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RELATION BETWEEN THE ROW LEFT RANK OF A QUATERNION UNIT GAIN GRAPH AND THE RANK OF ITS UNDERLYING GRAPH*

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Abstract. Let $\Phi = (G, U(\mathbb{Q}), \varphi)$ be a quaternion unit gain graph (or $U(\mathbb{Q})$-gain graph), where G is the underlying graph of Φ, $U(\mathbb{Q}) = \{z \in \mathbb{Q} : |z| = 1\}$ is the *circle group*, and $\varphi : \overrightarrow{E} \to U(\mathbb{Q})$ is the gain function such that $\varphi(e_{ij}) = \varphi(e_{ji})^{-1} = \overline{\varphi(e_{ji})}$. Let $A(\Phi)$ be the adjacency matrix of Φ and $r(\Phi)$ be the row left rank of Φ. In this paper, we prove that $-2c(G) \leq r(\Phi) - r(G) \leq 2c(G)$, where $r(G)$ and $c(G)$ are the rank and the dimension of cycle space of G, respectively. All corresponding extremal graphs are characterized. The results will generalize the corresponding results of signed graphs (Lu *et al.* [20] and Wang [33]), mixed graphs (Chen *et al.* [7]), and complex unit gain graphs (Lu *et al.* [21]).

Key words. Quaternion unit gain graph, Rank, Dimension of cycle space.

AMS subject classifications. 05C35, 05C50.

1. Introduction. In this paper, we consider only the graphs without multiedges and loops. Let $G = (V(G), E(G))$ be a simple graph, where $V(G)$ and $E(G)$ are the vertex set and the edge set of G, respectively. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, the *adjacency matrix* $A(G)$ of G is the symmetric $n \times n$ matrix with entries $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$ otherwise. Whenever $v_iv_j \in E$, let e_{ij} denote the ordered pair (v_i, v_j). Thus, e_{ij} and e_{ji} are considered to be distinct. Let $\overrightarrow{E}$ denote the set of $\{e_{ij}, e_{ji} : v_iv_j \in E\}$. The *rank* (resp., *nullity*) of G is the rank (resp., nullity) of $A(G)$, denoted by $r(G)$ (resp., $\eta(G)$).

Collatz *et al.* [10] first proposed to characterize all graphs of order n with $r(G) < n$. Until today, this problem is still unsolved. In the past decade, lots of research work have been done on bounding the nullities (or ranks) of graphs with given order in terms of various graph parameters (and identifying the extremal graphs) such as: the matching number (see [11, 17, 24, 27, 28, 32]); the number of pendant vertices (see [4, 6, 25, 30]); the maximum degree (see [9, 29, 31, 34, 39]); and the girth (see [5, 40]), etc.

Let $\mathbb{R}$ and $\mathbb{C}$ be the fields of the real numbers and complex numbers, respectively. Let $\mathbb{Q}$ be a four-dimensional vector space over $\mathbb{R}$ with an ordered basis, denoted by 1, i, j, and k. A *real quaternion*, simply called *quaternion*, is a vector $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{Q}$, where x_0, x_1, x_2, x_3 are real numbers and i, j, k satisfy the following conditions:

$$i^2 = j^2 = k^2 = -1;$$

$$ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

If a, b are any real numbers, while $\mathbf{u}, \mathbf{v}$ are any two of i, j, k, then $(a\mathbf{u})(b\mathbf{v}) = (ab)(\mathbf{uv})$.

From [38], we know that if x, y, and z are three different quaternions, then $(xy)^{-1} = y^{-1}x^{-1}$ and $(xy)z = x(yz)$. (Note that $xy \neq yx$, in general)

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Let $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{Q}$. The *conjugate* $\bar{q}$ (or q^*) of q is $\bar{q} = x_0 - x_1 i - x_2 j - x_3 k$. The *modulus* of q is $|q| = \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$. If $q \neq 0$, then the *inverse* of q is $q^{-1} = \frac{\bar{q}}{|q|^2}$. The *real part* of q is $Re(q) = x_0$. The *imaginary part* of q is $Im(q) = x_1 i + x_2 j + x_3 k$. The *row left (right) rank* of a quaternion matrix $A \in \mathbb{Q}^{m\times n}$ is the maximum number of rows of A that are left (right) linearly independent. The *column left (right) rank* of a quaternion matrix $A \in \mathbb{Q}^{m\times n}$ is the maximum number of columns of A that are left (right) linearly independent. The *rank* of a quaternion matrix $A \in \mathbb{Q}^{m\times n}$ is defined to be the row left rank of A. For a quaternion matrix $A = (h_{st})_{m\times n}$, the *conjugate transpose* of A is $A^* = (\overline{h_{ts}})_{n\times m}$.

Belardo *et al.* [1] studied the spectra of quaternion unit gain graphs. They defined the adjacency, Laplacian, and incidence matrices for a quaternion unit gain graph and study their properties. A *gain graph* is a graph with the additional structure that each orientation of an edge is given a group element, called a *gain*, which is the inverse of the group element assigned to the opposite orientation. Denote by $\Phi = (G, U(\mathbb{Q}), \varphi)$ (or G^φ for short) a quaternion unit gain graph (or $U(\mathbb{Q})$-gain graph), where G is the *underlying graph* of Φ, $U(\mathbb{Q}) = \{q \in \mathbb{Q} : |q| = 1\}$ is the *circle group*, and $\varphi : \overrightarrow{E} \to U(\mathbb{Q})$ is the gain function such that $\varphi(e_{ij}) = \varphi(e_{ji})^{-1} = \overline{\varphi(e_{ji})}$. For convenience, $\varphi(e_{ij})$ is also written as $\varphi_{v_iv_j}$ for $v_iv_j \in E(\Phi)$. The *adjacency matrix* of G^φ is the Hermitian matrix $A(G^\varphi) = (h_{ij})_{n\times n}$, where $h_{ij} = \varphi(e_{ij}) = \varphi_{v_iv_j}$ if $v_iv_j \in E(\Phi)$, and $h_{ij} = 0$ otherwise. The row left rank $r(G^\varphi)$ of $A(G^\varphi)$ is called the *rank* of G^φ.

Recently, people have turned to extend the research of the rank (or nullity) of simple graphs to signed graphs (see [13, 20, 23, 33, 35]) and complex unit gain graphs (see [12, 14, 15, 18, 19, 21, 22, 26, 36, 37]). Cavaleri *et al.* [3] introduce a switching operation to obtain pairs of G-cospectral gain graphs. They also define a Godsil–McKay switching for the right spectrum of quaternion unit gain graphs. Note that quaternion unit gain graphs will generalize simple graphs ($\varphi(\overrightarrow{E}) \subset \{1\}$), signed graphs ($\varphi(\overrightarrow{E}) \subset \{1, -1\}$), mixed graphs ($\varphi(\overrightarrow{E}) \subset \{1, i, -i\}$), and complex unit gain graphs ($\varphi(\overrightarrow{E}) \subset \{T \in \mathbb{C} : |T| = 1\}$). Hence, this paper is a work in this direction.

