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SEMIPOSITIVITY WITH RESPECT TO THE LORENTZ CONE*

ARITRA NARAYAN HISABIA† AND MANIDEEPA SAHA†

Abstract. Lorentz cone in the Euclidean space $\mathbb{R}^n$ is defined as $\mathcal{L}_+^n := \{x \in \mathbb{R}^n : x_n \geq 0, \sum_{i=1}^{n-1} x_i^2 \leq x_n^2\}$. The paper aims to study semipositivity of matrices with respect to $\mathcal{L}_+^n$. A $n \times n$ real matrix A is $\mathcal{L}_+^n$-semipositive if there exists $x \in \mathcal{L}_+^n$ such that $Ax \in \operatorname{int}(\mathcal{L}_+^n)$ (the topological interior of $\mathcal{L}_+^n$). $\mathcal{L}_+^n$-positive matrices ($A(\mathcal{L}_+^n \setminus \{0\}) \subseteq \mathcal{L}_+^n$) and minimally $\mathcal{L}_+^n$-semipositive matrices ($A^{-1}(\mathcal{L}_+^n) \subseteq \mathcal{L}_+^n$) are two important subclasses of $\mathcal{L}_+^n$-semipositive matrices. In this paper, we establish the existence of bases for the real vector space of $n \times n$ matrices, consisting of $\mathcal{L}_+^n$-positive matrices and of minimally $\mathcal{L}_+^n$-semipositive matrices. Sufficient conditions are determined for $\mathcal{L}_+^n$-semipositivity, in terms of the length of rows(columns) of the matrices. Furthermore, we discuss properties of $\mathcal{L}_+^n$-semipositive matrices involving product of matrices. At last, $\mathcal{L}_+^2$-semipositive matrices are described via entries of the matrices and equivalent $\mathcal{L}_+^n$-semipositive matrices are studied.

Key words. Lorentz cone, $\mathcal{L}_+^n$-semipositive matrix, Minimally $\mathcal{L}_+^n$-semipositive matrix, $\mathcal{L}_+^n$-positive matrix.

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1. Introduction.

The Lorentz cone in the Euclidean space $\mathbb{R}^n$ is defined as

$$\mathcal{L}_+^n := \{x \in \mathbb{R}^n : x_n \geq 0, \sum_{i=1}^{n-1} x_i^2 \leq x_n^2\} = \left\{x : \ \|x\| \leq \sqrt{2}x_n\right\},$$

where $\|.\|$ is the standard norm in $\mathbb{R}^n$. In literature, Lorentz cones are also known as "ice cream cones" or "second-order cones." Lorentz cones are one of the important cones in optimization problems, mainly due to their presence in linear second-order cone programming (SOCP) problems, the standard form of which is given by,

$$\begin{aligned} &\min_x \, c^T x \\ &\text{subject to } Ax = b \\ &\qquad\qquad x \in \mathcal{L}_+^n. \end{aligned}$$

SOCP problems arise in many mathematical models in robust optimization, plant location problems and investment portfolio management and in many engineering challenges, such as truss design, filter design, antenna array weight design, robotic grasping force optimization, etc, and for further details, we refer [1, 4, 16] and the references therein. In this paper, we study the semipositivity of matrices with respect to the Lorentz cone. To begin with, we first discuss semipositivity of matrices with respect to a proper cone K in $\mathbb{R}^n$ and some related properties.

Throughout the article, $\mathbb{R}^{m\times n}$ refers to the collection of all matrices of order $m \times n$ having real entries, and the inequality symbols used with vectors and matrices are meant to show inequality at the entry-level.

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†Department of Mathematics, National Institute of Technology Meghalaya, Shillong, Meghalaya 793003, India (ahisabia@nitm.ac.in, aritrahisabia@gmail.com, manideepa.saha@nitm.ac.in).

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A set $K \subseteq \mathbb{R}^n$ is said to be a *cone* if $K + K \subseteq K$ and $\alpha K \subseteq K$, for all $\alpha \geq 0$. The topological interior and boundary of the cone K with the standard topology in $\mathbb{R}^n$ are respectively denoted by $\text{int}(K)$ and ∂K. A cone K is called a *proper cone* if it is closed, pointed ($K \cap (-K) = \{0\}$), and solid (non-empty interior). The dual of a cone $K \in \mathbb{R}^n$ is denoted by K^* and is defined as

$$K^* = \{x \in \mathbb{R}^n : \langle x, y \rangle \geq 0 \text{ for all } y \in K\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in the Euclidean space. A cone K is called *self-dual* if $K = K^*$.

A proper cone K always generates a partial order in $\mathbb{R}^n$ via $y \overset{K}{\leq} x$ if and only if $x - y \in K$.

Definition 1.1. [3] Let K and $F \subseteq K$ be pointed closed cones. Then, F is called a *face* of K if

$$x \in F, \quad 0 \overset{K}{\leq} y \overset{K}{\leq} x \quad \text{implies} \quad y \in F.$$

The face F is non-trivial if $F \neq \{0\}$ and $F \neq K$.

We now discuss the concept of nonnegativity, positivity and monotonicity of matrices with respect to a cone. Several researches have been carried out finding properties of these matrices, mainly due to their occurrence in many problems in optimization, in particular, in linear complementarity problems (LCP). Throughout the paper, we work on the matrices over the real field $\mathbb{R}$. A matrix $A \in \mathbb{R}^{m \times n}$ is called nonnegative(positive) if all the entries are nonnegative (positive), and a matrix $A \in \mathbb{R}^{n \times n}$ is called monotone if $Ax \geq 0$ implies $x \geq 0$. We now define the nonnegative, positive, and monotone matrices with respect to a cone K.

Definition 1.2. [3] *For a cone K in $\mathbb{R}^n$, a matrix $A \in \mathbb{R}^{n \times n}$ is said to be*

(a) K-nonnegative *if for all $x \in K$, $Ax \in K$. The set of K-nonnegative matrices are denoted as $\pi(K)$, that is,*

$$\pi(K) = \{A \in \mathbb{R}^{n \times n} : A(K) \subseteq K\},$$

(b) K-positive *if $A(K \setminus \{0\}) \subseteq \text{int}(K)$,*

(c) K-monotone, *if $Ax \in K$ implies $x \in K$.*

One can observe that the nonnegative, positive and monotone matrices in $\mathbb{R}^n$ are a particular case of Definition 1.2 by taking $K = \mathbb{R}^n_+$. The following result is an important characterization of K-monotone matrices.

Theorem 1.3. [3] *A matrix A is K-monotone if and only if A is non-singular and $A^{-1} \in \pi(K)$, the collection of $n \times n$ matrices that leaves K invariant.*

We now consider another important class of matrices related to LCP, known as semipositive matrices. These matrices are initially introduced by Fielder and Pták in [7]. A matrix $A \in \mathbb{R}^{m \times n}$ is said to be *semipositive* if there exists an $x \geq 0$ and $x \neq 0$ such that $Ax > 0$. Some important subclasses of semipositive matrices are symmetric positive definite matrices, non-singular M-matrices, P-matrices, etc. Besides their involvement in LCP, semipositive matrices are crucial in a variety of problems, such as characterizing non-singular M-matrices, matrix stability, game theory, etc, due to which several researchers have developed many algebraic, geometrical, and spectral features of semipositive matrices, which may be found in [5, 6, 9, 12, 13, 14, 15].

