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THE MATRIX INVERSE YOUNG INEQUALITY*

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Abstract. An inverse Young inequality is established for positive definite matrices.

Key words. Singular value, Eigenvalue, Young's inequality.

AMS subject classifications. 15A42.

1. Introduction. For an $n \times n$ matrix M and k in $\{1, \dots, n\}$, let $\sigma_k(M)$ denotes its k^{th} singular value, and if M is Hermitian, let $\lambda_k(M)$ denote the k^{th} eigenvalue. We enumerate both singular values and eigenvalues in decreasing order.

Young's inequality states that

$$ab \le \frac{a^p}{p} + \frac{b^q}{q},$$

for $a, b > 0$ and p, q conjugate exponents, that is real numbers $p, q > 1$ such that $p^{-1} + q^{-1} = 1$.

The matrix version of Young's inequality was proved by Ando [1]. His result asserts that

$$\sigma_k(AB) \le \lambda_k\left(\frac{A^p}{p} + \frac{B^q}{q}\right),$$

for A, B positive definite matrices and p, q conjugate exponents.

There is also an inverse version of Young's inequality namely

$$ab \ge \nu b^{\frac{1}{\nu}} + (1-\nu) a^{\frac{1}{1-\nu}},$$

for $a, b > 0$ and $\nu > 1$.

A matrix version of this inverse inequality was attempted by Manjegani and Norouzi [6], but their paper is flawed. Their main result ([6, Theorem 2.4]) is that (2.1) below holds for A and B positive definite $n \times n$ matrices and all $\nu > 1$. In fact, this is false for $\nu = 2$ (see the addendum), and the 'proof' given in [6] is invalid in the entire range $1 < \nu < \infty$. Already, equation (10) in Manjegani and Norouzi [6] is incorrect. In fact, it contains a typological error. It should read

$$A^2 \le \lambda^2 B^{-1} P B^{-1} \qquad \text{on } N,$$

and is deduced from $\lambda^2 P \ge BA^2B$ on M. But N is the range of $B^{-1}P$, not BP.

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2. The result.

THEOREM 1. *Let A and B be positive definite $n\times n$ matrices. Let $1<\nu\le\frac{3}{2}$ and k in $\{1,\dots,n\}$. Then,*

$$\sigma_k(AB)\ge\lambda_k(\nu B^{\frac{1}{\nu}}+(1-\nu)A^{\frac{1}{1-\nu}}). \tag{2.1}$$

We are aware of counterexamples to (2.1) for ν only slightly larger than $\frac{3}{2}$. In case $k=1$, (2.1) holds for all $\nu>1$. We have $B^2\le\sigma_1(AB)^2A^{-2}$. It follows that $B^{\frac{1}{\nu}}\le\sigma_1(AB)^{\frac{1}{\nu}}A^{-\frac{1}{\nu}}$ and the result follows easily.

3. Basic lemmas. For Hermitian matrices X and Y of the same size, we interpret $X\le Y$ or $Y\ge X$ by the Loewner ordering, that is, $Y-X$ is positive semidefinite,

LEMMA 1. *Let X and Y be positive definite matrices, A a positive semidefinite matrix and Q be an orthogonal projection all of the same size. Then,*

(a) [4, 5] *If $X\le Y$ and $0\le r\le 1$, then $X^r\le Y^r$.*
(b) [3] *If $1\le r\le 2$, then $QA^rQ\ge(QAQ)^r$.*

The proofs can be found in [2, Theorems V.2.3 & V.2.5]. See also [7].

LEMMA 2. *Let A be a positive semidefinite matrix and Q an orthogonal projection of the same size. If $r\ge 1$, then $QAQ\le(QA^rQ)^{\frac{1}{r}}$.*

Proof. We prove the result in the range $2^m\le r\le 2^{m+1}$ by induction on $m=0,1,\dots$. The case $m=0$ follows from Lemma 1 parts (a) and (b) since

$$(QAQ)^r\le QA^rQ \text{ and } QAQ=\Big((QAQ)^r\Big)^{\frac{1}{r}}\le\Big(QA^rQ\Big)^{\frac{1}{r}}. \tag{3.2}$$

We now assume the result for r in the interval $[2^m,2^{m+1}]$ and prove it for $2^{m+1}\le r\le 2^{m+2}$. For B positive semidefinite, we have $QBQ\le(QB^{\frac{r}{2}}Q)^{\frac{2}{r}}$ since $\frac{r}{2}\in[2^m,2^{m+1}]$. Replacing B by A^2 gives $QA^2Q\le(QA^rQ)^{\frac{2}{r}}$. Then by applying Lemma 1(a) with $r=\frac{1}{2}$, we have $(QA^2Q)^{\frac{1}{2}}\le(QA^rQ)^{\frac{1}{r}}$. Now, by (3.2) in case $r=2$ we obtain $QAQ\le(QA^2Q)^{\frac{1}{2}}\le(QA^rQ)^{\frac{1}{r}}$. ∎

We can now present the proof of the main result.

