Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 436-448, August 2010

ELA

THE LINEAR INDEPENDENCE OF SETS OF TWO AND THREE CANONICAL ALGEBRAIC CURVATURE TENSORS*

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Abstract. We generalize the construction of canonical algebraic curvature tensors by self-adjoint endomorphisms of a vector space to arbitrary endomorphisms. Provided certain basic rank requirements are met, we establish a converse of the classical fact that if A is symmetric, then R_A is an algebraic curvature tensor. This allows us to establish a simultaneous diagonalization result in the event that three algebraic curvature tensors are linearly dependent. We use these results to establish necessary and sufficient conditions that a set of two or three algebraic curvature tensors be linearly independent. We present the proofs of these results using elementary methods.

Key words. Algebraic curvature tensor, Linear independence, Simultaneous diagonalization.

AMS subject classifications. 15A21, 15A63.

1. Introduction. Let V be a real vector space of finite dimension n, and let $V^* = \operatorname{Hom}(V, \mathbb{R})$ be its dual. An object $R \in \otimes^4 V^*$ is an *algebraic curvature tensor* [11] if it satisfies the following three properties for all $x, y, z, w \in V$:

$$
\begin{aligned}
R(x,y,z,w) &= -R(y,x,z,w), \\
R(x,y,z,w) &= R(z,w,x,y), \text{and} \\
0 &= R(x,y,z,w) + R(x,z,w,y) \\
&\quad +R(x,w,y,z)\,.
\end{aligned}
\tag{1.a}
$$

The last property is known as the *Bianchi identity*. Let $\mathcal{A}(V)$ be the vector space of all algebraic curvature tensors on V.

Suppose φ is a symmetric bilinear form on V which is nondegenerate, and let A be an endomorphism of V. Let A^* be the adjoint of A with respect to φ, characterized by the equation $\varphi(Ax, y) = \varphi(x, A^* y)$. We say that A is *symmetric* if $A^* = A$, and we say that A is *skew-symmetric* if $A^* = -A$. For the remainder of this paper, we will always consider the adjoint A^* of a linear endomorphism A of V with respect to the form φ.

*Received by the editors January 28, 2010. Accepted for publication July 30, 2010. Handling Editor: Raphael Loewy.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 436-448, August 2010

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Definition 1.1. If $A : V \to V$ is a linear map, then we may create the element $R_A \in \otimes^4 V^*$ by

$$R_A(x, y, z, w) = \varphi(Ax, w)\varphi(Ay, z) - \varphi(Ax, z)\varphi(Ay, w). \tag{1.b}$$

In the event that A is the identity map, we simply denote R_A as R_φ. The object R_A satisfies the first property in Equation (1.a), although one requires A to be symmetric to ensure $R_A \in \mathcal{A}(V)$. In the event that $A^* = -A$, there is a different construction [11]. A spanning set is known for the set of algebraic curvature tensors.

Theorem 1.2 (Fiedler [7, 8]). $\mathcal{A}(V) = \mathrm{Span}\{R_A | A^* = A\}$.

Given a pseudo-Riemannian manifold (M, g), one may use the Levi-Civita connection to construct the Riemann curvature tensor R_g, and upon restriction to any point $P \in M$, we have $R_g(P) \in \mathcal{A}(T_P M)$. In fact, if A is symmetric, then a classical differential geometric fact is that R_A is realizable as the curvature tensor of an embedded hypersurface in Euclidean space [11]. Hence there has been much interest in determining the structure of the vector space $\mathcal{A}(V)$, in addition to the generating tensors R_A when A is symmetric. By Theorem 1.2, the tensors R_A are referred to as *canonical* curvature tensors [12].

The study of algebraic curvature tensors has been approached in several ways. For instance, B. Fiedler [7] investigated $\mathcal{A}(V)$ using methods from algebraic combinatorics, and later used group representation theory to construct generators for $\mathcal{A}(V)$ and the vector space of algebraic covariant derivative tensors [8]. More of the general structure of $\mathcal{A}(V)$ has been studied using representation theory (for example, see [12], [13], and [15]). There has been a strong connection that has been studied between algebraic curvature tensors and corresponding geometrical consequences, and large areas of mathematics depend, in part, on these connections. The list of references is much too long to include here, although some representative examples are [1, 2, 5, 6, 10, 12].

The authors in [4] use the Nash embedding theorem [14] to show that, given an arbitrary algebraic curvature tensor R on V, one needs no more than $\frac{1}{2}n(n+1)$ symmetric endomorphisms $A_1, \ldots, A_{\frac{1}{2}n(n+1)}$ so that R is a linear combination of the tensors R_{A_i}. This is a remarkable result, since the dimension of $\mathcal{A}(V)$ is $\frac{1}{12}n^2(n^2-1)$ [11]. (A similar result with the same bound of $\frac{1}{2}n(n+1)$ is known for the vector space of algebraic covariant derivative curvature tensors as well [3].) The authors in [4] additionally show that if $n = 3$, then one needs at most two canonical algebraic curvature tensors to express any $R \in \mathcal{A}(V)$, and establish when one may express one canonical algebraic curvature tensor as the linear combination of two others. Thus, the bound of $\frac{1}{2}n(n+1)$ is not optimal, and, in an effort to improve upon this bound, given symmetric endomorphisms φ, ψ of V, it is an interesting question as to when there exists another symmetric endomorphism τ of V for which $\pm R_\varphi \pm R_\psi = R_\tau$. This is a question of linear dependence, and Section 5 is devoted to this study when

Electronic Journal of Linear Algebra ISSN 1081-3810
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$n \geq 4$.

