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ON TWO–SIDED INTERPOLATION FOR UPPER TRIANGULAR MATRICES*

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Abstract. The space of upper triangular matrices with Hilbert–Schmidt norm can be viewed as a finite dimensional analogue of the Hardy space $\mathbf{H}_2$ of the unit disk when one introduces the adequate notion of "point" evaluation. A bitangential interpolation problem in this setting is studied. The description of all solution in terms of Beurling–Lax representation is given.

Key words. Interpolation, matrices.

AMS subject classifications. 47A57, 47A48

1. Introduction. In this paper we pursue our study of bitangential interpolation in analogues of the Hardy space of the unit disk $\mathbf{H}_2$. To start with we recall the classical setting which has been considered in [8].

Let $\mathbf{H}_2^{m\times k}$ denote the Hilbert space of $m\times k$–matrix valued functions of the form

$$H(z)=\sum_{j=0}^{\infty}H_jz^j,\qquad \sum_{j=0}^{\infty}\operatorname{Tr}H_j^*H_j<\infty,$$

endowed with the $L_2^{m\times k}$ inner product

$$\langle H,\ G\rangle_{L_2^{m\times k}}=\langle H,\ G\rangle_{\mathbf{H}_2^{m\times k}}=\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Tr}G(e^{it})^*H(e^{it})dt=\sum_{j=0}^{\infty}\operatorname{Tr}G_j^*H_j. \tag{1}$$

The $\mathbf{H}_2^{m\times k}$-functions are analytic in the open unit disk $\mathbb{D}$ and have boundary values almost everywhere on the unit circle $\mathbb{T}$. The space $\mathbf{H}_2^{m\times k}$ is a Hilbert module (see, e.g, [18] and Section 2 of [1]) with respect to the Hermitian matrix-valued forms

$$\{H,\ G\}_{\mathbf{H}_2^{m\times k}}=\frac{1}{2\pi}\int_0^{2\pi}H(e^{it})G(e^{it})^*dt \tag{2}$$

and

$$[H,\ G]_{\mathbf{H}_2^{m\times k}}=\frac{1}{2\pi}\int_0^{2\pi}G(e^{it})^*H(e^{it})dt \tag{3}$$

and has the reproducing kernel property with the kernels

$$k_\omega^{\wedge}(z)=\frac{I_m}{1-z\omega^*}\quad\text{and}\quad k_\omega^{\triangle}(z)=\frac{I_k}{1-z\omega^*}$$

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in the sense that

$$H(\omega)A^* = \left\{H,\ Ak_\omega^{\triangle}\right\}_{\mathbf{H}_2^{m\times k}} \quad \text{and} \quad A^*H(\omega) = [H,\ k_\omega^{\wedge}A]_{\mathbf{H}_2^{m\times k}} \tag{4}$$

for every choice of a point $\omega \in \mathbb{D}$ and of a $m \times k$ matrix A. Note that the latter relations express Cauchy's formula for $\mathbf{H}_2^{m\times k}$-functions. In [8] we considered a general bitangential Nudelman type problem for $\mathbf{H}_2^{m\times k}$ functions with norm constraints. The multivariable analogue of the problem referred to in the previous paragraph in the setting of the polydisk was considered in [4].

It is well known (see [10], [11], [12], [15]) that there are deep analogies between analytic functions and upper (or lower) triangular matrices. In [2] we looked at the analogue of the problem referred to in the previous paragraph, for double infinite upper triangular matrices. In this paper we focus on the case of finite matrices. In fact, one could try to obtain the case of finite matrices from our previous paper [2] using time varying coefficient spaces in the spirit of [13, Section 12] or [14]. This does not seem to us natural, since the problem considered here is finite dimensional; furthermore, the approach presented in this paper is purely algebraic and much more explicit.

Other situations are also possible, such as the case of lower triangular integral Hilbert Schmidt operators (the continuous time varying case analogue of [2]). This was carried out in [3].

Consider $\mathcal{X}^{m\times k}$, the set of all $m\times k$ matrices which is a Hilbert space with respect to the inner product

$$\langle H,\ G\rangle = \operatorname{Tr} G^*H \qquad (H,\, G \in \mathcal{X}^{m\times k}), \tag{5}$$

which is the analogue of (1).

A matrix $H = [h_ij]_{i=1,\dots,m}^{j=1,\dots,k}$ is called *diagonal* if $h_{ij} = 0$ for $i \neq j$. It is called *upper* (*lower*) *triangular* if $h_{ij} = 0$ for $i > j$ ($i < j$, respectively). The symbols $\mathcal{D}^{m\times k}$, $\mathcal{U}^{m\times k}$ and $\mathcal{L}^{m\times k}$ will be used for the spaces of diagonal, upper triangular and lower triangular $m \times k$ matrices, respectively.

Let Z_m denote the $m \times m$ shift matrix defined by

$$Z_m = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & & 1 \\ 0 & & & \cdots & 0 \end{bmatrix} \tag{6}$$

and let Z_k be the $k \times k$ shift matrix defined similarly. We denote by p_+, p_0, p_- the orthogonal projections of $\mathcal{X}^{m\times k}$ onto $\mathcal{U}^{m\times k}Z_k$, $\mathcal{D}^{m\times k}$, $Z_m^*\mathcal{L}^{m\times k}$, respectively. We also use the projections of $\mathcal{X}$ onto its upper and lower parts, and denote them by

$$p = p_0 + p_+ \quad \text{and} \quad q = p_0 + p_-,$$

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respectively. The space $\mathcal{X}^{m\times k}$ is a Hilbert module with respect to Hermitian matrix-valued forms

$$\{H,\ G\}_{\mathcal{X}^{m\times k}} = p_0(HG^*) \quad \text{and} \quad [H,\ G]_{\mathcal{X}^{m\times k}} = p_0(G^*H)$$

which are the analogues of (2) and (3).

For a fixed $H \in \mathcal{U}^{m\times k}$ these forms make sense for $k\times k$ and $m\times m$ matrices G, respectively, and define two different "evaluation" maps for upper triangular matrices by

$$\text{(7)}\quad F^{\wedge}(W) = p_0\left((I_m - WZ_m^*)^{-1}F\right) \quad \text{and} \quad F^{\triangle}(V) = p_0\left(F\,(I_k - Z_k^*V)^{-1}\right),$$

where $F\in\mathcal{U}^{m\times k}$ and W and V are diagonal $m\times m$ and $k\times k$ matrices, which usually play the role of points in the nonstationary setting. The transformations $F^{\wedge}(W)$ and $F^{\triangle}(V)$ are noncommutative analogues of the point evaluation (4). We also define

$$F^{\sharp}(W,V) = p_0\left((I_m - WZ_m^*)^{-1}F\,(I_k - Z_k^*V)^{-1}Z_k^*\right).$$

Remark 1.1. It follows from the definition (6) of Z_m that for every choice of $W\in\mathcal{D}^{m\times m}$, the matrix $I_m - WZ_m^*$ is lower triangular with all diagonal entries equal to 1. In particular, $\det(I_m - WZ_m^*) = 1$ and the matrix is invertible. From the same reason the matrix $I_k - Z_k^*V$ is invertible for every choice of $V\in\mathcal{D}^{k\times k}$.

Note that if $m = k$, then for any $F\in\mathcal{U}^{m\times k}$ there exist uniquely defined diagonal matrices $F_{[j]}$ and $F_{\{j\}}$ which satisfy

$$F_{[j]} = Z_m^{*j}Z_m^jF_{[j]}, \qquad F_{\{j\}} = F_{\{j\}}Z_k^jZ_k^{*j}$$

and are such that

$$F = \sum_{j=0}^{m-1} Z_m^jF_{[j]} \quad \text{and} \quad F = \sum_{j=0}^{k-1} F_{\{j\}}Z_k^j.$$

The latter "polynomial" representations allows us to express evaluations (7) as

$$\text{(8)}\qquad F^{\wedge}(W) = \sum_{j=0}^{m-1}(WZ_m^*)^jZ_m^jF_{[j]} \quad \text{and} \quad F^{\triangle}(V) = \sum_{j=0}^{k-1}F_{\{j\}}Z_k^j(Z_k^*V)^j.$$

These formulas appear in [7]. If $m > k$ ($m < k$), the first (the second) formula in (8) is true. In general, for $m < k$ ($m > k$) the first (the second) formula in (8) is not valid, and for this reason we shall use formulas (7).

The maps (8) have been introduced in [5] and [6] for bounded upper triangular operators in infinite dimensional Hilbert spaces. For more on this setting and on the related interpolation problems for uper triangular contractions, see [9], [13], [16], [17].

Now we present the analogue of the classical Lagrange interpolation problem, which is posed below in terms of the "point" evaluations (7).

Problem 1.2. *Given matrices*

$$\begin{array}{llll} W_j\in\mathcal{D}^{\ell_j\times\ell_j}, & \xi_j\in\mathcal{D}^{\ell_j\times m}, & \eta_j\in\mathcal{D}^{\ell_j\times k} & (j=1,\dots,n)\\ V_i\in\mathcal{D}^{r_i\times r_i}, & \zeta_i\in\mathcal{D}^{r_i\times m}, & \nu_i\in\mathcal{D}^{r_i\times k} & (i=1,\dots,\ell)\end{array}$$

and $\Gamma_{ji}\in\mathcal{D}^{\ell_i\times\ell_j}$, *find all* $H\in\mathcal{U}^{m\times k}$ *such that*

$$\left(\xi_j^*H\right)^{\wedge}(W_j)=\eta_j^*,\qquad (H\nu_i)^{\triangle}(V_i)=\zeta_i\quad\text{and}\quad \left(\xi_j^*H\nu_i\right)^{\sharp}(W_j,V_i)=\Gamma_{ji}\tag{9}$$

for $j=1,\dots,n$ *and* $i=1,\dots,\ell$.

The paper consists of seven sections. To set the problem precisely we first need some notations and definitions. The general problem (which includes Problem 1.2 as a particular case) is stated in the second section. The description of all its solutions (formula (31)) is presented in the third section. The proof relies on the analysis of two special cases, namely right sided and left sided interpolation problems, which are considered in details respectively in Sections 4 and 5. The general two sided problem is studied in Section 6. The last section deals with the structure of the minimal norm solution.

2. Formulation of the problems and preliminary remarks. In this section we introduce the bitangential interpolation problem to be studied, and which includes Problem 1.2 as a particular case. Given two sets of positive integers $\{\ell_i\}$ and $\{r_j\}$, let

$$n_R=r_1+\ldots+r_t,\qquad n_L=\ell_1+\ldots+\ell_s,\tag{10}$$

let Z_{r_j} and Z_{ℓ_i} be the shift matrices defined via (6) and let

$$Z_\pi=\text{diag}\,(Z_{r_1},\dots,Z_{r_t})\in\mathbb{C}^{n_R\times n_R}\quad\text{and}\quad Z_\zeta=\text{diag}\,(Z_{\ell_1},\dots,Z_{\ell_s})\in\mathbb{C}^{n_L\times n_L}.$$

The relations

$$Z_\pi(I-Z_\pi^*Z_\pi)=0,\quad Z_\pi^*Z_\pi D=DZ_\pi^*Z_\pi,\tag{11}$$

$$(I-Z_\zeta Z_\zeta^*)Z_\zeta=0,\quad Z_\zeta Z_\zeta^*F=FZ_\zeta Z_\zeta^*,\tag{12}$$

$$(I-Z_\pi^*Z_\pi)(I-Z_\pi^*D)^{-1}=(I-Z_\pi^*Z_\pi),\quad\text{and}\tag{13}$$

$$(I-Z_\zeta Z_\zeta^*)(I-Z_\zeta F)^{-1}=(I-Z_\zeta Z_\zeta^*),\tag{14}$$

which hold for every choice of block diagonal matrices $D\in\mathcal{D}^{n_R\times n_R}$ and $F\in\mathcal{D}^{n_L\times n_L}$, will be useful.