Proof of Theorem 1. Let P be the orthogonal projection on the spectral subspace of BA^2B corresponding to the eigenvalues $\lambda_k,\dots,\lambda_n$ and let Q be the orthogonal projection on $\mathcal{M}$, the range of $B^{-1}P$. Then as shown in [1], (see also [7]) QB^2Q and $B^{-1}PB^{-1}$ preserve $\mathcal{M}$ and are inverses when restricted to $\mathcal{M}$. In particular, $QBP=QB$.

We have $B^{-1}A^{-2}B^{-1}\ge\sigma_k(AB)^{-2}P$ since this holds on the spectral subspace of BA^2B corresponding to $\lambda_1,\dots,\lambda_{k-1}$ and also on that corresponding to $\lambda_k,\dots,\lambda_n$. Let $T=\nu B^{\frac{1}{\nu}}-(\nu-1)A^{-\frac{1}{\nu-1}}$. Then by the minmax characterization,

$$\lambda_k(H)=\inf_{S\in\mathcal{S}_{n+1-k}}\sup_{\substack{x\in S\\ \|x\|=1}}x^*Hx=\sup_{S\in\mathcal{S}_k}\inf_{\substack{x\in S\\ \|x\|=1}}x^*Hx,$$

of the eigenvalues of a Hermitian matrix H in terms of the manifold of complex linear subspaces $\mathcal{S}_k$ of dimension k, it will suffice to show that

$$\sigma_k(AB)Q\ge QTQ. \tag{3.3}$$

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We have $A^{-2} \geq \sigma_k(AB)^{-2}BPB$ and hence

$$QA^{-2}Q \geq \sigma_k(AB)^{-2}QBPBQ = \sigma_k(AB)^{-2}QB^2Q. \tag{3.4}$$

By Lemma 1(b), we have

$$QB^{\frac{1}{\nu}}Q \leq (QB^2Q)^{\frac{1}{2\nu}}, \tag{3.5}$$

since $0 < \frac{1}{2\nu} < 1$. Next, by Lemma 1(a) and (3.4) we have

$$(QB^2Q)^{\frac{1}{2\nu}} \leq \sigma_k(AB)^{\frac{1}{\nu}}(QA^{-2}Q)^{\frac{1}{2\nu}}, \tag{3.6}$$

also since $0 < \frac{1}{2\nu} < 1$. Now by Lemma 2 with $r = \frac{1}{2(\nu-1)} > 1$, we have

$$(QA^{-2}Q) \leq (QA^{-\frac{1}{\nu-1}}Q)^{2(\nu-1)}.$$

Then, applying by Lemma 1(a) with $r = \frac{1}{2\nu}$ we get

$$(QA^{-2}Q)^{\frac{1}{2\nu}} \leq (QA^{-\frac{1}{\nu-1}}Q)^{\frac{\nu-1}{\nu}}. \tag{3.7}$$

Combining (3.5), (3.6) and (3.7) we get

$$QTQ \leq \nu\sigma_k(AB)^{\frac{1}{\nu}}(QA^{-\frac{1}{\nu-1}}Q)^{\frac{\nu-1}{\nu}} - (\nu-1)(QA^{-\frac{1}{\nu-1}}Q).$$

But the function $t \mapsto \nu\sigma_k(AB)^{\frac{1}{\nu}}t^{\frac{\nu-1}{\nu}} - (\nu-1)t$ is bounded above by $\sigma_k(AB)$ on $t \geq 0$. Hence, exploiting the commutativity of $(QA^{-\frac{1}{\nu-1}}Q)^{\frac{\nu-1}{\nu}}$ and $(QA^{-\frac{1}{\nu-1}}Q)$ we have $\sigma_k(AB)I - QTQ \geq 0$ and the result follows. ☐

4. Addendum.

EXAMPLE 1. *The inequality* (2.1) *fails for* $\nu = 2$.

Proof. Let

$$B^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{16} \end{pmatrix}, \qquad A = \begin{pmatrix} 5.6383 & 14.3907 \\ 14.3907 & 47.5744 \end{pmatrix},$$

$$BA = \begin{pmatrix} 5.6383 & 14.3907 \\ 0.89941875 & 2.9734 \end{pmatrix}, \qquad 2B^{\frac{1}{2}} - A^{-1} \approx \begin{pmatrix} 1.221960285 & 0.2353479209 \\ 0.2353479209 & 0.4077902963 \end{pmatrix}.$$

We have

$$\lambda_2(|BA|) = \sigma_2(BA) = \sigma_2(AB) \approx 0.2424 < 0.3446 \approx \lambda_2(2B^{\frac{1}{2}} - A^{-1}).$$

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