0 \end{bmatrix}$$

FIGURE 3. *Two irreducible 3×3 sign patterns with allow sequences $q_{\text{seq}}(\mathcal{Z}_1) = \langle 1, 0, 1 \rangle$ and $q_{\text{seq}}(\mathcal{Z}_2) = \langle 1, 1, 1 \rangle$, respectively.*

$$\mathcal{Y}_1 = \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ - & 0 & 0 \end{bmatrix} \quad \mathcal{Y}_2 = \begin{bmatrix} - & + & 0 \\ 0 & 0 & + \\ + & - & 0 \end{bmatrix} \quad \mathcal{Y}_3 = \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & - & 0 \end{bmatrix}$$

$$\mathcal{Y}_4 = \begin{bmatrix} + & + & 0 \\ - & - & + \\ 0 & + & 0 \end{bmatrix} \quad \mathcal{Y}_5 = \begin{bmatrix} - & + & + \\ - & 0 & 0 \\ + & 0 & - \end{bmatrix} \quad \mathcal{Y}_6 = \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & + & - \end{bmatrix}$$

FIGURE 4. *Irreducible 3×3 sign patterns whose superpatterns all have allow sequence $\langle 1, 1, 1 \rangle$.*

$$\mathcal{S}_{1,1} = \begin{bmatrix} + & + & 0 \\ - & \oplus & - \\ \oplus & - & \ominus \end{bmatrix} \quad \mathcal{S}_{1,2} = \begin{bmatrix} \oplus & + & 0 \\ - & + & - \\ + & \ominus & \ominus \end{bmatrix} \quad \mathcal{S}_{1,3} = \begin{bmatrix} \oplus & + & 0 \\ - & + & - \\ \oplus & - & \ominus \end{bmatrix} \quad \mathcal{S}_{1,4} = \begin{bmatrix} \oplus & + & 0 \\ - & \oplus & - \\ \oplus & - & - \end{bmatrix}$$

$$\mathcal{S}_{1,5} = \begin{bmatrix} \oplus & + & 0 \\ - & \oplus & - \\ + & - & \ominus \end{bmatrix} \quad \mathcal{S}_{2,1} = \begin{bmatrix} + & + & - \\ - & 0 & \ominus \\ + & \ominus & 0 \end{bmatrix} \quad \mathcal{S}_{2,2} = \begin{bmatrix} \oplus & + & - \\ - & 0 & - \\ + & - & 0 \end{bmatrix}$$

$$\mathcal{V}_1 = \begin{bmatrix} + & + & 0 \\ 0 & * & + \\ + & 0 & \circledast \end{bmatrix} \quad \mathcal{V}_2 = \begin{bmatrix} - & + & 0 \\ 0 & \circledast & + \\ + & 0 & \circledast \end{bmatrix} \quad \mathcal{W} = \begin{bmatrix} \circledast & + & \oplus \\ + & \circledast & + \\ + & \oplus & \circledast \end{bmatrix}$$

FIGURE 5. *Irreducible 3×3 patterns all of whose fixed signings have allow sequence $\langle 0, 1, 1 \rangle$.*

$$\mathcal{F}_1 = \begin{bmatrix} \circledast & + & 0 \\ + & \circledast & + \\ 0 & + & \circledast \end{bmatrix} \quad \mathcal{F}_2 = \begin{bmatrix} 0 & + & \ominus \\ \ominus & 0 & + \\ + & \ominus & 0 \end{bmatrix} \quad \mathcal{F}_3 = \begin{bmatrix} + & + & 0 \\ 0 & 0 & + \\ + & \ominus & 0 \end{bmatrix} \quad \mathcal{F}_4 = \begin{bmatrix} 0 & + & 0 \\ - & 0 & + \\ 0 & - & 0 \end{bmatrix}$$

FIGURE 6. *Irreducible 3×3 patterns all of whose fixed signings have allow sequence $\langle 0, 0, 1 \rangle$.*

Appendix B. 3×3 and 4×4 matrices with a repeated eigenvalue.

This appendix gives examples of 3×3 and 4×4 matrices having 1 as an eigenvalue with multiplicity 2 and satisfying the conditions (i)-(iii) of Theorem 3.7. For $1 \leq i \leq 17$, we define $\mathcal{H}_i = \operatorname{sgn}(H_i)$. The digraphs $D(\mathcal{H}_i)$ are provided in Figs. 10 and 11.

