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A PERMANENT INEQUALITY FOR POSITIVE SEMIDEFINITE MATRICES*

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Abstract. In this paper, we prove an inequality involving the permanent of a positive semidefinite matrix and its leading submatrices. We obtain a result in the similar spirit of Bapat–Sunder per-max conjecture.

Key words. Positive semidefinite matrix, Permanent, Gram matrix, Tensor product.

AMS subject classifications. 15A15, 15F10.

1. Introduction. Since its first introduction by Cauchy [6] and its first use in the current sense by Muir [21], the matrix permanent has been extensively studied and numerous interesting results have been obtained. Even though somehow it lacks the geometric interpretation of determinant, it appeared in several applications from graph theory [2, 5] to quantum mechanics and quantum information theory [1, 7, 22]. The permanent becomes particularly interesting when we consider a positive semidefinite (PSD) matrix. Several well-known results for determinants of PSD matrices appear to be reversed with permanents, or they are no longer correct. For instance, Oppenheim's determinant inequality(see [16]) states that for two PSD matrices A and B, $Det(A \circ B) \geq Det(A) \prod_{i=1}^{n} b_{ii}$, where $A \circ B$ is the Hadamard (Schur) product and b_{ii} are diagonal entries of matrix B. In [3] and [4], Bapat and Sunder raised the question whether a permanent equivalence of Oppenheim's inequality in the form of $Per(A \circ B) \leq Per(A) \prod_{i=1}^{n} b_{ii}$ exists. Bapat–Sunder conjecture is proven to be false by Drury in 2016 [10]. On a different note, there are also many interesting conjectures about the permanents of PSD matrices which are still open. A conjecture by Chollet [8] which proposes that for two PSD matrices A and B, $Per(A \circ B) \leq Per(A)Per(B)$ remains open although partial affirmative results are obtained in [11] and [12]. In 2016, Drury conjectured that for a $n \times n$ PSD matrix A,

$$(a_{11} Per(A_{11}))^2 + \left(\sum_{k=2}^{n} |a_{1k}|^2 Per(A_{kk}) \right)^2 \leq (Per(A))^2. \tag{1.1}$$

where A_{kk} is obtained from A by removing kth row and column [23]. Since A_{kk} is also PSD matrix, this is a well-defined question. If holds, this conjecture implies the Chollet conjecture. However, in 2017, Hutchinson [13] showed that Drury's permanent conjecture is false by providing a counterexample with a 4×4 complex matrix. The limited number of results involving permanents of PSD matrices and their submatrices has motivated the author to study such relations. Our main result for a $n \times n$ PSD matrix A states that

$$Per(A) \leq \sum_{i=1}^{n} a_{ii} Per(A_{ii}). \tag{1.2}$$

This results is as an addition to the limited number of results in this fashion.

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Remark 1.1. *The following matrix is a counterexample to Drury's permanent conjecture in real case. We thank the anonymous referee for providing this counterexample during the review process.*

$$A = \begin{bmatrix} 1 & -\frac{3}{5} & -\frac{3}{5} & \frac{3}{5} & \frac{3}{5} \\ -\frac{3}{5} & 1 & \frac{3}{20} & -\frac{3}{20} & -\frac{3}{20} \\ -\frac{3}{5} & \frac{3}{20} & 1 & -\frac{3}{20} & -\frac{3}{20} \\ \frac{3}{5} & -\frac{3}{20} & -\frac{3}{20} & 1 & \frac{3}{20} \\ \frac{3}{5} & -\frac{3}{20} & -\frac{3}{20} & \frac{3}{20} & 1 \end{bmatrix}. \tag{1.3}$$

For this matrix, the left-hand side of (1.1) is $\frac{237705630252001}{16000000000000}$ *and* $(Per(A))^2 = \frac{207315794356225}{16000000000000}$.