Let G be a graph. Denoted by $c(G) = |E(G)| - |V(G)| + \omega(G)$, the *dimension of cycle space* of G, where $\omega(G)$ is the number of connected components of G. A connected graph G is called *unicyclic* if $c(G) = 1$. Let G be a graph with pairwise vertex-disjoint cycles, and let $\mathcal{C}(G)$ denote the set of cycles in G. By compressing each cycle O of G into a vertex t_O, we obtain an acyclic graph T_G (see Fig. 1) from G. More definitely, the vertex set $V(T(G))$ is taken to be $U \cup C_G$, where U consists of all vertices of G that do not lie on any cycle and C_G consists of vertex t_O that is obtained by compressing a cycle O, that is, $C_G = \{t_O : O \in \mathcal{C}(G)\}$. Two vertices in U are adjacent in T_G if and only if they are adjacent in G; a vertex $u \in U$ is adjacent to a vertex $t_O \in C_G$ if and only if u is adjacent (in G) to a vertex in the cycle O; Vertices t_{O_1}, t_{O_2} are adjacent in T_G if and only if there exists an edge in G joining a vertex of $O_1 \in \mathcal{C}(G)$ to a vertex of $O_2 \in \mathcal{C}(G)$. It is clear that T_G is always acyclic. Observe the graph $T_G - C_G$ (obtained from T_G by deleting vertices in C_G and the incident edges) is the same as the graph obtained from G by deleting the vertices of all cycles and the incident edges, and the resultant graph is denoted by $[T_G]$ (see Fig. 1).

Lu *et al.* [20] and Wang [33] obtained the relationship between the rank of a signed graph and the rank of its underlying graph, respectively. Chen *et al.* [7] studied the relationship between the H-rank of a mixed graph and the rank of its underlying graph. Lu *et al.* [21] generalized the corresponding results on the complex unit gain graph. Motivated by these results, in this paper, we obtain the relationship between the row left rank of a quaternion unit gain graph G^φ and the rank of its underlying graph: $r(G) - 2c(G) \leq r(G^\varphi) \leq r(G) + 2c(G)$. Moreover, all corresponding extremal graphs are characterized.

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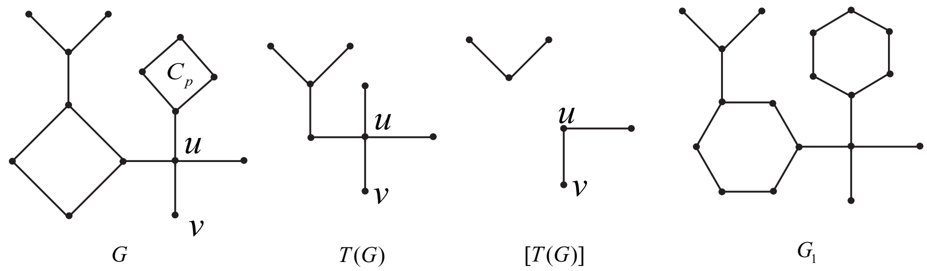

FIG. 1. *Examples.*

The rest of this paper is organized as follows. In Section 2, we give some known lemmas and results. In Section 3, we characterize the relations between the row left rank of a quaternion unit gain graph and the rank of its underlying graph.

2. Preliminaries. The *degree* of $x \in V(G)$ is the number of vertices adjacent to x in G, denoted by $d_G(x)$. A vertex of degree 1 in G is called *pendant vertex* (see v in Fig. 1). The neighbor of a pendant vertex in G is *quasi-pendant vertex* (see u in Fig. 1). Denote by P_n and C_n a path and a cycle of order n, respectively.

Let G be a graph, $S \subseteq V(G)$ and $S \neq \emptyset$. Denote by $G - S$ the *induced subgraph* obtained from G by deleting each vertex in S and its incident edges. For convenience, if $S = \{x\}$, we write $G - x$ instead of $G - \{x\}$. For a subgraph H of G, denote by $G - H$, the subgraph obtained from G by deleting all vertices of H and all incident edges. We use $H + x$ to denote the subgraph of G induced by the vertex set $V(H) \cup \{x\}$. A subgraph C_p of G is called a *pendant cycle* if C_p (see C_p in Fig. 1) is a cycle, which has a unique vertex of degree 3 in G and all other vertices of degree 2 in G.

Now, we list some known results for use later on.

LEMMA 1. ([8]) *Let P_n be a path. Then $r(P_n) = n - 1$ if n is odd and $r(P_n) = n$ if n is even.*

LEMMA 2. ([33]) *Let C_n be a cycle. Then $r(C_n) = n - 2$ if $n \equiv 0 \pmod 4$ and $r(C_n) = n$ otherwise.*

LEMMA 3. ([2]) *If v is a vertex of a graph G, then $r(G) - 2 \leq r(G - v) \leq r(G)$.*

LEMMA 4. ([33]) *Let G be a graph with $x \in V(G)$. Then*

(a) $c(G) = c(G - x)$ if x lies outside any cycle of G;
(b) $c(G - x) \leq c(G) - 1$ if x lies on a cycle of G;
(c) $c(G - x) \leq c(G) - 2$ if x is a common vertex of distinct cycles of G.

LEMMA 5. ([20]) *Let G be a graph containing a pendant vertex x with the unique neighbor y. Then $r(G) = r(G - \{x, y\}) + 2$.*

For a quaternion q, we have that

$$q = x_0 + x_1 i + x_2 j + x_3 k = (x_0 + x_1 i) + (x_2 + x_3 i)j = \gamma_1 + \gamma_2 j.$$

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So, every quaternion q is uniquely represented by a pair of complex numbers (γ_1, γ_2). By this fact, a quaternion matrix A can be uniquely written as $A = A_1 + A_2 j$, where $A_1, A_2 \in \mathbb{C}^{n\times n}$. The *complex adjoint matrix* of A is the matrix:

$$f(A) = f(A_1 + A_2 j) = \begin{pmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{pmatrix} \in \mathbb{C}^{2n\times 2n}.$$

Lemma 6. ([38]) *The rank of a quaternion matrix A is r if and only if the rank of its complex adjoint matrix $f(A)$ is $2r$.*

Lemma 7. ([16]) *The row left rank of a quaternion matrix A equals the column right rank of A. The row right rank of a quaternion matrix A equals the column left rank of A.*

Note that quaternions do not satisfy the commutative law of multiplication, the row left rank of a quaternion matrix is not necessarily equal to the row right rank. Let A_1 be the adjacency matrix of a quaternion unit gain graph with underlying graph C_4, where

$$A_1 = \begin{pmatrix} 0 & 1 & -j & 0 \\ 1 & 0 & 0 & i \\ j & 0 & 0 & k \\ 0 & -i & -k & 0 \end{pmatrix}.$$

The row left rank of A_1 is equal to 4, the row right rank of A_1 is equal to 2.

Lemma 8. ([38]) *Let $A \in M_{m\times n}(\mathbb{Q})$, $B \in M_{n\times s}(\mathbb{Q})$ be two quaternion matrices. Then $(AB)^* = B^* A^*$.*

We call a square quaternion matrix $A \in M_n(\mathbb{Q})$ *invertible* if $AB = BA = I$ for some $B \in M_n(\mathbb{Q})$.

