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THE EXPLICIT REPRESENTATIONS OF THE DRAZIN INVERSES OF A CLASS OF BLOCK MATRICES*

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Abstract. Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a 2×2 block matrix, where $ABC = 0$ and either $DC = 0$ or $BD = 0$. This paper gives the explicit representations of M^D in terms of $A, B, C, D, A^D, (BC)^D$ and D^D.

Key words. Block matrix, Drazin inverse, Index.

AMS subject classifications. 15A09, 65F20.

1. Introduction.

Let A be a complex square matrix. The Drazin inverse (see [1]) of A is the matrix A^D satisfying

$$A^{l+1}A^D = A^l, \quad A^D A A^D = A^D, \quad AA^D = A^D A \quad \text{for all integers } l \geq k, \tag{1.1}$$

where $k = Ind(A)$ is the index of A, the smallest nonnegative integer such that $rank(A^k) = rank(A^{k+1})$. Note that the definition of the Drazin inverse is equivalent to the existence a nonnegative integer l' such that

$$A^{l'+1}A^D = A^{l'}, \quad A^D A A^D = A^D, \quad AA^D = A^D A.$$

Applications of the Drazin inverse to singular differential equations and singular difference equations, to Markov chains and iterative methods, to structured matrices, and to perturbation bounds for the relative eigenvalue problem can be found in [1, 2, 3, 4, 5, 6, 7].

In 1977, Meyer (see [8]) gave the formula of the Drazin inverse of complex block matrix $\left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$, with A and C being square. Then in 1979, Campbell and Meyer proposed an open problem (see [1]) to find an explicit representation of the Drazin inverse of block matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, with A and D being square, in terms of A, B, C and

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D. Because of the difficulty of this problem, there are some results under some special conditions (see [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]).

It is known that A^D is existent and unique (see [1, 2]). Throughout this paper, $A^\pi = I - AA^D$, $s(i)$ and $i\%2$ denote the projection on the kernel of A^k along the range of A^k, the integer part of $i/2$ and the remainder of i divided by 2, respectively, where I is the identity matrix, and we assume that $\sum_{i=j}^{k} = 0$ if $k < j$ and $A^0 = I$ for any square matrix A. Here we state a result given in [10, Theorem 2.1] which will be used to prove our main results:

Let A and B be complex square matrices of the same size, $Ind(A) = i_A$, $Ind(B) = i_B$. If $AB = 0$, then

$$(A+B)^D = \sum_{i=0}^{k} B^\pi B^i \left(A^D\right)^{i+1} + \sum_{i=0}^{k} \left(B^D\right)^{i+1} A^i A^\pi, \tag{1.2}$$

where $max\{Ind(A), Ind(B)\} \leq k \leq Ind(A) + Ind(B)$. By the definition of the Drazin inverse, we have $A^\pi A^j = A^j A^\pi = 0$, if $j \geq Ind(A)$, so (1.2) is equivalent to [11, Corollary2.12]:

$$(A+B)^D = \sum_{i=0}^{i_B-1} B^\pi B^i \left(A^D\right)^{i+1} + \sum_{i=0}^{i_A-1} \left(B^D\right)^{i+1} A^i A^\pi. \tag{1.3}$$

Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a 2×2 block matrix, where $ABC = 0$ and either $DC = 0$ or $BD = 0$. In this paper, we mainly give the explicit representations of M^D in terms of $A, B, C, D, A^D, (BC)^D$ and D^D. The results about the representation of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)^D$ under special conditions below are all corollaries of the main results of this paper: in [8, Theorem3.2], $C = 0$; in [13, Theorem5.3], $BC = BD = DC = 0$.

2. Main results.

THEOREM 2.1. *Let* $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ *be a complex block matrix, with* A *and* 0 *being square,* $Ind(A) = i_A$, $Ind(BC) = i_{BC}$. *If* $ABC = 0$, *then*

$$\left(M^D\right)^l = \begin{pmatrix} U(l) & V(l)B \\ CU(l+1) & CV(l+1)B \end{pmatrix}, \tag{2.1}$$