In the class $\mathcal{U}^{m\times k}$ we consider the interpolation problem whose data set is an ordered collection

$$\Omega=\{C_+,\ C_-,\ A_\pi,\ A_\zeta,\ B_+,\ B_-,\ \Gamma\}\tag{15}$$

of seven matrices

$$C_+\in\mathcal{D}^{m\times n_R},\quad C_-\in\mathcal{D}^{k\times n_R},\quad B_+\in\mathcal{D}^{m\times n_L},\quad B_-\in\mathcal{D}^{k\times n_L},$$

$$A_\pi \in \mathcal{D}^{n_R\times n_R}, \quad A_\zeta \in \mathcal{D}^{n_L\times n_L}, \quad \Gamma \in \mathcal{D}^{n_R\times n_L}.$$

REMARK 2.1. By abuse of notation, by $A_\pi \in \mathcal{D}^{n_R\times n_R}$ and $A_\zeta \in \mathcal{D}^{n_L\times n_L}$ we mean that A_ζ and A_π are $n_R\times n_R$ and $n_L\times n_L$ matrices with diagonal block entries

$$(A_\zeta)_{ij} \in \mathcal{D}^{\ell_i\times \ell_j} \quad (i,j=1,\ldots,s) \quad \text{and} \quad (A_\pi)_{ij} \in \mathcal{D}^{r_i\times r_j} \quad (i,j=1,\ldots,t).$$

Similarly, by $C_+ \in \mathcal{D}^{m\times n_R}$ we mean that C_+ is a block row and every $m\times r_i$ block is diagonal (in other words, we reserve the symbols n_L and n_R to indicate block decomposition of the underlying matrices). We shall refer to such matrices as to *block diagonal*. The same convention holds with $\mathcal{U}$ and $\mathcal{L}$ instead of $\mathcal{D}$ for *block upper triangular* and for *block lower triangular* matrices. The other notations (such as $\mathcal{D}^{m\times n_R}$ or $\mathcal{D}^{k\times n_L}$) should be clear.

We say that the data (15) is *admissible* if

$$\text{(16)} \qquad \operatorname{Span}\left\{\operatorname{Ran}\left((A_\pi^* Z_\pi)^j\, C_-^*\right),\ j=0,\ldots,n_R-1\right\} = \mathbb{C}^{n_R},$$

$$\text{(17)} \qquad \operatorname{Span}\left\{\operatorname{Ran}\left(\left(A_\zeta^* Z_\zeta^*\right)^j B_+^*\right),\ j=0,\ldots,n_L-1\right\} = \mathbb{C}^{n_L},$$

$$\text{(18)} \qquad \Gamma = \Gamma Z_\pi Z_\pi^*,$$

and the Sylvester equality holds

$$\text{(19)} \qquad A_\zeta^* Z_\zeta^* \Gamma Z_\pi - \Gamma A_\pi = B_-^* C_- - B_+^* C_+.$$

We denote by **IP** the following two–sided interpolation problem.

PROBLEM 2.2. *Given an admissible data set Ω, find all $H \in \mathcal{U}^{m\times k}$ such that*

$$\text{(20)} \qquad p_0\left\{HC_-\left(I - Z_\pi^* A_\pi\right)^{-1}\right\} = C_+,$$

$$\text{(21)} \qquad p_0\left\{H^* B_+ \left(I - Z_\zeta A_\zeta\right)^{-1}\right\} = B_-,$$

$$\text{(22)} \qquad P_0\left\{\left(I - A_\zeta^* Z_\zeta^*\right)^{-1} B_+^* H C_- \left(I - Z_\pi^* A_\pi\right)^{-1} Z_\pi^*\right\} = \Gamma,$$

where in the first equation, p_0 is the orthogonal projection of $\mathcal{X}^{m\times n_R}$ onto $\mathcal{D}^{m\times n_R}$, in the second equation p_0 denotes the orthogonal projection of $\mathcal{X}^{k\times n_L}$ onto $\mathcal{D}^{k\times n_L}$, and finally, in the third equation,

$$P_0 = p_{01} \oplus \cdots \oplus p_{0s}$$

and p_{0i} is the orthogonal projection of $\mathcal{X}^{\ell_i\times \ell_i}$ onto $\mathcal{D}^{\ell_i\times \ell_i}$.

Note that by Remark 1.1 and in view of the block structure of A_π, A_ζ, Z_π and Z_ζ, the matrices $I - Z_\pi^* A_\pi$ and $I - Z_\zeta A_\zeta$ are invertible.

REMARK 2.3. Conditions (20) and (21) generalize the Nevanlinna–Pick conditions (9) and coincide with the last ones for the special choice of

$$B_+ = (\xi_1,\ldots,\xi_n), \quad B_- = (\eta_1,\ldots,\eta_n), \quad C_+ = (\zeta_1,\ldots,\zeta_\ell), \quad C_- = (\nu_1,\ldots,\nu_\ell),$$

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$$A_\zeta = \begin{bmatrix} W_1^* & & 0 \\ & \ddots & \\ 0 & & W_n^* \end{bmatrix}, \qquad A_\pi = \begin{bmatrix} V_1 & & 0 \\ & \ddots & \\ 0 & & V_\ell \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} \Gamma_{11} & \cdots & \Gamma_{1\ell} \\ \vdots & & \vdots \\ \Gamma_{n1} & \cdots & \Gamma_{n\ell} \end{bmatrix}.$$

The next lemma shows that conditions (20) and (21) contain more information about a solution H of the interpolation problem.

LEMMA 2.4. *Let H belong to $\mathcal{U}^{m\times k}$ and satisfy* (20), (21). *Then*

$$q\left\{HC_-\left(I - Z_\pi^* A_\pi\right)^{-1}\right\} = C_+\left(I - Z_\pi^* A_\pi\right)^{-1} \tag{23}$$

and

$$p\left\{H^* B_+\left(I - Z_\zeta A_\zeta\right)^{-1}\right\} = B_-\left(I - Z_\zeta A_\zeta\right)^{-1}. \tag{24}$$

Proof. For every $\Psi \in \mathcal{X}^{m\times n_R}$, it holds that $q\Psi = \sum_{j=0}^{n_R-1}\left(p_0 \Psi Z_\pi^j\right) Z_\pi^{*j}$ and therefore,

$$q\left\{HC_-\left(I - Z_\pi^* A_\pi\right)^{-1}\right\} = \sum_{j=0}^{n_R-1}\left(p_0\left\{HC_-\left(I - Z_\pi^* A_\pi\right)^{-1} Z_\pi^j\right\}\right) Z_\pi^{*j}. \tag{25}$$

The first term on the right hand side of the equality

$$\left(I - Z_\pi^* A_\pi\right)^{-1} Z_\pi^j = \sum_{\ell=0}^{j-1}\left(Z_\pi^* A_\pi\right)^\ell Z_\pi^j + \left(I - Z_\pi^* A_\pi\right)^{-1}\left(Z_\pi^* A_\pi\right)^j Z_\pi^j$$

is strictly block upper triangular, and thus, for every upper triangular H,

$$p_0\left\{HC_-\left(I - Z_\pi^* A_\pi\right)^{-1} Z_\pi^j\right\} = p_0\left\{HC_-\left(I - Z_\pi^* A_\pi\right)^{-1}\left(Z_\pi^* A_\pi\right)^j Z_\pi^j\right\}.$$

Since the operator $\left(Z_\pi^* A_\pi\right)^j Z_\pi^j$ is block diagonal, it follows from (20) that

$$\begin{aligned} p_0\left\{HC_-\left(I - Z_\pi^* A_\pi\right)^{-1}\left(Z_\pi^* A_\pi\right)^j Z_\pi^j\right\} &= \left(p_0\left\{HC_-\left(I - Z_\pi^* A_\pi\right)^{-1}\right\}\right) \times \\ &\quad \times \left(Z_\pi^* A_\pi\right)^j Z_\pi^j \\ &= C_+\left(Z_\pi^* A_\pi\right)^j Z_\pi^j. \end{aligned}$$

Since A_π is block diagonal,

$$\left(Z_\pi^* A_\pi\right)^j Z_\pi^j Z_\pi^{*j} = \left(Z_\pi^* A_\pi\right)^j, \qquad j = 0, 1, \ldots$$

Making use of the three last equalities we deduce (23) from (25):

$$\begin{aligned} q\left\{HC_-\left(I - Z_\pi^* A_\pi\right)^{-1}\right\} &= \sum_{j=0}^{n_R-1} C_+\left(Z_\pi^* A_\pi\right)^j Z_\pi^j Z_\pi^{*j} \\ &= \sum_{j=0}^{n_R-1} C_+\left(Z_\pi^* A_\pi\right)^j \\ &= C_+\left(I - Z_\pi^* A_\pi\right)^{-1}. \end{aligned}$$

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Taking advantage of the relation

$$p\left\{H^*B_+\left(I-Z_\zeta A_\zeta\right)^{-1}\right\}=\sum_{j=0}^{n_L-1}\left(p_0\left\{H^*B_+\left(I-Z_\zeta A_\zeta\right)^{-1}Z_\zeta^{*j}\right\}\right)Z_\zeta^j,$$

it can be checked in much the same way that (21) implies (24). ▯

Corollary 2.5. *Conditions* (20) *and* (21) *are equivalent to* (23) *and* (24) *respectively.*

Indeed, applying p_0 to both sides of (23) and (24) we obtain (20), (21). The rest follows from Lemma 2.4.

Sometimes it will be convenient to use conditions (20) and (21) in the following "adjoint" forms

$$(26)\quad P_0\left\{\left(I-A_\pi^*Z_\pi\right)^{-1}C_-^*H^*\right\}=C_+^*\quad\text{and}\quad P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*H\right\}=B_-^*.$$

Remark 2.6. Conditions (20), (21) are equivalent to conditions (26).

Proof. Taking $H\in\mathcal{U}^{m\times k}$ in the form $H=\sum_{j=0}^{k-1}H_jZ_k^j$ with $H_j\in\mathcal{D}^{m\times k}$, we get

$$p_0\left\{HC_-\left(I-Z_\pi^*A_\pi\right)^{-1}\right\}=\sum_{j=0}^{k-1}H_jZ_k^jC_-\left(Z_\pi^*A_\pi\right)^j,$$

$$P_0\left\{\left(I-A_\pi^*Z_\pi\right)^{-1}C_-^*H^*\right\}=\sum_{j=0}^{k-1}\left(A_\pi^*Z_\pi\right)^jC_-^*Z_k^{*j}H_j.$$

Comparing right hand sides in two last equalities we conclude that

$$\left(p_0\left\{HC_-\left(I-Z_\pi^*A_\pi\right)^{-1}\right\}\right)^*=P_0\left\{\left(I-A_\pi^*Z_\pi\right)^{-1}C_-^*H^*\right\}$$

and therefore, (20) is equivalent to the first condition in (26). The equivalence of (21) and the second condition in (26) is checked in much the same way. ▯

Remark 2.7. The Sylvester identity (19) follows from (20)–(22) and is therefore, a necessary condition for the problem **IP** to be solvable.