In this paper, it is our goal to present new results related to the linear independence of sets of two and three algebraic curvature tensors defined by symmetric operators, as in Definition 1.1. To make these results more accessible, we present the proofs of these new results using elementary methods.

A brief outline of the paper is as follows. Throughout, we assume that ψ and τ are symmetric endomorphisms, so that $R_\psi, R_\tau \in \mathcal{A}(V)$. In Section 2, we prove the following result which will be of later use. We note Theorem 1.3 is a generalization of Lemma 1.8.6 of [11]; here we present a proof using different methods, and that does not require that $\ker A$ have no vectors x so that $\varphi(x,x) > 0$.

THEOREM 1.3. *Let $A : V \to V$, and $R_A \in \mathcal{A}(V)$. If* Rank $A \geq 3$*, then $A^* = A$.*

We will use Theorem 1.3 to establish the following:

THEOREM 1.4. *Let $A : V \to V$, and $R_A = R_\psi \in \mathcal{A}(V)$. If* Rank $A \geq 3$*, then A is symmetric, and $A = \pm\psi$.*

After some brief introductory remarks, in Section 3 we prove our main result regarding the linear independence of two algebraic curvature tensors–it will be more convenient to state these conditions in terms of linear dependence, rather than linear independence. We will prove

THEOREM 1.5. *Suppose* Rank $\varphi \geq 3$*. The set $\{R_\varphi, R_\psi\}$ is linearly dependent if and only if $R_\psi \neq 0$, and $\varphi = \lambda\psi$ for some $\lambda \in \mathbb{R}$.*

We begin our study of linear independence of three curvature tensors in Section 4. The following will be a result crucial to our study of linear independence of three (or fewer) algebraic curvature tensors. Using Theorem 1.4, we establish the following:

THEOREM 1.6. *Suppose φ is positive definite,* Rank $\tau = n$*, and* Rank $\psi \geq 3$*. If $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent, then ψ and τ are simultaneously orthogonally diagonalizable with respect to φ.*

The proof of the result above will first establish that the operators ψ and τ commute, and it will follow that ψ and τ are simultaneously diagonalizable [9]. Of course, there are large areas of mathematics that are concerned with the commutativity of linear operators, although recently there has been an interest in the commutativity of certain operators associated to the Riemann curvature tensor in differential geometry. Tsankov proved [16] that if (M, g) is a Riemannian hypersurface in Euclidean space, then $\mathcal{J}(x)\mathcal{J}(y) = \mathcal{J}(y)\mathcal{J}(x)$ for $x \perp y$ if and only if R_g has constant sectional curvature, where $\mathcal{J}(x)$ is the Jacobi operator. This gave rise to a subsequent study of a study of the Tsankov condition in pseudo-Riemannian geometry, along with other

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 436-448, August 2010

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related notions. For an excellent survey on these and studies related to commuting curvature operators, see [2].

In Section 5, we use Theorem 1.6 to prove our main result regarding the linear independence of three algebraic curvature tensors. As in Theorem 1.5, it is more convenient to state necessary and sufficient conditions in terms of linear dependence, rather than linear independence. If A is an endomorphism of V, we denote the *spectrum* of A, $\mathrm{Spec}(A)$, as the set of eigenvalues of A, repeated according to multiplicity, and $|\mathrm{Spec}(A)|$ as the number of distinct elements of $\mathrm{Spec}(A)$. It is understood that if any of the quantities do not make sense in Condition (2) below, then Condition (2) is not satisfied.

THEOREM 1.7. *Suppose* $\dim(V) \geq 4$, *φ is positive definite,* $\mathrm{Rank}\ \tau = n$, *and* $\mathrm{Rank}\ \psi \geq 3$. *The set $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent if and only if one of the following is true:*

1. $|\mathrm{Spec}(\psi)| = |\mathrm{Spec}(\tau)| = 1$.
2. $\mathrm{Spec}(\tau) = \{\eta_1, \eta_2, \eta_2, \ldots\}$, *and* $\mathrm{Spec}(\psi) = \{\lambda_1, \lambda_2, \lambda_2, \ldots\}$, *with* $\eta_1 \neq \eta_2$, $\lambda_2^2 = \epsilon(\delta\eta_2^2 - 1)$, *and* $\lambda_1 = \frac{\epsilon}{\lambda_2}(\delta\eta_1\eta_2 - 1)$, *where* ε, $\delta = \pm 1$.

It is interesting to note that in all of our results except Theorem 1.7, we assume the rank of a certain object to be at least 3. However, in Theorem 1.7 we require that $\dim(V) \geq 4$, but there is no corresponding requirement that all objects involved have a rank of at least 4. Indeed, there exists examples in dimension 3 where Theorem 1.7 does not hold, although the situation is more complicated. See Theorem 5.1 in Section 5 for a detailed description of this situation.