This paper is organized as follows. In the next section, we give some preliminaries and revisit the known results. Then, using the tensor product space, certain symmetrizing operators, and Gram matrices, we obtain the main theorem of the paper. Lastly, we present a few corollaries and some additional results on permanent inequalities.

2. Preliminaries. Let $A = \{a_{ij}\}$ be an $n \times n$ Hermitian PSD matrix with entries in $\mathbb{C}$. Throughout this work, PSD matrices will be Hermitian. Such a matrix is the Gram matrix for some set of vectors $u_1, \dots u_n$ in a vector space V. These vectors are called *vector realization* of A. More explicitly, $a_{ij} = \langle u_i, u_j \rangle$, where $\langle , \rangle$ is an inner product on V. Since A is PSD, for any vector $\mathbf{x} \in \mathbb{C}^n$ we have $\mathbf{x}^T A \mathbf{x} \geq 0$.

Definition 2.1. For a $n \times n$ square matrix A the matrix permanent, $Per\ A$, is defined by:

$$Per\ (A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma i}. \tag{2.4}$$

where S_n is the permutation group on n letters.

We include the elementary properties of matrix permanent in the following lemma without proof. Interested reader can consult standard resources on this topics such as [14].

Lemma 2.2. *Let A be an $n \times n$ matrix. The following holds.*

1. *$Per\ (A)$= $Per\ (A^T)$,*
2. *$Per\ (A^*) = \overline{Per\ (A)}$, where A^* is the conjugate transpose of A.*
3. *$Per\ PAQ$= $Per\ (A)$, where P and Q are permutation matrices.*
4. *For a fixed $i \in \{1, \dots, n\}$, $Per\ (A) = \sum_{j=1}^n a_{ij} Per\ (A_{ij})$, where A_{ij} is the submatrix of A obtained by erasing ith row and jth column.*
5. *For a fixed $j \in \{1, \dots, n\}$, $Per\ (A) = \sum_{i=1}^n a_{ij} Per\ (A)_{ij}$*
6. *If A is PSD, then $Per\ (A) \geq 0$.*

It is worth mentioning that if A is a Hermitian $n \times n$ matrix, then $A_{ji} = A_{ij}^*$. Therefore, by Lemma 2.2, we have Per $(A_{ji}) = \overline{\text{Per}\ (A_{ij})}$.

Let $V^{\otimes n} = V \otimes \cdots \otimes V$ be the n-copy tensor product of V. Tensor product space $V^{\otimes n}$ has an inner product induced by the inner product of V given by:

$$\langle u_1 \otimes \cdots \otimes u_n, v_1 \otimes \cdots \otimes v_n \rangle = \prod_{i=1}^{n} \langle u_i, v_i \rangle. \tag{2.5}$$

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We define a linear operator on $V^{\otimes n}$ by:

$$T(u_1 \otimes \cdots \otimes u_n) = \frac{1}{n!} \sum_{\sigma \in S_n} u_{\sigma(1)} \otimes \ldots \otimes u_{\sigma(n)}. \tag{2.6}$$

The operator T permutes the vectors in the indecomposible tensor product $u_1 \otimes \cdots \otimes u_n$. The following lemma can be found in [14], Page 21.

LEMMA 2.3. *The operator T satisfies*

1. $T^2 = T$,
2. $T^* = T$. *That is, T is self-adjoint relative to the inner product on $V^{\otimes n}$.*

Marcus and Newman showed that the operator T can be used to compute the matrix permanent for a Gram matrix. We present this result without proof below.

THEOREM 2.4 ([14, 15]). *Let A be a $n \times n$ Hermitian PSD matrix with vector realization $u_1, \ldots, u_n$. Then,*

$$Per\ (A) = n! \langle u_1 \otimes \cdots \otimes u_n, T(u_1 \otimes \cdots \otimes u_n) \rangle. \tag{2.7}$$

3. Main results. In this section, we prove a permanent inequality for PSD matrices that involves the permanents of leading submatrices of A.