Proof. First we note the equality

$$(27)\qquad P_0\left(Z_\zeta^*M\left(I-Z_\pi^*Z_\pi\right)\right)=0,$$

which holds for every $M\in\mathcal{X}^{n_R\times n_L}$. Indeed, taking M in the form

$$M=\sum_{j=0}^{n_L-1}Z_\zeta^{*j}M_j+\sum_{i=1}^{n_R-1}M_{n_L-1+i}Z_\pi^i\quad\text{with}\quad M_\ell\in\mathcal{D}^{n_R\times n_L},$$

we get

$$Z_\zeta^*M\left(I-Z_\pi^*Z_\pi\right)=Z_\zeta^*\sum_{j=0}^{n_L-1}Z_\zeta^{*j}M_j\left(I-Z_\pi^*Z_\pi\right)+\left(\sum_{i=0}^{n_R-2}Z_\zeta^{*j}M_j\right)Z_\pi\left(I-Z_\pi^*Z_\pi\right)$$

and come to (27), since the first term on the right hand side in the latter equality is strictly block lower triangular and the second term is equal to zero due to (12).

Let H belong to $\mathcal{U}^{m\times k}$ and satisfy (20)–(22) (or equivalently, (22) and (26)). Upon applying (27) to

$$M=\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*HC_-\left(I-Z_\pi^*A_\pi\right)^{-1},$$

taking into account that the matrix A_ζ is block diagonal and making use of the equality

$$Z_\zeta^*(P_0X)Z_\pi=P_0Z_\zeta^*XZ_\pi,$$

which holds for every $X\in\mathcal{X}^{n_L\times n_L}$, we get

$$\begin{aligned}
A_\zeta^*Z_\zeta^*\Gamma Z_\pi = {} & A_\zeta^*Z_\zeta^*\left(P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*HC_-\left(I-Z_\pi^*A_\pi\right)^{-1}Z_\pi^*\right\}\right)Z_\pi\\
= {} & P_0\left(A_\zeta^*Z_\zeta^*\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*HC_-\left(I-Z_\pi^*A_\pi\right)^{-1}Z_\pi^*Z_\pi\right)\\
= {} & P_0\left(A_\zeta^*Z_\zeta^*\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*HC_-\left(I-Z_\pi^*A_\pi\right)^{-1}\right)\\
& -A_\zeta^*P_0\left(Z_\zeta^*\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*HC_-\left(I-Z_\pi^*A_\pi\right)^{-1}\left(I-Z_\pi^*Z_\pi\right)\right)\\
= {} & P_0\left\{\left(\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}-I\right)B_+^*HC_-\left(I-Z_\pi^*A_\pi\right)^{-1}\right\}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\Gamma A_\pi = {} & P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*HC_-\left(I-Z_\pi^*A_\pi\right)^{-1}Z_\pi^*A_\pi\right\}\\
= {} & P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*HC_-\left(\left(I-Z_\pi^*A_\pi\right)^{-1}-I\right)\right\}
\end{aligned}$$

and since B_+ and C_- are block diagonal, it follows from (26) that

$$\begin{aligned}
A_\zeta^*Z_\zeta^*\Gamma Z_\pi-\Gamma A_\pi &= P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*HC_-\right\}-P_0\left\{B_+^*HC_-\left(I-Z_\pi^*A_\pi\right)^{-1}\right\}\\
&=\left(P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*H\right\}\right)C_- - B_+^*P_0\left\{HC_-\left(I-Z_\pi^*A_\pi\right)^{-1}\right\}\\
&=B_-^*C_- - B_+^*C_+,
\end{aligned}$$

and this completes the proof. □

It will be shown in Section 6 that the problem **IP** always has a solution. Let us denote by $\mathbf{IP}_\gamma$ the problem **IP** to which has been added the norm constraint

$$\langle H,\ H\rangle_{\mathcal{X}^{m\times k}}\stackrel{\text{def}}{=}\operatorname{Tr}\left(H^*H\right)\le\gamma \tag{28}$$

for some preassigned number $\gamma\ge 0$. Because of the Hilbert space structure, we will see that there exists a unique solution $H_{\min}$ of the problem **IP** with minimal norm.

The value $\|H_{\min}\|_{\mathbf{HS}}$ (as well as $H_{\min}$ itself) depends only on the problem data (15) and the condition $\gamma \geq \|H_{\min}\|_{\mathbf{HS}}$ is necessary and sufficient for the problem $\mathbf{IP}_\gamma$ to be solvable. The explicit formula for $H_{\min}$ and a description of all solutions of the problem $\mathbf{IP}_\gamma$ will be given in Section 6.

We denote by $\mathbf{IP}_R(\Upsilon_R)$ the right–sided problem (20) (i.e., when conditions (21) and (22) are not in force) to which has been added the "matrix norm" constraint

$$\{H,\ H\} \stackrel{\text{def}}{=} p_0(HH^*) \leq \Upsilon_R \tag{29}$$

for some preassigned nonnegative matrix $\Upsilon_R \in \mathcal{D}^{m\times m}$.

Similarly, we denote by $\mathbf{IP}_L(\Upsilon_L)$ the left–sided problem (21) to which has been added the matrix "norm" constraint

$$[H,\ H] \stackrel{\text{def}}{=} p_0(H^*H) \leq \Upsilon_L \tag{30}$$

for some preassigned nonnegative matrix $\Upsilon_L \in \mathcal{D}^{k\times k}$.

It turns out that the constraints (29) and (30) do not suit the left–sided condition (21) and the right–sided condition (20) respectively. That is why we consider a two–sided problem only under the constraint (28) which on account of

$$\langle H,\ H\rangle = \mathrm{Tr}\ \{H,\ H\} = \mathrm{Tr}\ [H,\ H]$$

suits to left conditions as well as to right ones.

3. Statement of the main result and first formulas. The main result of the paper is now stated:

THEOREM 3.1. *The set of all solutions of Problem* **IP** *is given by*

$$H = H_{\min} + \Theta_L h \Theta_R \tag{31}$$

where $H_{\min} \in \mathcal{U}^{m\times k}$ is the minimal Hilbert–Schmidt norm solution, $\Theta_R \in \mathcal{U}^{(k+n_R)\times k}$ and $\Theta_L \in \mathcal{U}^{m\times(m+n_L)}$ are two partial isometries with upper triangular block entries, built from the interpolation data and h is a free parameter from $\mathcal{U}^{(m+n_L)\times(k+n_R)}$.

In this section we construct explicitly Θ_L and Θ_R (see formulas (43) and (42)), while the formula for $H_{\min}$ will be given in Section 7. We begin with preliminary lemmas.

LEMMA 3.2. *The Stein equations*

$$\mathbb{P}_R - A_\pi^* Z_\pi \mathbb{P}_R Z_\pi^* A_\pi = C_-^* C_- \quad \text{and} \quad \mathbb{P}_L - A_\zeta^* Z_\zeta^* \mathbb{P}_L Z_\zeta A_\zeta = B_+^* B_+ \tag{32}$$

are uniquely solvable, and that their solutions are the block diagonal matrices given by

$$\mathbb{P}_R = \sum_{j=0}^{m-1} (A_\pi^* Z_\pi)^j C_-^* C_- (Z_\pi^* A_\pi)^j \in \mathcal{D}^{n_R\times n_R}$$

$$\mathbb{P}_L = \sum_{j=0}^{m-1} (A_\zeta^* Z_\zeta^*)^j B_+^* B_+ (Z_\zeta A_\zeta)^j \in \mathcal{D}^{n_L\times n_L}.$$

Conditions (16) *and* (17) *are necessary and sufficient that the operators* $\mathbb{P}_R$ *and* $\mathbb{P}_L$ *are boundedly invertible.*

The proof is straightforward and will be omitted.

LEMMA 3.3. *Let* $\mathbb{P}_R$ *and* $\mathbb{P}_L$ *are solutions of the Stein equations* (32). *Then the matrices*

$$\mathbf{Q}_R = C_-\left(I - Z_\pi^* A_\pi\right)^{-1}(I - Z_\pi^* Z_\pi)\mathbb{P}_R^{-1}(I - Z_\pi^* Z_\pi)\left(I - A_\pi^* Z_\pi\right)^{-1} C_-^*, \tag{33}$$

$$\mathbf{T}_R = \begin{bmatrix} Z_\pi Z_\pi^* & 0 \\ 0 & I_k \end{bmatrix} - \begin{bmatrix} Z_\pi \mathbb{P}_R^{\frac{1}{2}} Z_\pi^* A_\pi \\ C_- \end{bmatrix} \mathbb{P}_R^{-1}\left(A_\pi^* Z_\pi \mathbb{P}_R^{\frac{1}{2}} Z_\pi^*,\ C_-^*\right), \tag{34}$$

$$\mathbf{Q}_L = B_+\left(I - Z_\zeta A_\zeta\right)^{-1}(I - Z_\zeta Z_\zeta^*)\mathbb{P}_L^{-1}(I - Z_\zeta Z_\zeta^*)\left(I - A_\zeta^* Z_\zeta^*\right)^{-1} B_+^*, \tag{35}$$

$$\mathbf{T}_L = \begin{bmatrix} Z_\zeta^* Z_\zeta & 0 \\ 0 & I_m \end{bmatrix} - \begin{bmatrix} Z_\zeta^* \mathbb{P}_L^{\frac{1}{2}} Z_\zeta A_\zeta \\ B_+ \end{bmatrix} \mathbb{P}_L^{-1}\left(A_\zeta^* Z_\zeta^* \mathbb{P}_L^{\frac{1}{2}} Z_\zeta,\ B_+^*\right) \tag{36}$$

are orthogonal projections.

Proof. The matrices defined in (33)–(36) are evidently selfadjoint. To show that $\mathbf{Q}_R$ is a projection, we start with the equality

$$\begin{aligned} &\left(I - A_\pi^* Z_\pi\right)^{-1} C_-^* C_-\left(I - Z_\pi^* A_\pi\right)^{-1} = \\ &= \mathbb{P}_R\left(I - Z_\pi^* A_\pi\right)^{-1} + \left(I - A_\pi^* Z_\pi\right)^{-1} A_\pi^* Z_\pi \mathbb{P}_R, \end{aligned} \tag{37}$$

which is an immediate consequence of the first equation in (32). It follows from (33) that

$$\mathbf{Q}_R^2 = C_-\left(I - Z_\pi^* A_\pi\right)^{-1} \mathbf{L}\left(I - A_\pi^* Z_\pi\right)^{-1} C_-, \tag{38}$$

where, on account of (37),

$$\begin{aligned} \mathbf{L} = {}& (I - Z_\pi^* Z_\pi)\mathbb{P}_R^{-1}(I - Z_\pi^* Z_\pi)\left\{\mathbb{P}_R\left(I - Z_\pi^* A_\pi\right)^{-1} + \left(I - A_\pi^* Z_\pi\right)^{-1} A_\pi^* Z_\pi \mathbb{P}_R\right\} \\ & \times(I - Z_\pi^* Z_\pi)\mathbb{P}_R^{-1}(I - Z_\pi^* Z_\pi). \end{aligned}$$