2. A study of the tensors R_A.

Our main objective for this section is to establish Theorem 1.4. We begin with the lemma below.

LEMMA 2.1. *Suppose* $A, B, \bar{A} : V \to V$.

1. $R_{A+B} + R_{A-B} = 2R_A + 2R_B$.
2. *If* $R_A \in \mathcal{A}(V)$, *then* $R_{A^*} \in \mathcal{A}(V)$, *and* $R_A = R_{A^*}$.
3. *If* $\bar{A}^* = -\bar{A}$ *and* $R_{\bar{A}} \in \mathcal{A}(V)$, *then* $R_{\bar{A}}(x, y, z, w) = \varphi(\bar{A}x, y)\varphi(\bar{A}w, z)$.

Proof. Assertion (1) follows from direct computation using Definition 1.1. To prove Assertion (2), let $x, y, z, w \in V$, and we compute

$$\begin{aligned}
R_A(x, y, z, w) &= R_A(z, w, x, y) \\
&= \varphi(Az, y)\varphi(Aw, x) - \varphi(Az, x)\varphi(Aw, y) \\
&= \varphi(A^*y, z)\varphi(A^*x, w) - \varphi(A^*x, z)\varphi(A^*y, w) \\
&= R_{A^*}(x, y, z, w).
\end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 436-448, August 2010

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Now we prove Assertion (3). Note that since $\bar{A}^* = -\bar{A}$, for all $u, v \in V$ we have $\varphi(\bar{A}v, u) = -\varphi(v, \bar{A}u) = -\varphi(\bar{A}u, v)$. We use the Bianchi identity to see that

$$\begin{aligned} 0 &= R_{\bar{A}}(x,y,z,w) + R_{\bar{A}}(x,w,y,z) + R_{\bar{A}}(x,z,w,y) \\ &= \quad \varphi(\bar{A}x,w)\varphi(\bar{A}y,z) - \varphi(\bar{A}x,z)\varphi(\bar{A}y,w) \\ &\quad +\varphi(\bar{A}x,z)\varphi(\bar{A}w,y) - \varphi(\bar{A}x,y)\varphi(\bar{A}w,z) \\ &\quad +\varphi(\bar{A}x,y)\varphi(\bar{A}z,w) - \varphi(\bar{A}x,w)\varphi(\bar{A}z,y) \\ &= 2\varphi(\bar{A}x,w)\varphi(\bar{A}y,z) - 2\varphi(\bar{A}x,z)\varphi(\bar{A}y,w) \\ &\qquad\qquad +2\varphi(\bar{A}x,y)\varphi(\bar{A}z,w) \\ &= 2R_{\bar{A}}(x,y,z,w) + 2\varphi(\bar{A}x,y)\varphi(\bar{A}z,w). \end{aligned}$$

It follows that $R_{\bar{A}}(x,y,z,w) = -\varphi(\bar{A}x,y)\varphi(\bar{A}z,w) = \varphi(\bar{A}x,y)\varphi(\bar{A}w,z)$. ☐

Remark 2.2. With regards to Assertion (2) of Lemma 2.1, we will only require that $R_{A^*} \in \mathcal{A}(V)$ if $R_A \in \mathcal{A}(V)$. The fact that $R_{A^*} = R_A$ is not needed, although it simplifies some calculations (see Equation (2.a)).

We now present a proof of Theorem 1.3.

Proof. (Proof of Theorem 1.3.) Define $\bar{A} = A - A^*$. Then $\bar{A}^* = -\bar{A}$. Then using $B = A^*$ in Assertion (1) of Lemma 2.1 we have $R_A = R_{A^*} \in \mathcal{A}(V)$ and

$$R_{\bar{A}} = 4R_A - R_{A+A^*}. \tag{2.a}$$

Since $(A + A^*)^* = A + A^*$, we have $R_{A+A^*} \in \mathcal{A}(V)$, and so, as the linear combination of algebraic curvature tensors, $R_{\bar{A}} \in \mathcal{A}(V)$. Thus by Lemma 2.1, we conclude that $R_{\bar{A}}(x,y,z,w) = \varphi(\bar{A}x,y)\varphi(\bar{A}w,z)$.

Since $\bar{A}$ is skew-symmetric with respect to φ, Rank $\bar{A}$ is even. We note that if Rank $\bar{A} = 0$, then $\bar{A} = 0$, and $A = A^*$. So we break the remainder of the proof up into two cases: Rank $\bar{A} \geq 4$, and Rank $\bar{A} = 2$.

Suppose Rank $\bar{A} \geq 4$. Then there exist $x, y, z, w \in V$ with

$$\varphi(\bar{A}x,y) = \varphi(\bar{A}w,z) = 1, \text{ and } \varphi(\bar{A}x,z) = \varphi(\bar{A}x,w) = 0.$$

Then we compute $R_{\bar{A}}(x,y,z,w)$ in two ways. First, we use Definition 1.1, and next we use Assertion (3) of Lemma 2.1:

$$\begin{aligned} R_{\bar{A}}(x,y,z,w) &= \varphi(\bar{A}x,w)\varphi(\bar{A}y,z) - \varphi(\bar{A}x,z)\varphi(\bar{A}y,w) \\ &= 0. \\ R_{\bar{A}}(x,y,z,w) &= \varphi(\bar{A}x,y)\varphi(\bar{A}w,z) \\ &= 1. \end{aligned}$$

This contradiction shows that Rank $\bar{A}$ is not 4 or more.

