



ON THE NUMERICAL RANGE OF KAC-SYLVESTER MATRICES*

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Abstract. In this paper, the boundary generating curves and the numerical range of Kac–Sylvester matrices up to the order 9 are characterized. Based on the obtained results and on several computational experiments performed with the Mathematica and MatLab programs, we conjecture that the found types of algebraic curves, namely ellipses and ovals, will appear for an arbitrary order.

Key words. Numerical range, Kac–Sylvester matrices, Plane algebraic curves.

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1. Introduction. Let M_n stand for the associative algebra of $n \times n$ complex matrices, and I_k be the identity matrix of order k . Let the space \mathbb{C}^n be endowed with the standard *inner product*: $\langle x, y \rangle = y^*x$, $x, y \in \mathbb{C}^n$.

The *numerical range* (NR), also called *field of values*, of a matrix $A \in M_n$ is denoted and defined by

$$W(A) = \{ \langle Ax, x \rangle : x \in \mathbb{C}^n, \langle x, x \rangle = 1 \}.$$

This concept was introduced a century ago in pioneering works by O. Toeplitz [19] and F. Hausdorff [12], and since then it has been intensively investigated due to its theoretical interest and applications. The numerical range is a closed, bounded, connected subset of \mathbb{C} , since it is the image of the unit sphere of \mathbb{C}^n under the continuous mapping $f_A(x) = \langle Ax, x \rangle$. Moreover, it is well known that $W(A)$ contains $\sigma(A)$, the spectrum of A , and it is a convex set, as asserted by the famous Toeplitz–Hausdorff Theorem [12, 19].

A related concept is the numerical *radius* of $A \in M_n$, which is the radius of the smallest circular disc centered at the origin, containing the numerical range of A , i.e.,

$$w(A) = \max \{ |z| : z \in W(A) \} = \max_{\|x\|=1} |\langle Ax, x \rangle|.$$

This defines a norm in M_n equivalent to the operator norm [11].

A *supporting line* of a convex set $S \subset \mathbb{C}$ is a line containing a boundary point of S and defining two half-planes, such that one of them does not contain S . A boundary point of $W(A)$ belonging to more than one of its supporting lines is called a *corner* of $W(A)$. If z_0 is a corner of $W(A)$, then z_0 is an eigenvalue

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of A . If $A \in M_n$ is Hermitian, then $W(A)$ is a closed line segment, joining the smallest and the greatest eigenvalues of A .

For any square matrix $A \in M_n$, we have that

$$\Re(A) = \frac{A + A^*}{2} \quad \text{and} \quad \Im(A) = \frac{A - A^*}{2i},$$

are the Hermitian components of the Cartesian decomposition of A , which is given by $A = \Re(A) + i\Im(A)$. It can be easily seen that the orthogonal projections of $W(A)$ onto the real and imaginary axes are $W(\Re(A))$ and $W(\Im(A))$, respectively. Therefore, if $\lambda_1 \geq \dots \geq \lambda_n$ and $\mu_1 \geq \dots \geq \mu_n$ are the eigenvalues of $\Re(A)$ and $\Im(A)$, respectively, then we clearly have

$$W(A) \subset [\lambda_n, \lambda_1] \times [\mu_n, \mu_1].$$

For $A \in M_n$ and every angle $\theta \in \mathbb{R}$, let $H_\theta(A) = \Re(e^{-i\theta}A)$. The matrix

$$(1) \quad H_\theta(A) = \Re(A) \cos \theta + \Im(A) \sin \theta,$$

is Hermitian, so its eigenvalues are real and we consider them arranged in nonincreasing order. The characteristic polynomial of $H_\theta(A)$ is given by

$$p_{A,\theta}(z) = \det(\Re(A) \cos \theta + \Im(A) \sin \theta - zI_n)$$

and it plays an important role in the characterization of $W(A)$ and it is here called the *NR generating polynomial* of A .

Moreover, the *Kippenhahn polynomial* of $A \in M_n$ is the degree n homogeneous real ternary polynomial given by

$$p_A(x, y, z) = \det(x\Re(A) + y\Im(A) + zI_n).$$

The roots of $p_A(1, i, -z) = 0$ are the eigenvalues of A . The convex hull of the real points of the dual curve of the order n algebraic curve Γ_A defined by $p_A(x, y, z) = 0$ is the numerical range of A . In fact, to any matrix $A \in M_n$, through the equation $p_A(x, y, z) = 0$, is associated a *class n algebraic curve* in homogeneous line coordinates. The supporting lines of $W(A)$ are generating elements of this algebraic curve. Its real part, called the *boundary generating curve* of $W(A)$ and denoted by $C(A)$, generates $W(A)$ as its convex hull. More details on the generating curve can be found in [14, 15].

A matrix $A = (a_{ij}) \in M_n$ is *centrosymmetric* if it remains unchanged when reflected vertically and horizontally, that is, if $a_{ij} = a_{n-i+1, n-j+1}$ for all i, j . Let J_n be the *backward identity* or *exchange matrix* of order n , obtained from the identity matrix I_n by reversing the order of its columns, that is, the matrix with (i, j) entry 1 if $i + j = n + 1$ and 0 elsewhere. Clearly, $A \in M_n$ is centrosymmetric if and only if $A = J_n A J_n$. A matrix $A = (a_{ij}) \in M_n$ is *tridiagonal* if $a_{ij} = 0$ for $|i - j| > 0$. Centrosymmetric and tridiagonal matrices appear naturally in many places and have several applications.