Taking advantage of (12) (with $D = \mathbb{P}_R^{-1}$) and of (11) and (13) (with $D = A_\pi$) successively, we get

$$\begin{aligned} \mathbf{L} &= (I - Z_\pi^* Z_\pi)\mathbb{P}_R^{-1}\left\{\mathbb{P}_R\left(I - Z_\pi^* A_\pi\right)^{-1} + \left(I - A_\pi^* Z_\pi\right)^{-1} A_\pi^* Z_\pi \mathbb{P}_R\right\}\mathbb{P}_R^{-1}(I - Z_\pi^* Z_\pi) \\ &= (I - Z_\pi^* Z_\pi)\left\{\left(I - Z_\pi^* A_\pi\right)^{-1}\mathbb{P}_R^{-1} + \mathbb{P}_R^{-1}\left(I - A_\pi^* Z_\pi\right)^{-1} A_\pi^* Z_\pi\right\}(I - Z_\pi^* Z_\pi) \\ &= (I - Z_\pi^* Z_\pi)\mathbb{P}_R^{-1}(I - Z_\pi^* Z_\pi), \end{aligned}$$

which together with (38) leads to $\mathbf{Q}_R^2 = \mathbf{Q}_R$. To show that $\mathbf{T}_R$ is a projection, we first note that

$$\mathbb{P}_R^{-1}\left(A_\pi^* Z_\pi \mathbb{P}_R^{\frac{1}{2}} Z_\pi^*,\ C_-^*\right)\begin{bmatrix} Z_\pi \mathbb{P}_R^{\frac{1}{2}} Z_\pi^* A_\pi \\ C_- \end{bmatrix}\mathbb{P}_R^{-1} = \mathbb{P}_R^{-1}. \tag{39}$$

Indeed, making use of (32), we transform the left hand side as

$$A_\pi^* Z_\pi \mathbb{P}_R^{\frac{1}{2}} Z_\pi^* Z_\pi \mathbb{P}_R^{\frac{1}{2}} Z_\pi^* A_\pi + C_-^* C_- = \mathbb{P}_R - A_\pi^* Z_\pi \mathbb{P}_R^{\frac{1}{2}} (I - Z_\pi^* Z_\pi) \mathbb{P}_R^{\frac{1}{2}} Z_\pi^* A_\pi$$

and obtain (39), since in view of (11),

$$Z_\pi \mathbb{P}_R^{\frac{1}{2}} (I - Z_\pi^* Z_\pi) = Z_\pi (I - Z_\pi^* Z_\pi) \mathbb{P}_R^{\frac{1}{2}} = 0. \tag{40}$$

The equality $\mathbf{T}_R^2 = \mathbf{T}_R$ follows easily from (39) and relations

$$(Z_\pi Z_\pi^*)^2 = Z_\pi Z_\pi^*, \qquad \begin{bmatrix} Z_\pi Z_\pi^* & 0 \\ 0 & I_k \end{bmatrix} \begin{bmatrix} Z_\pi \mathbb{P}_R^{\frac{1}{2}} Z_\pi^* A_\pi \\ C_- \end{bmatrix} = \begin{bmatrix} Z_\pi \mathbb{P}_R^{\frac{1}{2}} Z_\pi^* A_\pi \\ C_- \end{bmatrix}$$

which are consequences of (12). Equalities $\mathbf{Q}_L^2 = \mathbf{Q}_L$ and $\mathbf{T}_L^2 = \mathbf{T}_L$ are verified quite similarly with the help of (11)–(14) and

$$\begin{aligned} \left(I - A_\zeta^* Z_\zeta^*\right)^{-1} B_+^* B_+ \left(I - Z_\zeta A_\zeta\right)^{-1} = \mathbb{P}_L \left(I - Z_\zeta A_\zeta\right)^{-1} + \\ + \left(I - A_\zeta^* Z_\zeta^*\right)^{-1} A_\zeta^* Z_\zeta^* \mathbb{P}_L \end{aligned} \tag{41}$$

which in turn, follows immediately from the second equation in (32). □

LEMMA 3.4. *The matrices* $\Theta_R \in \mathbb{C}^{(k+n_R)\times k}$ *and* $\Theta_L \in \mathbb{C}^{p\times(m+n_L)}$ *defined by*

$$\Theta_R = \begin{bmatrix} 0 \\ I_k \end{bmatrix} + \begin{bmatrix} Z_\pi \mathbb{P}_R^{\frac{1}{2}} (I - Z_\pi^* A_\pi) \\ -C_- \end{bmatrix} \mathbb{P}_R^{-1} \left(I - A_\pi^* Z_\pi\right)^{-1} C_-^* \tag{42}$$

and

$$\Theta_L = \begin{bmatrix} 0, & I_m \end{bmatrix} + B_+ \left(I - Z_\zeta A_\zeta\right)^{-1} \mathbb{P}_L^{-1} \left(\left(I - A_\zeta^* Z_\zeta^*\right) \mathbb{P}_L^{\frac{1}{2}} Z_\zeta, \; -B_+^*\right) \tag{43}$$

respectively, are block upper triangular and satisfy

$$\Theta_R^* \Theta_R = I_k - \mathbf{Q}_R, \qquad \Theta_R \Theta_R^* = \mathbf{T}_R, \tag{44}$$

and

$$\Theta_L \Theta_L^* = I_m - \mathbf{Q}_L, \qquad \Theta_L^* \Theta_L = \mathbf{T}_L, \tag{45}$$

where $\mathbf{Q}_R$, $\mathbf{T}_R$, $\mathbf{Q}_L$ *and* $\mathbf{T}_L$ *are the orthogonal projections defined in* (33)–(36).

Proof. The upper triangular structure of Θ_R and Θ_L follows immediately from their definitions (42) and (43). The upper triangular structure here is meant in the sense of Remark 2.1: according to partitions (10), Θ_R and Θ_L are $(t+1)\times t$ and $s\times(s+1)$ block matrices, respectively, and each block entry is an upper triangular matrix in a usual sense. The verification of (44) and (45) is quite straightforward: by (42),

$$\Theta_R^* \Theta_R = I_k - C_- \left(I - Z_\pi^* A_\pi\right)^{-1} \mathbb{P}_R^{-1} \mathbf{M} \mathbb{P}_R^{-1} \left(I - A_\pi^* Z_\pi\right)^{-1} C_-^*, \tag{46}$$

where

$$\mathbf{M} = \mathbb{P}_R\,(I - Z_\pi^* A_\pi) + \\ + (I - A_\pi^* Z_\pi)\,\mathbb{P}_R - (I - A_\pi^* Z_\pi)\,\mathbb{P}_R^{\frac{1}{2}} Z_\pi^* Z_\pi \mathbb{P}_R^{\frac{1}{2}}\,(I - Z_\pi^* A_\pi) - C_-^* C_-.$$

In view of (32),

$$\mathbf{M} = (I - A_\pi^* Z_\pi)\,\mathbb{P}_R^{\frac{1}{2}}(I - Z_\pi^* Z_\pi)\mathbb{P}_R^{\frac{1}{2}}\,(I - Z_\pi^* A_\pi)$$

Taking advantage of the first two relations in (12) (with $D = \mathbb{P}_R^{\frac{1}{2}}$), we come to

$$\mathbf{M} = (I - Z_\pi^* Z_\pi)\mathbb{P}_R(I - Z_\pi^* Z_\pi),$$

which being substituted into (46) leads to the first equality in (44). Furthermore, on account of (42),

$$\begin{aligned} \Theta_R\Theta_R^* = {} & \begin{bmatrix} 0 & 0 \\ 0 & I_k \end{bmatrix} + \begin{bmatrix} 0 \\ C_- \end{bmatrix} (I - Z_\pi^* A_\pi)^{-1}\,\mathbb{P}_R^{-1}\left[(I - A_\pi^* Z_\pi)\,\mathbb{P}_R^{\frac{1}{2}} Z_\pi^*,\ -C_-^*\right] \\ & + \begin{bmatrix} Z_\pi \mathbb{P}_R^{\frac{1}{2}}\,(I - Z_\pi^* A_\pi) \\ -C_- \end{bmatrix} \mathbb{P}_R^{-1}\,(I - A_\pi^* Z_\pi)^{-1}\left(0,\ C_-^*\right) \\ & + \begin{bmatrix} Z_\pi \mathbb{P}_R^{\frac{1}{2}}\,(I - Z_\pi^* A_\pi) \\ -C_- \end{bmatrix} \mathbb{P}_R^{-1}\,(I - A_\pi^* Z_\pi)^{-1}\,C_-^* C_-\,(I - Z_\pi^* A_\pi)^{-1} \\ & \times \mathbb{P}_R^{-1}\left[(I - A_\pi^* Z_\pi)\,\mathbb{P}_R^{\frac{1}{2}} Z_\pi^*,\ -C_-^*\right]. \end{aligned} \tag{47}$$

Substituting (37) into the third term on the right hand side of (47) and taking into account (34), we get

$$\begin{aligned} \Theta_R\Theta_R^* = {} & \begin{bmatrix} 0 & 0 \\ 0 & I_k \end{bmatrix}\bigg) + \begin{bmatrix} Z_\pi \mathbb{P}_R^{\frac{1}{2}}\,(I - Z_\pi^* A_\pi) \\ -C_- \end{bmatrix} \mathbb{P}_R^{-\frac{1}{2}}\,(Z_\pi^*,\ 0) \\ & + \begin{bmatrix} Z_\pi \\ 0 \end{bmatrix} \mathbb{P}_R^{-\frac{1}{2}}\left[(I - A_\pi^* Z_\pi)\,\mathbb{P}_R^{\frac{1}{2}} Z_\pi^*,\ -C_-^*\right] \\ & - \begin{bmatrix} Z_\pi \mathbb{P}_R^{\frac{1}{2}}\,(I - Z_\pi^* A_\pi) \\ -C_- \end{bmatrix} \mathbb{P}_R^{-1}\left[(I - A_\pi^* Z_\pi)\,\mathbb{P}_R^{\frac{1}{2}} Z_\pi^*,\ -C_-^*\right] \\ = {} & \begin{bmatrix} Z_\pi Z_\pi^* & 0 \\ 0 & I_k \end{bmatrix} - \begin{bmatrix} Z_\pi \mathbb{P}_R^{\frac{1}{2}} Z_\pi^* A_\pi \\ C_- \end{bmatrix} \mathbb{P}_R^{-1}\left(A_\pi^* Z_\pi \mathbb{P}_R^{\frac{1}{2}} Z_\pi,\ C_-^*\right), \end{aligned}$$

which proves the second equality in (44). The equalities (45) are verified in much the same way with help of (43) and (41). □

REMARK 3.5. The matrices Θ_R and Θ_L defined via (42) and (43) admit the representations

$$\Theta_R = \mathbf{T}_R \begin{bmatrix} Z_\pi \mathbb{P}_R^{-\frac{1}{2}}\,(I - A_\pi^* Z_\pi)^{-1}\,C_-^* \\ I_k \end{bmatrix}, \tag{48}$$

$$\Theta_L = \left(B_+\,(I - Z_\zeta A_\zeta)^{-1}\,\mathbb{P}_L^{-\frac{1}{2}} Z_\zeta,\ I_m\right)\mathbf{T}_L. \tag{49}$$

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Proof. In view of (34) and (48),

$$\mathbf{T}_R \begin{bmatrix} Z_\pi \mathbb{P}_R^{-\frac{1}{2}} (I - A_\pi^* Z_\pi)^{-1} C_-^* \\ I_k \end{bmatrix} = \begin{bmatrix} Z_\pi \mathbb{P}_R^{-\frac{1}{2}} (I - A_\pi^* Z_\pi)^{-1} C_-^* \\ I_k \end{bmatrix}$$
$$- \begin{bmatrix} Z_\pi \mathbb{P}_R^{\frac{1}{2}} Z_\pi^* A_\pi \\ C_- \end{bmatrix} \mathbb{P}_R^{-1} \left(A_\pi^* Z_\pi \mathbb{P}_R^{\frac{1}{2}} Z_\pi^* Z_\pi \mathbb{P}_R^{-\frac{1}{2}} (I - A_\pi^* Z_\pi)^{-1} + I \right) C_-^*.$$