Lemma 9. ([38]) *Let A, B be two quaternion matrices. If $AB = I$, then $BA = I$.*

Lemma 10. ([38]) *For any invertible quaternion matrices P and Q of suitable sizes, the quaternion matrices A and PAQ have the same rank.*

Since quaternions do not satisfy the commutative law of multiplication, when calculating the row left (right) rank of a quaternion matrix by elementary row operations, we can only multiply a nonzero quaternion on the left (right) side of a row and add it to other rows. Similarly, when calculating the column left (right) rank of a quaternion matrix by elementary column operations, we can only multiply a nonzero quaternion on the left (right) side of a column and add it to other columns. An example is as follows. Let

$$A' = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

Then the row left, the row right, the column left, and the column right ranks of A' are equal to 1. Consider now the matrix A obtained by multiply j on the left side of 2-th row of A' and add it to 1-th row of A'. That is,

$$A = \begin{pmatrix} 1 - ji & j + i \\ -i & 1 \end{pmatrix}.$$

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The row left rank

$$r(A) = r\begin{pmatrix} 1 - ji + (j+i)i & j + i - (j+i) \\ -i & 1 \end{pmatrix} = r\begin{pmatrix} 0 & 0 \\ -i & 1 \end{pmatrix} = 1.$$

The row right rank

$$r(A) = r\begin{pmatrix} 1 - ji + i(j+i) & j + i - (j+i) \\ -i & 1 \end{pmatrix} = r\begin{pmatrix} 2ij & 0 \\ -i & 1 \end{pmatrix} = 2.$$

The column left rank

$$r(A) = r\begin{pmatrix} 1 - ji + i(j+i) & j + i \\ -i + i & 1 \end{pmatrix} = r\begin{pmatrix} 2ij & j+i \\ 0 & 1 \end{pmatrix} = 2.$$

The column right rank

$$r(A) = r\begin{pmatrix} 1 - ji + (j+i)i & j + i \\ -i + i & 1 \end{pmatrix} = r\begin{pmatrix} 0 & j+i \\ 0 & 1 \end{pmatrix} = 1.$$

From above examples, we can know that when we multiply a nonzero quaternion on the left side of a row and add it to other rows of a quaternion matrix A, and the row left rank of A maintains unchanged; however, the row right rank of A may be various. Combining with this fact and Lemma 10, in this paper, when we calculate the row left rank of a quaternion matrix by elementary row (or column) operations, we only multiply a nonzero quaternion on the left side of a row and add it to other rows (or right side of a column and add it to other columns).

For a $U(\mathbb{Q})$-gain cycle, we list the following definition and lemmas.

Definition 11. ([1]) *Let C_n^φ be a $U(\mathbb{Q})$-gain cycle with vertices $v_1, v_2, \ldots, v_n$ in turn. For any vertex $v_k \in V(C_n)$, $1 \le k \le n$, we define*

$$\begin{aligned} \varphi(\overrightarrow{C_n}(v_k)) &= \varphi(v_k v_{k+1} \cdots v_n v_1 \cdots v_k) \\ &= \varphi(v_k v_{k+1})\varphi(v_{k+1} v_{k+2}) \cdots \varphi(v_n v_1) \cdots \varphi(v_{k-1} v_k). \end{aligned}$$

Lemma 12. ([1]) *Let C_n^φ be a $U(\mathbb{Q})$-gain cycle with vertices $v_1, v_2, \ldots, v_n$ in turn. Then $Re(\varphi(\overrightarrow{C_n}(v_i))) = Re(\varphi(\overrightarrow{C_n}(v_j)))$, $i \neq j$.*

By fundamental matrix theory, we can derive the following lemma.

Lemma 13. *(a) Let $G^\varphi = G_1^\varphi \cup G_2^\varphi \cup \cdots \cup G_t^\varphi$, where $G_1^\varphi, G_2^\varphi, \ldots, G_t^\varphi$ are connected components of a $U(\mathbb{Q})$-gain graph G^φ. Then $r(G^\varphi) = \Sigma_{i=1}^t r(G_i^\varphi)$.*

(b) Let G^φ be a $U(\mathbb{Q})$-gain graph with n vertices. Then $r(G^\varphi) = 0$ if and only if G^φ is a graph without edges.

3. Relation between the row left rank of G^φ and the rank of its underlying graph. In this section, we will obtain some bounds for the row left rank of a quaternion unit gain graph in terms of the rank of its underlying graph.

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For a complex matrix M, if M_1 is a complex matrix obtained by deleting a row (or a column) of M, then we have that $r(M)-1 \leq r(M_1) \leq r(M)$. By this fact and Lemma 6, we can obtain the following lemma.

LEMMA 14. *Let G^{φ} be a $U(\mathbb{Q})$-gain graph and v is a vertex of G. Then $r(G^{\varphi})-2 \leq r(G^{\varphi}-v) \leq r(G^{\varphi})$.*

Similar to Lemma 5, for a $U(\mathbb{Q})$-gain graph, we have the following lemma.

LEMMA 15. *Let G^{φ} be a $U(\mathbb{Q})$-gain graph. If x is a pendant vertex of G and y is its unique neighbor in G, then $r(G^{\varphi}) = r(G^{\varphi}-\{x,y\})+2$.*

Proof. We label all vertices of G in return by $x_1, x_2, \ldots, x_n$ with $x_1 = x, x_2 = y$. Let $h_{ij} = \varphi(e_{ij}), 1 \leq i,j \leq n$ and $A(G^{\varphi})$ be the adjacency matrix of G^{φ}. Then,

$$A(G^{\varphi}) = \begin{pmatrix} 0 & h_{12} & 0 & \cdots & 0 \\ h_{21} & 0 & h_{23} & \cdots & h_{2n} \\ 0 & h_{32} & 0 & \cdots & h_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & h_{n2} & h_{n3} & \cdots & 0 \end{pmatrix}.$$

We use elementary row and column operations on $A(G^{\varphi})$, then

$$r(G^{\varphi}) = r\begin{pmatrix} 0 & h_{12} & 0 & \cdots & 0 \\ h_{21} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & h_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & h_{n3} & \cdots & 0 \end{pmatrix} = 2 + r(G^{\varphi}-\{x,y\}).$$

This completes the proof. ∎

The gain of a path $P_n = e_{12}e_{23}\ldots e_{(n-1)n}$ is $\varphi(P_n) = \varphi(e_{12})\varphi(e_{23})\ldots\varphi(e_{(n-1)n})$.

Suppose $\theta : V \to U(\mathbb{Q})$ is a switching function. Switching the $U(\mathbb{Q})$-gain graph G^{φ} by θ means forming a new quaternion unit gain graph $G^{\varphi^{\theta}}$, whose underlying graph is the same as G^{φ}, but whose gain function is defined on an edge uv by $\varphi^{\theta}(uv) = \theta(u)^{-1}\varphi(uv)\theta(v)$. Two quaternion unit gain graphs G^{φ_1} and G^{φ_2} are called *switching equivalent*, denoted by $G^{\varphi_1} \leftrightarrow G^{\varphi_2}$, if there exists a switching function θ such that $G^{\varphi_2} = G^{\varphi_1^{\theta}}$. Note that two switching equivalent quaternion unit gain graphs have the same rank.