In this paper, we focus our study on the class of *Kac–Sylvester matrices of order $n \geq 2$* , which are n -square tridiagonal matrices with zero main diagonal, superdiagonal $(1, 2, 3, \dots, n - 1)$ and subdiagonal $(n - 1, \dots, 3, 2, 1)$, that is, matrices of type

numerical range due to Chien et al. [5, Theorem 1]. For that, let the eigenvalues of $H_\theta(A)$ be labeled as follows:

$$\lambda_1(H_\theta(A)) \geq \lambda_2(H_\theta(A)) \geq \cdots \geq \lambda_n(H_\theta(A)).$$

PROPOSITION 2.2. *Let $A \in M_n$, $a, b > 0$. Then $W(A)$ is an elliptical disc, centered at the origin, with horizontal major semi-axis of length a and vertical minor semi-axis of length b if and only if*

$$\lambda_1(H_\theta(A)) = \sqrt{a^2 - (a^2 - b^2) \sin^2 \theta},$$

for any $0 \leq \theta < 2\pi$.

Kippenhahn [14] classified the boundary generating curves of the numerical range of matrices of order 3 as follows.

THEOREM 2.3. *A matrix $A \in M_3$ can only possess the following types of curves as its boundary generating curve:*

1. *three points;*
2. *a point and a nondegenerate ellipse;*
3. *a curve of order 4 with a double tangent and a cusp;*
4. *a proper curve of order 6, consisting of an oval and a curve with three cusps lying in its interior.*

This means that for $A \in M_3$, the shape of $W(A)$ is triangular (possibly degenerate), if $p_A(x, y, z)$ factors into three real linear factors; the convex hull of a nondegenerate elliptical disc and a point (possibly contained in the disc), if $p_A(x, y, z)$ factors into a real linear factor and an irreducible quadratic factor; has a smooth boundary curve with a flat portion, if $p_A(x, y, z)$ is irreducible and Γ_A has a real node; is ovular, if $p_A(x, y, z)$ is irreducible and Γ_A has no singular point [6].

Moreover, for matrices $A \in M_4$, Chien and Nakazato [6] classified the boundary generating curves of $W(A)$, via the factorability of the homogeneous ternary polynomial $p_A(x, y, z)$, as follows.

THEOREM 2.4. *The boundary generating curve of the numerical range of a matrix $A \in M_4$ falls into one of the following cases:*

1. *the vertices of a (possibly degenerate) quadrilateral;*
2. *a (nondegenerate) ellipse and two points, one or two of these points may be contained in the elliptical disc;*
3. *two (nondegenerate) ellipses, that may take arbitrary relative position;*
4. *the dual curve of an irreducible cubic curve and a point, which may be contained in the convex hull of the dual curve;*
5. *the dual of an irreducible quartic curve.*

Now, we recall that the spectral radius of a square matrix A , denoted by $\rho(A)$, is the maximum of the absolute values of its eigenvalues.

Concerning the numerical radius of the Kac–Sylvester matrices K_n , we have the following result.

PROPOSITION 2.5. *Let $n \geq 2$. The numerical radius of K_n is*

$$w(K_n) = n \cos \frac{\pi}{n+1}.$$

where E_{ij} denotes the m -square elementary matrix with 1 at the (i, j) entry and zero elsewhere, or

$$K_{2m+1} = \begin{bmatrix} R_m^o & m e_m & O \\ (m+1) e_m^T & 0 & (m+1) e_1^T \\ O & m e_1 & J R_m^o J \end{bmatrix}.$$

THEOREM 3.1. For $m \in \mathbb{N}$, let R_m^e and R_m^o be the matrices defined in (5).

(a) If $n = 2m$, then $W(K_n)$ is the convex hull of

$$W(R_m^e - mE_{mm}) \quad \text{and} \quad W(R_m^e + mE_{mm}).$$

(b) If $n = 2m + 1$, then $W(K_n)$ is the convex hull of $W(R_m^o)$ and $W(S_{m+1})$, where

$$S_{m+1} = \begin{bmatrix} 0 & (m+1)\sqrt{2}e_m^T \\ m\sqrt{2}e_m & R_m^o \end{bmatrix}.$$

Proof. We easily see that each Kac–Sylvester matrix is orthogonally similar to a block diagonal matrix. In fact,

$$Q = \frac{\sqrt{2}}{2} \begin{bmatrix} I_m & I_m \\ -J & J \end{bmatrix},$$

is an orthogonal matrix, such that

$$Q^T K_{2m} Q = \begin{bmatrix} R_m^e - JE_{1m} & O \\ O & R_m^e + JE_{1m} \end{bmatrix},$$

and $JE_{1m} = mE_{mm}$. Then

$$W(K_{2m}) = W((R_m^e - mE_{mm}) \oplus (R_m^e + mE_{mm})).$$

On the other hand,

$$Q = \frac{\sqrt{2}}{2} \begin{bmatrix} I_m & 0_m & I_m \\ 0_m^T & \sqrt{2} & 0_m^T \\ -J & 0_m & J \end{bmatrix},$$

is an orthogonal matrix, such that

$$Q^T K_{2m+1} Q = \begin{bmatrix} R_m^o & O \\ O & S_{m+1} \end{bmatrix}.$$

Then $W(K_{2m+1}) = W(R_m^o \oplus S_{m+1})$. Therefore, by property **W3**, we conclude that $W(K_{2m})$ is the convex hull of $W(R_m^e - mE_{mm})$ and $W(R_m^e + mE_{mm})$, whereas $W(K_{2m+1})$ is the convex hull of $W(R_m^o)$ and $W(S_{m+1})$. \square

3.1. Elliptical boundary generating curves. In this section, the shapes of the numerical range of the Kac–Sylvester matrices of order $n \leq 5$ are obtained. We will see that their boundary generating curves are a point and an ellipse, if $n = 3$, or two nonconcentric ellipses, if $n = 4$, or a point and two concentric ellipses, if $n = 5$.

We start by the smaller cases of odd order, that is, $n = 3$ and $n = 5$, in which cases this set is an elliptical disc.