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The two disjoint subclasses of semipositive matrices are minimally semipositive matrices and redundantly semipositive matrices. A semipositive matrix A is referred to as *minimally semipositive matrix*(MSP) if there is no column deleted submatrix that is semipositive, and otherwise it is known as *redundantly semipositive matrix*(RSP). A square minimally semipositive matrix is characterized by an inverse nonnegative matrix [15]. More generally, in [12] the existence of a nonnegative left inverse is established as an equivalent requirement for a matrix to be minimally semipositive. Another interesting result for minimally semipositive matrices is established in [5], which provides a basis of minimally semipositive matrices for the real vector space $\mathbb{R}^{m\times n}$, the statement of which is given below:

Theorem 1.4. [5] *The vector space of real matrices of order $m \times n(m \geq n)$ has a basis of minimally semipositive matrices, and the basis $\mathcal{C} = \{C^{ij} : i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}\}$ is defined as follows:*

For $i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}$, if $i = sn + j$ for some integer $s \geq 0$,

$$(C^{ij})_{kl} = \begin{cases} 1 & \text{if } i \neq k \text{ and } k = tn + l \text{ for some integer } t \geq 0 \\ 2 & \text{if } i = k \text{ and } j = l \\ 0 & \text{otherwise,} \end{cases}$$

if $i \neq sn + j$,

$$(C^{ij})_{kl} = \begin{cases} 1 & \text{if } k = tn + l \text{ for some integer } t \geq 0 \\ -2 & \text{if } i = k \text{ and } j = l \\ 0 & \text{otherwise.} \end{cases}$$

They further studied the conditions for which product of two matrices is (minimally) semipositive and also discussed semipositivity or minimal semipositivity of certain intervals of matrices and of Schur complement of a matrix.

A generalization of semipositivity of a matrix with respect to proper cones is available in literature [3], which is defined below for our study.

Definition 1.5. [3] *Let K_1 and K_2 be two proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A matrix $A \in \mathbb{R}^{m\times n}$ is said to be* (K_1, K_2)-semipositive *if there exists an $x \in K_1$ such that $Ax \in \mathrm{int}(K_2)$. For brevity we write A is K-semipositive if $K_1 = K_2 = K$.*

By continuity of entries of A we can say that $A \in \mathbb{R}^{m\times n}$ is said to be (K_1, K_2)-semipositive if there exists $x \in \mathrm{int}(K_1)$ such that $Ax \in \mathrm{int}(K_2)$. In this case, such a vector x is called (K_1, K_2)-semipositivity vector *of A. We simply write x as a* semipositivity vector*, if the context is clear.*

One may observe that semipositive matrices are $(\mathbb{R}^n_+, \mathbb{R}^m_+)$-semipositive matrices. Also, it is clear that K-positive matrices are K-semipositive, and from [3], Page 114, we observe that K-monotone matrices are K-semipositive.

Next we define minimally K-semipositive matrices for any proper cone K.

Definition 1.6. [11] *Let K be a proper cone in $\mathbb{R}^n$. An invertible matrix $A \in \mathbb{R}^{n\times n}$ with a K-nonnegative inverse, is referred to as a* minimally K-semipositive *matrix.*

Following result is an immediate consequence of Theorem 1.3.

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Proposition 1.7. *Let K be a proper cone in $\mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. Then, the following are equivalent:*

(i) *A is K-monotone.*
(ii) *A is minimally K-Semipositive.*

The primary goal of our work is to study properties of $\mathcal{L}_+^n$-semipositive matrices. For our purpose, we restate the theorem of alternative for the Lorentz cone $\mathcal{L}_+^n$, which is given below:

Theorem 1.8. [3] *Let $A \in \mathbb{R}^{n\times n}$. Then exactly one of the following is true:*

(i) *There is an $x \in \text{int}(\mathcal{L}_+^n)$, such that $Ax \in \text{int}(\mathcal{L}_+^n)$, that is, A is an $\mathcal{L}_+^n$-semipositive matrix.*
(ii) *There is a $y \in -\mathcal{L}_+^n$ and $y \neq 0$ such that $A^T y \in \mathcal{L}_+^n$.*

We now discuss the significance of the vector e_n, the n-th column of the $n\times n$ identity matrix, to describe the interior and boundary of the Lorentz cone $\mathcal{L}_+^n$. From the definition of $\mathcal{L}_+^n$ one can notice the following:

Observation 1.9. *We have the following results:*

(i) $e_n \in \text{int}(\mathcal{L}_+^n)$.
(ii) $x \in \text{int}(\mathcal{L}_+^n)$ *if and only if* $\arccos(\langle x, e_n\rangle) < \dfrac{\pi}{4}$.
(iii) $\arccos(\langle x, e_n\rangle) = \dfrac{\pi}{4}$ *if and only if $x \in \partial(\mathcal{L}_+^n)$, the boundary of $\mathcal{L}_+^n$.*

In [10], authors have presented some necessary and other sufficient conditions for $\mathcal{L}_+^n$-semipositive matrices, and also provided characterization of orthogonal $\mathcal{L}_+^n$-semipositive matrices. In this paper, we present sufficient conditions for which a matrix is not an $\mathcal{L}_+^n$-semipositive matrix. A decomposition of $\mathcal{L}_+^n$-semipositive matrices is described in [2]. More specifically, it is shown that a $\mathcal{L}_+^n$-semipositive matrix A can be decomposed as $A = YX^{-1}$, where both X and Y are $\mathcal{L}_+^n$-positive and X is invertible. Motivated by this result, in this study, we provide a decomposition of a square matrix as a product of two $\mathcal{L}_+^n$-semipositive matrices. We further present sufficient conditions for $\mathcal{L}_+^n$-semipositivity of a matrix. We also provide a few algebraic features of $\mathcal{L}_+^n$-semipositive matrices.

Throughout the paper, we denote $\mathbb{R}^{m\times n}$ to represent $m\times n$ real matrices. We write $\mathbb{R}^n$ for $\mathbb{R}^{n\times 1}$ and $\mathbb{R}_+^n$ for nonnegative vectors in $\mathbb{R}^n$. The notation $\langle n\rangle$ is used to present the set $\{1, 2, \ldots, n\}$. The matrix (or vector) E^{ij} (or e_i) denotes the matrix (or vector) with (i, j)- (or i-) th entry 1 and others are zero. The symbol "$\geq$ $(>)$" is used to represent entrywise nonnegative matrix or vector. We write $R(A)$ for the range of the matrix A. For a given cone K in $\mathbb{R}^n$, we write, respectively, K^*, $\text{int}(K)$ and ∂K, to represent the dual, topological interior and boundary of K. $\pi(K)$ denotes the set of matrices A satisfying $A(K) \subseteq K$. $x \overset{K}{\leq} y$ means $x - y \in K$. For a given set $S \subseteq \mathbb{R}^n$, the set S^G denotes the collection of positive linear combinations of elements of S; $\text{span}(S)$ denotes the subspace of $\mathbb{R}^n$ spanned by S; $S^\perp$ means the orthogonal complement of S with respect to the standard inner product $\langle x, y\rangle = x^T y$ in $\mathbb{R}^n$.

The paper is structured as follows: In Section 2, existence of two bases of $\mathcal{L}_+^n$-positive matrices and of minimally $\mathcal{L}_+^n$-semipositive matrices, for the real vector space $\mathbb{R}^{n\times n}$, is established. Algebraic properties of $\mathcal{L}_+^n$-semipositivity are studied in Section 3. More specifically, sufficient conditions for $\mathcal{L}_+^n$-semipositivity in terms of lengths of columns (rows) are determined. In Section 4, $\mathcal{L}_+^n$-semipositivity of products of two matrices, one of which is either $\mathcal{L}_+^n$-semipositive or minimally $\mathcal{L}_+^n$-semipositive, or diagonal, is discussed. Section 5 is focused on algebraic properties of equivalent $\mathcal{L}_+^n$-semipositive matrices and of $\mathcal{L}_+^2$-semipositive matrices. At last, we end with a concluding remark.

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2. $\mathcal{L}_+^n$-semipositive bases for the real vector space of square matrices. In Theorem 1.4, Choudhury et al. describe a basis of $\mathbb{R}^{m\times n}$ of minimally semipositive matrices, which motivates us to study bases of $\mathbb{R}^{n\times n}$ of $\mathcal{L}_+^n$-semipositive matrices. In this section, we present bases of $\mathbb{R}^{n\times n}$ of $\mathcal{L}_+^n$-positive matrices and of minimally $\mathcal{L}_+^n$-semipositive matrices. In order to prove the same, we first to show that interior of $\mathcal{L}_+^n$-positive and minimally $\mathcal{L}_+^n$-positive matrices are non-empty. In fact, we prove it for any proper cone K.