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A publication of the International Linear Algebra Society
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Finally, we assume Rank $\bar{A} = 2$. There exists a basis $\{e_1, e_2, \ldots, e_n\}$ that is orthonormal with respect to φ, where $\ker \bar{A} = \text{Span}\{e_3, \ldots, e_n\}$, and we have the relations $\varphi(\bar{A}e_2, e_1) = -\varphi(\bar{A}e_1, e_2) = \lambda \neq 0$. Let $A_{ij} = \varphi(Ae_i, e_j)$ be the (j,i) entry of the matrix A with respect to this basis, similarly for $\bar{A}$, A^*, and $A + A^*$. With respect to this basis, the only nonzero entries $\bar{A}_{ij}$ are $A_{12} - A_{12} = \bar{A}_{12} = -\bar{A}_{21} = \lambda$. Thus, unless $\{i, j\} = \{1, 2\}$, we have $A_{ij} = A_{ji}$, and in such a case, we have $(A + A^*)_{ij} = 2A_{ij}$.

Now suppose that $\{i, j\} \not\subseteq \{1, 2\}$. We compute $R_{\bar{A}}(e_i, e_2, e_j, e_1)$ in two ways. According to Assertion (3) of Lemma 2.1, we have

$$R_{\bar{A}}(e_i, e_2, e_j, e_1) = \varphi(\bar{A}e_i, e_2)\varphi(\bar{A}e_1, e_j) = 0. \tag{2.b}$$

Now according to Equation (2.a), we have

$$\begin{aligned} R_{\bar{A}}(e_i, e_2, e_j, e_1) &= (4R_A - R_{A+A^*})(e_i, e_2, e_j, e_1) \\ &= 4A_{i1}A_{2j} - 4A_{ij}A_{21} - (2A_{i1})(2A_{2j}) + (2A_{ij})(A_{21} + A_{12}) \\ &= 2A_{ij}(A_{12} - A_{21}) \\ &= 2\lambda A_{ij}. \end{aligned} \tag{2.c}$$

Comparing Equations (2.b) and (2.c) and recalling that $\lambda \neq 0$, we see that $A_{ij} = 0$ if one or both of i or j exceed 2. This shows that Rank $A \leq 2$, which is a contradiction to our original hypothesis. ∎

Remark 2.3. If Rank $A = 1$ or 0, then $R_A = 0$, and in the rank 1 case, A^* need not equal A. In addition, if A is *any* rank 2 endomorphism, then there exists examples of $0 \neq R_A \in \mathcal{A}(V)$ where $A^* \neq A$. Thus, Theorem 1.3 can fail if Rank $A \leq 2$.

We may now prove Theorem 1.4 as a corollary to Theorem 1.3. The following result found in [12] will be of use, and we state it here for completeness.

Lemma 2.4. *If* Rank $\varphi \geq 3$, *then* $R_\varphi = R_\psi$ *if and only if* $\varphi = \pm\psi$.

Proof. (Proof of Theorem 1.4.) According to Theorem 1.3, we conclude A is symmetric. Since Rank $A \geq 3$ we apply Lemma 2.4 to conclude $A = \pm\psi$. ∎

3. Linear independence of two algebraic curvature tensors. This section begins our study of linear independence of algebraic curvature tensors. Some preliminary remarks are in order before we begin this study.

Let $A, A_i : V \to V$ be a collection of symmetric endomorphisms. It is easy to verify that for any real number c, we have $cR_A = \epsilon R_{|c|^{\frac{1}{2}}A}$, where $\epsilon = \text{sign}(c) = \pm 1$. Let $c_i \in \mathbb{R}$, and let $\epsilon_i = \text{sign}(c_i) = \pm 1$. By replacing A_i with $B_i = |c|^{\frac{1}{2}}A_i$, we may

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 436-448, August 2010

express any linear combination of algebraic curvature tensors

$$\sum_{i=1}^{k} c_i R_{A_i} = \sum_{i=1}^{k} \epsilon_i R_{B_i}.$$

Thus the study of linear independence of algebraic curvature tensors amounts to a study of when a sum or difference of R_{A_i} equal another canonical algebraic curvature tensor. This would always be the case if each of the A_i are multiples of one another–this possibility is discussed here. Proceeding systematically from the case of two algebraic curvature tensors, we would assume that each of the constants c_i are nonzero, leading us to study the equation $R_{A_1} \pm R_{A_2} = 0$ in this section, and, for ϵ and δ a choice of signs, $R_{A_1} + \epsilon R_{A_2} = \delta R_{A_3}$ in Section 5.

We begin with a lemma exploring the possibility that $R_\varphi = -R_\psi$. The proof follows similarly to the proof in [12].