THEOREM 3.2. *Let $n \leq 5$ be odd. The numerical range of K_n is an elliptical disc centered at the origin, with foci at $1 - n$ and $n - 1$, with horizontal major axis and vertical minor axis of lengths:*

- i. $3\sqrt{2}$ and $\sqrt{2}$, respectively, if $n = 3$;*
- ii. $5\sqrt{3}$ and $\sqrt{11}$, respectively, if $n = 5$.*

Proof. i. Let $n = 3$ and $m = 1$. In this case, $R_1^o = 0$ and, by Theorem 3.1, the set $W(K_3)$ is the convex hull of the origin and $W(S_2)$, where

$$S_2 = \begin{bmatrix} 0 & 2\sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix}.$$

The eigenvalues of S_2 are $-2, 2$ and $\text{Tr}(S_2^T S_2) = 10$. By the Elliptical Range Theorem, $W(S_2)$ is an elliptical disc with foci at $-2, 2$, horizontal major axis and vertical minor axes of lengths $3\sqrt{2}$ and $\sqrt{2}$, respectively. Thus, the boundary generating curves of $W(K_3)$ are a point, the origin, which is one of the eigenvalues of K_3 , and the previous ellipse, whose foci are at the remaining eigenvalues of K_3 . Therefore, the set $W(K_3)$ is bounded by the ellipse with Cartesian equation

$$\frac{x^2}{\frac{9}{2}} + \frac{y^2}{\frac{1}{2}} = 1.$$

ii. Let $n = 5$ and $m = 2$. By Theorem 3.1, the set $W(K_5)$ is the convex hull of $W(R_2^o)$ and $W(S_3)$, where

$$R_2^o = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad S_3 = \begin{bmatrix} 0 & 0 & 3\sqrt{2} \\ 0 & 0 & 1 \\ 2\sqrt{2} & 4 & 0 \end{bmatrix}.$$

The eigenvalues of R_2^o are $-2, 2$ and $\text{Tr}((R_2^o)^T R_2^o) = 17$. By the Elliptical Range Theorem, the set $W(R_2^o)$ is an elliptical disc with foci at $-2, 2$, horizontal major axis and vertical minor axis of lengths 5 and 3, respectively. Thus, one of the boundary generating curves of $W(K_5)$ is the ellipse that bounds $W(R_2^o)$.

The remaining eigenvalues of K_5 , which are $-4, 0, 4$, are those of S_3 and

$$\begin{aligned} H_\theta(S_3) &= \frac{5}{2} \begin{bmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 1 \\ \sqrt{2} & 1 & 0 \end{bmatrix} \cos \theta + \frac{1}{2i} \begin{bmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & -3 \\ -\sqrt{2} & 3 & 0 \end{bmatrix} \sin \theta \\ &= \frac{1}{2} \begin{bmatrix} 0 & 0 & \sqrt{2}(5 \cos \theta - i \sin \theta) \\ 0 & 0 & 5 \cos \theta + 3i \sin \theta \\ \sqrt{2}(5 \cos \theta + i \sin \theta) & 5 \cos \theta - 3i \sin \theta & 0 \end{bmatrix}. \end{aligned}$$

The NR generating polynomial of S_3 is given by

$$p_{S_3, \theta}(z) = -z \left(z^2 - 3 \frac{5^2}{2^2} \cos^2 \theta - \frac{2 + 3^2}{2^2} \sin^2 \theta \right),$$

with roots $z = 0$ and

$$z = \pm \frac{1}{2} \sqrt{3 \times 5^2 \cos^2 \theta + 11 \sin^2 \theta}.$$

Then the maximum eigenvalue of $H_\theta(S_3)$ is

$$\lambda_1(H_\theta(S_3)) = \frac{1}{2} \sqrt{3 \times 5^2 - (3 \times 5^2 - 11) \sin^2 \theta}.$$

By Proposition 2.2, the set $W(S_3)$ is an elliptical disc, with horizontal major axes and vertical minor axes of lengths $5\sqrt{3}$ and $\sqrt{11}$, respectively.

In this case, the boundary generating curves of $W(K_5)$ are a point, the origin, and two concentric ellipses, with foci at $-2, 2$ and $-4, 4$, respectively, defined by the Cartesian equations

$$\frac{x^2}{\left(\frac{5}{2}\right)^2} + \frac{y^2}{\left(\frac{3}{2}\right)^2} = 1 \quad \text{and} \quad \frac{x^2}{\frac{75}{4}} + \frac{y^2}{\frac{11}{4}} = 1.$$

Then the numerical range of K_5 is bounded by the outer ellipse. □

Next, we see that the numerical range of the Kac–Sylvester matrix of order 4 is the convex hull of two nonconcentric ellipses.

THEOREM 3.3. *The numerical range of the Kac–Sylvester matrix of order 4 is the convex hull of two ellipses, centered at the points -1 and 1 , with foci at $-3, 1$ and $-1, 3$, with horizontal major and minor semi-axes of lengths $\sqrt{5}$ and 1 , respectively, that is, the convex hull of the ellipses with Cartesian equations:*

$$\frac{(x+1)^2}{5} + y^2 = 1 \quad \text{and} \quad \frac{(x-1)^2}{5} + y^2 = 1.$$

Proof. Let $n = 4$ and $m = 2$. By Theorem 3.1, the set $W(K_4)$ is the convex hull of $W(R_2^e - 2E_{22})$ and $W(R_2^e + 2E_{22})$, where

$$R_2^e \pm 2E_{22} = \begin{bmatrix} 0 & 1 \\ 3 & \pm 2 \end{bmatrix}.$$

Concerning their spectra, we have

$$\sigma(R_2^e - 2E_{22}) = \{-3, 1\}, \quad \sigma(R_2^e + 2E_{22}) = \{-1, 3\},$$

and

$$\text{Tr}((R_2^e \pm 2E_{22})^T (R_2^e \pm 2E_{22})) = 14.$$

By the Elliptical Range Theorem, $W(R_2^e - 2E_{22})$ and $W(R_2^e + 2E_{22})$ are bounded by the ellipses, centered at -1 and 1 , with foci at $-3, 1$ and $-1, 3$, respectively, both with horizontal major axis of length $2\sqrt{5}$ and vertical minor axis of length 2 . Thus, the boundary generating curves of $W(K_4)$ are the previous ellipses and their convex hull gives $W(K_4)$. □

We remark that the convex hull of the ellipses generating the boundary of $W(K_4)$ yields a pair of horizontal flat portions on its boundary, namely the line segments defined by $-1 \leq x \leq 1$ and $y = \pm 1$.