It is known that from [3] [page 4, Exercise 2.18] that $\mathrm{int}(\pi(K)) = \{A \in \mathbb{R}^{n\times n} : A(K \setminus \{0\}) \subseteq \mathrm{int}(K)\}$, which implies that $\mathrm{int}(\pi(K))$ is the collection of all K-positive matrices, and therefore, interior of the collection of all K-positive matrices is $\mathrm{int}(\pi(K))$, which is non-empty, and hence, we have the following result.

Theorem 2.1. *For any proper cone K, the set of all K-positive matrices has non-empty interior.*

We further know that the minimally K-semipositive matrices are the inverse K-nonnegative matrices. Therefore, if $\mathcal{M}_K$ is the collection of all minimally K-semipositive matrices, then

$$\mathrm{int}\,(\mathcal{M}_K) = \{A \in \mathbb{R}^{n\times n} : A^{-1}(K \setminus \{0\}) \subseteq \mathrm{int}(K)\} \neq \emptyset.$$

We now show that $\mathcal{M}_K$ has non-empty interior. For this purpose, following results are required.

Proposition 2.2. [3] *For a given proper cone, K we have*

(i) *If F is face of K, then $F \subseteq \partial K$.*
(ii) *For any $x \in K$, $F_x = \{y \in K :$ there exists a positive α such that $\alpha y \overset{K}{\leq} x\}$ is the smallest face containing x, and it is non-trivial if and only if $0 \neq x \in \partial K$.*

Lemma 2.3. *Let K be a proper cone and $x \in \partial K$ and $y \in \mathrm{int}(K)$. Then for any $\alpha, \beta > 0$, we have $\alpha x + \beta y \in \mathrm{int}(K)$. In fact, the result holds if $x \in \mathrm{int}(K)$.*

Proof. Choose $z = \alpha x + \beta y$ so that $z \in K$. If $z \in \partial K$. Then by Proposition 2.2(ii), there exists a non-trivial face F_z of K such that $z \in F_z$. Again, $0 \overset{K}{\leq} \beta y \overset{K}{\leq} z$, implies $y \in F_z$. Therefore, by Proposition 2.2(i), $y \in \partial K$, which is a contradiction. Thus $z \in \mathrm{int}(K)$. ∎

Theorem 2.4. *Given a proper cone K, the class of minimally K-semipositive matrices has non-empty interior in $\mathbb{R}^{n\times n}$.*

Proof. As $\mathrm{int}(K)$ and $\mathrm{int}(K^*)$ are non-empty, choose $u \in \mathrm{int}(K)$ and $v \in \mathrm{int}(K^*)$ and consider the matrix $C = I + uv^T$. From the choice of u, v, it is clear that $1 + v^T u \neq 0$, and hence by Sherman-Morrison formula, $B = I + uv^T$ is invertible. We take $A = B^{-1}$. For any $x \in K \setminus \{0\}$, we have

$$A^{-1}x = (I + uv^T)x = x + (v^T x)u.$$

Since $x \in K$, $x \neq 0$, $u \in \mathrm{int}(K)$ and $v^T x > 0$, by Lemma 2.3, $A^{-1}x \in \mathrm{int}(K)$. Thus, $A^{-1} \in \mathrm{int}(\mathcal{M}_K)$, and hence, the result follows. ∎

Because of Theorem 2.1 and 2.4, for a given cone K in $\mathbb{R}^n$, we can conclude the following:

Theorem 2.5. *Given a proper cone K, there exist bases of $\mathbb{R}^{n\times n}$ consisting of K-positive and minimally K-semipositive matrices.*

We now explicitly find out bases of $\mathbb{R}^{n\times n}$ consisting of $\mathcal{L}_+^n$-positive matrices and of minimally $\mathcal{L}_+^n$-positive matrices.

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Theorem 2.6. *There is a basis of $\mathcal{L}^n_+$-positive matrices for the real vector space $\mathbb{R}^{n\times n}$ of $n\times n$ matrices..*

Proof. Define the set of matrices $\mathcal{A} = \{A^{kl} : k,l \in \langle n\rangle\}$, as follows:

If $l \neq n$ or $k \neq n$

$$(A^{kl})_{ij} = \begin{cases} 1 & \text{if } i=k, j=l \\ 0 & \text{if } i \neq k \text{ or}, j \neq l \\ 2 & \text{if } i=n, j=n \end{cases}, \tag{2.1}$$

and if $l = k = n$

$$(A^{nn})_{ij} = \begin{cases} 0 & \text{if } i \neq n \text{ or}, j \neq n \\ 3 & \text{if } i=n, j=n \end{cases}. \tag{2.2}$$

For any $x \in \mathbb{R}^n$ and $k,l = \{1,2,\dots,n\}$, we have that

$$A^{kl}x = \begin{cases} [\overbrace{0,\dots,0}^{(k-1)\text{ nos.}}, x_l, 0, \dots, 2x_n]^T & \text{if } k \neq n \\ [0,\dots,0,x_l+2x_n]^T & \text{if } k = n \end{cases}.$$

It is easy to verify that $A^{kl}x \in \operatorname{int}(\mathcal{L}^n_+)$ whenever $x \in \mathcal{L}^n_+$ and $x \neq 0$, and hence, each A^{kl} is $\mathcal{L}^n_+$-positive. Further, note that $A^{kl} = E^{kl} + 2E^{nn}$, where $\{E^{kl} :\ k,l \in \langle n\rangle\}$ is the standard basis of $\mathbb{R}^{n\times n}$. Therefore, $\{A^{kl} : k,l \in \langle n\rangle\}$ is linearly independent and so serves as a basis of $\mathbb{R}^{n\times n}$. ◻

Remark 2.7. *In the definition of the basis elements A^{kl} in Theorem 2.6, the numerics 2 and 3 can be replaced by any real number $\alpha > 1$.*

For verification purpose, we now explicitly show the basis defined in Theorem 2.6 for $\mathbb{R}^{3\times 3}$ of $\mathcal{L}^3_+$-positive matrices.

Example 2.8. *We write the $\mathcal{L}^3_+$-positive matrices defined in (2.1) and (2.2):*

$$A^{11} = \begin{bmatrix} 1&0&0\\0&0&0\\0&0&2 \end{bmatrix},\quad A^{12} = \begin{bmatrix} 0&1&0\\0&0&0\\0&0&2 \end{bmatrix},\quad A^{13} = \begin{bmatrix} 0&0&1\\0&0&0\\0&0&2 \end{bmatrix}$$

$$A^{21} = \begin{bmatrix} 0&0&0\\1&0&0\\0&0&2 \end{bmatrix},\quad A^{22} = \begin{bmatrix} 0&0&0\\0&1&0\\0&0&2 \end{bmatrix},\quad A^{23} = \begin{bmatrix} 0&0&0\\0&0&1\\0&0&2 \end{bmatrix}$$

$$A^{31} = \begin{bmatrix} 0&0&0\\0&0&0\\1&0&2 \end{bmatrix},\quad A^{32} = \begin{bmatrix} 0&0&0\\0&0&0\\0&1&2 \end{bmatrix},\quad A^{33} = \begin{bmatrix} 0&0&0\\0&0&0\\0&0&3 \end{bmatrix}$$

Next, we describe another basis of $\mathbb{R}^{n\times n}$ consisting of minimally $\mathcal{L}^n_+$-semipositive matrices, which are essentially $\mathcal{L}^n_+$-monotone.