LEMMA 16. *Let T^{φ} be a $U(\mathbb{Q})$-gain tree of order n. Then $A(T^{\varphi})$ and $A(T)$ have the same rank.*

Proof. We shall verify that T^{φ} and T are switching equivalent. Let x be a vertex of T^{φ}. We will define a switching function θ satisfying $\theta(x) = 1, \theta(y) = \varphi(P_{xy})^{-1}$ for any vertex y of T^{φ}, where P_{xy} is the path from x to y. Let z_1, z_2 be any two adjacent vertices in T^{φ}, without loss of generality, assume that z_1, z_2 on the path P_{xy} satisfy $d(x,z_1) = d(x,z_2)-1$, then

$$\begin{aligned}\varphi^{\theta}(e_{z_1z_2}) &= \theta(z_1)^{-1}\varphi(e_{z_1z_2})\theta(z_2) \\ &= \varphi(P_{xz_1})\varphi(e_{z_1z_2})\varphi(P_{xz_2})^{-1} \\ &= \varphi(e_{xy_1})\varphi(e_{y_1y_2})\ldots\varphi(e_{y_mz_1})\varphi(e_{z_1z_2})(\varphi(e_{xy_1})\varphi(e_{y_1y_2})\ldots\varphi(e_{y_mz_1}) \\ &\quad \varphi(e_{z_1z_2}))^{-1} \\ &= 1.\end{aligned}$$

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So, $A(T) = Q^{-1}A(T^{\varphi})Q$, where

$$Q = \begin{pmatrix} \theta(z_1) & & & \\ & \theta(z_2) & & \\ & & \ddots & \\ & & & \theta(z_n) \end{pmatrix}.$$

By Lemma 10, $A(T^{\varphi})$ and $A(T)$ have the same rank. □

Let $C_n^{\varphi}(n \geq 3)$ be a quaternion unit gain cycle and $V(G) = \{v_1, v_2, \ldots, v_n\}$. By Lemma 9, $\varphi_{v_1v_2}\varphi_{v_2v_3}\cdots\varphi_{v_{n-1}v_n}\varphi_{v_nv_1} = c$ if and only if

$$\varphi_{v_lv_{l+1}}\varphi_{v_{l+1}v_{l+2}}\cdots\varphi_{v_{n-1}v_n}\cdots\varphi_{v_{l-1}v_l} = c,$$

where c is constant. By this fact and Lemma 12, we have the following definition.

Definition 17. *Let $C_n^{\varphi}(n \geq 3)$ be a quaternion unit gain cycle, denote by*

$$\varphi(C_n^{\varphi}) = \varphi_{v_1v_2}\varphi_{v_2v_3}\cdots\varphi_{v_{n-1}v_n}\varphi_{v_nv_1}.$$

Then C_n^{φ} is said to be:

$$\begin{cases} \text{Type 1,} & \text{if } \varphi(C_n^{\varphi}) = (-1)^{n/2} \text{ and } n \text{ is even;} \\ \text{Type 2,} & \text{if } \varphi(C_n^{\varphi}) \neq (-1)^{n/2} \text{ and } n \text{ is even;} \\ \text{Type 3,} & \text{if } Re\left((-1)^{(n-1)/2}\varphi(C_n^{\varphi})\right) \neq 0 \text{ and } n \text{ is odd;} \\ \text{Type 4,} & \text{if } Re\left((-1)^{(n-1)/2}\varphi(C_n^{\varphi})\right) = 0 \text{ and } n \text{ is odd.} \end{cases}$$

Lemma 18. *Let C_n^{φ} be a $U(\mathbb{Q})$-gain cycle. Then*

$$r(C_n^{\varphi}) = \begin{cases} n-2, & if\ C_n^{\varphi}\ is\ of\ Type\ 1; \\ n, & if\ C_n^{\varphi}\ is\ of\ Type\ 2\ or\ 3; \\ n-1, & if\ C_n^{\varphi}\ is\ of\ Type\ 4. \end{cases}$$

Proof. Let $\{u_1, u_2, \ldots, u_n\}$ be the vertex set of C_n^{φ} and $e_{u_ku_{k+1}} \in E(C_n)(1 \leq k \leq n-1), e_{u_1u_n} \in E(C_n)$. Let $h_k = \varphi(e_{u_ku_{k+1}})(1 \leq k \leq n-1)$ and $h_n = \varphi(e_{u_nu_1})$. Then

$$A(C_n^{\varphi}) = \begin{pmatrix} 0 & h_1 & 0 & 0 & \cdots & 0 & \bar{h}_n \\ \bar{h}_1 & 0 & h_2 & 0 & & & 0 \\ 0 & \bar{h}_2 & 0 & h_3 & & & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \bar{h}_{n-2} & 0 & h_{n-1} \\ h_n & 0 & 0 & \cdots & 0 & \bar{h}_{n-1} & 0 \end{pmatrix}.$$

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Now we discuss two cases.

Case 1. n is even.

By the elementary row and column operations,

$$r(C_n^{\varphi}) = r\begin{pmatrix} 0 & h_1 & 0 & 0 & \cdots & 0 & \bar{h}_n \\ \bar{h}_1 & 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & h_3 & & & -\bar{h}_2\bar{h}_1\bar{h}_n \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \bar{h}_{n-2} & 0 & h_{n-1} \\ h_n & 0 & -h_nh_1h_2 & \cdots & 0 & \bar{h}_{n-1} & 0 \end{pmatrix}$$

$$= r\begin{pmatrix} 0 & h_1 & 0 & 0 & \cdots & 0 & 0 \\ \bar{h}_1 & 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & h_3 & & & -\bar{h}_2\bar{h}_1\bar{h}_n \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \bar{h}_{n-2} & 0 & h_{n-1} \\ 0 & 0 & -h_nh_1h_2 & \cdots & 0 & \bar{h}_{n-1} & 0 \end{pmatrix}$$

$$= 2 + r\begin{pmatrix} 0 & h_3 & 0 & 0 & \cdots & 0 & -\bar{h}_2\bar{h}_1\bar{h}_n \\ \bar{h}_3 & 0 & h_4 & 0 & & & 0 \\ 0 & \bar{h}_4 & 0 & h_5 & & & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \bar{h}_{n-2} & 0 & h_{n-1} \\ -h_nh_1h_2 & 0 & 0 & \cdots & 0 & \bar{h}_{n-1} & 0 \end{pmatrix}$$

$$= 2+$$

$$r\begin{pmatrix} 0 & h_3 & 0 & 0 & \cdots & 0 & 0 \\ \bar{h}_3 & 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & h_5 & & & (-1)^2\bar{h}_4\bar{h}_3\bar{h}_2\bar{h}_1\bar{h}_n \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \bar{h}_{n-2} & 0 & h_{n-1} \\ 0 & 0 & (-1)^2h_nh_1h_2h_3h_4 & \cdots & 0 & \bar{h}_{n-1} & 0 \end{pmatrix}$$

$$= \cdots$$

$$= n - 2 + r\begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix},$$

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where $a = h_{n-1} + (-1)^{\frac{n-2}{2}} \bar{h}_{n-2}\bar{h}_{n-3} \ldots \bar{h}_2\bar{h}_1\bar{h}_n$.