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for all integers $l \geq 1$, where

$$\begin{aligned} U(l) =& X(A^D)^{l-1} - \sum_{k=1}^{s(l-1)} \left((BC)^D\right)^k (A^D)^{l-2k} \\ &+ \left((BC)^D\right)^{s(l-1)} YAA^{l\%2} + \left((BC)^D\right)^{s(l+1)} A^{l\%2}A^{\pi}, \\ V(l) =& X(A^D)^{l} - \sum_{k=1}^{s(l-1)} \left((BC)^D\right)^k (A^D)^{l+1-2k} \\ &+ \left((BC)^D\right)^{s(l-1)} YA^{l\%2} + (-1)^{l+1} \left((BC)^D\right)^{s(l+1)} (A^{\pi})^{l\%2}(A^D)^{(l+1)\%2}, \end{aligned}$$

$$\begin{aligned} X &= \sum_{i=0}^{i_{BC}-1} (BC)^{\pi}(BC)^i(A^D)^{2i+1}, \\ Y &= \sum_{i=1}^{s(i_A)} ((BC)^D)^{i+1}A^{\pi}A^{2i-1}. \end{aligned}$$

Proof. We prove $G = \begin{pmatrix} U(1) & V(1)B \\ CU(2) & CV(2)B \end{pmatrix} = M^D$ by the definition of the Drazin inverse.

First, we prove $MG = GM$ holds.

Let

$$MG = \begin{pmatrix} AU(1) + BCU(2) & AV(1)B + BCV(2)B \\ CU(1) & CV(1)B \end{pmatrix}$$

and

$$GM = \begin{pmatrix} U(1)A + V(1)BC & U(1)B \\ CU(2)A + CV(2)BC & CU(2)B \end{pmatrix}.$$

From $ABC = 0$, we have $A^DBC = (A^D)^2ABC = 0$, similarly we have $A(BC)^D = 0$ and $A^D(BC)^D = 0$. Further we have $AX = AA^D$, $AY = 0$, $XAA^D = X$, $YBC = 0$, $AU(1) = A(X + YA^2 + (BC)^DAA^{\pi}) = AA^D$, $V(1)BC = (XA^D + YA + (BC)^DA^{\pi})BC = BC(BC)^D$, $AV(1) = A(XA^D + YA + (BC)^DA^{\pi}) = A^D$ and $V(2)BC = (X(A^D)^2 + Y - (BC)^DA^D)BC = 0$. Using these equalities, we get

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$$\begin{aligned}
AU(1)+BCU(2) =& AA^D+BCXA^D+BCYA+BC(BC)^DA^\pi\\
=& \sum_{i=1}^{i_{BC}-1}(BC)^\pi(BC)^i(A^D)^{2i}+BCYA+AA^D+BC(BC)^DA^\pi\\
=& XA+BCYA+BC(BC)^D,\\
U(1)A+V(1)BC =& XA+YA^3+(BC)^DA^2A^\pi+BC(BC)^D\\
=& XA+\sum_{i=1}^{s(i_A)}((BC)^D)^iA^\pi A^{2i}+BC(BC)^D\\
=& XA+BCYA+BC(BC)^D.
\end{aligned}$$

$$\begin{aligned}
AV(1)B+BCV(2)B =& A^DB+BCX(A^D)^2B+BCYB-BC(BC)^DA^DB\\
=& (BC)^\pi A^DB+\sum_{i=1}^{i_{BC}-1}(BC)^\pi(BC)^i(A^D)^{2i+1}B+BCYB\\
=& \sum_{i=0}^{i_{BC}-1}(BC)^\pi(BC)^i(A^D)^{2i+1}B+BCYB\\
=& XB+BCYB,\\
U(1)B =& XB+YA^2B+(BC)^DAA^\pi B\\
=& XB+\sum_{i=2}^{s(i_A)}((BC)^D)^iA^\pi A^{2i-1}B+(BC)^DAA^\pi B\\
=& XB+\sum_{i=1}^{s(i_A)}((BC)^D)^iA^\pi A^{2i-1}B\\
=& XB+BCYB.
\end{aligned}$$

From $XAA^D=X$, $V(2)BC=0$ and $V(1)=U(2)$, we get

$$CU(1)=CX+CYA^2+C(BC)^DAA^\pi=CU(2)A+CV(2)BC$$

and

$$CV(1)B=CU(2)B=CXA^DB+CYAB+C(BC)^DA^\pi B.$$

Thus $MG=GM$.

Second, we prove $(GM)G=G$ holds.