Theorem 2.9. *The class of minimally $\mathcal{L}^n_+$-semipositive matrices contains a basis for $\mathbb{R}^{n\times n}$.*

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Proof. We define a set of matrices $\mathcal{B} = \{B^{kl} : k,l \in \{1,2,\ldots,n\}\}$, as follows:

If $l \neq k,\ l \neq n,\ k \neq n$,

$$(B^{kl})_{ij} = \begin{cases} -1 & \text{if } i=k, j=l \\ 1 & \text{if } i=j \neq n \\ \frac{1}{3} & \text{if } \ i=j=n \\ 0 & \text{elsewhere} \end{cases}. \tag{2.3}$$

If $l \neq k$ and ($k=n$ or $l=n$),

$$(B^{kl})_{ij} = \begin{cases} -\frac{1}{3} & \text{if } i=k, j=l \\ 1 & \text{if } i=j \neq n \\ \frac{1}{3} & \text{if } \ i=j=n \\ 0 & \text{elsewhere} \end{cases}. \tag{2.4}$$

For $l = k \neq n$,

$$(B^{kl})_{ij} = \begin{cases} 1 & \text{if } i=j \neq k \\ \frac{1}{2} & \text{if } \ i=j=k \\ \frac{1}{3} & \text{if } i=j=n \\ 0 & \text{elsewhere} \end{cases}. \tag{2.5}$$

For $l = k = n$,

$$(B^{kl})_{ij} = \begin{cases} 1 & \text{if } i=j \neq n \\ \frac{1}{4} & \text{if } i=j=n \\ 0 & \text{elsewhere} \end{cases}. \tag{2.6}$$

In order to show that each B^{kl} is minimally $\mathcal{L}^n_+$-semipositive, it suffices to prove that each B^{kl} is invertible and an inverse $\mathcal{L}^n_+$-nonnegative matrix. Notice that each B^{kl} is a triangular matrix with positive diagonal entries and hence is invertible. We now calculate the inverses of these matrices. In order to accomplish this, we first describe each B^{kl} as a linear combination of the standard basis matrices E^{kl}. From the definition of B^{kl}, we observe that

$$B^{kl} = \begin{cases} \sum\limits_{i\neq n} E^{ii} + \frac{1}{3}E^{nn} - E^{kl} & \text{if } l \neq k,\ l \neq n,\ k \neq n \\ \sum\limits_{i\neq n} E^{ii} + \frac{1}{3}E^{nn} - \frac{1}{3}E^{kl} & \text{if } l \neq k \text{ and } (k=n \text{ or } l=n) \\ \sum\limits_{i\neq k,n} E^{ii} + \frac{1}{3}E^{nn} + \frac{1}{2}E^{kk} & \text{if } k=l\neq n \\ \sum\limits_{i\neq n} E^{ii} + \frac{1}{4}E^{nn} & \text{if } k=l=n \end{cases}. \tag{2.7}$$

Using the fact that

$$E^{ij}E^{pq} = \begin{cases} E^{iq} & \text{if } j=p \\ 0 & \text{otherwise} \end{cases},$$

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and from the expression (2.7), one can easily prove that for any k, l,

$$\left(B^{kl}\right)^{-1} = I + E^{kl} + 2E^{nn} = I + A^{kl},$$

where A^{kl} is the basis matrix defined in Theorem 2.6. As A^{kl} is $\mathcal{L}^n_+$-positive, so for any $x \in \mathcal{L}^n_+$, we have $A^{kl}x \in \operatorname{int}(\mathcal{L}^n_+)$ which implies

$$(B^{kl})^{-1}x = (I + A^{kl})x = x + A^{kl}x \in \mathcal{L}^n_+. \qquad \square$$

Thus, $(B^{kl})^{-1}$ is $\mathcal{L}^n_+$-nonnegative and by Proposition 1.7, each B^{kl} is minimally $\mathcal{L}^n_+$-semipositive. Also from (2.7), it is easy to verify that the set $\{B^{kl} : k, l \in \langle n \rangle\}$ is linearly independent and hence forms a basis.

Remark 2.10. *Each basis matrix B^{kl} is defined in (2.7) in Theorem 2.9, is inverse $\mathcal{L}^n_+$-positive.*

In the next example, we write explicitly the basis elements for $\mathbb{R}^{3\times 3}$ of minimally $\mathcal{L}^3_+$-semipositive matrices, defined in Theorem 2.9.

Example 2.11. *Following matrices are minimally $\mathcal{L}^3_+$-semipositive and form a basis of $\mathbb{R}^{3\times 3}$.*

$$B^{11} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, \quad B^{12} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, \quad B^{13} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$B^{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, \quad B^{22} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, \quad B^{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$B^{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}, \quad B^{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}, \quad B^{33} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

In [5], Choudhury et al. proved that any matrix can be expressed as difference of two minimally semipositive matrices, the statement of which is given below:

Theorem 2.12. [5] *Let $A \in \mathbb{R}^{m\times n}$, $m \geq n$. Then, there exist minimally semipositive matrices $B, C \in \mathbb{R}^{m\times n}$ such that $A = B - C$.*

We now furnish a similar result for $\mathcal{L}^n_+$-semipositive matrices.

Theorem 2.13. *Any matrix $A \in \mathbb{R}^{n\times n}$ can be written as $A = B - C$, where B and C both are $\mathcal{L}^n_+$-semipositive.*

Proof. Let x be the last column of A. Observe that x can be written as $x = y - z$, where y_n and z_n are positive numbers and large enough so that $y, z \in \operatorname{int}(\mathcal{L}^n_+)$. Next, we define $B = (b_{ij})$ and $C = (c_{ij})$ by : for $i = 1, 2, \ldots, n$

$$b_{ij} = \begin{cases} a_{ij} & \text{if } j = 1, 2, \ldots, n-1 \\ y_i & \text{if } j = n \end{cases},$$

and

$$c_{ij} = \begin{cases} 0 & \text{if } j = 1, 2, \ldots, n-1 \\ z_i & \text{if } j = n \end{cases}.$$

It follows that $A = B - C$. Again as $Be_n (= y), Ce_n (= z) \in \operatorname{int}(\mathcal{L}^n_+)$, we have B, C are $\mathcal{L}^n_+$-semipositive with semipositivity vector e_n. $\square$

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Example 2.14. *Consider the 3×3 matrix A defined by*

$$A = \begin{bmatrix} 1 & -2 & 6 \\ 5 & 10 & -6 \\ 0 & -8 & 2 \end{bmatrix}.$$

Then, we can write $A = B - C$ where $B = \begin{bmatrix} 1 & -2 & 6 \\ 5 & 10 & -6 \\ 0 & -8 & 10 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}$, both are $\mathcal{L}^3_+$-semipositive matrices.

3. Properties of $\mathcal{L}^n_+$-semipositive matrices via rows and columns. In this section, we study sufficient conditions for $\mathcal{L}^n_+$-semipositivity in terms of lengths of columns (rows) of the matrix. Notice that if the last column of a square matrix A is in $\operatorname{int}(\mathcal{L}^n_+)$, then the matrix is $\mathcal{L}^n_+$-semipositive with e_n as a semipositivity vector. However, similar result may not hold for rows, and the following example illustrates this fact.

Example 3.1. *Consider the 3×3 matrix A defined as*

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 4 & 8 \\ 1 & 1 & 2 \end{bmatrix}.$$

Note that the transpose of the last row $[1, 1, 2]^T$ is in $\operatorname{int}(\mathcal{L}^3_+)$ but A is not $\mathcal{L}^3_+$-semipositive, which follows from Theorem 1.8, since $x = [0, \frac{1}{2}, -1]^T \in -\mathcal{L}^3_+$, and $A^T x = [1, 1, 2]^T \in \operatorname{int}(\mathcal{L}^3_+)$.

With the assumption that the transpose of the last row of the matrix A is in the interior of $\mathcal{L}^n_+$, we provide a sufficient condition for $\mathcal{L}^n_+$-semipositivity.

Theorem 3.2. *Let $A = [a_1^T, \dots, a_n^T]^T$ with $a_n \in \operatorname{int}(\mathcal{L}^n_+)$. If*

$$\|a_1\|^2 + \ldots + \|a_{n-1}\|^2 < \|a_n\|^2. \tag{3.8}$$

Then A is $\mathcal{L}^n_+$-semipositive.