If $a = 0$, then

$$h_{n-1} + (-1)^{\frac{n-2}{2}} \bar{h}_{n-2}\bar{h}_{n-3} \ldots \bar{h}_2\bar{h}_1\bar{h}_n = 0.$$

Since quaternion does not satisfy commutative law of multiplication, then we multiply both left sides of the equation by $h_1h_2 \ldots h_{n-3}h_{n-2}$, the equation is equivalent to

$$h_1h_2 \ldots h_{n-2}h_{n-1} + (-1)^{\frac{n-2}{2}} \bar{h}_n = 0.$$

We multiply both right sides of the equation by h_n, then the equation is equivalent to

$$h_1h_2 \ldots h_{n-2}h_{n-1}h_n = (-1)^{\frac{n}{2}}.$$

That is, $\varphi(C_n^\varphi) = (-1)^{\frac{n}{2}}$, then we have that $r(C_n^\varphi) = n-2$. Similarly, $a \neq 0$ is equivalent to $\varphi(C_n^\varphi) \neq (-1)^{\frac{n}{2}}$, then we have that $r(C_n^\varphi) = n$.

Case 2. n is odd.

Using the same method as Case 1, we have

$$\begin{aligned} r(C_n^\varphi) &= n - 3 + r\begin{pmatrix} 0 & h_{n-2} & b_0 \\ \bar{h}_{n-2} & 0 & h_{n-1} \\ \bar{b}_0 & \bar{h}_{n-1} & 0 \end{pmatrix} \\ &= n - 3 + r\begin{pmatrix} 0 & h_{n-2} & 0 \\ \bar{h}_{n-2} & 0 & 0 \\ 0 & 0 & b \end{pmatrix} \\ &= n - 1 + r(b), \end{aligned}$$

where

$$b_0 = (-1)^{\frac{n-3}{2}} \bar{h}_{n-3}\bar{h}_{n-4} \ldots \bar{h}_2\bar{h}_1\bar{h}_n,$$

and

$$b = (-1)^{\frac{n-1}{2}} \bar{h}_{n-1}\bar{h}_{n-2} \ldots \bar{h}_2\bar{h}_1\bar{h}_n + (-1)^{\frac{n-1}{2}} h_nh_1h_2 \ldots h_{n-2}h_{n-1}.$$

By Lemmas 8 and 12,

$$\begin{gathered} (h_nh_1h_2 \ldots h_{n-2}h_{n-1})^* \\ = \bar{h}_{n-1}\bar{h}_{n-2} \ldots \bar{h}_2\bar{h}_1\bar{h}_n, Re(\varphi(C_n)) = Re(\varphi(h_nh_1h_2 \ldots h_{n-2}h_{n-1})). \end{gathered}$$

So,

$$b = 2Re((-1)^{\frac{n-1}{2}} h_nh_1h_2 \ldots h_{n-1}) = 2Re((-1)^{\frac{n-1}{2}} \varphi(C_n)).$$

If $b = 0$, that is $Re((-1)^{\frac{n-1}{2}} \varphi(C_n)) = 0$, then $r(C_n^\varphi) = n - 1$.

If $b \neq 0$, that is $Re((-1)^{\frac{n-1}{2}} \varphi(C_n)) \neq 0$, then $r(C_n^\varphi) = n - 1 + 1 = n$. □

Now, we will characterize the relationship between the row left rank of a quaternion unit gain graph and the rank of its underlying graph.

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THEOREM 19. *Let G^{φ} be a $U(\mathbb{Q})$-gain graph. Then*

$$r(G) - 2c(G) \leq r(G^{\varphi}) \leq r(G) + 2c(G).$$

Proof. We get the inequalities by induction on $c(G)$. If $c(G) = 0$, then G^{φ} is a $U(\mathbb{Q})$-gain acyclic graph, $r(G^{\varphi}) = r(G)$ holds by Lemmas 13 and 16. Now we assume that the results hold for each $U(\mathbb{Q})$-gain graph whose dimension of cycle space is less than $c(G)$. Let u be a vertex on some cycle of G^{φ}. By Lemma 4, we have

$$c(G-u) \leq c(G) - 1.$$

Then by induction hypothesis,

$$r(G-u) - 2c(G-u) \leq r(G^{\varphi} - u) \leq r(G-u) + 2c(G-u).$$

By Lemmas 3 and 14, we have

$$r(G^{\varphi} - u) \leq r(G^{\varphi}) \leq r(G^{\varphi} - u) + 2, r(G-u) \leq r(G) \leq r(G-u) + 2.$$

So

$$r(G^{\varphi}) \geq r(G^{\varphi} - u) \geq r(G-u) - 2c(G-u) \geq r(G) - 2 - 2(c(G) - 1) = r(G) - 2c(G),$$

and

$$r(G^{\varphi}) \leq r(G^{\varphi} - u) + 2 \leq r(G-u) + 2c(G-u) + 2 \leq r(G) + 2(c(G) - 1) + 2 = r(G) + 2c(G).$$

This completes the proof. ∎

For convenience, we call a $U(\mathbb{Q})$-gain graph G^{φ} *lower-optimal* (resp., *upper-optimal*) if G^{φ} attains the lower bound (resp., upper bound) in Theorem 19.

LEMMA 20. *Let G^{φ} be a $U(\mathbb{Q})$-gain graph and let u be a vertex on some cycle of G^{φ}.*

(a) If G^{φ} is lower-optimal, then $r(G^{\varphi}) = r(G^{\varphi} - u), r(G^{\varphi} - u) = r(G-u) - 2c(G-u), r(G-u) = r(G) - 2, c(G) = c(G-u) + 1$.
(b) If G^{φ} is upper-optimal, then $r(G^{\varphi}) = r(G^{\varphi} - u) + 2, r(G^{\varphi} - u) = r(G-u) + 2c(G-u), r(G-u) = r(G), c(G) = c(G-u) + 1$.
(c) If G^{φ} is lower-optimal (or upper-optimal), then u lies on just one cycle of G and u is not a quasi-pendant vertex in G.

Proof. By the proof of Theorem 19, we can obtain the (a) and (b). For (c), when G^{φ} is lower-optimal. If u lies on at least two cycles of G^{φ}, by Lemma 4(c),

$$c(G-u) \leq c(G) - 2,$$

which contradicts $c(G) = c(G-u) + 1$ in (a). If u is a quasi-pendant vertex of G^{φ}, then by Lemma 15,

$$r(G^{\varphi} - u) = r(G^{\varphi}) - 2,$$

which contradicts $r(G^{\varphi} - u) = r(G^{\varphi})$ in (a).