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We have got $AU(1) = AA^D$, $XAA^D = X$ and $AV(1) = A^D$ before. It is easy to get $YA^D = 0$, $BC(BC)^DU(1) = BC(BC)^D(X + YA^2 + (BC)^DAA^\pi) = YA^2 + (BC)^DAA^\pi$, $XBC = 0$, $YBC = 0$, $XU(1) = X(X + YA^2 + (BC)^DAA^\pi) = XA^D$, $BC(BC)^DV(1) = YA + (BC)^DA^\pi$, $XV(1) = X(XA^D + YA + (BC)^DA^\pi) = X(A^D)^2$ and $BC(BC)^DV(2) = BC(BC)^D(X(A^D)^2 + Y - (BC)^DA^D) = Y - (BC)^DA^D$. Using these equalities we can get

$$\begin{aligned}
(GM)G =& \begin{pmatrix} XA + BCYA + BC(BC)^D & XB + BCYB \\ CX + CYA^2 + C(BC)^DAA^\pi & CXA^DB + CYAB + C(BC)^DA^\pi B \end{pmatrix} \\
& \begin{pmatrix} U(1) & V(1)B \\ CU(2) & CV(2)B \end{pmatrix} \\
=& \begin{pmatrix} X + YA^2 + (BC)^DAA^\pi & \left(XA^D + BC(BC)^DV(1)\right)B \\ C\left(XA^D + BC(BC)^DU(2)\right) & C\left(X(A^D)^2 + BC(BC)^DV(2)\right)B \end{pmatrix} \\
=& \begin{pmatrix} U(1) & V(1)B \\ CU(2) & CV(2)B \end{pmatrix} \\
=& G.
\end{aligned}$$

Last, we prove there exists a nonnegative integers l' such that $M^{l'+1}G = M^{l'}$.

It is easy to get

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^{l'} = \begin{pmatrix} \sum\limits_{i=0}^{s(l')} (BC)^iA^{l'-2i} & \sum\limits_{i=0}^{s(l'-1)} (BC)^iA^{l'-1-2i}B \\ \sum\limits_{i=0}^{s(l'+1)-1} C(BC)^iA^{l'-1-2i} & \sum\limits_{i=0}^{s(l')-1} C(BC)^iA^{l'-2-2i}B \end{pmatrix}, \quad (2.2)$$

for all positive integers l'.

Here we only prove that there exists an l' such that $(M^{l'+1}G)_{11} = (M^{l'})_{11}$, since the proofs of $(M^{l'+1}G)_{12} = (M^{l'})_{12}$, $(M^{l'+1}G)_{21} = (M^{l'})_{21}$ and $(M^{l'+1}G)_{22} = (M^{l'})_{22}$ are similar.

From $AU(1) = AA^D$ and $ABC = 0$, we have

$$\begin{aligned}
(M^{l'+1}G)_{11} =& \sum_{i=0}^{s(l'+1)} (BC)^iA^{l'+1-2i}U(1) + \sum_{i=0}^{s(l')} (BC)^iA^{l'-2i}BCU(2) \\
=& \sum_{i=0}^{s(l'+1)-1} (BC)^iA^{l'+1-2i}A^D + (BC)^{s(l'+1)}A^{l'+1-2s(l'+1)}U(1) \\
&+ \frac{1+(-1)^{l'}}{2}(BC)^{s(l')}A^{l'-2s(l')}BCU(2).
\end{aligned}$$

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If l' is odd, we can prove $(M^{l'+1}G)_{11} = (M^{l'})_{11}$ holds when $l' \geq 2s(i_A)+2i_{BC}+1$; if l' is even, we can prove $(M^{l'+1}G)_{11} = (M^{l'})_{11}$ holds when $l' \geq 2s(i_A) + 2i_{BC}$. So $(M^{l'+1}G)_{11} = (M^{l'})_{11}$ holds for all $l' \geq 2s(i_A) + 2i_{BC}$. Without loss of generality, we prove the case $l' = 2s(i_A) + 2i_{BC} + 1$:

$$
\begin{aligned}
(M^{l'+1}G)_{11} &= \sum_{i=0}^{(l'-1)/2} (BC)^i A^{l'+1-2i} A^D + (BC)^{(l'+1)/2} U(1) \\
&= \sum_{i=0}^{(l'-1)/2} (BC)^i A^{l'+1-2i} A^D + \sum_{i=0}^{i_{BC}-1} (BC)^{\pi} (BC)^{i+(l'+1)/2} (A^D)^{2i+1} \\
&\quad + \sum_{i=1}^{s(i_A)} (BC)^{(l'+1)/2} ((BC)^D)^{i+1} A^{\pi} A^{2i+1} + (BC)^{(l'+1)/2} (BC)^D A A^{\pi} \\
&= \sum_{i=0}^{(l'-1)/2} (BC)^i A^{l'+1-2i} A^D + 0 + \sum_{i=0}^{s(i_A)} (BC)^{(l'-1)/2-i} A^{\pi} A^{2i+1} \\
&= \sum_{i=0}^{(l'-1)/2} (BC)^i A^{l'+1-2i} A^D + \sum_{i=(l'-1)/2-s(i_A)}^{(l'-1)/2} (BC)^i A^{\pi} A^{l'-2i} \\
&= \sum_{i=0}^{(l'-3)/2-s(i_A)} (BC)^i A^{l'+1-2i} A^D \\
&\quad + \sum_{i=(l'-1)/2-s(i_A)}^{(l'-1)/2} (BC)^i \left(A^{\pi} A^{l'-2i} + A^{l'+1-2i} A^D \right) \\
&= \sum_{i=0}^{(l'-3)/2-s(i_A)} (BC)^i A^{l'-2i} + \sum_{i=(l'-1)/2-s(i_A)}^{(l'-1)/2} (BC)^i A^{l'-2i} \\
&= \sum_{i=0}^{(l'-1)/2} (BC)^i A^{l'-2i}.
\end{aligned}
$$

Thus $G = M^D$.

Now we prove (2.1) holds by induction. We have proved $G = M^D$, that is to say, (2.1) is true for $l = 1$. Assume (2.1) is true for $l = j \geq 1$, now we check it for

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$l = j+1$, that is to say, we prove

$$\begin{aligned}
&\begin{pmatrix} U(1) & V(1)B \\ CU(2) & CV(2)B \end{pmatrix}\begin{pmatrix} U(j) & V(j)B \\ CU(j+1) & CV(j+1)B \end{pmatrix} \\
=&\begin{pmatrix} U(1)U(j)+V(1)BCU(j+1) & (U(1)V(j)+V(1)BCV(j+1))\,B \\ C\,(U(2)U(j)+V(2)BCU(j+1)) & C\,(U(2)V(j)+V(2)BCV(j+1))\,B \end{pmatrix} \\
=&\begin{pmatrix} U(j+1) & V(j+1)B \\ CU(j+2) & CV(j+2)B \end{pmatrix}.
\end{aligned}$$

From $X^2 = XA^D$, $X(BC)^D = 0$, $YA^2X = 0$, $AX = AA^D$, $A^\pi A^D = 0$, $(BC)^D X = 0$ and $V(1)BC = BC(BC)^D$, we get

$$U(1)U(j)+V(1)BCU(j+1) = X(A^D)^j + BC(BC)^D U(j+1) = U(j+1),$$

$$U(1)V(j)+V(1)BCV(j+1) = X(A^D)^{j+1} + BC(BC)^D V(j+1) = V(j+1).$$

From $A^D X = (A^D)^2$, $A^D(BC)^D = 0$, $YAX = 0$, $A(BC)^D = 0$, $A^\pi X = X - A^D$, $V(2)BC = 0$, $(BC)^D X = 0$ and $s(j-1)+1 = s(j+1)$, we get

$$\begin{aligned}
&U(2)U(j)+V(2)BCU(j+1) = U(2)U(j) \\
=&X(A^D)^{j+1} - (BC)^D(A^D)^j - \sum_{k=1}^{s(j-1)} ((BC)^D)^{k+1}(A^D)^{j-2k} \\
&+ ((BC)^D)^{s(j-1)+1} YAA^{j\%2} + ((BC)^D)^{s(j+1)+1} A^{j\%2} A^\pi \\
=&X(A^D)^{j+1} - \sum_{k=1}^{s(j+1)} ((BC)^D)^{k}(A^D)^{j+2-2k} \\
&+ ((BC)^D)^{s(j+1)} YAA^{(j+2)\%2} + ((BC)^D)^{s(j+3)} A^{(j+2)\%2} A^\pi \\
=&U(j+2),
\end{aligned}$$