Proof. We prove A is $\mathcal{L}^n_+$-semipositive by showing that $x = a_n$ is a semipositivity vector. It is clear from the hypothesis that, $x \in \operatorname{int}(\mathcal{L}^n_+)$. Now $(Ax)_n = \langle a_n, a_n \rangle = \|a_n\|^2 > 0$ and

$$\sum_{i=1}^{n-1} (Ax)_i^2 = \sum_{i=1}^{n-1} \langle a_i, a_n \rangle \leq \sum_{i=1}^{n-1} \|a_n\|^2 \ \|a_i\|^2 = \|a_n\|^2 \sum_{i=1}^{n-1} \|a_i\|^2 < \|a_n\|^4 = (Ax)_n^2.$$

This shows that A is $\mathcal{L}^n_+$-semipositive. ∎

The example below illustrates that the assumption (3.8) in Theorem 3.2 cannot be withdrawn.

Example 3.3. *Let $A = \begin{bmatrix} -1 & 1 & 7 \\ 2 & 3 & 1 \\ 2 & 4 & -7 \end{bmatrix}$. Clearly, A satisfies the hypothesis in Theorem 4.2 except $a_3 \in \operatorname{int}(\mathcal{L}^3_+)$. Note that $x = -e_n \in -\mathcal{L}^n_+$ and $A^T x \in \operatorname{int}(\mathcal{L}^n_+)$. Thus by Theorem 1.8, A is not $\mathcal{L}^n_+$-semipositive.*

We now move forward in determining further sufficient conditions in terms of lengths of row and/or columns of the matrix. Following lemma is required for this purpose.

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LEMMA 3.4. *Let $x \notin \mathcal{L}^n_+ \cup -\mathcal{L}^n_+$. Then, there exists a $y \in \operatorname{int}(\mathcal{L}^n_+)$ such that $\langle x, y\rangle = 0$.*

Proof. As $x \notin \mathcal{L}^n_+ \cup -\mathcal{L}^n_+$, so from Observations 1.9 we have that $\arccos(\langle x, e_n\rangle) > \frac{\pi}{4}$ and $\arccos(\langle x, -e_n\rangle) > \frac{\pi}{4}$. Let $F = \operatorname{span}\{x, e_n\}$ be the two dimensional subspace spanned by $\{x, e_n\}$. Then, we have that

$$\frac{\pi}{4} < \arccos(\langle x, e_n\rangle) < \frac{3\pi}{4}.$$

Hence, there exists $y \in F$ such that $\langle x, y\rangle = 0$ and $\arccos(\langle y, e_n\rangle) < \frac{\pi}{4}$, that is $y \in \operatorname{int}(\mathcal{L}^n_+)$, and so the result follows. ∎

COROLLARY 3.5. *Let $x \notin \mathcal{L}^n_+ \cup -\mathcal{L}^n_+$. Then, there exists a $y \in -\operatorname{int}(\mathcal{L}^n_+)$ such that $\langle x, y\rangle = 0$.*

In the subsequent results, we provide another adequate criteria for a square matrix A to be $\mathcal{L}^n_+$-semipositive.

THEOREM 3.6. *Let $A = [a_1^T, \dots, a_n^T]^T$ with $a_n \neq 0$. Then, the following results hold:*

(i) *If*

$$\|a_1\|^2 + \ldots + \|a_{n-1}\|^2 < \|a_n\|^2 \text{ and } \sum_{i=1}^{n-1} a_{ni}^2 > \left(a_{nn} - \sqrt{2}\right)^2, \tag{3.9}$$

then A is $\mathcal{L}^n_+$-semipositive.

(ii) *If*

$$\sum_{i=1}^{n-1} a_{ni}^2 > \left(a_{nn} - \sqrt{2\sum_{i=1}^{n-1}\|a_i\|^2}\right)^2, \tag{3.10}$$

then A is $\mathcal{L}^n_+$-semipositive.

Proof. As for any $\alpha > 0$, αA is $\mathcal{L}^n_+$-semipositive if and only if A is $\mathcal{L}^n_+$-semipositive, so we may assume $\|a_n\| = 1$.

(i) As $\sum_{i=1}^{n-1} |a_{ni}|^2 > (a_{nn} - \sqrt{2})^2$, $a_n - \sqrt{2}e_n \notin \mathcal{L}^n_+ \cup -\mathcal{L}^n_+$. By Lemma 3.4, there exists $x \in \operatorname{int}(\mathcal{L}^n_+)$ such that $\langle a_n - \sqrt{2}e_n, x\rangle = 0$. That is, $\langle a_n, x\rangle = \sqrt{2}x_n$.

Next we show that $Ax \in \operatorname{int}(\mathcal{L}^n_+)$. In order to achieve this, we compute the terms $(Ax)_n$ and $(Ax)_1^2 + \ldots + (Ax)_{n-1}^2$. Now,

$$(Ax)_n = \langle a_n, x\rangle = \sqrt{2}x_n > 0.$$

Next, we have that

$$\begin{aligned}
\sum_{i=1}^{n-1}(Ax)_i^2 = \sum_{i=1}^{n-1}\langle a_i, x\rangle &\le \sum_{i=1}^{n-1}\|x\|^2\ \|a_i\|^2 \\
&\le \|x\|^2 \sum_{i=1}^{n-1}\|a_i\|^2 \\
&< 2x_n^2\|a_n\|^2 \qquad [\text{ since } x \in \mathcal{L}^n_+ \text{ and } (3.9)] \\
&= (\langle a_n, x\rangle)^2 = (Ax)_n^2.
\end{aligned}$$

∎

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This shows that A is $\mathcal{L}^n_+$-semipositive.

(ii) Let $\alpha = \sum_{i=1}^{n-1} \|a_i\|^2$, so that $\sum_{i=1}^{n-1} a_{ni}^2 > (a_{nn} - \sqrt{2\alpha})^2$, that is $a_n - \sqrt{2\alpha} e_n \notin \mathcal{L}^n_+ \cup -\mathcal{L}^n_+$. By Lemma 3.4, there exists $x \in \operatorname{int}(\mathcal{L}^n_+)$ such that $\langle a_n - \sqrt{2\alpha} e_n, x\rangle = 0$. That is, $\langle a_n, x\rangle = \sqrt{2\alpha} x_n$. Then as shown in the proof of (i), we can see that x is a semipositivity vector of A.

To validate the conditions mentioned in Theorem 3.6, let us consider the following two examples.

Example 3.7. *Let* $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & 3 \\ 3 & 3 & 1+\sqrt{2} \end{bmatrix}$. *Notice that the hypothesis in Theorem 3.6(i) is satisfied by A, with a semipositivity vector* $[-1, -1, 6]^T$.

Example 3.8. *Consider the matrix* $A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 1 \\ 2 & 2 & \sqrt{10}-2 \end{bmatrix}$. *It can be verified that the hypothesis in Theorem 3.6(ii) is fulfilled by A and* $[1, 1, 2]^T$ *is a semipositivity vector of A.*

At the beginning of the section, we have observed that if the last column of a matrix is in $\operatorname{int}(\mathcal{L}^n_+)$, then the matrix is $\mathcal{L}^n_+$-semipositive, whereas Example 3.1 shows that same may not be true if the last row belongs to $\operatorname{int}(\mathcal{L}^n_+)$. We now ask the same question here for Theorem 3.6, that is, if the columns of A satisfy the conditions (3.9) and (3.10), then whether the matrix A is $\mathcal{L}^n_+$-semipositive or not. However, it is intriguing to see that in that case A can't be a $\mathcal{L}^n_+$-semipositive matrix. Proposition 3.9, proves our claim.