3.2. Oval and flat portions on the boundary. In this section, we show that the numerical range of the Kac–Sylvester matrix of order 6 is the convex hull of two nonelliptical ovals, which yields a pair of horizontal flat portions on its boundary (cf. Figure 1).

THEOREM 3.4. *The numerical range of the Kac–Sylvester matrix of order 6 is the convex hull of two (nonelliptical) oval curves, symmetrically positioned with respect to the y -axis, which have horizontal tangent lines, containing flat portions of the boundary of $W(K_6)$, defined by*

$$(6) \quad -\frac{3}{10} \leq x \leq \frac{3}{10} \quad \text{and} \quad y = \pm\sqrt{5},$$

and vertical tangent lines given by $x = \pm w(K_6)$, with $w(K_6) \approx 5.4058$.

Proof. Let $n = 6$ and $m = 3$. By Theorem 3.1, the set $W(K_6)$ is the convex hull of $W(R_3^e - 3E_{33})$ and $W(R_3^e + 3E_{33})$, where

$$R_3^e \pm 3E_{33} = \begin{bmatrix} 0 & 1 & 0 \\ 5 & 0 & 2 \\ 0 & 4 & \pm 3 \end{bmatrix}.$$

Let $A_- = R_3^e - 3E_{33}$ and $A_+ = R_3^e + 3E_{33}$. Then

$$\sigma(A_-) = \{-5, -1, 3\}, \quad \sigma(A_+) = \{-3, 1, 5\},$$

and

$$\begin{aligned} H_\theta(A_\pm) &= 3 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & \pm 1 \end{bmatrix} \cos \theta + \frac{1}{i} \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \sin \theta \\ &= \frac{1}{2} \begin{bmatrix} 0 & 3 \cos \theta + 2i \sin \theta & 0 \\ 3 \cos \theta - 2i \sin \theta & 0 & 3 \cos \theta + i \sin \theta \\ 0 & 3 \cos \theta - i \sin \theta & \pm 3 \cos \theta \end{bmatrix}. \end{aligned}$$

The NR generating polynomials of the matrices A_\pm are

$$p_{A_\pm, \theta}(z) = -z^3 \pm 3z^2 \cos \theta + (18 \cos^2 \theta + 5 \sin^2 \theta)z \mp (27 \cos^3 \theta + 12 \cos \theta \sin^2 \theta).$$

We prove the theorem via the Kippenhahn polynomials of A_\pm :

$$p_{A_\pm}(x, y, z) = \det(x \Re(A_\pm) + y \Im(A_\pm) + zI_3),$$

that is,

$$p_{A_\pm}(x, y, z) = z^3 \pm 3xz^2 - (18x^2 + 5y^2)z \mp 3x(9x^2 + 4y^2).$$

We have $p_{A_-}(x, y, z) = p_{A_+}(-x, y, z)$ and these are irreducible polynomials. In fact, if p_{A_+} was reducible, then

$$p_{A_+}(x, y, z) = (ax + by + z)q_{A_+}(x, y, z),$$

for some real linear factor $ax + by + z$ and some quadratic homogeneous polynomial $q_{A_+}(x, y, z)$. Since the eigenvalues of A_+ are all real, from

$$p_{A_+}(1, i, -z) = (a + bi - z)q_{A_+}(1, i, -z),$$

we find $b = 0$ and $a \in \sigma(A_+)$. Then $ax + z$ would be a factor of $p_{A_+}(x, y, z)$ and $p_{A_+}(x, y, -ax)$ would be the zero polynomial, for some $a \in \{-3, 1, 5\}$. However, this is not true, as

$$p_{A_+}(x, y, 3x) = -27x(x^2 + y^2),$$

$$p_{A_+}(x, y, -x) = -7x(x^2 + y^2),$$

$$p_{A_+}(x, y, -5x) = 13x(x^2 + y^2).$$

For this reason, the boundary generating curves of the numerical range of K_6 cannot be either three points or a point and an ellipse.

We recall that the discriminant Δ of a cubic polynomial $az^3 + bz^2 + cz + d$ is given by

$$\Delta = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.$$

The discriminant $\Delta(x, y)$ of the polynomial $p_{A_+}(x, y, z)$ with respect to z is

$$\Delta(x, y) = 35721x^6 + 23814x^4y^2 + 4977x^2y^4 + 500y^6 > 0,$$

for every $(x, y) \in \mathbb{R}^2$, with $(x, y) \neq (0, 0)$. Hence, the equation $p_{A_+, \theta}(z) = 0$ has 3 distinct real roots, and the analogous holds for $p_{A_-, \theta}(z) = 0$. We may conclude that a curve of order 4 with a double tangent and a cusp is not a generating curve of $W(A_{\pm})$. By Theorem 2.3, the generating curves of the numerical range of each matrix A_{\pm} consist of an oval and a curve with three cusps lying in its interior, that is, a deltoid. Then each set $W(A_{\pm})$ is oval shaped. Thus, $W(K_6)$ is the convex hull of the oval shaped, symmetrically positioned with respect to the y -axis, numerical ranges of A_+ and A_- .