Lemma 3.1. *Suppose* Rank $\varphi \geq 3$. *There does not exist a symmetric* ψ *so that* $R_\varphi = -R_\psi$.

Proof. Suppose to the contrary that there is such a solution. By replacing φ with $-\varphi$ if need be, we may assume that there are vectors e_1, e_2, e_3 with the relations $\varphi(e_1, e_1) = \varphi(e_2, e_2) = \epsilon\varphi(e_3, e_3) = 1$, where $\epsilon = \pm 1$. Thus on $\pi = \text{Span}\{e_1, e_2\}$, the form $\varphi|_\pi$ is positive definite, and we may diagonalize $\psi|_\pi$ with respect to $\varphi|_\pi$. Therefore the matrix $[(\psi|_\pi)_{ij}]$ of $\psi|_\pi$ has $(\psi|_\pi)_{12} = (\psi|_\pi)_{21} = 0$, and $(\psi|_\pi)_{ii} = \lambda_i$ for $i = 1, 2, 3$. Now we compute

$$1 = R_\varphi(e_1, e_2, e_2, e_1) = -\lambda_1\lambda_2, \tag{3.a}$$

so λ_1 and $\lambda_2 \neq 0$. We compute

$$0 = R_\varphi(e_1, e_2, e_3, e_1) = -\lambda_1(\psi|_\pi)_{23},$$

so $(\psi|_\pi)_{23} = 0$. Similarly,

$$0 = R_\varphi(e_2, e_1, e_3, e_2) = -\lambda_2(\psi|_\pi)_{13},$$

and so $(\psi_\pi)_{13} = 0$. Now, for $j = 1, 2$, we have

$$\epsilon = R_\varphi(e_j, e_3, e_3, e_j) = -\lambda_j\lambda_3.$$

We conclude $\lambda_3 \neq 0$, and that $\lambda_1 = \lambda_2$. This contradicts Equation (3.a), since it would follow that $1 = -\lambda_1^2 < 0$. ☐

We now use Lemma 3.1 to establish Theorem 1.5.

Proof. (Proof of Theorem 1.5.) Suppose $c_1R_\varphi + c_2R_\psi = 0$, and at least one of c_1 or c_2 is not zero. Since $R_\psi \neq 0$, and φ is of rank 3 or more (which implies $R_\varphi \neq 0$),

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A publication of the International Linear Algebra Society
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we conclude that both $c_1, c_2 \neq 0$. Thus, we may write

$$c_1 R_\varphi + c_2 R_\psi = 0 \Leftrightarrow R_\varphi = \epsilon R_{\tilde{\lambda}\psi}$$

for some $\tilde{\lambda} \neq 0$, and for ϵ a choice of signs. If $\epsilon = 1$, then we use Lemma 2.4 to conclude that $\varphi = \pm\tilde{\lambda}\psi$, in which case $\varphi = \lambda\psi$ for $0 \neq \lambda = \pm\tilde{\lambda}$. Lemma 3.1 eliminates the possibility that $\epsilon = -1$.

Conversely, suppose $\varphi = \lambda\psi$ for some $\lambda \neq 0$. Then we have

$$\begin{aligned} R_\varphi + (-\lambda^2) R_\psi &= R_{\lambda\psi} + (-\lambda^2) R_\psi \\ &= \lambda^2 R_\psi + (-\lambda^2) R_\psi \\ &= 0. \end{aligned}$$

This demonstrates the linear dependence of the tensors R_φ and R_ψ and completes the proof. □

4. Commuting symmetric endomorphisms. Our main objective in this section is to establish Theorem 1.6 concerning the simultaneous diagonalization of the endomorphisms ψ and τ with respect to φ. The following lemma is easily verified using Definition 1.1.

LEMMA 4.1. *Suppose $\theta : V \to V$. For all $x, y, z, w \in V$, we have*

$$R_\theta(x, y, z, w) = R_\varphi(\theta x, \theta y, z, w) = R_\varphi(x, y, \theta^* z, \theta^* w).$$

We may now provide a proof to Theorem 1.6.

Proof. (Proof of Theorem 1.6.) Suppose $c_1 R_\varphi + c_2 R_\psi + c_3 R_\tau = 0$. According to the discussion at the beginning of the previous section, we reduce the situation to one of two cases. If one or more of the c_i are zero, then since none of φ, ψ or τ have a rank less than 3, Theorem 1.5 applies, and the result holds. Otherwise, we have all $c_i \neq 0$, and we are reduced to the case that $R_\varphi + \epsilon R_\psi = \delta R_\tau$, where ϵ and δ are a choice of signs.