By (40), $Z_\pi \mathbb{P}_R^{\frac{1}{2}} Z_\pi^* Z_\pi = Z_\pi \mathbb{P}_R^{\frac{1}{2}}$ and therefore,

$$\begin{aligned} &\mathbf{T}_R \begin{bmatrix} Z_\pi \mathbb{P}_R^{-\frac{1}{2}} (I - A_\pi^* Z_\pi)^{-1} C_-^* \\ I_k \end{bmatrix} \\ &= \begin{bmatrix} Z_\pi \mathbb{P}_R^{-\frac{1}{2}} (I - A_\pi^* Z_\pi)^{-1} C_-^* \\ I_k \end{bmatrix} - \begin{bmatrix} Z_\pi \mathbb{P}_R^{\frac{1}{2}} Z_\pi^* A_\pi \\ C_- \end{bmatrix} \mathbb{P}_R^{-1} (I - A_\pi^* Z_\pi)^{-1} C_-^* \\ &= \begin{bmatrix} 0 \\ I_k \end{bmatrix} + \begin{bmatrix} Z_\pi^* \mathbb{P}_R^{\frac{1}{2}} (I - Z_\pi^* A_\pi) \\ -C_- \end{bmatrix} \mathbb{P}_R^{-1} (I - A_\pi^* Z_\pi)^{-1} C_-^* = \Theta_R, \end{aligned}$$

which proves (48). The representation (49) is verified in much the same way with help of (36) and (43). ▯

4. Right–sided problem. In this section we describe all $H \in \mathcal{U}^{m \times k}$ satisfying condition (20). We first exhibit a particular solution, which will be shown in the sequel to be of the minimal Hilbert–Schmidt norm.

LEMMA 4.1. *Let H_R be the block upper triangular matrix given by*

$$H_R = C_+ \mathbb{P}_R^{-1} (I - A_\pi^* Z_\pi)^{-1} C_-^*. \tag{50}$$

Then H_R satisfies the condition (20) *and*

$$\{H_R,\ H_R\} = C_+ \mathbb{P}_R^{-1} C_+^*, \qquad \langle H_R,\ H_R \rangle \;=\; \mathrm{Tr}\ C_+ \mathbb{P}_R^{-1} C_+^*. \tag{51}$$

Proof. In view of (37),

$$\begin{aligned} H_R C_- (I - Z_\pi^* A_\pi)^{-1} &= C_+ \mathbb{P}_R^{-1} (I - A_\pi^* Z_\pi)^{-1} C_-^* C_- (I - Z_\pi^* A_\pi)^{-1} \\ &= C_+ (I - Z_\pi^* A_\pi)^{-1} + C_+ \mathbb{P}_R^{-1} (I - A_\pi^* Z_\pi)^{-1} A_\pi^* Z_\pi \mathbb{P}_R, \end{aligned} \tag{52}$$

and since

$$C_+^* \mathbb{P}_R^{-1} (I - A_\pi^* Z_\pi)^{-1} A_\pi^* Z_\pi \mathbb{P}_R \in \mathcal{U}^{m \times n_R} Z_\pi, \tag{53}$$

condition (23) (which is equivalent to (20) by Corollary 2.5) follows from (52). Furthermore, multiplying (52) by $\mathbb{P}_R^{-1} C_+^*$ on the right we get

$$H_R H_R^* = C_+ (I - Z_\pi^* A_\pi)^{-1} \mathbb{P}_R^{-1} C_+^* + C_+ \mathbb{P}_R^{-1} (I - A_\pi^* Z_\pi)^{-1} A_\pi^* Z_\pi C_+^*. \tag{54}$$

By (53), the second term on the right hand side of (54) is strictly block upper triangular while the first one is upper triangular and thus,

$$\{H_R,\ H_R\} \stackrel{\text{def}}{=} p_0(H_R H_R^*) = p_0\left\{C_+\left(I - Z_\pi^* A_\pi\right)^{-1}\mathbb{P}_R^{-1}C_+^*\right\} = C_+\mathbb{P}_R^{-1}C_+^*,$$

which proves the first equality in (51). The second equality is an immediate consequence of the first one. □

Note that every $H \in \mathcal{U}^{m\times k}$ satisfying the condition (20) is of the form

$$H = H_R + \Psi \tag{55}$$

where Ψ is an element from $\mathcal{U}^{m\times k}$ such that

$$p_0\left\{\Psi C_-\left(I - Z_\pi^* A_\pi\right)^{-1}\right\} = 0. \tag{56}$$

Lemma 4.2. *The matrix Ψ belongs to $\mathcal{U}^{m\times k}$ and satisfies* (56) *if and only if it admits a representation*

$$\Psi = \widehat{H}\Theta_R \tag{57}$$

where Θ_R is given by (42) *and $\widehat{H} \in \mathcal{U}^{m\times(k+n_R)}$.*

Proof. Let $\widehat{H}$ be in $\mathcal{U}^{m\times(k+n_R)m}$ and let Ψ be of the form (57). Since Θ_R is block upper triangular, $\Psi \in \mathcal{U}^{m\times k}$. Furthermore, in view of (42) and (37),

$$\begin{aligned}
&\Theta_R C_-\left(I - Z_\pi^* A_\pi\right)^{-1}\\
&= \left(\begin{bmatrix} 0 \\ I_k \end{bmatrix} + \begin{bmatrix} Z_\pi \mathbb{P}_R^{\frac{1}{2}}\left(I - Z_\pi^* A_\pi\right) \\ -C_- \end{bmatrix}\mathbb{P}_R^{-1}\left(I - A_\pi^* Z_\pi\right)^{-1}C_-^*\right)C_-\left(I - Z_\pi^* A_\pi\right)^{-1}\\
&= \begin{bmatrix} 0 \\ C_- \end{bmatrix}\left(I - Z_\pi^* A_\pi\right)^{-1}\\
&\qquad + \begin{bmatrix} Z_\pi \mathbb{P}_R^{\frac{1}{2}}\left(I - Z_\pi^* A_\pi\right) \\ -C_- \end{bmatrix}\left(\left(I - Z_\pi^* A_\pi\right)^{-1} + \mathbb{P}_R^{-1}\left(I - A_\pi^* Z_\pi\right)^{-1}A_\pi^* Z_\pi \mathbb{P}_R\right)\\
&= \begin{bmatrix} Z_\pi \mathbb{P}_R^{\frac{1}{2}} \\ 0 \end{bmatrix} + \begin{bmatrix} Z_\pi \mathbb{P}_R^{\frac{1}{2}}\left(I - Z_\pi^* A_\pi\right) \\ -C_- \end{bmatrix}\mathbb{P}_R^{-1}\left(I - A_\pi^* Z_\pi\right)^{-1}A_\pi^* Z_\pi \mathbb{P}_R.
\end{aligned}$$

It is easily seen from the last equality that the matrix $\Theta_R C_-\left(I - Z_\pi^* A_\pi\right)^{-1}$ is strictly block upper triangular and therefore

$$p_0\left\{\Psi C_-\left(I - Z_\pi^* A_\pi\right)^{-1}\right\} = p_0\left\{\widehat{H}\Theta_R C_-\left(I - Z_\pi^* A_\pi\right)^{-1}\right\} = 0$$

for every element $\widehat{H} \in \mathcal{U}^{(k+n_R)\times m}$.

Conversely, let Ψ belongs to $\mathcal{U}^{m\times k}$ and satisfy (56). By Corollary 2.5,

$$q\left\{\Psi C_-\left(I - Z_\pi^* A_\pi\right)^{-1}\right\} = 0 \tag{58}$$

which means that the matrix $\Psi C_-\left(I - Z_\pi^* A_\pi\right)^{-1}$ is strictly block upper triangular and therefore

$$\Psi C_-\left(I - Z_\pi^* A_\pi\right)^{-1}\left(I - Z_\pi^* Z_\pi\right) = 0.$$

It follows from the last relation and from (33) that $\Psi \mathbf{Q}_R = 0$, which together with the first equality in (44) implies that

$$\Psi \Theta_R^* \Theta_R = \Psi\left(I - \mathbf{Q}_R\right) = \Psi.$$

The latter equality means that Ψ admits a representation (57) with $\widehat{H} = \Psi \Theta_R^*$.

It remains to note that by (42),

$$\widehat{H} = \Psi\left\{(0,\ I_k) + C_-\left(I - Z_\pi^* A_\pi\right)^{-1} \mathbb{P}_R^{-1}\left(\left(I - A_\pi^* Z_\pi\right)\mathbb{P}_R^{\frac{1}{2}} Z_\pi^*,\ -C_-^*\right)\right\}$$

which implies, in view of (58), that $\widehat{H}$ is block upper triangular: $\widehat{H} \in \mathcal{U}^{m\times(k+n_R)}$. □

Using (55) and Lemma 4.2 we obtain the following result.

THEOREM 4.3. *All $H \in \mathcal{U}^{m\times k}$ which satisfy* (20) *are parametrized by the formula*

$$H = H_R + \widehat{H}\Theta_R \tag{59}$$

where H_R and Θ_R are given by (50) *and* (42), *respectively and $\widehat{H}$ is a free parameter from $\mathcal{U}^{m\times(k+n_R)}$.*

Now we can describe the set of all solutions of the problem $\mathbf{IP}_R(\Upsilon_R)$.

LEMMA 4.4. *The representation* (59) *is orthogonal: for every $\widehat{H} \in \mathcal{U}^{m\times(k+n_R)}$*

$$\{\widehat{H}\Theta_R,\ H_R\} = 0 \quad \text{and} \quad \langle \widehat{H}\Theta_R,\ H_R\rangle = 0. \tag{60}$$

Proof. It follows from (36) that $\mathbf{T}_R\begin{bmatrix} Z_\pi \mathbb{P}_R^{\frac{1}{2}} Z_\pi^* A_\pi \\ C_- \end{bmatrix} = 0$. Making use of this equality together with (37), (48) and (50) we obtain

$$\begin{aligned}
\Theta_R H_R^* &= \mathbf{T}_R\begin{bmatrix} Z_\pi \mathbb{P}_R^{-\frac{1}{2}}\left(I - A_\pi^* Z_\pi\right)^{-1} C_-^* \\ I_k \end{bmatrix} C_-\left(I - Z_\pi^* A_\pi\right)^{-1}\mathbb{P}_R^{-1} C_+^* \\
&= \mathbf{T}_R\begin{bmatrix} Z_\pi \mathbb{P}_R^{-\frac{1}{2}}\left\{\mathbb{P}_R Z_\pi^* A_\pi\left(I - Z_\pi^* A_\pi\right)^{-1} + \left(I - A_\pi^* Z_\pi\right)^{-1}\mathbb{P}_R\right\}\mathbb{P}_R^{-1} C_+^* \\ C_-\left(I - Z_\pi^* A_\pi\right)^{-1}\mathbb{P}_R^{-1} C_+^* \end{bmatrix} \\
&= \mathbf{T}_R\begin{bmatrix} Z_\pi \mathbb{P}_R^{\frac{1}{2}} Z_\pi^* A_\pi \\ C_- \end{bmatrix}\left(I - Z_\pi^* A_\pi\right)^{-1}\mathbb{P}_R^{-1} C_+^* + \mathbf{T}_R\begin{bmatrix} Z_\pi \mathbb{P}_R^{-\frac{1}{2}}\left(I - A_\pi^* Z_\pi\right)^{-1} C_+^* \\ 0 \end{bmatrix} \\
&= \mathbf{T}_R\begin{bmatrix} Z_\pi \mathbb{P}_R^{-\frac{1}{2}}\left(I - A_\pi^* Z_\pi\right)^{-1} C_+^* \\ 0 \end{bmatrix}.
\end{aligned}$$

It is readily seen from (34) that the projection $\mathbf{T}_R$ is block diagonal. Then it follows from the last equation that $\Theta_R H_R^*$ is strictly block upper triangular. Thus, for every upper triangular $\widehat{H}$, the matrix $\widehat{H}\Theta_R H_R^*$ is strictly block upper triangular and therefore,

$$\left\{\widehat{H}\Theta_R,\ H_R\right\} = p_0\left(\widehat{H}\Theta_R H_R^*\right) = 0$$

which in turn, implies

$$\langle\widehat{H}\Theta_R,\ H_R\rangle = \mathrm{Tr}\ \left(\widehat{H}\Theta_R H_R^*\right) = 0$$

and finishes the proof of the lemma. □

THEOREM 4.5. *All solutions H of the problem $\mathbf{IP}_R(\Upsilon_R)$ are parametrized by the formula* (59) *the parameter $\widehat{H}$ varies in $\mathcal{U}^{m\times(k+n_R)}$ and is subject to*

$$\left\{\widehat{H}\mathbf{T}_R,\ \widehat{H}\mathbf{T}_R\right\} \le \Upsilon_R - C_+\mathbb{P}_R^{-1}C_+^*, \tag{61}$$

where $\mathbf{T}_R$ is the orthogonal projection defined in (36).