In this case, $W(K_6)$ has a pair of horizontal flat portions on its boundary. In fact, from $p_{A_{\pm}}(0, 1, -z) = z(z^2 - 5)$, we can see that

$$\sigma(\mathfrak{S}(A_{\pm})) = \{-\sqrt{5}, 0, \sqrt{5}\},$$

and so $\mathfrak{S}(K_6)$ has three multiple eigenvalues. Moreover, when $x_{\pm} = (-2, \pm\sqrt{5}i, 1)$, that is, x_{\pm} is an eigenvector of $\mathfrak{S}(A_+) = \mathfrak{S}(A_-)$ associated to the maximum/minimum eigenvalue $\pm\sqrt{5}$, we get

$$\frac{\langle \Re(A_-)x_{\pm}, x_{\pm} \rangle}{\langle x_{\pm}, x_{\pm} \rangle} = -\frac{3}{10}, \quad \frac{\langle \Re(A_+)x_{\pm}, x_{\pm} \rangle}{\langle x_{\pm}, x_{\pm} \rangle} = \frac{3}{10}.$$

Therefore, we may conclude that the line segments given by (6) define the flat portions of the boundary of the set $W(K_6)$. Finally, it is easy to see that the vertical tangent lines of the set $W(K_6)$ are $x = \pm w(K_6)$ with $w(K_6) = 6 \cos \frac{\pi}{7} \approx 5.4058$, by Proposition 2.5. \square

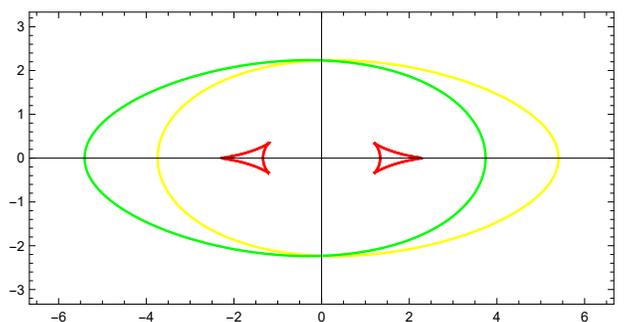


FIGURE 1. Boundary generating curves of $W(K_6)$.

In general, a pair of horizontal flat portions occurs on the boundary of $W(K_n)$ if and only if $n \geq 4$ is even. Otherwise, no flat portions exist. This was observed in [3, Example 11], as a consequence of a criterion of existence of line segments on the boundary of the numerical range of tridiagonal matrices in [3, Theorem 10] (see also the simpler reworded [16, Theorem 1]) applied in the case $\theta = 0$. In fact, that equivalently follows from the existence of multiple eigenvalues $\pm\lambda$ of the matrix $\mathfrak{S}(K_n)$ which are the maximum/minimum of its spectrum, and of more than one point in the sets

$$\left\{ \frac{\langle \Re(K_n)x, x \rangle}{\langle x, x \rangle} : x \in \mathcal{E}_{\pm\lambda} \right\},$$

where $\mathcal{E}_{\pm\lambda}$ denotes the eigenspace of $\mathfrak{S}(K_n)$ associated to $\pm\lambda$, which is true whenever $n \geq 4$ is even.

3.3. Oval-shaped numerical range. In this section, we first show that the numerical range of the Kac–Sylvester matrix of order 7 has a nonelliptical ovular shape. We will afterward see that this oval shape of the numerical range persists, by presenting some examples, for Kac–Sylvester matrices of higher odd orders.

THEOREM 3.5. *The numerical range of the Kac–Sylvester matrix of order 7 has an oval shape and*

$$w(K_7) = \frac{7}{2}\sqrt{2 + \sqrt{2}}.$$

The vertical and horizontal tangent lines of $W(K_7)$ are

$$(7) \quad x = \pm \frac{7}{2}\sqrt{2 + \sqrt{2}} \quad \text{and} \quad y = \pm \frac{1}{2}\sqrt{18 + \sqrt{274}}.$$

The set $W(K_7)$ is contained in the elliptical disc bounded by the ellipse defined by

$$(8) \quad \frac{x^2}{\frac{49}{4}(2 + \sqrt{2})} + \frac{y^2}{\frac{9}{2} + \frac{1}{4}\sqrt{274}} = 1.$$

Proof. By Proposition 2.5, the numerical radius of K_7 is

$$w(K_7) = 7 \cos \frac{\pi}{8} = \frac{7}{2}\sqrt{2 + \sqrt{2}}.$$

Let $n = 7$ and $m = 3$. By Theorem 3.1, the set $W(K_7)$ is the convex hull of $W(R_3^o)$ and $W(S_4)$, where

$$R_3^o = \begin{bmatrix} 0 & 1 & 0 \\ 6 & 0 & 2 \\ 0 & 5 & 0 \end{bmatrix} \quad \text{and} \quad S_4 = \begin{bmatrix} 0 & 0 & 0 & 4\sqrt{2} \\ 0 & 0 & 1 & 0 \\ 0 & 6 & 0 & 2 \\ 3\sqrt{2} & 0 & 5 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} H_\theta(R_3^o) &= \frac{7}{2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cos \theta + \frac{1}{2i} \begin{bmatrix} 0 & -5 & 0 \\ 5 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix} \sin \theta \\ &= \frac{1}{2} \begin{bmatrix} 0 & 7 \cos \theta + 5i \sin \theta & 0 \\ 7 \cos \theta - 5i \sin \theta & 0 & 7 \cos \theta + 3i \sin \theta \\ 0 & 7 \cos \theta - 3i \sin \theta & 0 \end{bmatrix}, \end{aligned}$$

and the NR generating polynomial of the matrix R_3^o is given by

$$p_{R_3^o, \theta}(z) = -\frac{1}{2}z(2z^2 - 7^2 \cos^2 \theta - 17 \sin^2 \theta).$$

The greatest eigenvalue of $H_\theta(R_3^o)$ is

$$\lambda_1(H_\theta(R_3^o)) = \frac{\sqrt{2}}{2} \sqrt{7^2 - (7^2 - 17) \sin^2 \theta}.$$

By Proposition 2.2, the set $W(R_3^c)$ is an elliptical disc, centered at the origin, with horizontal major axes and vertical minor axes of lengths $7\sqrt{2}$ and $\sqrt{34}$, respectively. The foci of the ellipse are $-4, 4$, and the length of its major semi-axis is smaller than $w(K_7)$.