THEOREM 3.1. *Let A be an $n \times n$ Hermitian PSD matrix. Then,*

$$Per\ (A) \leq \sum_{i=1}^{n} a_{ii} Per\ (A_{ii}). \tag{3.8}$$

The equality holds if and only if A has rank 1.

Proof. Let the vectors $u_1, \ldots, u_n$ be a vector realization of the matrix A. Define a matrix $P = \{p_{ij}\}$ where $p_{ij} = a_{ij} Per(A_{ij})$. That is,

$$P = \begin{bmatrix} a_{11}Per(A_{11}) & a_{12}Per(A_{12}) & \ldots & a_{1n}Per(A_{1n}) \\ a_{21}Per(A_{21}) & a_{22}Per(A_{22}) & \ldots & a_{2n}Per(A_{2n}) \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1}Per(A_{n1}) & a_{n2}Per(A_{n2}) & \ldots & a_{nn}Per(A_{nn}). \end{bmatrix}. \tag{3.9}$$

Note that the matrix P can be written as the Hadamard product of the matrix A and the matrix $B = \{b_{ij}\}$ where $b_{ij} = Per(A_{ij})$. That is, $P = A \circ B$. Since A is Hermitian, by Lemma 2.2 the matrix P is also Hermitian. Our goal is to show that the matrix B is a PSD matrix. Define the vectors

$$v_i = \sqrt{(n-1)!}\, T(u_1 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots u_n) = \sqrt{(n-1)!}\, T(u_1 \otimes \cdots \otimes u_{i-1} \otimes u_{i+1} \cdots \otimes u_n),$$

for $i = 1, \ldots, n$. The vectors v_i are in $V^{\otimes(n-1)}$, and T is the operator defined by (2.6) acting on the space $V^{\otimes(n-1)}$. With the inner product $\langle , \rangle$ on $V^{\otimes(n-1)}$, we compute

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$$
\begin{aligned}
\langle v_i, v_j\rangle &= (n-1)!\,\langle T(u_1\otimes\cdots\otimes\widehat{u_i}\otimes\cdots\otimes u_n), T(u_1\otimes\cdots\otimes\widehat{u_j}\otimes\cdots\otimes u_n)\rangle\\
&= (n-1)!\langle u_1\otimes\cdots\otimes\widehat{u_i}\otimes\cdots\otimes u_n, T^2(u_1\otimes\cdots\otimes\widehat{u_j}\otimes\cdots\otimes u_n)\rangle\\
&= (n-1)!\langle u_1\otimes\cdots\otimes\widehat{u_i}\otimes\cdots\otimes u_n, T(u_1\otimes\cdots\otimes\widehat{u_j}\otimes\cdots\otimes u_n)\rangle\\
&= \langle u_1\otimes\cdots\otimes\widehat{u_i}\otimes\cdots\otimes u_n, \sum_{\sigma\in S_{n-1}} u_{\sigma(1)}\otimes\cdots\otimes\widehat{u_{\sigma(j)}}\otimes\cdots u_{\sigma(n)}\rangle\\
&= \sum_{\sigma\in S_{n-1}} \langle u_1, u_{\sigma(1)}\rangle\cdots\widehat{\langle u_i, u_{\sigma(i)}\rangle}\cdots\widehat{\langle u_j, u_{\sigma(j)}\rangle}\cdots\langle u_n, u_{\sigma(n)}\rangle\\
&= \sum_{\sigma\in S_{n-1}} a_{1\sigma(1)}\cdots\widehat{a_{i\sigma(i)}}\cdots\widehat{a_{j\sigma(j)}}\cdots a_{n\sigma(n)}.
\end{aligned}
$$

The product in the last row above contains no entry from ith row and jth column. We consider S_{n-1} as the set of permutations on the set $\{1,\ldots,n\}$ which fix j. Consequently, we obtain $\langle v_i, v_j\rangle = Per(A_{ij})$. This indicates that the matrix B has a vector realization $v_1,\ldots v_n$, and therefore, it is also PSD. Finally, as a consequence of Schur product theorem, the Hadamard product $P = A\circ B$ becomes a PSD matrix.