Proof. In view of (44),

$$\left\{\widehat{H}\Theta_R,\ \widehat{H}\Theta_R\right\} = p_0\ \left(\widehat{H}\Theta_R\Theta_R^*\widehat{H}^*\right) = p_0\ \left(\widehat{H}\mathbf{T}_R\widehat{H}^*\right) = \left\{\widehat{H}\mathbf{T}_R,\ \widehat{H}\mathbf{T}_R\right\}$$

which together with (51), (59) and (60) leads to

$$\begin{aligned}\{H,\ H\} = \left\{H_R + \widehat{H}\Theta_R,\ H_R + \widehat{H}\Theta_R\right\} = &\ \ \{H_R,\ H_R\} + \left\{\widehat{H}\Theta_R,\ \widehat{H}\Theta_R\right\}\\ = &\ \ C_+\mathbb{P}_R^{-1}C_+^* + \left\{\widehat{H}\mathbf{T}_R,\ \widehat{H}\mathbf{T}_R\right\},\end{aligned}$$

and thus, the matrix H of the form (59) satisfies (29) if and only if the corresponding parameter $\widehat{H}$ satisfies (61). □

5. Left–sided problem. In this section we describe all $H \in \mathcal{U}^{m\times k}$ satisfying the condition (21).

LEMMA 5.1. *Let H_L be the block upper triangular matrix given by*

$$H_L = B_+\left(I - Z_\zeta A_\zeta\right)^{-1}\mathbb{P}_L^{-1}B_-^*. \tag{62}$$

Then H_L satisfies the condition (21) *and*

$$[H_L,\ H_L] = B_-\mathbb{P}_L^{-1}B_-^*, \qquad \langle H_L,\ H_L\rangle\ =\ \mathrm{Tr}\ B_-\mathbb{P}_L^{-1}B_-^*. \tag{63}$$

Proof. Making use of (62) and (41) we get

$$H_L^*B_+\left(I - Z_\zeta A_\zeta\right)^{-1} = B_-\left(I - Z_\zeta A_\zeta\right)^{-1} + B_-\mathbb{P}_L^{-1}\left(I - A_\zeta^*Z_\zeta^*\right)^{-1}A_\zeta^*Z_\zeta^*\mathbb{P}_L \tag{64}$$

and since the second term on the right hand side in the latter equality is strictly block lower triangular, the condition (21) follows from (64). Furthermore, multiplying (64) by $\mathbb{P}_L^{-1}B_-^*$ from the right we get

$$H_L^*H_L = B_-\left(I - Z_\zeta A_\zeta\right)^{-1}\mathbb{P}_L^{-1}B_-^* + B_-\mathbb{P}_L^{-1}\left(I - A_\zeta^*Z_\zeta^*\right)^{-1}A_\zeta^*Z_\zeta^*B_-^*,$$

which implies

$$[H_L,\ H_L]=p_0\left(H_L^*H_L\right)=p_0\left\{B_-\left(I-Z_\zeta A_\zeta\right)^{-1}\mathbb{P}_L^{-1}B_-^*\right\}=B_-\mathbb{P}_L^{-1}B_-^*,$$

and in particular the second equality in (63) is in force. □

It will be shown that H_L has minimal Hilbert–Schmidt norm among all $\mathcal{U}^{m\times k}$–solutions of the problem $\mathbf{IP}_L$. The rest is similar to considerations from the previous section: every $H\in\mathcal{U}^{m\times k}$ satisfying the condition (21) is of the form

$$H=H_L+\Psi \tag{65}$$

where Ψ is an element from $\mathcal{U}^{m\times k}$ such that

$$p_0\left\{\Psi^*B_+\left(I-Z_\zeta A_\zeta\right)^{-1}\right\}=0. \tag{66}$$

Lemma 5.2. *The matrix Ψ belongs to $\mathcal{U}^{m\times k}$ and satisfies* (66) *if and only if it admits a representation*

$$\Psi=\Theta_L\widehat{H} \tag{67}$$

where Θ_L is given by (43) *and $\widehat{H}\in\mathcal{U}^{(m+n_L)\times k}$.*

Proof. Let $\widehat{H}$ be in $\mathcal{U}^{(m+n_L)\times k}$ and let Ψ be of the form (67). Since Θ_L is block upper triangular, $\Psi\in\mathcal{U}^{m\times k}$. Furthermore, in view of (43) and (41),

$$\begin{aligned}
&\Theta_L^*B_+\left(I-Z_\zeta A_\zeta\right)^{-1}\\
&=\left(\begin{bmatrix}0\\ I_k\end{bmatrix}+\begin{bmatrix}Z_\zeta^*\mathbb{P}_L^{\frac12}\left(I-Z_\zeta A_\zeta\right)\\ -B_+\end{bmatrix}\mathbb{P}_L^{-1}\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*\right)B_+\left(I-Z_\zeta A_\zeta\right)^{-1}\\
&=\begin{bmatrix}Z_\zeta^*\mathbb{P}_L^{\frac12}\\ 0\end{bmatrix}+\begin{bmatrix}Z_\zeta^*\mathbb{P}_L^{\frac12}\left(I-Z_\zeta A_\zeta\right)\\ -B_+\end{bmatrix}\mathbb{P}_L^{-1}\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}A_\zeta^*Z_\zeta^*\mathbb{P}_L.
\end{aligned}$$

Therefore, the matrix $\Theta_L^*B_+\left(I-Z_\zeta A_\zeta\right)^{-1}$ is strictly block lower triangular and thus,

$$p_0\left\{\Psi^*B_+\left(I-Z_\zeta A_\zeta\right)^{-1}\right\}=p_0\left\{\widehat{H}^*\Theta_L^*B_+\left(I-Z_\zeta A_\zeta\right)^{-1}\right\}=0$$

for every block upper triangular $\widehat{H}\in\mathcal{U}^{(m+n_L)\times k}$.

Conversely, let Ψ belongs to $\mathcal{U}^{m\times k}$ and satisfy (66). By Corollary 2.5,

$$p\left\{\Psi^*B_+\left(I-Z_\zeta A_\zeta\right)^{-1}\right\}=0$$

and thus, the operator $\Psi^*B_+\left(I-Z_\zeta A_\zeta\right)^{-1}$ is strictly block lower triangular. Therefore

$$\Psi^*B_+\left(I-Z_\zeta A_\zeta\right)^{-1}\left(I-Z_\zeta Z_\zeta^*\right)=0$$

and now it follows from (35) that $\Psi^*\mathbf{Q}_L=0$. In view of (45),

$$\Psi^*\Theta_L\Theta_L^*=\Psi^*(I-\mathbf{Q}_L)=\Psi^*.$$

Taking adjoints in the last equality we conclude that Ψ admits a representation (67) with $\widehat{H}:=\Theta_L^*\Psi$, which belongs to $\mathcal{U}^{(p+n_L)\times k}$, by (43). $\square$

Using (65) and Lemma 5.2 we obtain the following result.

THEOREM 5.3. *All $H\in\mathcal{U}^{m\times k}$ which satisfy* (21) *are parameterized by the formula*

$$H=H_L+\Theta_L\widehat{H} \tag{68}$$

where H_L and Θ_L are given by (62) *and* (43) *respectively and $\widehat{H}$ is a parameter from $\mathcal{U}^{(m+n_L)\times k}$.*

Now we describe the set of all solutions of the problem $\mathbf{IP}_L(\Upsilon_L)$.

LEMMA 5.4. *The representation* (68) *is orthogonal: for every $\widehat{H}\in\mathcal{U}^{(m+n_L)\times k}$,*

$$[\Theta_L\widehat{H},\ H_L]=0 \quad \text{and} \quad \langle\Theta_L\widehat{H},\ H_L\rangle\ =0. \tag{69}$$

Proof. It follows from (36) that $\left(A_\zeta^*Z_\zeta^*\mathbb{P}_L^{\frac{1}{2}}Z_\zeta,\ B_+^*\right)\mathbf{T}_L=0$. Taking advantage of the last equality together with (41), (49) and (62) we get

$$\begin{aligned}
H_L^*\Theta_L = {} & B_-\mathbb{P}_L^{-1}\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*\left(B_+\left(I-Z_\zeta A_\zeta\right)^{-1}\mathbb{P}_L^{-\frac{1}{2}}Z_\zeta,\ I_m\right)\mathbf{T}_L\\
= {} & B_-\mathbb{P}_L^{-1}\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}\left(A_\zeta^*Z_\zeta^*\mathbb{P}_L^{\frac{1}{2}}Z_\zeta,\ B_+^*\right)\mathbf{T}_L\\
& +B_-\left(I-Z_\zeta A_\zeta\right)^{-1}\mathbb{P}_L^{-\frac{1}{2}}Z_\zeta\left(I_{n_L},\ 0\right)\mathbf{T}_L\\
= {} & B_-\left(I-Z_\zeta A_\zeta\right)^{-1}\mathbb{P}_L^{-\frac{1}{2}}Z_\zeta\left(I_{n_L},\ 0\right)\mathbf{T}_L
\end{aligned}$$

which shows (since the projection $\mathbf{T}_L$ is block diagonal) that $H_L^*\Theta_L$ is strictly block upper triangular. Therefore, for every block upper triangular $\widehat{H}$, the matrix $H_L^*\Theta_L\widehat{H}$ is also strictly block upper triangular and hence,

$$[\Theta_L\widehat{H},\ H_L]=p_0\ (H_L^*\Theta_L\widehat{H})=0.$$

In particular,

$$\langle\Theta_L\widehat{H},\ H_L\rangle=\mathrm{Tr}\ (H_L^*\Theta_L\widehat{H})=0,$$

which ends the proof. $\square$

THEOREM 5.5. *All solutions H of the problem $\mathbf{IP}_L(\Upsilon_L)$ are parameterized by the formula* (68) *the parameter $\widehat{H}$ varies in $\mathcal{U}^{(m+n_L)\times k}$ and is subject to*

$$\left[\mathbf{T}_L\widehat{H},\ \mathbf{T}_L\widehat{H}\right]\le\Upsilon_L-B_-\mathbb{P}_L^{-1}B_-^*. \tag{70}$$