Let $x, y, z, w \in V$. By hypothesis, τ^{-1} exists. Note first that τ is self-adjoint with respect to φ, so that according to Lemma 4.1

$$\begin{aligned} R_\varphi(\tau x, \tau y, \tau^{-1} z, \tau^{-1} w) &= R_\varphi(x, y, z, w), \text{and} \\ R_\tau(\tau x, \tau y, \tau^{-1} z, \tau^{-1} w) &= R_\varphi(\tau x, \tau y, z, w) \\ &= R_\tau(x, y, z, w). \end{aligned}$$

Note that τ is self-adjoint with respect to φ if and only if τ^{-1} is self-adjoint with

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A publication of the International Linear Algebra Society
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respect to φ. Now we use the hypothesis $R_\varphi + \epsilon R_\psi = \delta R_\tau$ and Lemma 4.1 to see that

$$\begin{aligned}
\delta R_\tau(x,y,z,w) &= \delta R_\tau(\tau x, \tau y, \tau^{-1}z, \tau^{-1}w) \\
&= R_\varphi(\tau x, \tau y, \tau^{-1}z, \tau^{-1}w) + \epsilon R_\psi(\tau x, \tau y, \tau^{-1}z, \tau^{-1}w) \\
&= R_\varphi(x,y,z,w) + \epsilon R_\varphi(\psi\tau x, \psi\tau y, \tau^{-1}z, \tau^{-1}w) \\
&= R_\varphi(x,y,z,w) + \epsilon R_\varphi(\tau^{-1}\psi\tau x, \tau^{-1}\psi\tau y, z, w) \\
&= R_\varphi(x,y,z,w) + \epsilon R_{\tau^{-1}\psi\tau}(x,y,z,w).
\end{aligned}$$

It follows that $R_{\tau^{-1}\psi\tau} = R_\psi$. Now Rank $\tau^{-1}\psi\tau =$ Rank $\psi \geq 3$, so using $A = \tau^{-1}\psi\tau$ in Theorem 1.4 gives us $\tau^{-1}\psi\tau = \pm\psi$. We show presently that $\tau^{-1}\psi\tau = -\psi$ is not possible.

Suppose $\tau^{-1}\psi\tau = -\psi$. We diagonalize τ with respect to φ with the basis $\{e_1, \ldots, e_n\}$. Suppose i, j, and k are distinct indices. With respect to this basis, for all $v \in V$ we have

$$R_\varphi(e_i, e_k, v, e_j) = R_\tau(e_i, e_k, v, e_j) = 0.$$

Let ψ_{ij} be the (j,i) entry of ψ with respect to this basis. Since τ and ψ anti-commute, and τ is diagonal, $\psi_{ii} = 0$. Thus there exists an entry $\psi_{ij} \neq 0$. Fix this i and j for the remainder of the proof. We must have $i \neq j$. Then for indices i, j, k, ℓ with i, j, k distinct, we have

$$\begin{aligned}
0 = \delta R_\tau(e_i, e_k, e_\ell, e_j) &= (R_\varphi + \epsilon R_\psi)(e_i, e_k, e_\ell, e_j) \\
&= \epsilon R_\psi(e_i, e_k, e_\ell, e_j) \\
&= \epsilon(\psi_{ij}\psi_{k\ell} - \psi_{i\ell}\psi_{kj}).
\end{aligned} \tag{4.a}$$

If $\ell = i$, then Equation (4.a) with $\psi_{ii} = 0$ and $\psi_{ij} \neq 0$ shows that $\psi_{ki} = \psi_{ik} = 0$ for all $k \neq j$. Exchanging the roles of i and j and setting $\ell = j$ shows that $\psi_{jk} = \psi_{kj} = 0$ as well. Finally, for i, j, k, ℓ distinct, we use Equation (4.a) again to see that $\psi_{k\ell} = \psi_{\ell k} = 0$. Thus, under the assumption that there is at least one nonzero entry in the matrix $[\psi_{ab}]$ leads us to the conclusion that there are at most two nonzero entries $(\psi_{ij} = \psi_{ji} \neq 0)$ in $[\psi_{ab}]$. This contradicts the assumption that Rank $\psi \geq 3$. We conclude that ψ and τ must not anticommute.

Otherwise, we have $\tau^{-1}\psi\tau = \psi$, and so $\psi\tau = \tau\psi$. Thus, we may simultaneously diagonalize ψ and τ. ☐

5. Linear independence of three algebraic curvature tensors. We may now use our previous results to establish our main results concerning the linear independence of three algebraic curvature tensors. This section is devoted to the proof of Theorem 1.7, and to the description of the exceptional setting when $\dim(V) = 3$.

Proof. (Proof of Theorem 1.7.) We assume first that $\{R_\varphi, R_\psi, R_\tau\}$ is linearly dependent, and show that one of Conditions (1) or (2) must be satisfied. As such,

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A publication of the International Linear Algebra Society
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we suppose there exist c_i (not all zero) so that $c_1R_\varphi + c_2R_\psi + c_3R_\tau = 0$. As in the proof of Theorem 1.6 in Section 4, if any of the c_i are zero, then this case reduces to Theorem 1.5, and all of the forms involved are real multiples of one another. Namely, $|\text{Spec}(\psi)| = |\text{Spec}(\tau)| = 1$, and Condition (1) holds.