Concerning the matrix S_4 , we have

$$H_\theta(S_4) = \frac{7}{2} \begin{bmatrix} 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \sqrt{2} & 0 & 1 & 0 \end{bmatrix} \cos \theta + \frac{1}{2i} \begin{bmatrix} 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & -5 & 0 \\ 0 & 5 & 0 & -3 \\ -\sqrt{2} & 0 & 3 & 0 \end{bmatrix} \sin \theta.$$

The Kippenhahn polynomial of S_4 is given by

$$p_{S_4}(x, y, z) = \begin{vmatrix} z & 0 & 0 & \frac{\sqrt{2}}{2}(7x - iy) \\ 0 & z & \frac{1}{2}(7x + 5iy) & 0 \\ 0 & \frac{1}{2}(7x - 5iy) & z & \frac{1}{2}(7x + 3iy) \\ \frac{\sqrt{2}}{2}(7x + iy) & 0 & \frac{1}{2}(7x - 3iy) & z \end{vmatrix},$$

that is,

$$p_{S_4}(x, y, z) = z^4 - (7^2x^2 + 3^2y^2)z^2 + \frac{1}{2^3}(7^2x^2 + 5y^2)^2 + 2 \times 7^2x^2y^2,$$

which is an irreducible polynomial. In fact, suppose that

$$p_{S_4}(x, y, z) = (ax + by + z) c_{S_4}(x, y, z),$$

for some real linear factor $ax + by + z$ and some cubic homogeneous polynomial $c_{S_4}(x, y, z)$. As the spectrum of S_4 is real, we have $b = 0$ and $a \in \sigma(S_4)$. Then $p_{S_4}(x, y, -ax)$ would be the zero polynomial, for some $a \in \{-6, -2, 2, 6\}$, which can be easily seen not to hold. On the other hand, suppose that

$$p_{S_4}(x, y, z) = q_1(x, y, z)q_2(x, y, z),$$

for some quadratic homogeneous polynomials of type

$$q_1(x, y, z) = ax^2 + bxy + cxz + dy^2 + eyz + z^2,$$

$$q_2(x, y, z) = gx^2 + hxy - cxz + ky^2 - eyz + z^2,$$

due to the null coefficient in z^3 of $p_{S_4}(x, y, z)$. But $p_{S_4}(x, y, z)$ has the null coefficient in z too, that is, we would have

$$(9) \quad (cx + ey)((g - a)x^2 + (h - b)xy + (k - d)y^2) = 0,$$

which implies either $c = e = 0$ or $g = a$ and $h = b$ and $k = d$. Then the coefficient of $q_1(x, y, z)q_2(x, y, z)$ associated with z^2 has, respectively, one of the following forms:

$$(a + g)x^2 + (b + h)xy + (d + k)y^2 \quad \text{or} \quad 2(ax^2 + bxy + dy^2) - (cx + ey)^2.$$

The first one implies $g = -a - 49$ and $h = -b$ and $k = -d - 9$, while the second one implies $2a = -49$ and $2d = -9$ and $b = c = e = 0$; but in both cases, the independent terms of $p_{S_4}(x, y, z)$ and $q_1(x, y, z)q_2(x, y, z)$ would not be equal.

Then the boundary generating curve of $W(S_4)$ is the dual of an irreducible quartic curve.

The discriminant $\Delta(x, y)$ of a quartic curve of the form $az^4 + bz^2 + c = 0$ is $16ac(b^2 - 4ac)^2$. We compute the discriminant $\Delta(x, y)$ of the polynomial $p_{S_4}(x, y, z)$ with respect to z and obtain

$$\Delta(x, y) = \frac{1}{2}(7^2x^2 + y^2)(7^2x^2 + 5^2y^2)(2401x^4 + 490x^2y^2 + 137y^4)^2 > 0,$$

for every $(x, y) \in \mathbb{R}^2$, with $(x, y) \neq (0, 0)$. Then $p_{S_4, \theta}(z)$ has four real distinct roots and the greatest one is equal to

$$\lambda_1(H_\theta(S_4)) = \frac{1}{2}\sqrt{58 + 40 \cos(2\theta) + \sqrt{2026 + 2264 \cos(2\theta) + 512 \cos(4\theta)}}.$$

The equation of a supporting line of $W(S_4)$ perpendicular to the direction $\theta \in [0, 2\pi)$ is

$$(10) \quad x \cos \theta + y \sin \theta = \lambda_1(H_\theta(S_4)).$$

The envelope of this family of supporting lines with θ ranging over $\theta \in [0, 2\pi)$, gives the boundary generating curve of $W(S_4)$. To compute this envelope, note that it has parametric equations given by

$$\begin{cases} x = \lambda_1(H_\theta(S_4)) \cos \theta - \lambda_1'(H_\theta(S_4)) \sin \theta \\ y = \lambda_1(H_\theta(S_4)) \sin \theta + \lambda_1'(H_\theta(S_4)) \cos \theta \end{cases}.$$

After some heavy and lengthy computations for eliminating the parameter θ , performed with the program Mathematica, we find that the boundary of $W(S_4)$ is the outer oval curve depicted in Figure 2.

Since the elliptical boundary of $W(R_3^2)$ and the oval boundary of $W(S_4)$ are nested curves, the numerical range of K_7 is bounded by the outer oval, that is, we have $W(K_7) = W(S_4)$.

Moreover, the maximum eigenvalues of $\Re(S_4)$ and $\Im(S_4)$ are given by

$$(11) \quad \lambda_1(H_0(S_4)) = \frac{7}{2}\sqrt{2 + \sqrt{2}} \quad \text{and} \quad \lambda_1(H_{\frac{\pi}{2}}(S_4)) = \frac{1}{2}\sqrt{18 + \sqrt{274}},$$

whereas the minimum eigenvalues of $\Re(S_4)$ and $\Im(S_4)$ are the corresponding symmetric values. Then the vertical and horizontal tangent lines of $W(K_7)$ are defined by (7). Further, since

$$\lambda_1(H_\theta(S_4)) \leq \sqrt{\lambda_1^2(H_0(S_4)) \cos^2(\theta) + \lambda_1^2(H_{\frac{\pi}{2}}(S_4)) \sin^2(\theta)},$$

for every $\theta \in [0, 2\pi)$, we may conclude that $W(K_7)$ is contained in the disc, bounded by the ellipse, centered at the origin, with horizontal major and vertical minor semi-axes of lengths (11), respectively, that is, with the Cartesian equation (8). \square

In Figure 3, we can see the oval boundary of $W(K_7)$, as well as the ellipse defined by (8), containing $W(K_7)$, and we may conclude how well it approximates the first oval curve.