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When G^φ is upper-optimal. If u lies on at least two cycles of G^φ, by Lemma 4(c),

$$c(G-u) \leq c(G) - 2,$$

which contradicts $c(G) = c(G-u) + 1$ in (b). If u is a quasi-pendant vertex of G, then by Lemma 5,

$$r(G) = r(G-u) + 2,$$

which contradicts $r(G-u) = r(G)$ in (b). So we obtain the (c) of this lemma. □

LEMMA 21. *Let G^φ be a $U(\mathbb{Q})$-gain graph which contains a pendant vertex x with its unique neighbor y and $H^\varphi = G^\varphi - \{x, y\}$. If G^φ is lower-optimal (upper-optimal), then y does not lie on any cycle of G, and H^φ is also lower-optimal (upper-optimal).*

Proof. We can obtain that y does not lie on any cycle of G from Lemma 20(c). Since G^φ is lower-optimal, by Lemmas 5 and 15,

$$r(H^\varphi) + 2 = r(G^\varphi) = r(G) - 2c(G) = r(G - \{x, y\}) + 2 - 2c(G - x - y).$$

Thus, we have

$$r(H^\varphi) = r(G - \{x, y\}) - 2c(G - \{x, y\}) = r(H) - 2c(H).$$

Similarly, if G^φ is upper-optimal, then H^φ is also upper-optimal. □

For a $U(\mathbb{Q})$-gain unicyclic graph, we have the following lemma.

LEMMA 22. *Let G^φ be a $U(\mathbb{Q})$-gain unicyclic graph with unique cycle C_l^φ.*

(a) If G^φ is lower-optimal, then C_l^φ is of Type 1 with $l \equiv 2 \pmod 4$;
(b) If G^φ is upper-optimal, then C_l^φ is of Type 2 with $l \equiv 0 \pmod 4$.

Proof. For (a), if G^φ contains no pendant vertices, then G^φ is cycle C_l^φ or the union of cycle C_l^φ and some isolated vertices. By Lemmas 2, 13, and 18, the result holds naturally.

If G^φ contains pendant vertices, by Lemma 21, then we know that the quasi-pendant vertices of G^φ do not lie on cycle C_l. Denote by H^φ the $U(\mathbb{Q})$-gain unicyclic graph obtained from deleting all pendant vertices and quasi-pendant vertices of G^φ. If H^φ has pendant vertices, then we repeat the above steps (delete pendant vertices and quasi-pendant vertices). After several steps, we can obtain a $U(\mathbb{Q})$-gain unicyclic graph H_m^φ such that H_m^φ is cycle C_l^φ or the union of cycle C_l^φ and some isolated vertices. By Lemma 21, H_m^φ is lower-optimal. So by Lemmas 2, 13, and 18, C_l^φ is of Type 1 with $l \equiv 2 \pmod 4$. This completes the proof of (a).

Similarly, we can obtain the (b) of this lemma. □

LEMMA 23. *Let G^φ be a $U(\mathbb{Q})$-gain graph obtained by identifying a vertex of a $U(\mathbb{Q})$-gain cycle C_n^φ with a vertex of a $U(\mathbb{Q})$-gain graph G_1^φ of order $m (m \geq 1) (V(C_n) \cap V(G_1) = u)$ and $G_2^\varphi = G_1^\varphi - u$. Then*

$$\begin{cases} r(G^\varphi) = n - 2 + r(G_1^\varphi), & if\ C_n^\varphi\ is\ of\ Type\ 1, \\ r(G^\varphi) = n + r(G_2^\varphi), & if\ C_n^\varphi\ is\ of\ Type\ 2, \\ r(G^\varphi) = n - 1 + r(G_1^\varphi), & if\ C_n^\varphi\ is\ of\ Type\ 4, \\ n - 1 + r(G_2^\varphi) \leq r(G^\varphi) \leq n + r(G_1^\varphi), & if\ C_n^\varphi\ is\ of\ Type\ 3. \end{cases}$$

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Proof. Let $V(C_n) = u_1, u_2, \ldots, u_n$ and $u_i u_{i+1} \in E(C_n)(1 \leq i \leq n), u_1 u_n \in E(C_n)$. Let $h_i = \varphi(u_i u_{i+1})(1 \leq i \leq n-1)$ and $h_n = \varphi(u_n u_1)$. Without loss of generality, we assume that $V(C_n) \cap V(G_1) = u_n$. Now we discuss the following two cases.

Case 1. When n is even, we have that

$$A(G_n^{\varphi}) = \begin{pmatrix} 0 & h_1 & 0 & 0 & \cdots & 0 & \bar{h}_n & & & \\ \bar{h}_1 & 0 & h_2 & 0 & & & 0 & & & \\ 0 & \bar{h}_2 & 0 & h_3 & & & 0 & & 0 & \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \vdots & & & \\ \vdots & & & \ddots & \ddots & \ddots & 0 & & & \\ 0 & 0 & \cdots & 0 & \bar{h}_{n-2} & 0 & h_{n-1} & 0 & \cdots & 0 \\ h_n & 0 & 0 & \cdots & 0 & \bar{h}_{n-1} & 0 & \varphi(\beta_1) & \cdots & \varphi(\beta_{m-1}) \\ & & & & & 0 & \overline{\varphi(\beta_1)} & & & \\ & & 0 & & & \vdots & \vdots & & M & \\ & & & & & 0 & \overline{\varphi(\beta_{m-1})} & & & \end{pmatrix},$$

where $\beta_i \in E(G_1)$, $i = 1, 2, \ldots, m-1$, $M = A(G_2^{\varphi})$.

Observe that $h_i \bar{h}_i = 1$, then by elementary row and column transformations, we have that

$$r(G^{\varphi}) = r \begin{pmatrix} A_1 & & & & & & & \\ & \ddots & & & & & & \\ & & A_{\frac{n-2}{2}} & & & & & \\ & & & 0 & a & 0 & \cdots & 0 \\ & & & \overline{a} & 0 & \varphi(\beta_1) & \cdots & \varphi(\beta_{m-1}) \\ & & & 0 & \overline{\varphi(\beta_1)} & & & \\ & & & \vdots & \vdots & & M & \\ & & & 0 & \overline{\varphi(\beta_{m-1})} & & & \end{pmatrix},$$

where

$$A_i = \begin{pmatrix} 0 & h_{2i-1} \\ \bar{h}_{2i-1} & 0 \end{pmatrix}, a = h_{n-1} + (-1)^{\frac{n-2}{2}} \bar{h}_{n-2} \bar{h}_{n-3} \cdots \bar{h}_1 \bar{h}_n,$$

for $i = 1, 2, \ldots, \frac{n-2}{2}$. Then we can obtain that

(i) If C_n^{φ} is of Type 1, that is

$$h_1 h_2 \cdots h_n = (-1)^{\frac{n}{2}},$$

$$\Longleftrightarrow h_{n-1} + (-1)^{\frac{n-2}{2}} \bar{h}_{n-2} \bar{h}_{n-3} \cdots \bar{h}_1 \bar{h}_n = 0,$$

then $a = 0$ and so

$$r(G^{\varphi}) = n - 2 + r(G_1^{\varphi}).$$

(ii) If C_n^{φ} is of Type 2, then $a \neq 0$ and so

$$r(G^{\varphi}) = n + r(G_2^{\varphi}).$$

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Case 2. When n is odd, by elementary row and column operations, we have that

$$r(G^{\varphi}) = r\begin{pmatrix} A_1 & & & & & & \\ & \ddots & & & & & \\ & & A_{\frac{n-1}{2}} & & & & \\ & & & a & \varphi(\beta_1) & \cdots & \varphi(\beta_{m-1}) \\ & & & \overline{\varphi(\beta_1)} & & & \\ & & & \vdots & & M & \\ & & & \overline{\varphi(\beta_{m-1})} & & & \end{pmatrix},$$

where

$$A_i = \begin{pmatrix} 0 & h_{2i-1} \\ \bar{h}_{2i-1} & 0 \end{pmatrix}, a = 2Re((-1)^{\frac{n-1}{2}} h_1 h_2 \dots h_n),$$

for $i = 1, 2, \dots, \frac{n-1}{2}$.