$$\begin{aligned}
&U(2)V(j)+V(2)BCV(j+1) = U(2)V(j) \\
=&X(A^D)^{j+2} - (BC)^D(A^D)^{j+1} - \sum_{k=1}^{s(j-1)} ((BC)^D)^{k+1}(A^D)^{j+1-2k} \\
&+ ((BC)^D)^{s(j-1)+1} YA^{j\%2} + (-1)^{j+1}((BC)^D)^{s(j+1)+1}(A^\pi)^{j\%2}(A^D)^{(j+1)\%2} \\
=&X(A^D)^{j+2} - \sum_{k=1}^{s(j+1)} ((BC)^D)^{k}(A^D)^{j+3-2k} \\
&+ ((BC)^D)^{s(j+1)} YA^{(j+2)\%2} + (-1)^{j+3}((BC)^D)^{s(j+3)}(A^\pi)^{(j+2)\%2}(A^D)^{(j+3)\%2} \\
=&V(j+2).
\end{aligned}$$

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Thus (2.1) is true for $l = j + 1$. □

Now we give our main results.

THEOREM 2.2. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a complex block matrix, with* A *and* D *being square,* $Ind(A) = i_A$, $Ind(BC) = i_{BC}$, $Ind\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = i_{ABC}$ *and* $Ind(0 \oplus D) = i_D$ *(as* 0 *is not absent). If* $ABC = 0$ *and* $DC = 0$, *then*

$$M^D = \begin{pmatrix} U(1) & T_1 \\ CU(2) & T_2 \end{pmatrix}, \tag{2.3}$$

where

$$\begin{aligned} T_1 =& -(A^D + BCV(2))BD^D + \sum_{i=0}^{i_D-1} V(i+1)BD^iD^\pi \\ &+ \sum_{i=1}^{i_{ABC}-1} \left((A^\pi - BCU(2))\,R(i) - \left(A^D + BCV(2)\right)BS(i)\right), \\ T_2 =& (I - CV(1)B)D^D + \sum_{i=0}^{i_D-1} CV(i+2)BD^iD^\pi \\ &+ \sum_{i=1}^{i_{ABC}-1} \left((-CU(1))\,R(i) + (I - CV(1)B)\,S(i)\right), \\ R(i) =& \sum_{k=0}^{s(i-1)} (BC)^k A^{i-1-2k} B (D^D)^{i+1}, \\ S(i) =& \sum_{k=0}^{s(i)-1} C(BC)^k A^{i-2-2k} B (D^D)^{i+1}, \end{aligned}$$

$U(i), V(i), X, Y$ *are all the same as those in Theorem 2.1, and let* $\sum_{i=j}^{k} = 0$ *if* $k < j$.

Proof. Let

$$M = E + F,$$

where

$$E = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, \quad F = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}.$$

Since $DC = 0$, we have $EF = 0$, then we can use (1.3) to get

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$$\begin{aligned} M^D = &\sum_{i=0}^{i_{ABC}-1} \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^{\pi} \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^{i} \begin{pmatrix} 0 & 0 \\ 0 & D^D \end{pmatrix}^{i+1} \\ &+ \sum_{i=0}^{i_D-1} \left(\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^{D} \right)^{i+1} \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}^{i} \begin{pmatrix} I & 0 \\ 0 & D^{\pi} \end{pmatrix}. \end{aligned} \tag{2.4}$$

From $ABC = 0$, recall that $AU(1) = AA^D$ and $AV(1) = A^D$, thus by Theorem 2.1, we have

$$\left(\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^{D} \right)^{i+1} = \begin{pmatrix} U(i+1) & V(i+1)B \\ CU(i+2) & CV(i+2)B \end{pmatrix} \tag{2.5}$$

and

$$\begin{aligned} \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^{\pi} =& I - \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} U(1) & V(1)B \\ CU(2) & CV(2)B \end{pmatrix} \\ =& I - \begin{pmatrix} AA^D + BCU(2) & (A^D + BCV(2))B \\ CU(1) & CV(1)B \end{pmatrix} \\ =& \begin{pmatrix} A^{\pi} - BCU(2) & -(A^D + BCV(2))B \\ -CU(1) & I - CV(1)B \end{pmatrix}. \end{aligned} \tag{2.6}$$

After we substitute (2.2), (2.5) and (2.6) into (2.4), we get the representation of M^D as shown in (2.3). □