Analogously, we may prove the next result for the order 9. We sketch the main steps of the proof below, being the more involved computational details omitted, as the arguments used are similar to those of the previous proof. By Proposition 2.5 for the numerical radius, we get

$$w(K_9) = 9 \cos \frac{\pi}{10} = \frac{9}{4}\sqrt{10 + 2\sqrt{5}}.$$

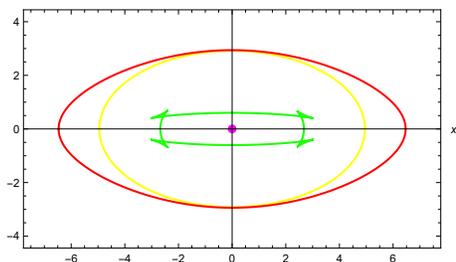


FIGURE 2. Boundary generating curves of $W(K_7)$.

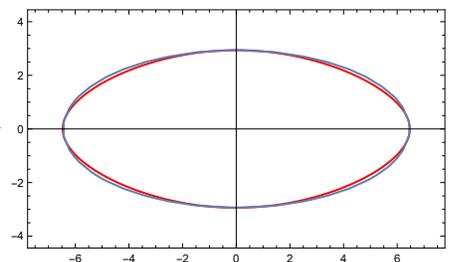


FIGURE 3. Elliptical disc approximating $W(K_7)$.

By Theorem 3.1, the set $W(K_9)$ is the convex hull of $W(R_4^o)$ and $W(S_5)$. We can prove that

$$W(R_4^o) \subset W(S_5),$$

and the Kippenhahn polynomial of S_5 is given by the product of $-z$ and

$$z^4 - \left(\frac{405}{4}x^2 + 85y^2\right)z^2 + \frac{1}{16}(32805x^2 + 17010x^2y^2 + 589y^4),$$

which is an irreducible quartic polynomial. Therefore, we obtain the oval boundary of $W(K_9)$, its vertical and horizontal tangent lines, having in mind that

$$\lambda_1(H_\theta(S_5)) = \frac{1}{2}\sqrt{\frac{245}{2} + 80\cos(2\theta)} + \frac{3}{2}\sqrt{1581 + 1552\cos(2\theta) + 512\cos(4\theta)},$$

as well as the elliptical disc, containing $W(K_9)$, given in the next theorem.

THEOREM 3.6. *The numerical ranges of the Kac-Sylvester matrix of order 9 has an oval shape and its numerical radius is*

$$w(K_9) = \frac{9}{4}\sqrt{10 + 2\sqrt{5}}.$$

The vertical and horizontal tangent lines of $W(K_9)$ are

$$x = \pm\frac{9}{4}\sqrt{10 + 2\sqrt{5}} \quad \text{and} \quad y = \pm\frac{1}{2}\sqrt{\frac{85}{2} + \frac{3}{2}\sqrt{541}}.$$

The set $W(K_9)$ is contained in the elliptical disc bounded by the ellipse defined by

$$\frac{x^2}{\frac{81}{8}(5 + \sqrt{5})} + \frac{y^2}{\frac{1}{8}(85 + 3\sqrt{541})} = 1.$$

In Figure 4, the boundary generating curves of $W(K_9)$ are presented. In Figure 5, we can see the oval boundary of $W(K_9)$, as well as the elliptical disc approximating $W(K_9)$.

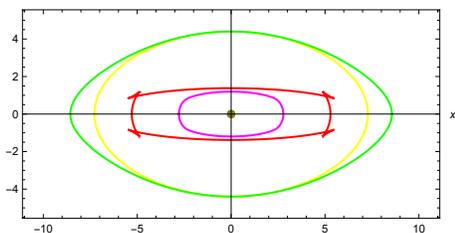


FIGURE 4. Boundary generating curves of $W(K_9)$.

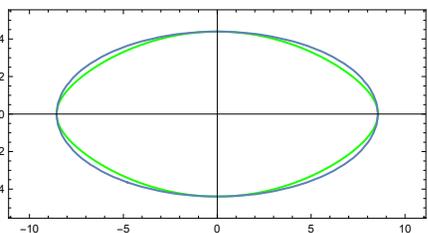


FIGURE 5. Elliptical disc approximating $W(K_9)$.

4. Final comments. Concerning the numerical range of the Kac–Sylvester matrix of order $2m$, $m \geq 4$, by numerical calculations for some examples of even order, we can see that it is the convex hull of two nonelliptical ovals, those defined by the numerical ranges of the matrices

$$R_m^e \pm mE_{mm} = \begin{bmatrix} 0 & 1 & & & & \\ 2m-1 & 0 & 2 & & & \\ & 2m-2 & \ddots & \ddots & & \\ & & \ddots & 0 & m-1 & \\ & & & m+1 & \pm m & \end{bmatrix}.$$

As previously observed, the boundary of $W(K_{2m})$, $m \geq 2$, has a pair of horizontal flat portions. In fact, since $\mathfrak{S}(R_m^e \pm mE_{mm}) = \mathfrak{S}(R_m^e)$, for $m \geq 2$ we can see that $\mathfrak{S}(K_{2m})$ is unitarily similar to the direct sum of two copies of

$$\mathfrak{S}(R_m^e) = i \begin{bmatrix} 0 & m-1 & & & \\ 1-m & 0 & m-2 & & \\ & 2-m & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ & & & -1 & 0 \end{bmatrix},$$

which has simple eigenvalues by [4, Corollary 7]. Thus, if $m \geq 2$, a closer inspection of the behavior of the matrix $\Re(K_{2m}) = mT_m$ on the 2-dimensional eigenspace of $\mathfrak{S}(K_{2m})$ associated to the maximum eigenvalue λ of $\mathfrak{S}(R_m^e)$ and the symmetry of $W(K_{2m})$ with respect to the x -axis, stated in Proposition 2.6, lead to the two flat portions on the lines $y = \pm\lambda$.