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$$H_1 = \begin{bmatrix} \boxed{2} & \boxed{1} & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad H_2 = \begin{bmatrix} \boxed{2} & \boxed{1} & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad H_3 = \begin{bmatrix} \boxed{1/2} & \boxed{2} & 0 \\ -1/2 & 0 & 2 \\ 0 & 3/2 & 0 \end{bmatrix} \quad H_4 = \begin{bmatrix} \boxed{-1} & \boxed{4} & 0 \\ 2 & 0 & 3 \\ 0 & -1 & 0 \end{bmatrix}$$

$$H_5 = \begin{bmatrix} \boxed{3/2} & \boxed{1/2} & 0 \\ 0 & 0 & 2 \\ -1/2 & 0 & 0 \end{bmatrix} \quad H_6 = \begin{bmatrix} 0 & \boxed{2} & 0 \\ 0 & 0 & \boxed{1} \\ -1 & 3 & 0 \end{bmatrix} \quad H_7 = \begin{bmatrix} \boxed{4} & \boxed{1} & -5 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

FIGURE 7. *For $1 \le i \le 7$, H_i has spectrum $\{1, 1, \lambda_i\}$ for some $\lambda_i \notin \{0, 1\}$ and satisfies (i)-(iii) of Theorem 3.7 with $\lambda = 1$, $m = 2$ and positions of x_1, x_2 indicated by boxed entries.*

$$H_8 = \begin{bmatrix} 0 & \boxed{1} & \boxed{1} & 1 \\ -9/2 & 4 & 0 & 0 \\ -2/5 & 0 & 0 & 0 \\ -1/10 & 0 & 0 & 0 \end{bmatrix} \quad H_9 = \begin{bmatrix} 0 & \boxed{-1} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & \boxed{3} \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad H_{10} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & \boxed{4} & \boxed{-5} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

FIGURE 8. *For $8 \le i \le 10$, H_i has spectrum $\{1, 1, \lambda_i, 0\}$ for some $\lambda_i \notin \{0, 1\}$ and satisfies (i)-(iii) of Theorem 3.7 with $\lambda = 1$, $m = 2$ and positions of x_1, x_2 indicated by boxed entries.*

$$H_{11} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \boxed{-1} & \boxed{2} & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad H_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \boxed{-1} & \boxed{2} & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad H_{13} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \boxed{1} & \boxed{-1} & -2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad H_{14} = \begin{bmatrix} 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & \boxed{-1/2} \\ 0 & 0 & 1/4 & 5/4 \end{bmatrix}$$

$$H_{15} = \begin{bmatrix} 0 & \boxed{1} & 0 & 0 \\ \boxed{-3} & 0 & 4 & 0 \\ 4 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad H_{16} = \begin{bmatrix} \boxed{4/3} & \boxed{1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1/3 & 0 & 0 & 0 \end{bmatrix} \quad H_{17} = \begin{bmatrix} 0 & \boxed{3/2} & 0 & 0 \\ 0 & 0 & \boxed{2} & 0 \\ 0 & 0 & 0 & 2 \\ -1/2 & 1 & 0 & 0 \end{bmatrix}$$

FIGURE 9. *For $11 \le i \le 17$, H_i has spectrum $\{1, 1, \lambda_i, \mu_i\}$ for some $\lambda_i, \mu_i \notin \{0, 1\}$, $\lambda_i \neq \mu_i$ and satisfies (i)-(iii) of Theorem 3.7 with $\lambda = 1$, $m = 2$ and positions of x_1, x_2 indicated by boxed entries.*

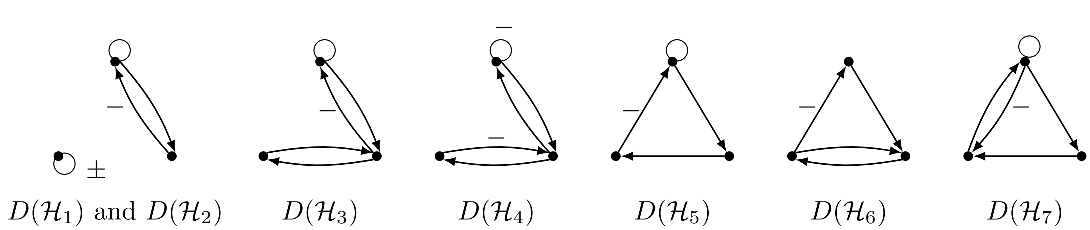

FIGURE 10. *The order 3 digraphs $D(\mathcal{H}_i)$, $1 \le i \le 7$. Unlabeled arcs are assumed to be positively signed.*

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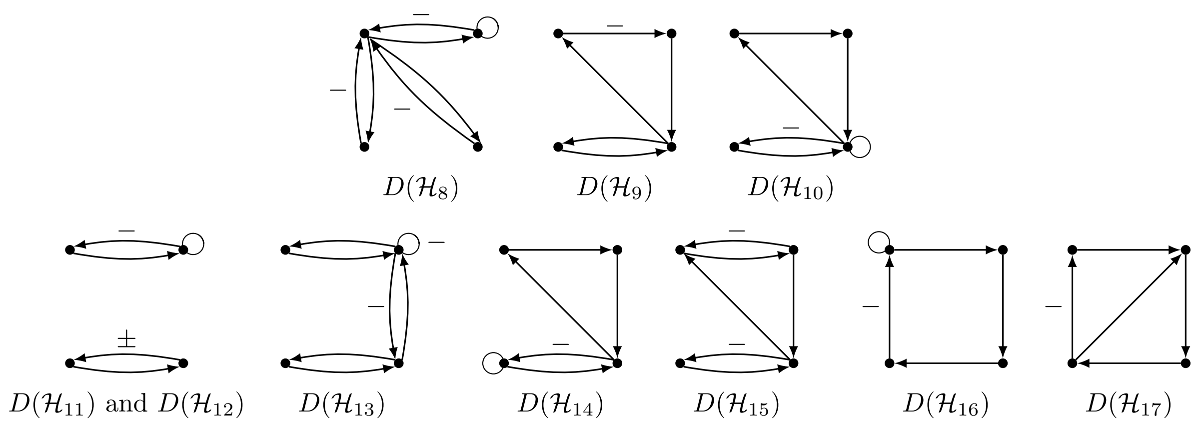