Proposition 3.9. *Let* $A = [\widetilde{a_1}, \ldots, \widetilde{a_n}]$ *with* $\widetilde{a_n} \neq 0$. *Then, the following results hold:*

(i) *Suppose that*

$$\|\widetilde{a_1}\|^2 + \ldots + \|\widetilde{a}_{n-1}\|^2 < \|\widetilde{a_n}\|^2 \text{ and } \sum_{i=1}^{n-1} a_{in}^2 > \left(a_{nn} - \sqrt{2}\right)^2. \tag{3.11}$$

Then, A is not a $\mathcal{L}^n_+$-semipositive matrix.

(ii) *Let*

$$\sum_{i=1}^{n-1} a_{in}^2 > \left(a_{nn} - \sqrt{2\sum_{i=1}^{n-1} \|\widetilde{a_i}\|^2}\right)^2. \tag{3.12}$$

Then, A is not a $\mathcal{L}^n_+$-semipositive matrix.

Proof. (i) If we take $\beta = \sum_{i=1}^{n-1} \|\widetilde{a_i}\|^2$, then $\widetilde{a}_n - \sqrt{2\beta} e_n \notin \mathcal{L}^n_+ \cup -\mathcal{L}^n_+$. By Corollary 3.5, we can choose $x \in -\operatorname{int}(\mathcal{L}^n_+)$ such that $\langle a_n - \sqrt{2\beta} e_n, x\rangle = 0$. Then similar to the proof of Theorem 3.6, it can be shown that $A^T x \in \operatorname{int}(\mathcal{L}^n_+)$. Thus by theorem of alternative (Theorem 1.8), A is not a $\mathcal{L}^n_+$-semipositive matrix.

The proof of the second part is similar to that of Theorem 3.6 (i) and hence is skipped.

The following concepts are necessary to carry on our study.

Definition 3.10. *Given a non-zero vector $a \in \mathbb{R}^n$ and a scalar $\alpha \in \mathbb{R}$, we define*

(a) *a* hyperplane *as the set* $\{x \in \mathbb{R}^n : \langle a, x\rangle = \alpha\}$.

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(b) *a (closed)* halfspace *as the set* $\{x \in \mathbb{R}^n : \langle a, x\rangle \geq \alpha\}$
(c) *a* hyperspace *as the set* $\{x \in \mathbb{R}^n : \langle a, x\rangle = 0\}$.

Notice that, the particular hyperplane $H = \{x \in \mathbb{R}^n : \langle a, x\rangle = 0\}$ is a subspace of $\mathbb{R}^n$ of dimension $n-1$ and is known as *hyperspace*. Also, the vector a is perpendicular to the hyperspace H, in fact, $H^\perp = \text{span}\{a\}$.

In the remaining part of this section, we present a geometrical condition for a matrix A to be $\mathcal{L}^n_+$-semipositive. The following lemma is crucial in this context.

LEMMA 3.11. *Let H be a hyperspace of $\mathbb{R}^n$ such that $\mathcal{L}^n_+ \cap H = \{0\}$. Then, there exists a $y \in \text{int}(\mathcal{L}^n_+)$ such that $y \in H^\perp$.*

Proof. Let $H = \{x \in \mathbb{R}^n : \langle z, x\rangle = 0\}$ so that $H^\perp = \text{span}\{z\}$. It is known that H splits $\mathbb{R}^n$ into two closed halfspaces $H_+ = \{x \in \mathbb{R}^n : \langle y, x\rangle \geq 0\}$ and $H_- = \{x \in \mathbb{R}^n : \langle y, x\rangle \leq 0\}$ with the origin in each of their boundaries. As $\mathcal{L}^n_+$ is convex and $\mathcal{L}^n_+ \cap H = \{0\}$, the hyperspace H separates the cones $\mathcal{L}^n_+$ and $-\mathcal{L}^n_+$. Let $\mathcal{L}^n_+$ and y be in the same halfspace otherwise we take $-y$. In that case, $\arccos(\langle y, e_n\rangle) \leq \dfrac{\pi}{2}$.

We claim: $y \in \text{int}(\mathcal{L}^n_+)$. Suppose that $y \notin \text{int}(\mathcal{L}^n_+)$. Then $\arccos(\langle y, e_n\rangle) \geq \dfrac{\pi}{4}$. Let $F = \text{span}\{y, e_n\}$. As $y, e_n \in F$ and $\dfrac{\pi}{4} \leq \arccos(\langle y, e_n\rangle) \leq \dfrac{\pi}{2}$, there exists $z \in F$ and $z \neq 0$ such that $\langle y, z\rangle = 0$ and $\arccos(\langle z, e_n\rangle) \leq \dfrac{\pi}{4}$. Then, $z \in H$ and $z \in \mathcal{L}^n_+$, which is a contradiction to the given hypothesis $\mathcal{L}^n_+ \cap H = \{0\}$. ∎

THEOREM 3.12. *Let $A = [a_1^T, \ldots, a_n^T]^T$ be square matrix such that $\{a_1, \ldots, a_{n-1}\}$ contained in a separating hyperspace $H = \{x \in \mathbb{R}^n : \langle y, x\rangle = 0\}$ of $\mathcal{L}^n_+$ and $-\mathcal{L}^n_+$. If a_n belongs to the relative interior of the halfspace containing $\mathcal{L}^n_+$ and y, then A is $\mathcal{L}^n_+$-semipositive.*

Proof. By Lemma 3.11, $y \in \text{int}(\mathcal{L}^n_+)$. As $\{a_1, \ldots, a_{n-1}\} \in H$, $\langle y, a_i\rangle = 0,\ \forall\ i = 1, 2, \ldots, n-1$ and as a_n belongs to the interior of the halfspace containing y, $\langle y, a_n\rangle > 0$. Hence, $(Ay)_n > 0$ and $\|Ay\| = \sqrt{\sum_{i=1}^{n} \langle y, a_i\rangle^2} = \langle y, a_n\rangle = (Ay)_n < \sqrt{2}(Ay)_n$. Again as $a_n \in H_+ = \{x \in \mathbb{R}^n : \langle y, x\rangle \geq 0\}$, $(Ay)_n \geq 0$. Hence, the result follows. ∎

4. $\mathcal{L}^n_+$-semipositivity involving matrix product. In [5], the semipositivity of the product of two matrices is investigated in depth. In this section, we explore similar results for $\mathcal{L}^n_+$-semipositive matrices. More specifically, we furnish conditions under which product of two matrices is a (minimally) $\mathcal{L}^n_+$-semipositive matrix. In this regard, the conditions are provided in terms of diagonal, $\mathcal{L}^n_+$-semipositive, $\mathcal{L}^n_+$-positive and minimally $\mathcal{L}^n_+$-semipositive matrices.

We begin with an example to show that the product of two $\mathcal{L}^n_+$-semipositive matrices is not necessarily a $\mathcal{L}^n_+$-semipositive matrix.

EXAMPLE 4.1. *Consider the matrices $A = \begin{bmatrix} 1 & 4 \\ 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Both A and B are $\mathcal{L}^2_+$-semipositive with semipositivity vectors $[\frac{1}{2}, 1]^T$ and $[0, 1]^T$, respectively. However, $AB = \begin{bmatrix} 0 & 4 \\ 0 & 3 \end{bmatrix}$ is not $\mathcal{L}^2_+$-semipositive.*

The result below demonstrates that $\mathcal{L}^n_+$-semipositivity is preserved under pre-multiplication by a $\mathcal{L}^n_+$-nonnegative matrix and post-multiplication by a minimally $\mathcal{L}^n_+$-semipositive matrix. Further, we prove that minimally $\mathcal{L}^n_+$-semipositive property is preserved under matrix multiplication.