Proof. In view of (44),

$$\left[\Theta_L\widehat{H},\ \Theta_L\widehat{H}\right]=p_0\ \left(\widehat{H}^*\Theta_L^*\Theta_L\widehat{H}\right)=p_0\ \left(\widehat{H}^*\mathbf{T}_L\widehat{H}\right)=\left[\mathbf{T}_L\widehat{H},\ \mathbf{T}_L\widehat{H}\right]$$

which together with (63), (68) and (69) leads to

$$\begin{aligned}
[H,\ H]=\left[H_L+\Theta_L\widehat{H},\ H_L+\Theta_L\widehat{H}\right] = {} & [H_L,\ H_L]+\left[\Theta_L\widehat{H},\ \Theta_L\widehat{H}\right]\\
= {} & B_-\mathbb{P}_L^{-1}B_-^*+\left[\mathbf{T}_L\widehat{H},\ \mathbf{T}_L\widehat{H}\right]
\end{aligned}$$

and thus, the matrix H of the form (68) satisfies (30) if and only if the corresponding parameter $\widehat{H}$ satisfies (70). ◻

6. Solutions of the two–sided problem. Using the results from the two previous sections we now describe the set of all solutions of problems **IP** and $\mathbf{IP}_\gamma$. By Theorem 4.3 all operators $H \in \mathcal{U}^{m\times k}$ satisfying the condition (20) are given by the formula (59). It is possible to restrict the set of parameters $\widehat{H}$ in (59) in such way that the operator H of the form (59) would satisfy also (21) and (22). We begin with the following auxiliary result; for the proof see [2, Lemma 6.1].

LEMMA 6.1. *For every choice of matrices $\Phi \in \mathcal{U}^{n_L\times k}$ and $\Psi \in \mathcal{U}^{n_L\times m}$, it holds that*

$$P_0\left\{\left(I - A_\zeta^* Z_\zeta^*\right)^{-1}\Phi\Psi\right\} = P_0\left\{\left(I - A_\zeta^* Z_\zeta^*\right)^{-1} L\Psi\right\}$$

where $L \in \mathcal{D}^{n_L\times k}$ is defined by $L = P_0\left\{\left(I - A_\zeta^* Z_\zeta^*\right)^{-1}\Phi\right\}$.

LEMMA 6.2. *A matrix H of the form (59) belongs to $\mathcal{U}^{m\times k}$ and satisfies conditions (21), (22) if and only if the corresponding parameter $\widehat{H}$ belongs to $\mathcal{U}^{m\times(k+n_R)}$ and satisfies the condition*

$$p_0\left\{\mathbf{T}_R \widehat{H}^* B_+ \left(I - Z_\zeta A_\zeta\right)^{-1}\right\} = \widehat{B}_- \tag{71}$$

where $\widehat{B}_- \in \mathcal{D}^{n_R\times n_L}$ is defined by

$$\widehat{B}_- = \mathbf{T}_R \begin{bmatrix} Z_\pi \mathbb{P}_R^{-\frac{1}{2}} Z_\pi^* \Gamma \\ B_- \end{bmatrix} \tag{72}$$

Proof. Let H be of the form (59). Multiplying (59) by Θ_R^* on the right and using (44) we get

$$\widehat{H}\mathbf{T}_R = \left(H - H_R\right)\Theta_R^*.$$

Making use of (48), we rewrite the last equality as

$$\begin{aligned}\widehat{H}\mathbf{T}_R = {} & H\left(C_-\left(I - Z_\pi^* A_\pi\right)^{-1}\mathbb{P}_R^{-\frac{1}{2}} Z_\pi^*,\ I_k\right)\mathbf{T}_R \\ & - \left(C_+\left(I - Z_\pi^* A_\pi\right)^{-1}\mathbb{P}_R^{-\frac{1}{2}} Z_\pi^*,\ 0\right)\mathbf{T}_R.\end{aligned} \tag{73}$$

Since $\mathbf{T}_R$ is block diagonal, the second term on the right hand side in (73) is strictly block lower triangular and therefore,

$$P_0\left\{\left(I - A_\zeta^* Z_\zeta^*\right)^{-1} B_+^*\left(C_+\left(I - Z_\pi^* A_\pi\right)^{-1}\mathbb{P}_R^{-\frac{1}{2}} Z_\pi^*,\ 0\right)\mathbf{T}_R\right\} = 0.$$

Then it follows from (73) that

$$\begin{aligned}& P_0\left(I - A_\zeta^* Z_\zeta^*\right)^{-1} B_+^*\widehat{H}\mathbf{T}_R = \\ & = P_0\left(I - A_\zeta^* Z_\zeta^*\right)^{-1} B_+^* H\left(C_-\left(I - Z_\pi^* A_\pi\right)^{-1}\mathbb{P}_R^{-\frac{1}{2}} Z_\pi^*,\ I_k\right)\mathbf{T}_R.\end{aligned} \tag{74}$$

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Suppose that H satisfies (22) and the second equality from (26) (which is equivalent to (21) by Remark 2.6). Then, as $\mathbb{P}_R^{-\frac{1}{2}}Z_\pi^* = Z_\pi^* Z_\pi \mathbb{P}_R^{-\frac{1}{2}} Z_\pi^*$, it follows from (74) that

$$\begin{aligned}
&P_0\left\{\left(I - A_\zeta^* Z_\zeta^*\right)^{-1} B_+^* \widehat{H}\mathbf{T}_R\right\}\\
&= \left[P_0\left\{\left(I - A_\zeta^* Z_\zeta^*\right)^{-1} B_+^* H\left(C_-\left(I - Z_\pi^* A_\pi\right)^{-1} Z_\pi^* Z_\pi \mathbb{P}_R^{-\frac{1}{2}} Z_\pi^*,\ I_k\right)\right\}\right]\mathbf{T}_R\\
&= \left(\Gamma Z_\pi \mathbb{P}_R^{-\frac{1}{2}} Z_\pi^*,\ B_-^*\right)\mathbf{T}_R.
\end{aligned}$$

Taking adjoints in the last equality, using Remark 2.6 and (72), we get (71):

$$p_0\left\{\mathbf{T}_R \widehat{H}^* B_+\left(I - Z_\zeta A_\zeta\right)^{-1}\right\} = \mathbf{T}_R\begin{bmatrix} Z_\pi \mathbb{P}_R^{-\frac{1}{2}} Z_\pi^* \Gamma^* \\ B_- \end{bmatrix} = \widehat{B}_-.$$

Conversely, let $\widehat{H}$ satisfy (71) and let H be of the form (59). Then

$$P_0\left\{\left(I - A_\zeta^* Z_\zeta^*\right)^{-1} B_+^* H\right\} = P_0\left\{\left(I - A_\zeta^* Z_\zeta^*\right)^{-1} B_+^*\left(H_R + \widehat{H}\Theta_R\right)\right\}. \tag{75}$$

Applying Lemma 6.1 to $\Phi = B_+^*\widehat{H}$ and $\Psi = \Theta_R$ and taking into account (71) we obtain

$$P_0\left\{\left(I - A_\zeta^* Z_\zeta^*\right)^{-1} B_+^* \widehat{H}\Theta_R\right\} = P_0\left\{\left(I - A_\zeta^* Z_\zeta^*\right)^{-1} \widehat{B}_-^* \Theta_R\right\}. \tag{76}$$

It follows from (42), (48) and (72), that

$$\begin{aligned}
\widehat{B}_-^*\Theta_R &= \left(\Gamma Z_\pi \mathbb{P}_R^{-\frac{1}{2}} Z_\pi^*,\ B_-^*\right)\mathbf{T}_R\begin{bmatrix} Z_\pi \mathbb{P}_R^{-\frac{1}{2}}\left(I - A_\pi^* Z_\pi\right)^{-1} C_-^* \\ I_k \end{bmatrix}\\
&= \left(\Gamma Z_\pi \mathbb{P}_R^{-\frac{1}{2}} Z_\pi^*,\ B_-^*\right)\begin{bmatrix} Z_\pi \mathbb{P}_R^{-\frac{1}{2}}\left(I - A_\pi^* Z_\pi\right)^{-1} C_-^* \\ I_k \end{bmatrix}\\
&\quad - \left(\Gamma A_\pi + B_-^* C_-\right)\mathbb{P}_R^{-1}\left(A_\pi^* Z_\pi \mathbb{P}_R^{\frac{1}{2}} Z_\pi^*,\ C_-^*\right)\begin{bmatrix} Z_\pi \mathbb{P}_R^{-\frac{1}{2}}\left(I - A_\pi^* Z_\pi\right)^{-1} C_-^* \\ I_k \end{bmatrix}\\
&= B_-^* + \left\{\Gamma\left(Z_\pi - A_\pi\right) - B_-^* C_-\right\}\mathbb{P}_R^{-1}\left(I - A_\pi^* Z_\pi\right)^{-1} C_-^*. && (77)
\end{aligned}$$

Furthermore, in view of (50),

$$B_+^* H_R = B_+^* C_+ \mathbb{P}_R^{-1}\left(I - A_\pi^* Z_\pi\right)^{-1} C_-^*$$

which being added to (77) leads, on account of (19), to

$$\begin{aligned}
&B_+^* H_R + B_-^*\Theta_R =\\
&\quad = B_-^* + \left\{\Gamma\left(Z_\pi - A_\pi\right) - B_-^* C_- + B_+^* C_+\right\}\mathbb{P}_R^{-1}\left(I - A_\pi^* Z_\pi\right)^{-1} C_-^*\\
&\quad = B_-^* + \left\{\Gamma Z_\pi - A_\zeta^* Z_\zeta^* \Gamma Z_\pi\right\}\mathbb{P}_R^{-1}\left(I - A_\pi^* Z_\pi\right)^{-1} C_-^*\\
&\quad = B_-^* + \left(I - A_\zeta^* Z_\zeta^*\right)\Gamma Z_\pi \mathbb{P}_R^{-1}\left(I - A_\pi^* Z_\pi\right)^{-1} C_-^*. && (78)
\end{aligned}$$

Substituting (76) into (75) and using (78) we obtain

$$\begin{aligned} P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*H\right\} &= P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}\left(B_+^*H_R+B_-^*\Theta_R\right)\right\} \\ &= P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_-^*+P_0\Gamma Z_\pi \mathbb{P}_R^{-1}\left(I-A_\pi^*Z_\pi\right)^{-1}C_-^*\right\}. \end{aligned}$$

Since the matrix $\Gamma Z_\pi \mathbb{P}_R^{-1}\left(I-A_\pi^*Z_\pi\right)^{-1}C_-^*$ is strictly block upper triangular, the second term on the right hand side in the last equality equals to zero and thus,

$$P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*H\right\} = P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*\right\} = B_-^*$$

which is equivalent to (21), by Remark 2.6.