FIGURE 11. *The order 4 digraphs $D(\mathcal{H}_i)$, $8 \le i \le 17$. Unlabeled arcs are assumed to be positively signed.*

Appendix C. Proof of Lemma 3.12.

Here we provide a proof of Lemma 3.12.

Proof. Let $1 \le r \le m$ and $b_r = \prod_{j \ne r} \frac{\ell_j}{\ell_j - \ell_r}$. Let V be the $m \times m$ Vandermonde matrix with $V_{rk} = \ell_k^{r-1}$. The inverse of V (see, for example, [16]) is given by

$$V_{rk}^{-1} = \frac{(-1)^{k+1}\sigma_{rk}(\ell_1, \dots, \ell_m)}{\prod\limits_{j \ne r}(\ell_j - \ell_r)},$$

where $\sigma_{rk}(\ell_1, \dots, \ell_m)$ is the sum of all the products of $m-k$ distinct elements from $\{\ell_1, \dots, \ell_m\} \backslash \{\ell_r\}$ with $\sigma_{rm}(\ell_1, \dots, \ell_m) = 1$. Note that $V_{r1}^{-1} = b_r$ and

$$V_{rm}^{-1} = \frac{(-1)^{m+1}}{\prod\limits_{j \ne r}(\ell_j - \ell_r)} = \frac{(-1)^{m+1} b_r}{\prod\limits_{j \ne r} \ell_j}. \tag{C.8}$$

Let $\mathbf{b} = (b_1, b_2, \dots, b_m)^T$. Since $\mathbf{b}$ is the first column of the inverse of V, it follows that $V\mathbf{b} = (1, 0, \dots, 0)^T$. Thus, $\sum_{r=1}^m b_r = 1$ and $\sum_{r=1}^m \ell_r^t b_r = 0$ for $1 \le t \le m-1$. Now $(n - \ell_r)_i = (n)_i + \sum_{t=1}^i \ell_r^t f_t(n)$ for some polynomials $f_t(n)$ of degree $m-t$, for $1 \le t \le i$ and particularly $f_i(n) = (-1)^i$. Let

$$S_i = \sum_{r=1}^m (n - \ell_r)_i \prod_{j \ne r} \frac{\ell_j}{\ell_j - \ell_r}.$$

Note that

$$S_i = \sum_{r=1}^m (n - \ell_r)_i b_r = \sum_{r=1}^m \left[(n)_i + \sum_{t=1}^i \ell_r^t f_t(n) \right] b_r = (n)_i \sum_{r=1}^m b_r + \sum_{r=1}^m \sum_{t=1}^i f_t(n) \ell_r^t b_r = (n)_i + \sum_{t=1}^i f_t(n) \sum_{r=1}^m \ell_r^t b_r.$$

Therefore $S_i = (n)_i$ when $1 \le i < m$ since $\sum_{r=1}^m \ell_r^t b_r = 0$ for $1 \le t < m$ and

$$S_m = (n)_m + f_m(n) \sum_{r=1}^m \ell_r^m b_r.$$

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Note that $f_m(n) = (-1)^m$. We claim that

$$\sum_{r=1}^{m} \ell_r^m b_r = (-1)^{m+1} \ell_1 \ldots \ell_m, \tag{C.9}$$

and therefore $S_m \neq (n)_m$.

To prove the claim, let D be a diagonal matrix with $D_{r,r} = \ell_r$ and let $\mathbf{x} = (x_1, x_2, \ldots, x_m)^T$ be $\mathbf{x} = VD\mathbf{b}$. Note that x_m is the left hand side of (C.9). Let $W = D^{-1}V^{-1}$. We will use Cramer's rule on the equation $W\mathbf{x} = \mathbf{b}$ to show that $x_m = (-1)^{m+1}\ell_1 \ldots \ell_m$. From (C.8), note that $W_m(\mathbf{b})$ can be obtained from W by multiplying the last column of W by $(-1)^{m+1}\ell_1 \ldots \ell_m$. Thus, $\det(W_m(\mathbf{b})) = (-1)^{m+1}\ell_1 \ldots \ell_m \det(W)$. Therefore, $x_m = (-1)^{m+1}\ell_1 \ldots \ell_m$ by Cramer's rule. ∎