Remark 3.2. *In the equation above, the appearance of $\sigma(j)$ is just for convenience, and since it is not a part of actual computation, it does not cause any problem.*

We now study the structure of the PSD matrix $P = \{p_{ij}\}$. Note that for any i, $\sum_{j=1}^n p_{ij} = Per(A)$, and for any j, $\sum_{i=1}^n p_{ij} = Per(A)$. That is, the row sum and the column sum for each row and column are the same and equal to $Per(A)$. This means, $\lambda = Per(A)$ is an eigenvalue associated with eigenvector $x = [1,\ldots,1]^T$. Since P is PSD, all eigenvalues of P are non-negative. This leads to

$$
Tr(P) = \sum_{i=1}^n a_{ii}Per(A_{ii}) = Per(A) + \sum\mu, \tag{3.10}
$$

where $\sum\mu$ is the sum of remaining nonnegative eigenvalues of P. Therefore, we have shown that

$$
Per(A) \le \sum_{i=1}^n a_{ii}Per(A_{ii}).
$$

Assume that A is a rank 1 PSD matrix. Then, we have

$$
A = x\overline{x}^T = \begin{bmatrix} x_1\\ \cdots\\ x_n\end{bmatrix}\begin{bmatrix}\overline{x_1} & \ldots & \overline{x_n}\end{bmatrix} = \begin{bmatrix} |x_1|^2 & x_1\overline{x_2} & \ldots & x_1\overline{x_n}\\ \vdots & \vdots & \vdots & \vdots\\ x_n\overline{x_1} & x_n\overline{x_2} & \ldots & |x_n|^2\end{bmatrix},
$$

where x is a vector in $\mathbb{C}^n$. The permanent of A and its leading submatrices can be computed as:

$$
\begin{aligned}
Per(A) &= n!\,|x_1|^2\cdots|x_n|^2,\\
a_{ii}Per(A_{ii}) &= (n-1)!\,|x_1|^2\cdots|x_n|^2.
\end{aligned}
$$

Hence, the equality holds.

Conversely, assume that the equality holds. We know that $Per(A)$ is an eigenvalue of P with eigenvector $[1,\ldots 1]^T$. By (3.10), all other eigenvalues of P are 0. Since the eigenvectors associated with distinct eigenvalues of P are orthogonal, each eigenvector $[x_1,\ldots,x_n]^T$ associated with eigenvalue $\mu = 0$ lies on the

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hyperplane $x_1 + \cdots + x_n = 0$. In particular, for $j \in \{1, \dots, n\}$, the vectors $z^{(j)} = [1, \dots, -1, \dots 0]^T$ where the jth component is -1 are eigenvectors for $\mu = 0$. For $z^{(j)}$, the matrix equation $Pz^{(j)} = 0$ gives

$$\begin{gathered} a_{11}Per(A_{11}) = a_{1j}Per(A_{1j}), \\ a_{21}Per(A_{21}) = a_{2j}Per(A_{2j}), \\ \vdots \\ a_{n1}Per(A_{n1}) = a_{nj}Per(A_{nj}). \end{gathered}$$

Considering different j values we see that for any i, each entry of ith row is equal to a number α_i. Moreover, since the row sum for each row is $Per(A)$, we have $n\alpha_i = Per(A)$ for all i. As a result, we conclude that all entries of the matrix P are the same and equal to $p_{ij} = a_{ij}Per(A_{ij}) = Per(A)/n$. We observe that for any i, j, we have

$$a_{ii}Per(A_{ij}) = a_{jj}Per(A_{jj}) = a_{ij}Per(A_{ij}).$$

Using this, we see

$$\begin{aligned} (a_{ij}Per(A_{ij}))^2 &= a_{ii}Per(A_{ii})a_{jj}Per(A_{jj}) \\ a_{ij}^2(Per(A_{ij}))^2 &= a_{ii}Per(A_{ii})a_{jj}Per(A_{jj}) \\ a_{ij}^2\langle v_i, v_j\rangle^2 &= a_{ii}a_{jj}\|v_i\|^2\|v_j\|^2 \\ |a_{ij}|^2|\langle v_i, v_j\rangle|^2 &= a_{ii}a_{jj}\|v_i\|^2\|v_j\|^2. \end{aligned}$$