(i) If C_n^{φ} is of Type 4, then $a = 0$ and so $r(G^{\varphi}) = n - 1 + r(G_1^{\varphi})$.

(ii) If C_n^{φ} is of Type 3, then $a \neq 0$. Let

$$C = \begin{pmatrix} a & \varphi(\beta_1) & \cdots & \varphi(\beta_{m-1}) \\ \overline{\varphi(\beta_1)} & & & \\ \vdots & & M & \\ \overline{\varphi(\beta_{m-1})} & & & \end{pmatrix}.$$

Then $r(G_2^{\varphi}) \leq r(C) \leq r(G_1^{\varphi}) + 1$. So $n - 1 + r(G_2^{\varphi}) \leq r(G^{\varphi}) \leq n + r(G_1^{\varphi})$. □

In order to characterize the lower-optimal and upper-optimal $U(\mathbb{Q})$-gain graphs, the following two lemmas are needed.

LEMMA 24. *Let G^{φ} be a $U(\mathbb{Q})$-gain graph of order n such that each cycle (if any) C_l^{φ} in G^{φ} is of Type 1 with $l \equiv 2 \pmod 4$. If G^{φ} is lower-optimal, then the following results hold*

(a) $r(G^{\varphi}) = r(T_G) + \Sigma_{C \in \mathcal{C}(G)} r(C^{\varphi})$;
(b) $r(G) = r([T_G]) + \Sigma_{C \in \mathcal{C}(G)} r(C)$.

Proof. Since G^{φ} is lower-optimal, by Lemma 20(c), we have that any two cycles of G have no common vertices. We shall apply induction on the order of G^{φ} to prove this lemma. If $n = 1$, the results hold naturally. Assume that the results hold for each lower-optimal $U(\mathbb{Q})$-gain graph of order less than n. If T_G has no edges, that is, G consists of disjoint cycles and some isolated vertices, then the results hold by Lemma 13. If T_G has at least one edge, then we will consider the following cases.

Case 1. G^{φ} contains a pendant vertex, say x.

Let y be the unique neighbor vertex of x in G and $G_1^{\varphi} = G^{\varphi} - \{x, y\}$. By Lemma 21, G_1^{φ} is lower-optimal and y does not lie on any cycle of G^{φ}. By induction hypothesis, we have that

$$r(G_1^{\varphi}) = r(T_{G_1}) + \Sigma_{C \in \mathcal{C}(G_1)} r(C^{\varphi}), r(G_1) = r([T_{G_1}]) + \Sigma_{C \in \mathcal{C}(G_1)} r(C).$$

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By Lemmas 5 and 15,

$$r(G^{\varphi}) = r(G_1^{\varphi}) + 2, r(G) = r(G_1) + 2, r(T_G) = r(T_{G_1}) + 2, r([T_G]) = r([T_{G_1}]) + 2.$$

Note that $\mathcal{C}(G) = \mathcal{C}(G_1)$. So,

$$r(G^{\varphi}) = r(T_G) + \Sigma_{C\in\mathcal{C}(G)} r(C^{\varphi}), r(G) = r([T_G]) + \Sigma_{C\in\mathcal{C}(G)} r(C).$$

Case 2. G^{φ} contains a pendant cycle, say C_l^{φ}.

Let $u \in C_l$ be the unique vertex of degree 3, $H^{\varphi} = G^{\varphi} - C_l^{\varphi}$ and $K^{\varphi} = H^{\varphi} + u$. Let v be a vertex of C_l adjacent to u. By Lemma 23,

$$r(G^{\varphi}) = l - 2 + r(K^{\varphi}).$$

By Lemmas 5 and 20(a),

$$r(G) = r(G - v) + 2 = l + r(K).$$

Since G^{φ} is lower-optimal, that is, $r(G^{\varphi}) = r(G) - 2c(G)$, we have

$$r(K^{\varphi}) = r(K) - 2(c(G) - 1) = r(K) - 2c(K).$$

Hence, K^{φ} is lower-optimal. By induction hypothesis, we obtain

$$r(K^{\varphi}) = r(T_K) + \Sigma_{C\in\mathcal{C}(K)} r(C^{\varphi}).$$

Note that $T_G \cong T_K$, by Lemma 18, $r(C_l^{\varphi}) = l - 2$. So

$$r(G^{\varphi}) = l - 2 + r(K^{\varphi}) = r(T_K) + \Sigma_{C\in\mathcal{C}(K)} r(C^{\varphi}) + l - 2 = r(T_G) + \Sigma_{C\in\mathcal{C}(G)} r(C^{\varphi}).$$

We obtain the (a) of this lemma.

For (b), by Lemmas 1, 13, 16, and 20(a),

$$r(G^{\varphi}) = r(G^{\varphi} - u) = l - 2 + r(H^{\varphi}), r(G) = r(G - u) + 2 = l + r(H).$$

Since G^{φ} is lower-optimal, that is, $r(G^{\varphi}) = r(G) - 2c(G)$, we have

$$r(H^{\varphi}) = r(H) - 2(c(G) - 1) = r(H) - 2c(H).$$

Hence, H^{φ} is lower-optimal. By induction hypothesis, we obtain

$$r(H) = r([T_H]) + \Sigma_{C\in\mathcal{C}(H)} r(C).$$

Note that $[T_H] \cong [T_G]$, by Lemma 2, $r(C_l) = l$. So

$$r(G) = l + r(H) = r([T_H]) + \Sigma_{C\in\mathcal{C}(H)} r(C) + l = r([T_G]) + \Sigma_{C\in\mathcal{C}(G)} r(C).$$

We obtain the (b) of this lemma. □

Using the same methods as Lemma 24, we can obtain the following lemma.

Lemma 25. *Let G^{φ} be a $U(\mathbb{Q})$-gain graph of order n such that each cycle (if any) C_l^{φ} in G^{φ} is of Type 2 with $l \equiv 0 \pmod 4$. If G^{φ} is upper-optimal, then the following results hold*

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(a) $r(G^{\varphi}) = r(T_G) + \Sigma_{C\in\mathcal{C}(G)} r(C^{\varphi})$*;*
(b) $r(G) = r([T_G]) + \Sigma_{C\in\mathcal{C}(G)} r(C)$*.*

Now, we will characterize the lower-optimal $U(\mathbb{Q})$-gain graph.

THEOREM 26. *Let G^{φ} be a $U(\mathbb{Q})$-gain graph of order n. Then G^{φ} is lower-optimal if and only if the following conditions all hold*

(a) any two cycles of G^{φ} share no common vertices;
(b) each cycle (if any) C_l^{φ} in G^{φ} is of Type 1 with $l \equiv 2 \pmod 4$;
(c) $r(G^{\varphi}) = r(T_G) + \Sigma_{C\in\mathcal{C}(G)} r(C^{\varphi})$, $r(G) = r([T_G]) + \Sigma_{C\in\mathcal{C}(G)} r(C)$ and $r(T_G) = r([T_G])$.