In particular, when $n = 8$ and $m = 4$, by Theorem 3.1, the set $W(K_8)$ is the convex hull of $W(R_4^e + 4E_{44})$ and $W(R_4^e - 4E_{44})$. The Kippenhahn polynomials of $R_4^e \pm 4E_{44}$ are given by

$$z^4 \mp 4xz^3 - (48x^2 + 14y^2)z^2 \pm (2^7x^3 + 52xy^2)z + 2^8x^4 + 160x^2y^2 + 3^2y^4,$$

which are irreducible quartic polynomials. Then, using arguments of the proofs presented in the previous sections, we may conclude that $W(K_8)$ is the convex hull of two nonelliptical oval curves, as shown is Figure 6. These oval curves are, as in Theorem 3.4 for the order 6, symmetrically positioned with respect to the y -axis. Now, after simple computations derived from the eigenvalues of $\mathfrak{S}(R_4^e)$, and the corresponding eigenvectors

associated to its maximum and minimum eigenvalues, we can see that the flat portions of the boundary of $W(K_8)$ are defined by

$$(12) \quad -1 + 3\frac{\sqrt{10}}{10} \leq x \leq 1 - 3\frac{\sqrt{10}}{10} \quad \text{and} \quad y = \pm\sqrt{7 + 2\sqrt{10}}.$$

We easily see that the vertical tangent lines of $W(K_8)$ are $x = \pm w(K_8)$, with

$$w(K_8) = 8 \cos \frac{\pi}{9} \approx 7.51754.$$

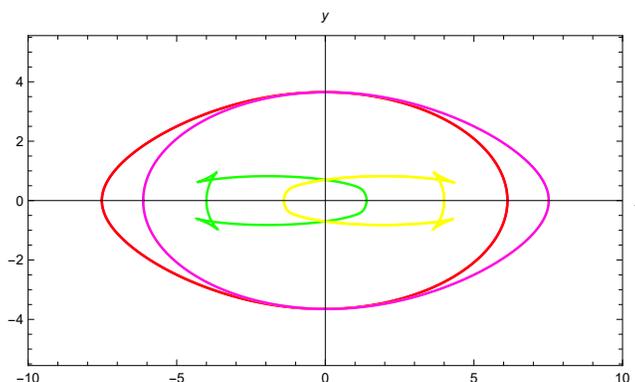


FIGURE 6. Boundary generating curves of $W(K_8)$.

We conjecture that this behavior is the same for any even order n , but the computations involved are much more intricate for $n \geq 10$. In fact, observe that the Kippenhahn polynomials of $R_m^e \pm mE_{mm}$ are irreducible quintic polynomials if $m = 5$ or irreducible polynomials of higher degree for greater values of m .

For instance, for the order 20, we have the boundary generating curves of $W(K_{20})$ presented in Figure 7.

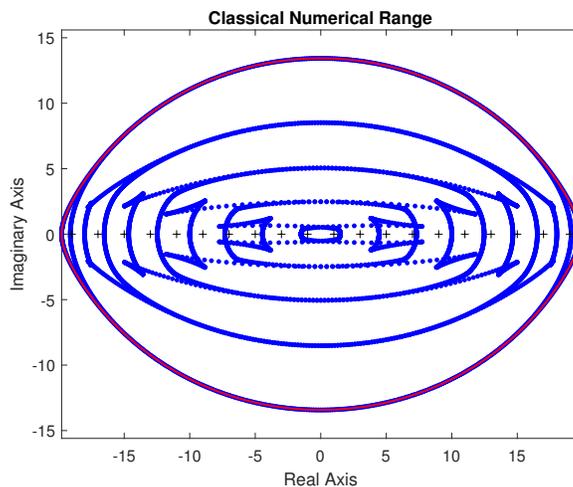


FIGURE 7. Boundary generating curves of $W(K_{20})$.

For Kac–Sylvester matrices of order $2m + 1$, $m \geq 5$, by experimental numerical computations performed with Mathematica or MatLab, we can observe that the nonelliptical oval shape of the numerical range persists. For instance, for the order 21, we have the boundary generating curves shown in Figure 8.

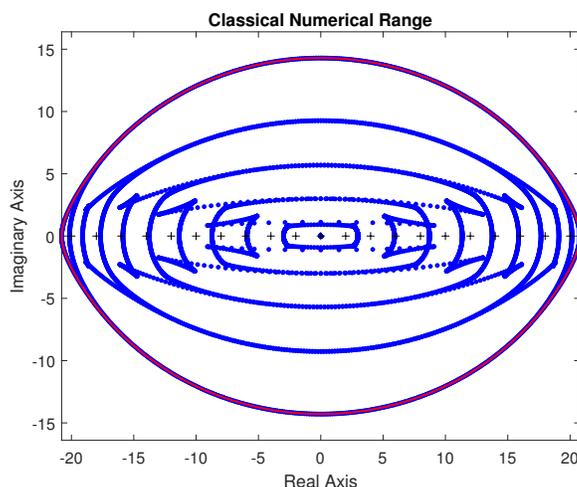


FIGURE 8. Boundary generating curves of $W(K_{21})$.

We also conjecture that this type of oval-shaped numerical range holds for Kac–Sylvester matrices of any odd order $n \geq 11$. Finally, it is interesting to notice that although not having numerical range ellipticity, we still have quasi-ellipticity for odd sizes.

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