Since $|a_{ij}|^2|\langle v_i, v_j\rangle|^2 \le |a_{ij}|^2\|v_i\|^2\|v_j\|^2$, we obtain

$$a_{ii}a_{jj} \le |a_{ij}|^2. \tag{3.11}$$

On the other hand, $a_{ij} = \langle u_i, u_j\rangle$. Then, the inequality (3.11) gives

$$\|u_i\|^2\|u_j\|^2 \le |\langle u_i, u_j\rangle|^2. \tag{3.12}$$

□

By applying Cauchy–Schwarz inequality to the pair u_i, u_j and using the inequality (3.12), we find u_i and u_j to be parallel for any i, j. In other words, we can take each u_i, $i = 2, 3, \dots, n$ as a scalar multiple of u_1 and consequently this implies that the matrix A is of rank 1.

Corollary 3.3. *Let A be a $n \times n$ PSD matrix such that $Per(A_{ii}) \le 1$. Then, $Per(A) \le Tr(A)$.*

Proof. If the condition of corollary is satisfied, by Theorem (3.1) we obtain

$$Per(A) \le \sum_{i=1}^{n} a_{ii} = Tr(A).$$

□

Corollary 3.4. *Let A be $n \times n$ PSD matrix and P be as in the proof of Theorem (3.1). Let μ be the largest eigenvalue of P. Then,*

$$\mu \le \sum_{i=1}^{n} a_{ii}Per(A_{ii}).$$

Proof. This is a direct application of Theorem (3.1). □

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Note that Corollary (3.4) is in similar fashion as Bapat–Sunder per-max conjecture ([4], Conjecture 3), which proposes that the maximum eigenvalue of P is $Per(A)$. This conjecture was proven to be false by Drury [10] and Tran [17]. With this corollary, we give an upper bound for the maximum eigenvalue of P.

In the following theorem, we investigate the relation between the determinant and permanent of the matrices A and P. To achieve this goal, we need the following theorem.

THEOREM 3.5 (see *[18]*). *Let A be a $n \times n$ PSD matrix, r_i denote the ith row sum of A, and $s(A)$ denote the sum of all entries of the matrix A. Then,*

$$Per(A) \geq \frac{n!}{s(A)^n} \prod_{i=1}^{n} |r_i|^2. \tag{3.13}$$

THEOREM 3.6. *Let A be a $n \times n$ PSD matrix and $P = A \circ B$ be the matrix given in the proof of Theorem 3.1. Then, we have the following inequalities:*

$$\begin{aligned} \frac{n!}{n^n}(Per(A))^n &\leq Per(P), \\ Det(P) &\leq Per(A)Per(B). \end{aligned}$$

Proof. In the proof of Theorem 3.1, we show that P is a PSD matrix with the property that each row sum is $Per(A)$. Therefore, we set $r_i = Per(A)$ and $s(A) = nPer(A)$. Applying Theorem 3.5, we compute

$$Per(P) \geq \frac{n!}{(nPer(A))^n} (Per(A))^{2n} = \frac{n!}{n^n}(Per(A))^n. \tag{3.14}$$

Marcus and Minc show that([19, 20]) for any $n \times n$ PSD matrix A with diagonal entries a_{ii}, we have

$$Per(A) \geq \prod_{i=1}^{n} a_{ii}, \quad Det(A) \leq \prod_{i=1}^{n} a_{ii}. \tag{3.15}$$