Proof. **Sufficiency:** By condition (b), Lemmas 2 and 18, we have $r(C^{\varphi}) - r(C) = -2$ for each cycle C^{φ} of G^{φ}. By conditions (a) and (c),

$$r(G^{\varphi}) - r(G) = \Sigma_{C\in\mathcal{C}(G)}(r(C^{\varphi}) - r(C)) = -2c(G),$$

that is, G^{φ} is lower-optimal.

Necessity: Since G^{φ} is lower-optimal, by Lemma 20(c), we have that any vertex of the cycles of G lies on only one cycle, that is, any two cycles of G^{φ} share no common vertices. So the condition (a) holds.

If G is an acyclic graph, the condition (b) holds naturally. If $c(G) = 1$, the condition (b) holds by Lemma 22(a). Now we assume that the condition (b) holds for the $U(\mathbb{Q})$-gain graph whose dimension of cycle space is smaller than G^{φ}. Let u be a vertex of some cycle C' in G. By Lemma 20(a), $G^{\varphi} - u$ is lower-optimal. By induction hypothesis, each cycle C_l^{φ} of $G^{\varphi} - u$ is of Type 1 with $l \equiv 2 \pmod 4$. Let v be a vertex of some cycle C'' (different from C') in G, by Lemma 20(a), $G^{\varphi} - v$ is lower-optimal. By induction hypothesis, each cycle C_l^{φ} of $G^{\varphi} - v$ is of Type 1 with $l \equiv 2 \pmod 4$. Combining with the above discussion, we have that each cycle (if any) C_l^{φ} in G^{φ} is of Type 1 with $l \equiv 2 \pmod 4$. So the condition (b) holds.

Let G^{φ} be a $U(\mathbb{Q})$-gain graph such that

$$r(G^{\varphi}) = r(G) - 2c(G).$$

By Lemma 24, we have that $r(G^{\varphi}) = r(T_G) + \Sigma_{C\in\mathcal{C}(G)} r(C^{\varphi})$ and $r(G) = r([T_G]) + \Sigma_{C\in\mathcal{C}(G)} r(C)$. So, $r(G^{\varphi}) - r(G) = r(T_G) - r([T_G]) + \Sigma_{C\in\mathcal{C}(G)}(r(C^{\varphi}) - r(C))$. Combining with the (b), Lemmas 2 and 18, we have that $r(C^{\varphi}) - r(C) = -2$ for each cycle C^{φ} of G^{φ}, then $r(G^{\varphi}) - r(G) = r(T_G) - r([T_G]) - 2c(G)$.

Hence, $r(T_G) = r([T_G])$, the condition (c) holds. ∎

Now, we will characterize the upper-optimal $U(\mathbb{Q})$-gain graph.

THEOREM 27. *Let G^{φ} be a $U(\mathbb{Q})$-gain graph of order n. Then G^{φ} is upper-optimal if and only if the following conditions all hold*

(a) any two cycles of G^{φ} share no common vertices;
(b) each cycle (if any) C_l^{φ} in G^{φ} is of Type 2 with $l \equiv 0 \pmod 4$;
(c) $r(G^{\varphi}) = r(T_G) + \Sigma_{C\in\mathcal{C}(G)} r(C^{\varphi})$, $r(G) = r([T_G]) + \Sigma_{C\in\mathcal{C}(G)} r(C)$ and $r(T_G) = r([T_G])$.

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Proof. **Sufficiency:** By condition (b), Lemmas 2 and 18, we have $r(C^{\varphi}) - r(C) = 2$ for each cycle C^{φ} of G^{φ}. By conditions (a) and (c),

$$r(G^{\varphi}) - r(G) = \Sigma_{C\in\mathcal{C}(G)}(r(C^{\varphi}) - r(C)) = 2c(G),$$

that is, G^{φ} is upper-optimal.

Necessity: Since G^{φ} is upper-optimal, by Lemma 20(c), we have that any vertex of the cycles of G lies on only one cycle, that is, any two cycles of G^{φ} share no common vertices. So the condition (a) holds.

If G is an acyclic graph, the condition (b) holds naturally. If $c(G) = 1$, the condition (b) holds by Lemma 22(b). Now we assume that the condition (b) holds for all $U(\mathbb{Q})$-gain graph whose dimension of cycle space is smaller than G^{φ}. Let u be a vertex of some cycle $C^{'}$ in G. By Lemma 20(b), $G^{\varphi} - u$ is upper-optimal. By induction hypothesis, each cycle C_l^{φ} of $G^{\varphi} - u$ is of Type 2 with $l \equiv 0 \pmod 4$. Let v be a vertex of some cycle $C^{''}$ (different from $C^{'}$) in G, by Lemma 20(b), $G^{\varphi} - v$ is upper-optimal. By induction hypothesis, each cycle C_l^{φ} of $G^{\varphi} - v$ is of Type 2 with $l \equiv 0 \pmod 4$. Combining with the above discussion, we have that each cycle (if any) C_l^{φ} in G^{φ} is of Type 2 with $l \equiv 0 \pmod 4$. So the condition (b) holds.

Let G^{φ} be a $U(\mathbb{Q})$-gain graph such that

$$r(G^{\varphi}) = r(G) + 2c(G).$$

By Lemma 25, we have that $r(G^{\varphi}) = r(T_G) + \Sigma_{C\in\mathcal{C}(G)}r(C^{\varphi})$ and $r(G) = r([T_G]) + \Sigma_{C\in\mathcal{C}(G)}r(C)$. So, $r(G^{\varphi}) - r(G) = r(T_G) - r([T_G]) + \Sigma_{C\in\mathcal{C}(G)}(r(C^{\varphi}) - r(C))$. Combining with the (b), Lemmas 2 and 18, we have that $r(C^{\varphi}) - r(C) = 2$ for each cycle C^{φ} of G^{φ}, then $r(G^{\varphi}) - r(G) = r(T_G) - r([T_G]) + 2c(G)$.

Hence, $r(T_G) = r([T_G])$, the condition (c) holds. ∎

Example 28.
- Let G be the underlying graph of a $U(\mathbb{Q})$-gain graph G^{φ}. If the two cycles of G^{φ} are of Type 2, then we have $r(G^{\varphi}) = 12$, $r(G) = 8$, $c(G) = 2$, and $r(G^{\varphi}) = r(G) + 2c(G)$. Hence, G^{φ} is upper-optimal.
- Let G_1 be the underlying graph of a $U(\mathbb{Q})$-gain graph G_1^{φ}. If the two cycles of G_1^{φ} are of Type 1, then we have $r(G_1^{\varphi}) = 12$, $r(G_1) = 16$, $c(G_1) = 2$, and $r(G_1^{\varphi}) = r(G_1) - 2c(G_1)$. Hence, G_1^{φ} is lower-optimal.

Remark 29. By Lemma 7 and the example after this lemma, in future, we will consider the following problems:

1. The relationship between the row right rank of a quaternion unit gain graph and the rank of its underlying graph;
2. The relationship between the row left rank and the row right rank of a quaternion unit gain graph.

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