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Proposition 4.2. *For an $\mathcal{L}^n_+$-semipositive matrix A, the following results hold.*

(i) *If B is $\mathcal{L}^n_+$-nonnegative, then BA is $\mathcal{L}^n_+$-semipositive.*
(ii) *If B is minimally $\mathcal{L}^n_+$-semipositive, then AB is $\mathcal{L}^n_+$-semipositive.*
(iii) *If A, B both are minimally $\mathcal{L}^n_+$-semipositive, then AB is also minimally $\mathcal{L}^n_+$-semipositive.*
(iv) *If B^{-1} exists and A, B^{-1} have a common semipositivity vector, then AB is $\mathcal{L}^n_+$-semipositive.*

Proof. As A is $\mathcal{L}^n_+$-semipositive matrix, choose $x \in \mathcal{L}^n_+$ such that $Ax \in \operatorname{int}(\mathcal{L}^n_+)$.

(i) Since $B \in \pi(\mathcal{L}^n_+)$, it is obvious that BA is $\mathcal{L}^n_+$-semipositive with x is a semipositivity vector.
(ii) Now $B^{-1} \in \pi(\mathcal{L}^n_+)$ implies that $B^{-1}x \in \operatorname{int}(\mathcal{L}^n_+)$. Thus AB is $\mathcal{L}^n_+$-semipositive with the semipositivity vector $B^{-1}x$.
(iii) The conclusion follows from the fact $A^{-1}, B^{-1} \in \pi(\mathcal{L}^n_+)$.
(iv) Let x be a common semipositivity vector, that is $x \in \operatorname{int}(\mathcal{L}^n_+)$ and $Ax, B^{-1}x \in \operatorname{int}(\mathcal{L}^n_+)$. If $y = Ax$ and $z = B^{-1}x$, so that $ABz = y$, where $x, z \in \operatorname{int}(\mathcal{L}^n_+)$. Thus, AB is $\mathcal{L}^n_+$-semipositive. □

In contrast to the previous result, we now investigate the $\mathcal{L}^n_+$-semipositivity property of the matrices whose product is given to be $\mathcal{L}^n_+$-semipositive.

Theorem 4.3. *Let BA be $\mathcal{L}^n_+$-semipositive for any $\mathcal{L}^n_+$-semipositive matrix B. Then, A is invertible and $\mathcal{L}^n_+$-semipositive.*

Proof. We first show that A is invertible. Assume that $x^TA = 0$, for some $x \neq 0$. Consider the matrix $B = [0^T, \ldots, 0^T, x^T]^T$. If $x_n \neq 0$, without loss of generality we assume $x_n > 0$. In that case, B is $\mathcal{L}^n_+$-semipositive with the semipositivity vector e_n. Again, if $x_n = 0$, then $x_i \neq 0$ for some i. Take $y = [0, \ldots, \alpha_i, \ldots, \beta]^T$, such that $\alpha_i x_i > 0$ and $|\alpha_i| < \beta$. Then, $y \in \operatorname{int}(\mathcal{L}^n_+)$ and $\langle x, y\rangle > 0$. Since $By = [0, \ldots, 0, \langle x, y\rangle]^T \in \operatorname{int}(\mathcal{L}^n_+)$, so B is $\mathcal{L}^n_+$-semipositive. Observe that the last row of BA is zero, which implies that BA is not $\mathcal{L}^n_+$-semipositive, which is a contradiction. Hence, A is invertible.

As I is an $\mathcal{L}^n_+$-semipositive matrix, from our hypothesis $IA = A$ is also an $\mathcal{L}^n_+$-semipositive matrix. □

It is trivial from the definition that a diagonal matrix $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ is $\mathcal{L}^n_+$-semipositive if and only if $d_n > 0$.

One may observe that a diagonal matrix is not necessary $\mathcal{L}^n_+$-nonnegative. We now provide a characterization of diagonal $\mathcal{L}^n_+$-nonnegative matrices.

Theorem 4.4. *Let $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ be a diagonal matrix.*

(i) *D is $\mathcal{L}^n_+$-nonnegative if and only if $d_n \geq 0$ and $|d_i| \leq d_n$ for all i.*
(ii) *If D is invertible, then D^{-1} is $\mathcal{L}^n_+$-nonnegative if and only if $d_n > 0$ and $|d_i| \geq d_n$ for all i.*

Proof. (i) Assume that $d_n \geq 0$ and $|d_i| \leq d_n$ for all i. If $d_n = 0$, $d_i = 0$ for all i, then D is $\mathcal{L}^n_+$-nonnegative. Let $d_n > 0$ and $x \in \mathcal{L}^n_+$. Since $\dfrac{d_i^2}{d_n^2} \leq 1$ for all i, we get

$$x_n^2 \geq \sum_{i=1}^{n-1} x_i^2 \geq \sum_{i=1}^{n-1} \frac{d_i^2 x_i^2}{d_n^2}$$
$$\implies d_n^2 x_n^2 \geq \sum_{i=1}^{n-1} d_i^2 x_i^2.$$

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Again as $d_n x_n \geq 0$, so $Dx \in \mathcal{L}^n_+$.

Conversely, let D be $\mathcal{L}^n_+$-nonnegative. If $d_n \leq 0$, then for every $x \in \mathcal{L}^n_+$, $(Dx)_n = d_n x_n < 0$, which is not possible as $Dx \in \mathcal{L}^n_+$, so $d_n > 0$.

Next suppose that $|d_k| > d_n$, for some $k \in \{1, 2, \ldots, n\}$. Choose $0 < x_k < 1$ such that $|d_k| x_k > d_n$. Take the vector $x \in \mathcal{L}^n_+$, whose k-th entry is x_k and n-th entry is 1, others are zero. Then $Dx \notin \mathcal{L}^n_+$.

(ii) follows from (i). ◻

The following results are immediate consequences of Proposition 4.2 and Theorem 4.4.

Corollary 4.5. *Let A be an $\mathcal{L}^n_+$-semipositive matrix and $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ be a diagonal matrix.*

(i) *If $d_n \geq 0$ and $|d_i| \leq d_n$ for all i, then DA is $\mathcal{L}^n_+$-semipositive.*
(ii) *If $d_n > 0$ and $|d_i| \geq d_n$ for all i, then AD is $\mathcal{L}^n_+$-semipositive*

In [2], authors presented a characterization of $\mathcal{L}^n_+$-semipositive matrices by exhibiting a decomposition of the form YX^{-1}, where both X and Y are $\mathcal{L}^n_+$-positive and X is invertible. The statement of which is given below:

Theorem 4.6. [2] *$A \in \mathbb{R}^{n\times n}$ is $\mathcal{L}^n_+$-semipositive if and only if there exist $X, Y \in \mathbb{R}^{n\times n}$ with X and Y are $\mathcal{L}^n_+$-positive, X is invertible and $A = YX^{-1}$.*

Factorization of any matrix into two semipositive matrices is discussed in [6]. Motivated by these results, we now provide a decomposition of any square matrix as product of two $\mathcal{L}^n_+$-semipositive matrices. For this purpose, we recall the following lemma.

Lemma 4.7. [6] *Let $m \geq 2$, $n \geq 1$, and suppose that $0 \neq C \in \mathbb{R}^{m\times n}$. If $\{x, y\} \in \mathbb{R}^m$ is a linearly independent set and $z \in \mathbb{R}^n$ such that $\{Cz, y\}$ is a linearly independent set, then there exists $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{m\times n}$ such that $Ax = y, Bz = y, C = AB$.*

Theorem 4.8. *If $n \geq 2$ and $C \in \mathbb{R}^{n\times n}$, then there exist $\mathcal{L}^n_+$-semipositive matrices $A, B \in \mathbb{R}^{n\times n}$ such that $C = AB$.*

Proof. Let $C \neq 0$. As $\mathcal{L}^n_+$ is solid and $\dim N(C) \leq n - 1$, there exists a $z \in \operatorname{int}(\mathcal{L}^n_+)$ such that $Cz \neq 0$.