THEOREM 2.3. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a complex block matrix, with* A *and* D *being square,* $Ind(A) = i_A$, $Ind(BC) = i_{BC}$, $Ind \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = i_{ABC}$ *and* $Ind(0 \oplus D) = i_D$ *(as* 0 *is not absent). If* $ABC = 0$ *and* $BD = 0$, *then*

$$M^D = \begin{pmatrix} (U(1))^t & \left(\sum_{i=0}^{i_D-1} D^{\pi} D^i CU(i+2) + \sum_{i=1}^{i_{ABC}} R(i) - D^D CU(1) \right)^t \\ (V(1)B)^t & \left(\sum_{i=0}^{i_D-1} D^{\pi} D^i CV(i+2)B + \sum_{i=1}^{i_{ABC}} S(i) + D^D (I - CV(1)B) \right)^t \end{pmatrix}^t, \tag{2.7}$$

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where

$$R(i) = \sum_{k=0}^{s(i+1)-1} (D^D)^{i+1} C(BC)^k A^{i-1-2k} A^\pi - \frac{1+(-1)^{i+1}}{2}(D^D)^{i+1}C(BC)^{s(i+1)}U(2) - \frac{1+(-1)^i}{2}(D^D)^{i+1}C(BC)^{s(i)}U(1),$$

$$S(i) = \sum_{k=0}^{s(i+1)-1} (D^D)^{i+1} C(BC)^k A^{i-1-2k} A^D B - \frac{1+(-1)^{i+1}}{2}(D^D)^{i+1}C(BC)^{s(i+1)}V(2)B - \frac{1+(-1)^i}{2}(D^D)^{i+1}C(BC)^{s(i)}V(1)B,$$

$U(i), V(i), X, Y$ are all the same as those in Theorem 2.1, A^t is the transpose of matrix A, and let $\sum_{i=j}^{k} = 0$ if $k < j$.

Proof. Let

$$M = E + F,$$

where

$$E = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}.$$

Since $BD = 0$, we have $EF = 0$. Then we can also use (1.3) to get the representation of M^D. The procedure of obtaining the representation of M^D is similar to that in Theorem 2.2, so we omit it. □

Corollary 2.4. *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a complex block matrix , with A and D being square, $Ind(A) = i_A$, $Ind(BC) = i_{BC}$, $Ind\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = i_{ABC}$ and $Ind(0 \oplus D) = i_D$ (as 0 is not absent). If $ABC = 0$, $DC = 0$ and $BD = 0$, then*

$$M^D = \begin{pmatrix} U(1) & V(1)B \\ CU(2) & CV(2)B + D^D \end{pmatrix}, \tag{2.8}$$

where $U(i), V(i), X, Y$ are all the same as those in Theorem 2.1, and let $\sum_{i=j}^{k} = 0$ if $k < j$.

Last, we give an example to illustrate Theorem 2.2.

Example 2.5. *Let*

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

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We can partition M as follows:

$$Case1: \quad A=1, B=\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, C=\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, D=\begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix};$$

$$Case2: \quad A=\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, B=\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, C=\begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, D=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix};$$

$$Case3: \quad A=\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, B=\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, C=\begin{pmatrix} -1 & -1 & 1 \end{pmatrix}, D=0.$$

It is easy to get that only the last two cases satisfy $ABC=0$ and $DC=0$. Next, we will use Theorem 2.2 to obtain the representation of M^D under the last two cases.

Case2: It is easy to get $BC=\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$, $(BC)^D=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $D^D=D=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $A^D=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $i_A=2$, $i_{BC}=2$, $i_{ABC}=4$ *and* $i_D=1$.

Then $X=Y=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. $U(i)=V(i)=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *for all integer* $i\geq 1$. $R(1)=BD^D$, $R(2)=ABD^D$, $R(3)=BCBD^D$ *and* $R(i)=0$ *for all integer* $i\geq 4$. $S(1)=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $S(2)=CBD^D$ *and* $S(i)=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *for all integer* $i\geq 3$.

Then

$$M^D=\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Case3: It is easy to get $BC=\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, $(BC)^D=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $D^D=0$, $A^D=\begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, $i_A=2$, $i_{BC}=2$, $i_{ABC}=3$ *and* $i_D=1$.

Then $X=\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $Y=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. $R(i)=S(i)=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $U(i)=X(A^D)^{i-1}$ *and* $V(i)=X(A^D)^i$ *for all integer* $i\geq 1$.

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Then

$$M^D = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

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