So we consider the situation that none of the c_i are zero. This question of linear dependence reduces to the equation

$$R_\varphi + \epsilon R_\psi = \delta R_\tau. \tag{5.a}$$

We use Theorem 1.6 to simultaneously diagonalize ψ and τ with respect to φ to find a basis $\{e_1, \ldots, e_n\}$ which is orthonormal with respect to φ. Therefore, if $\text{Spec}(\psi) = \{\lambda_1, \ldots, \lambda_n\}$, and $\text{Spec}(\tau) = \{\eta_1, \ldots, \eta_n\}$, evaluating Equation (5.a) at (e_i, e_j, e_j, e_i) gives us the equations

$$1 + \epsilon\lambda_i\lambda_j = \delta\eta_i\eta_j \tag{5.b}$$

for any $i \neq j$. The remainder of this portion of the proof eliminates all possibilities except those found in Conditions (1) and (2).

If $|\text{Spec}(\tau)| \geq 3$, then we permute the basis vectors so that η_1, η_2 and η_3 are distinct. Then we have, according to Equation (5.b) for i, j, and k distinct:

$$\begin{aligned} 1 + \epsilon\lambda_i\lambda_j &= \delta\eta_i\eta_j \\ 1 + \epsilon\lambda_i\lambda_k &= \delta\eta_i\eta_k, \text{ so subtracting,} \\ \epsilon\lambda_i(\lambda_j - \lambda_k) &= \delta\eta_i(\eta_j - \eta_k). \end{aligned} \tag{5.c}$$

All $\eta_i \neq 0$ since $\det\tau \neq 0$, and so since $\eta_j \neq \eta_k$, the above equation shows that all $\lambda_i \neq 0$, and that $\lambda_j \neq \lambda_k$ for $\{i, j, k\} = \{1, 2, 3\}$. Since $\dim(V) \geq 4$, we may compute

$$\begin{aligned} (1 + \epsilon\lambda_1\lambda_2)(1 + \epsilon\lambda_3\lambda_4) &= \eta_1\eta_2\eta_3\eta_4 \\ &= (1 + \epsilon\lambda_1\lambda_3)(1 + \epsilon\lambda_2\lambda_4). \end{aligned}$$

Multiplying the above and cancelling, we have $\lambda_1\lambda_2 + \lambda_3\lambda_4 = \lambda_1\lambda_3 + \lambda_2\lambda_4$. In other words, $\lambda_2(\lambda_1 - \lambda_4) = \lambda_3(\lambda_1 - \lambda_4)$. Since $\lambda_2 \neq \lambda_3$, we conclude $\lambda_1 = \lambda_4$. Performing the same manipulation, we have

$$\begin{aligned} (1 + \epsilon\lambda_1\lambda_4)(1 + \epsilon\lambda_2\lambda_3) &= \eta_1\eta_2\eta_3\eta_4 \\ &= (1 + \epsilon\lambda_1\lambda_3)(1 + \epsilon\lambda_2\lambda_4). \end{aligned}$$

One then concludes, similarly to above, that $\lambda_4 = \lambda_3$. This is a contradiction, since $\lambda_4 = \lambda_1 \neq \lambda_3 = \lambda_4$.

Now suppose that $|\text{Spec}(\tau)| = 2$, and that there are at least two pairs of repeated eigenvalues of τ. We may assume that $\text{Spec}(\tau) = \{\eta_1, \eta_1, \eta_3, \eta_3, \ldots\}$, and that $\eta_1 \neq \eta_3$. Proceeding as in Equation (5.b), we have

$$1 + \epsilon\lambda_1\lambda_3 = 1 + \epsilon\lambda_1\lambda_4 = \delta\eta_1\eta_3 = 1 + \epsilon\lambda_2\lambda_3 = 1 + \epsilon\lambda_2\lambda_4.$$

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Now, as in Equation (5.c) we have the equations

$$\begin{aligned} &\lambda_1(\lambda_3-\lambda_4)=0, \qquad \lambda_3(\lambda_1-\lambda_2)=0,\\ &\lambda_2(\lambda_3-\lambda_4)=0, \qquad \lambda_4(\lambda_1-\lambda_2)=0. \end{aligned} \tag{5.d}$$

Now according to Equation (5.b), we have $1+\epsilon\lambda_1\lambda_2=\delta\eta_1^2$, and $1+\epsilon\lambda_1\lambda_3=\delta\eta_1\eta_2$. Subtracting, we conclude $\epsilon\lambda_1(\lambda_2-\lambda_3)=\delta\eta_1(\eta_1-\eta_3)\neq 0$. Thus $\lambda_1\neq 0$, and $\lambda_2\neq\lambda_3$. We use a similar argument to conclude λ_2,λ_3, and $\lambda_4\neq 0$. According to Equation (5.d), we have $\lambda_3=\lambda_4$, and $\lambda_1=\lambda_2$.

We find a contradiction after performing one more calculation from Equation (5.b). Note that

$$(1+\epsilon\lambda_1^2)(1+\epsilon\lambda_3^2)=\eta_1^2\eta_1^2=(1+\epsilon\lambda_1\lambda_3)(1+\epsilon\lambda_1\lambda_3).$$

After multiplying out and cancelling the common constant and quartic terms, we conclude

$$\begin{aligned} &\lambda_1^2+\lambda_3^2=2\lambda_1\lambda_3\\ \Leftrightarrow\quad &(\lambda_1-\lambda_3)^2=0\\ \Leftrightarrow\quad &\lambda_1=\lambda_3. \end{aligned}$$

This is a contradiction since $\lambda_1=\lambda_2\neq\lambda_3$.