As $\mathcal{L}^n_+$ is solid (non-empty interior), we can find two linearly independent vectors $x, y \in \operatorname{int}(\mathcal{L}^n_+)$. This implies that for any $z \in \operatorname{int}(\mathcal{L}^n_+)$ with $Cz \neq 0$, we have either $\{Cz, x\}$ or $\{Cz, y\}$ is linearly independent. Without lose of generality, we may assume that $\{Cz, y\}$ is linearly independent. Hence by Lemma 4.7, there exist $\mathcal{L}^n_+$-semipositive matrices $A, B \in \mathbb{R}^{n\times n}$ such that $C = AB$. If $C = 0$, then we can choose matrices $A = E^{nn}$ and $B = E^{n1}$. Observe that both are $\mathcal{L}^n_+$-semipositive.

5. Algebraic properties of $\mathcal{L}^n_+$-semipositive matrices.

This section is divided into two subsections. At first, we emphasis on characterizing $\mathcal{L}^2_+$-semipositive matrices in terms of entries of the matrices. In the next section, we study equivalent $\mathcal{L}^n_+$-semipositive matrices.

5.1. $\mathcal{L}^2_+$-semipositive matrices.

This section is focused on characterizing $\mathcal{L}^2_+$-semipositive matrices. In [12], it is shown that a 2×2 semipositive matrices either have a positive column or take the form

$$\begin{bmatrix} p & -q \\ -r & s \end{bmatrix} \text{ or } \begin{bmatrix} -q & p \\ s & -r \end{bmatrix},$$

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with $p > 0$, $s > 0$, $q > 0$, $r \geq 0$, and $ps > qr$. This leads us to find the description of $\mathcal{L}^2_+$-semipositive matrices in terms of their entries.

A 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\mathcal{L}^2_+$-semipositive if and only if there exists $x = [x_1, 1]^T \in \mathcal{L}^2_+$ such that $Ax \in \operatorname{int} \mathcal{L}^2_+$, which is equivalent to write

$$|x_1| \leq 1, \quad b + d > -(a + c)x_1, \quad d - b > -(c - a)x_1, \text{ and } d > -cx_1 \tag{5.13}$$

Theorem 5.2 gives a description of $\mathcal{L}^2_+$-semipositive matrices with $a_{22} > 0$. For this purpose, we require the following trivial lemma, the proof of which is skipped.

Lemma 5.1. *Let $p, q \in \mathbb{R}$ with $p < 0$ and $|p| \geq |q|$, then no real x_1 with $|x_1| \leq 1$ satisfy $p > -qx_1$.*

We now state and proof the main result.

Theorem 5.2. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix with $d > 0$. Then, A is $\mathcal{L}^2_+$-semipositive if and only if $|c - a| > b - d$ and $|c + a| > -b - d$.*

Proof. Assume that A is $\mathcal{L}^2_+$-semipositive. Then, there exists an $x = [x_1, 1]^T$ satisfying (5.13).

If $|c+a| \leq -b-d$, then $b+d \leq 0$ and $|c+a| \leq |b+d|$. Then by Lemma 5.1, there is no such x_1 satisfying $|x_1| < 1$ and $b + d > -(a + c)x_1$.

Again if $|c - a| \leq b - d$, then $d - b \leq 0$ and $|c - a| \leq |d - b|$. Thus, no real x_1 satisfy $|x_1| < 1$ and $d - b > -(c - a)x_1$.

Conversely, let $|c - a| > b - d$ and $|c + a| > -b - d$. Note that, to show A is $\mathcal{L}^2_+$-semipositive we have to find an $x = [x_1, 1]^T$ satisfying (5.13). We now consider the following cases:

Case I: Let $|b| < d$. In that case, e_n is a semipositivity vector of A.

Case II: Let $|b| > d$, then $(b + d)$ and $(d - b)$ are of opposite sign.

First let $b+d > 0$ and $d-b < 0$. As $|c-a| > b-d$ we choose x_1 with $|x_1| \leq 1$ such that $d-b > -(c-a)x_1$. Observe that for such an x_1, the sign of $c - a$ and x_1 are the same. Any other x_1 with same sign and a smaller modulus value is also acceptable. So we choose x_1 such that $b + d > -(a + c)x_1$ and $d > -cx_1$ so that x satisfies (5.13).

Again if $b + d < 0, d - b > 0$, we get a semipositivity vector with the similar argument as before.

Case III: Let $|b| = d$, that is $b = d$ or $b = -d$. If $b = d$, then $b + d = 2d$ and $d - b = 0$. Choose x_1, with sufficiently small absolute value such that, $|x_1| < 1$ and $d > -cx_1$ and $2d = b + d > -(a + c)x_1$. And we choose the sign of x_1 same as the sign of $(c - a)$, so that $d - b = 0 > -(c - a)x_1$. In that case, x satisfies (5.13).

Again if $b = -d$, $b + d = 0$ and $d - b = 2d$. We choose x_1 with a similar argument as before. Hence, the result follows.

5.2. $\mathcal{L}^n_+$-semipositive matrices and equivalence theory. Two matrices A and B are known as equivalent matrices if there exists invertible matrices P and Q such that $B = PAQ$. In [6], such equivalency

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of semipositive matrices is explained. Motivated by their work, in this section, we study the equivalence relation between $\mathcal{L}^n_+$-semipositive matrices and other square matrices.

Theorem 5.3. *Every non-zero matrix $A \in \mathbb{R}^{n\times n}$ is equivalent to a $\mathcal{L}^n_+$-semipositive matrix.*

Proof. It is known that if the rank$(A) = k(\neq 0)$, then there exists non-singular (elementary)matrices S, T such that

$$A = S\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} T.$$

Applying appropriate permutation to rows and columns, one can easily get non-singular matrices P and Q such that

$$A = P\begin{bmatrix} 0 & 0 \\ 0 & I_k \end{bmatrix} Q.$$

Observe that the matrix $B = \begin{bmatrix} 0 & 0 \\ 0 & I_k \end{bmatrix}$ is an $\mathcal{L}^n_+$-semipositive matrix with the semipositivity vector e_n. Hence, the result follows. ∎

We say that B is *right equivalent* to A if there exists an invertible matrix P such that $B = AP$. We end this section by providing a characterization of right equivalent $\mathcal{L}^n_+$-semipositive matrices.

Theorem 5.4. *A matrix is right equivalent to a $\mathcal{L}^n_+$-semipositive if and only if the range of the given matrix intersects the interior of the Lorentz cone.*

Proof. Assume that A is right equivalent to a $\mathcal{L}^n_+$-semipositive matrix. Choose an invertible matrix P such that $AP = B$, where B is an $\mathcal{L}^n_+$-semipositive matrix. Then, there exists $x \in \text{int}(\mathcal{L}^n_+)$ such that $y = Bx \in \text{int}(\mathcal{L}^n_+)$. Thus, $y = Bx = A(Px) \in \text{int}(\mathcal{L}^n_+) \cap R(A)$.

Conversely, let $y \in \text{int}(\mathcal{L}^n_+)$ and $y = Ax$, for some x. Take any invertible matrix C with the last column is x, that is, $Ce_n = x$. Then, $ACe_n = y$, and hence, AC is $\mathcal{L}^n_+$-semipositive. Hence, the result follows.

6. Conclusion. In this article, we have introduced two new bases for $\mathbb{R}^{n\times n}$. One of these bases elements consisting of $\mathcal{L}^n_+$-positive matrices, whereas the elements of the other basis consist of minimally $\mathcal{L}^n_+$-semipositive matrices. We validated the existence of such bases with the help of examples. We further showed that any matrix can be decomposed as a difference and product of $\mathcal{L}^n_+$-semipositive matrices. Sufficient conditions involving length of rows are determined for $\mathcal{L}^n_+$-semipositivity. We also provided the conditions for which a matrix can't be $\mathcal{L}^n_+$-semipositive. Given two matrices A and B, (minimally) $\mathcal{L}^n_+$-semipositivity condition of their product AB is discussed. We further proved that any square matrix is equivalent to a $\mathcal{L}^n_+$-semipositive matrix. At last, we provided algebraic conditions on entries of 2×2 matrices to describe $\mathcal{L}^2_+$-semipositive matrices.

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