Using the inequalities in (3.15) and the fact that the matrix B is also PSD, we obtain

$$Det(P) \leq \prod_{i=1}^{n} a_{ii} Per(A_{ii}) = \Big(\prod_{i=1}^{n} a_{ii}\Big)\Big(\prod_{i=1}^{n} Per(A_{ii})\Big) \leq Per(A)Per(B), \tag{3.16}$$

as required. □

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REFERENCES

[1] S. Aaronson and A. Arkhipov. The computational complexity of linear optics. In: *Proceedings of the Forty-Third Annual ACM Symposium on Theory of Computing*, (2011) 333-342
[2] N. Alon and S. Friedland. The maximum number of perfect matchings in graphs with a given degree sequence. *Electron. J. Combin.*, 15(1): Note 13, 2, 2008.
[3] R.B. Bapat and V.S. Sunder. On majorization and Schur products. *Linear Algebra Appl.*, 72:107–117, 1985
[4] R.B. Bapat and V.S. Sunder. An extremal property of the permanent and the determinant. *Linear Algebra Appl.*, 76:153–163, 1986

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 71-77, February 2023.

[5] L.M. Brègman. Some properties of nonnegative matrices and their permanents. *Dokl. Akad. Nauk SSSR*, 211(1):27–30, 1973.

[6] A.L. Cauchy. Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérées entre les variables qu'elles renferment. *J. Ec. Polytech- Math.*, 10:91–169, 1815.

[7] L. Chakhmakhchyan, N.J. Cerf and R. Garcia-Patron Quantum-inspired algorithm for estimating the permanent of positive semidsefinite matrices. *Phys. Rev. A*, 96:022329, 2017.

[8] J. Chollet. Unsolved problems: Is there a permanental analogue to oppenheim's inequality? *Amer. Math. Monthly*, 89(1):57–58, 1982.

[9] S. Drury. A counterexample to a question of Bapat and Sunder. *Electron. J. Linear Algebra*, 31:69-70, 2016.

[10] S. Drury. A counterexample to a question of Bapat and Sunder. *Math. Inequal. Appl.*, 21(2):517–520, 2018

[11] R.J. Gregorac and I.R. Hentzel. A note on the analogue of Oppenheim's inequality for permanents. *Linear Algebra Appl.*, 94:109–112, 1987.

[12] G. Hutchinson. An elementary proof of Chollet's permanent conjecture for 4×4 real matrices. *Spec. Matrices*, 9:83–102, 2021.

[13] G. Hutchinson. A counterexample to Drury permanent conjecture, *Spec. Matrices*, 5:301–302, 2017.

[14] H. Minc. Permanents. *Encyclopedia of Mathematics and Its Applications.* Addison-Wesley, Reading, Massachusetts, 1978.

[15] M. Marcus and M. Newman. Inequalities for the permanent function, *Ann. Math*, 675 (1962) 47-62

[16] A. Oppenheim. Inequalities connected with definite Hermitian forms. *J. London Math. Soc.,* S1-5(2):114–119.

[17] H. A. Tran. A simple counterexample for the permanent-on-top conjecture *Math. Inequal. Appl.*, 25(1): 1–16, 2022.

[18] M. Marcus and H. Minc. Generalized matrix functions. *Trans. Amer. Math. Soc.*, 116:316–329, 1965.

[19] M. Marcus and H. Minc. The Pythagorean theorem in certain symmetry classes of tensors, *Trans. Amer. Math. Soc.*, 104:510–515, 1962.

[20] M. Marcus. The permanent analog of the Hadamard determinant theorem. *Bull. Amer. Math. Soc.*, 69:494–496, 1963.

[21] T. Muir. On a class of permanent symmetric functions. *Proc. Roy. Soc. Edinburgh*, 11:409–418, 1882.

[22] T. Wei and S. Severini. Matrix permanent and quantum entanglement of permutation invariant states. *J. Math. Phys.*, 51:092203, 2010.

[23] F. Zhang. An update on a few permanent conjecture. *Spec. Matrices*, 4:305–316, 2016.