In order to finish the proof of one implication in Theorem 1.7, we consider the case that $\mathrm{Spec}(\tau)=\{\eta_1,\eta_2,\eta_2,\ldots\}$. Using $i=1$ in Equation (5.b), we see that $\lambda_j=\lambda_2$ for all $j\geq 2$. In that event we may solve for λ_2 and λ_1 to be as given in Condition (2) of Theorem 1.7. This concludes the proof that if the set $\{R_\varphi,R_\psi,R_\tau\}$ is linearly dependent, then Condition (1) or Condition (2) must hold.

Conversely, we suppose one of Condition (1) or (2) from Theorem 1.7 holds, and show that the set $\{R_\varphi,R_\psi,R_\tau\}$ is linearly dependent. If Condition (1) is satisfied, then $\psi=\lambda\varphi$, and $\tau=\eta\varphi$ for some λ and η. The set $\{R_\varphi,R_\psi\}$ is a linearly dependent set by Theorem 1.5, and so it follows that $\{R_\varphi,R_\psi,R_\tau\}$ is linearly dependent as well. If Condition (2) holds, then the discussion already presented in the above paragraph shows that, for this choice of ψ and τ, that $R_\varphi+\epsilon R_\psi=\delta R_\tau$. $\square$

The following result shows that the assumption that $\dim(V)=4$ is necessary in Theorem 1.7 by exhibiting (in certain cases) a unique solution up to sign ψ of full rank in the case $\dim(V)=3$. Of course, our assumptions put certain restrictions on the eigenvalues η_i of τ for there to exist a solution, in particular, since we assume ψ and τ have full rank, none of their eigenvalues can be 0.

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A publication of the International Linear Algebra Society
Volume 20, pp. 436-448, August 2010

ELA

THEOREM 5.1. *Let ϕ be a positive definite symmetric bilinear form on a real vector space V of dimension 3. Suppose* $\mathrm{Spec}(\tau) = \{\eta_1, \eta_2, \eta_3\}$*, and* ε*,* $\delta = \pm 1$*. Set*

$$\eta(i,j,k) = (-\epsilon)\sqrt{\frac{(1-\delta\eta_i\eta_j)(1-\delta\eta_i\eta_k)}{(-\epsilon)(1-\delta\eta_j\eta_k)}}.$$

If $\mathrm{Rank}\ \psi = \mathrm{Rank}\ \tau = 3$*, and* $\mathrm{Spec}(\psi) = \{\lambda_1, \lambda_2, \lambda_3\}$*, where*

$$\lambda_1 = (-\epsilon)\eta(1,2,3), \quad \lambda_2 = (-\epsilon)\eta(2,3,1), \quad \lambda_3 = \eta(3,1,2),$$

then ψ *and* $-\psi$ *are the only solutions to the equation* $R_\varphi + \epsilon R_\psi = \delta R_\tau$*.*

Proof. If a solution exists, we use Theorem 1.6 that orthogonally diagonalizes ψ and τ with respect to φ. The diagonal entries λ_i of ψ and η_i of τ are their respective eigenvalues. We have the following equations for $i \neq j$:

$$\begin{aligned} \epsilon R_\psi(e_i,e_j,e_j,e_i) \quad = \quad \epsilon\lambda_i\lambda_j &= -R_\phi(e_i,e_j,e_j,e_i) + \delta R_\tau(e_i,e_j,e_j,e_i) \\ &= -1 + \delta\eta_i\eta_j. \end{aligned}$$

Since ψ has full rank, we know $\lambda_3 \neq 0$. Solving for λ_1 and λ_2 gives

$$\lambda_1 = \frac{-1+\delta\eta_1\eta_3}{\epsilon\lambda_3}, \text{and } \lambda_2 = \frac{-1+\delta\eta_2\eta_3}{\epsilon\lambda_3}.$$

Substituting into $R_\psi(e_1, e_2, e_2, e_1)$ gives

$$\epsilon\left(\frac{-1+\delta\eta_1\eta_3}{\epsilon\lambda_3}\right)\left(\frac{-1+\delta\eta_2\eta_3}{\epsilon\lambda_3}\right) = -1+\delta\eta_1\eta_2, \text{ so}$$

$$\frac{\epsilon(-1+\delta\eta_1\eta_3)(-1+\delta\eta_2\eta_3)}{\epsilon^2\lambda_3^2} = -1+\delta\eta_1\eta_2, \text{ and so}$$

$$\frac{(-1+\delta\eta_1\eta_3)(-1+\delta\eta_2\eta_3)}{(-\epsilon)(1-\delta\eta_1\eta_2)} = \eta(3,1,2)^2 = \lambda_3^2.$$

One checks that for these values of λ_1, λ_2, and λ_3, that ψ is a solution, and that these λ_i are completely determined in this way by the η_i, hence, they are the only solutions. ∎

Acknowledgments. It is a pleasure to thank R. Trapp for helpful conversations while this research was conducted. The authors would also like to thank the referee as well. This research was jointly funded by the NSF grant DMS-0850959, and California State University, San Bernardino.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 436-448, August 2010

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