The verification of (22) is done in much the same way: in view of (59),

$$\begin{aligned} &P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*HC_-\left(I-Z_\pi^*A_\pi\right)^{-1}Z_\pi^*\right\} \\ &= P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*\left(H_R+\widehat{H}\Theta_R\right)C_-\left(I-Z_\pi^*A_\pi\right)^{-1}Z_\pi^*\right\}. \end{aligned} \tag{79}$$

Applying Lemma 6.1 to $\Phi = B_+^*\widehat{H}$ and $\Psi = \Theta_R C_-\left(I-Z_\pi^*A_\pi\right)^{-1}Z_\pi^*$ (which is strictly block lower triangular) and taking into account (71) we obtain

$$\begin{aligned} &P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*\widehat{H}\Theta_R C_-\left(I-Z_\pi^*A_\pi\right)^{-1}Z_\pi^*\right\} \\ &= P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}\widehat{B}_-^*\Theta_R C_-\left(I-Z_\pi^*A_\pi\right)^{-1}Z_\pi^*\right\} \end{aligned}$$

which being substituted to (79), leads to

$$\begin{aligned} &P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*HC_-\left(I-Z_\pi^*A_\pi\right)^{-1}Z_\pi^*\right\} \\ &= P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}\left(B_+^*H_R+\widehat{B}_-^*\Theta_R\right)C_-\left(I-Z_\pi^*A_\pi\right)^{-1}Z_\pi^*\right\}. \end{aligned}$$

Upon substituting (78) into this equality and using (37) we get

$$\begin{aligned} &P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*HC_-\left(I-Z_\pi^*A_\pi\right)^{-1}Z_\pi^*\right\} \\ &= P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_-^*C_-\left(I-Z_\pi^*A_\pi\right)^{-1}Z_\pi^*\right\} \\ &\quad +P_0\left\{\Gamma Z_\pi \mathbb{P}_R^{-1}\left(I-A_\pi^*Z_\pi\right)^{-1}C_-^*C_-\left(I-Z_\pi^*A_\pi\right)^{-1}Z_\pi^*\right\} \\ &= P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_-^*C_-\left(I-Z_\pi^*A_\pi\right)^{-1}Z_\pi^*\right\} \\ &\quad +P_0\left\{\Gamma Z_\pi\left(I-Z_\pi^*A_\pi\right)^{-1}Z_\pi^*\right\}+P_0\left\{\Gamma Z_\pi \mathbb{P}_R^{-1}\left(I-A_\pi^*Z_\pi\right)^{-1}A_\pi^*Z_\pi \mathbb{P}_R\right\}. \end{aligned}$$

Since the matrices

$$\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_-^*C_-\left(I-Z_\pi^*A_\pi\right)^{-1}Z_\pi^* \quad \text{and} \quad \Gamma Z_\pi \mathbb{P}_R^{-1}\left(I-A_\pi^*Z_\pi\right)^{-1}A_\pi^*Z_\pi \mathbb{P}_R$$

are respectively, strictly block lower and block upper triangular, the first and the third terms on the right hand side in the last equality are equal to zero and by (18),

$$P_0\left\{\left(I-A_\zeta^*Z_\zeta^*\right)^{-1}B_+^*HC_-\left(I-Z_\pi^*A_\pi\right)^{-1}Z_\pi^*\right\} = P_0\left\{\Gamma Z_\pi\left(I-A_\pi Z_\pi^*\right)^{-1}Z_\pi^*\right\}$$
$$= \Gamma Z_\pi Z_\pi^* = \Gamma.$$

Therefore H satisfies the condition (22) which ends the proof of the lemma. □

According to Theorem 4.3, all operators $\widehat{H}\in\mathcal{U}^{(n_R+k)\times m}$ which satisfy (71) are of the form

$$\widehat{H} = \widehat{H_L} + \Theta_L h \tag{80}$$

where Θ_L is defined by (43),

$$\widehat{H_L} = B_+\left(I-Z_\zeta A_\zeta\right)^{-1}\mathbb{P}_L^{-1}\widehat{B}_-^* \tag{81}$$

and h is an arbitrary operator from $\mathcal{U}^{(k+n_R)\times(m+n_L)}$. Since $\mathbf{T}_R$ is the projection, it follows from (72) that

$$\widehat{H_L}\mathbf{T}_R = \widehat{H_L}. \tag{82}$$

Substituting (80) into (59) we come to the following theorem.

THEOREM 6.3. *All solutions H of the problem* **IP** *are parameterized by the formula*

$$H = H_R + \widehat{H_L}\Theta_R + \Theta_L h\Theta_R \tag{83}$$

where H_R, $\widehat{H_L}$, Θ_L and Θ_R are block upper triangular matrices defined by (50), (81), (43) *and* (42) *respectively and h is a free parameter from $\mathcal{U}^{(k+n_R)\times(m+n_L)}$.*

Note that equation (83) is in fact (31) with

$$H_{\min} = H_R + \widehat{H_L}\Theta_R. \tag{84}$$

According to Lemma 5.1 and in view of (82),

$$\langle\widehat{H}_L\mathbf{T}_R,\ \widehat{H}_L\mathbf{T}_R\rangle = \langle\widehat{H}_L,\ \widehat{H}_L\rangle = \operatorname{Tr}\widehat{B}_-\mathbb{P}_L^{-1}\widehat{B}_-^*$$

while Lemmas 4.4 and 5.4 ensure that the representation (83) is orthogonal with respect to the inner product (5). Due to (44) and (45),

$$\langle H,\ H\rangle = \langle H_R,\ H_R\rangle + \langle\widehat{H}_L\mathbf{T}_R,\ \widehat{H}_L\mathbf{T}_R\rangle + \langle\mathbf{T}_L h\mathbf{T}_R,\ \mathbf{T}_L h\mathbf{T}_R\rangle$$
$$= \operatorname{Tr} C_+\mathbb{P}_R^{-1}C_+^* + \operatorname{Tr}\widehat{B}_-\mathbb{P}_L^{-1}\widehat{B}_-^* + \langle\mathbf{T}_L h\mathbf{T}_R,\ \mathbf{T}_L h\mathbf{T}_R\rangle$$

This equality leads to the description of all the solutions to the problem $\mathbf{IP}_\gamma$ (i.e., under the additional norm constraint (28).

THEOREM 6.4. *(i) The problem $\mathbf{IP}_\gamma$ is solvable if and only if*

$$\operatorname{Tr} C_+\mathbb{P}_R^{-1}C_+^* + \operatorname{Tr}\widehat{B}_-\mathbb{P}_L^{-1}\widehat{B}_-^* \le \gamma.$$

(ii) All solutions of the problem $\mathbf{IP}_\gamma$ *are parameterized by the formula* (83) *when the parameter* $h \in \mathcal{U}^{(k+n_R)\times(m+n_L)}$ *is such that*

$$\langle \mathbf{T}_L h \mathbf{T}_R,\ \mathbf{T}_L h \mathbf{T}_R\rangle \ \le \gamma - \mathrm{Tr}\ C_+ \mathbb{P}_R^{-1} C_+^* - \mathrm{Tr}\ \widehat{B}_- \mathbb{P}_L^{-1} \widehat{B}_-^* = \gamma - \|H_{\min}\|^2_{\mathbf{HS}}.$$

Theorem 3.1 is clearly a consequence of these two last theorems.

7. The minimal norm solution. It was shown in the previous section that the minimal norm solution $H_{\min}$ of the problem **IP** is given by the formula (84) and contains as an additive term the minimal norm solution H_R of the right–sided problem (20). Of course, $\|H_{\min}\|_{\mathbf{HS}}$ is not less than $\|H_R\|_{\mathbf{HS}}$ and their difference is determined by the supplementary left–sided problem (71). From symmetry arguments the presence of the minimal norm solution H_L of the left–sided problem (21) as a term in the additive representation of $H_{\min}$ should be expected.

LEMMA 7.1. *The matrix* $H_{\min}$ *defined by* (84) *can be represented as*

$$H_{\min} = H_L + \Theta_L \widehat{H}_R \tag{85}$$

where H_L *and* Θ_L *are given by* (62) *and* (43) *respectively,*

$$\widehat{H}_R = \widehat{C}_+ \mathbb{P}_R^{-1} \left(I - A_\pi^* Z_\pi\right)^{-1} C_-^* \tag{86}$$

and where

$$\widehat{C}_+ = \mathbf{T}_L \begin{bmatrix} Z_\zeta^* \mathbb{P}_L^{-\frac{1}{2}} \Gamma Z_\pi \\ C_+ \end{bmatrix}. \tag{87}$$

Proof. First we note that the second equality in (87) follows immediately from (36). As a consequence of (87) and (49) we get

$$\begin{aligned}\Theta_L \widehat{C}_+ &= \left(B_+ \left(I - Z_\zeta A_\zeta\right)^{-1} \mathbb{P}_L^{-\frac{1}{2}} Z_\zeta,\ I_m\right) \mathbf{T}_L \begin{bmatrix} Z_\zeta^* \mathbb{P}_L^{-\frac{1}{2}} \Gamma Z_\pi \\ C_+ \end{bmatrix} \\ &= C_+ + B_+ \left(I - Z_\zeta A_\zeta\right)^{-1} \mathbb{P}_L^{-1} \left\{\left(I - A_\zeta^* Z_\zeta^*\right) \Gamma Z_\pi - B_+^* C_+\right\}.\end{aligned}$$

Multiplying this equality by $\mathbb{P}_R^{-1} \left(I - A_\pi^* Z_\pi\right)^{-1} C_-^*$ on the right and taking into account (50), (86), we obtain

$$\begin{aligned}\Theta_L \widehat{H}_R = H_R + \quad & B_+ \left(I - Z_\zeta A_\zeta\right)^{-1} \mathbb{P}_L^{-1} \left\{\left(I - A_\zeta^* Z_\zeta^*\right) \Gamma Z_\pi - B_+^* C_+\right\} \\ & \times \mathbb{P}_R^{-1} \left(I - A_\pi^* Z_\pi\right)^{-1} C_-^*.\end{aligned} \tag{88}$$

Furthermore, multiplying the equality (77) by $B_+ \left(I - Z_\zeta A_\zeta\right)^{-1} \mathbb{P}_L^{-1}$ from the left and taking into account (62), (81), we get

$$\begin{aligned}\widehat{H}_L \Theta_R = H_L + \quad & B_+ \left(I - Z_\zeta A_\zeta\right)^{-1} \mathbb{P}_L^{-1} \left\{\Gamma \left(Z_\pi - A_\pi\right) - B_-^* C_-\right\} \\ & \times Z_\pi^* \mathbb{P}_R^{-1} Z_\pi \left(I - A_\pi^* Z_\pi\right)^{-1} C_-^*.\end{aligned} \tag{89}$$

Subtracting this equality from (88) and using (19), we get

$$\Theta_L \widehat{H}_R - \widehat{H}_L \Theta_R = H_R - H_L$$

which means that (84) and (85) define the same operator $H_{\min}$. □

It follows from (84) and (89) that $H_{\min}$ can be represented as

$$H_{\min}(z) = H_L + H_C + H_R \tag{90}$$

where H_L, H_R are given by (62), (50) respectively and

$$H_C = B_+ (I - Z_\zeta A_\zeta)^{-1} \mathbb{P}_L^{-1} \left\{ \Gamma (Z_\pi - A_\pi) - B_-^* C_- \right\} \mathbb{P}_R^{-1} (I - A_\pi^* Z_\pi)^{-1} C_-^*.$$

The representation (90) is not orthogonal; nevertheless it turns out that

$$H_L \perp (H_C + H_R) \qquad \text{and} \qquad (H_L + H_C) \perp H_R.$$

Indeed, in view of (88) and (89),

$$H_C + H_R = \widehat{H}_L \Theta_R, \qquad H_L + H_C = \Theta_L \widehat{H}_R$$

and the claimed orthogonalities hold by Lemmas 4.4 and 5.4.

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