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# LINEAR MAPS THAT PRESERVE PARTS OF THE SPECTRUM ON PAIRS OF SIMILAR MATRICES*

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**Abstract.** In this paper, we characterize linear bijective maps $\varphi$ on the space of all $n \times n$ matrices over an algebraically closed field $\mathbb{F}$ having the property that the spectrum of $\varphi(A)$ and $\varphi(B)$ have at least one common eigenvalue for each similar matrices $A$ and $B$. Using this result, we characterize linear bijective maps having the property that the spectrum of $\varphi(A)$ and $\varphi(B)$ have common elements for each matrices $A$ and $B$ having the same spectrum. As a corollary, we also characterize linear bijective maps preserving the equality of the spectrum.

**Key words.** Linear preserver problems, Matrix spaces, Similar matrices, Spectrum.

**AMS subject classifications.** 15A86, 47B49.

**1. Introduction and statement of the main results.** Let $\mathbb{F}$ be an algebraically closed field, let $n \geq 2$ be a natural number, and let us denote by $\mathcal{M}_n$ the algebra of all $n \times n$ matrices over $\mathbb{F}$. For $T \in \mathcal{M}_n$, we shall denote by $\sigma(T)$ its spectrum, that is the set of all its eigenvalues without taking into account multiplicities. Also, $\operatorname{tr}(T) \in \mathbb{F}$ shall denote the trace and $T^t \in \mathcal{M}_n$ the transpose of the matrix $T \in \mathcal{M}_n$, and by $I_n \in \mathcal{M}_n$, we shall denote the $n \times n$ identity matrix. By $\mathrm{sl}_n$, we shall denote the set of all matrices in $\mathcal{M}_n$ with zero trace, that is the linear subspace of $\mathcal{M}_n$ generated by nilpotent matrices. For two matrices $A$ and $B$ in $\mathcal{M}_n$, we shall write $A \sim B$ if they are similar; that is, there exists an invertible $U \in \mathcal{M}_n$ such that $B = UAU^{-1}$.

When $\mathbb{F}$ is the complex field $\mathbb{C}$, the general form of linear maps on $\mathcal{M}_n$ preserving the similarity of matrices was obtained by Hiai in [5]. It is proved at [5, Theorem 1.1] that a linear map $\varphi : \mathcal{M}_n \to \mathcal{M}_n$ has the property that $A \sim B$ in $\mathcal{M}_n$ implies $\varphi(A) \sim \varphi(B)$ in $\mathcal{M}_n$ if and only if either there exist $c, d \in \mathbb{C}$ and an invertible $U \in \mathcal{M}_n$ such that $\varphi$ is either of the form $\varphi(T) = cUTU^{-1} + d(\operatorname{tr}(T))I_n$ for each $T \in \mathcal{M}_n$, or $\varphi(T) = cUT^tU^{-1} + d(\operatorname{tr}(T))I_n$ for each $T \in \mathcal{M}_n$, or there exists a fixed matrix $X \in \mathcal{M}_n$ such that $\varphi(T) = (\operatorname{tr}(T))X$ for each $T \in \mathcal{M}_n$. The result was further generalized by Lim in [6]: it is proved at [6, Theorem 2] that [5, Theorem 1.1] remains true with the same statement if $\mathbb{F}$ is an arbitrary infinite field such that $\mathrm{char}(\mathbb{F}) = 0$ or $\mathrm{char}(\mathbb{F})$ does not divide $n$ and with a slightly different formulation for the case when $\mathrm{char}(\mathbb{F})$ divides $n$ [6, Theorem 3]. Hiai obtained his result by relating the rank of each matrix to the dimension of the tangent space of its similarity orbit, which allowed him to use a result of Marcus and Moyls [7] on rank one preservers to obtain the general form for the map $\varphi$. Lim obtained his result by relating the study of similarity preserving linear maps to the study of bijective linear maps on $\mathrm{sl}_n$ preserving nilpotency, which allowed him to use a result of Botta, Pierce, and Watkins [3] giving the general form of such preserving maps.

The main purpose of the present paper is to obtain a corresponding result where instead of supposing

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that $\varphi(A) \sim \varphi(B)$ if $A \sim B$, we merely suppose that $\varphi(A)$ and $\varphi(B)$ have at least one common eigenvalue. When $\varphi$ preserves similarity, Hiai proved at [5, Lemma 1.2] that either $\ker\varphi \subseteq \mathbb{C}I_n$ or $\mathrm{sl}_n \subseteq \ker\varphi$. Therefore, either $\ker\varphi$ is quite small or quite big. In our case, we cannot hope to obtain a nice characterization for the map $\varphi$ with no bijectivity assumption on it: for example, any linear map $\varphi$ on $\mathcal{M}_n$ such that its range contains only singular matrices has the property that $\sigma(\varphi(A)) \cap \sigma(\varphi(B)) \neq \emptyset$ for any matrices $A$ and $B$.

THEOREM 1.1. *Let $\varphi : \mathcal{M}_n \to \mathcal{M}_n$ be a bijective linear map such that*

$$A \sim B \Longrightarrow \sigma(\varphi(A)) \cap \sigma(\varphi(B)) \neq \emptyset \qquad (A, B \in \mathcal{M}_n). \tag{1.1}$$

*Then, there exist $c \in \mathbb{F}\backslash\{0\}$ and $d \in \mathbb{F}$ with $c + dn \neq 0$ and an invertible matrix $U \in \mathcal{M}_n$ such that either*

$$\varphi(T) = cUTU^{-1} + d(tr(T))I_n \qquad (T \in \mathcal{M}_n), \tag{1.2}$$

*or*

$$\varphi(T) = cUT^tU^{-1} + d(tr(T))I_n \qquad (T \in \mathcal{M}_n). \tag{1.3}$$

*Conversely, any map $\varphi$ of the form (1.2) or (1.3) is linear and bijective on $\mathcal{M}_n$ and satisfies (1.1), provided that $c, d$, and $U$ satisfy the above conditions.*

For a nonzero scalar $c$ and an invertible matrix $U$, if the scalar $d$ satisfies the fact that $c + dn = 0$, then the map $\varphi$ given by either (1.2) or (1.3) is linear on $\mathcal{M}_n$ and satisfies (1.1). Also, the kernel of $\varphi$ is exactly $\mathbb{F}I_n$. Thus, $\varphi$ satisfies (1.1) and is not bijective on $\mathcal{M}_{n,}$ and its image contains also nonsingular matrices: for example, any invertible matrix with zero trace belongs to the image of the map $\varphi$.

For linear bijective maps $\varphi$ having the property that $\varphi(A)$ and $\varphi(B)$ have always at least one common eigenvalue for every matrices $A$ and $B$ with the same spectrum, we have the following characterization.

THEOREM 1.2. *Let $\varphi : \mathcal{M}_n \to \mathcal{M}_n$ be a bijective linear map such that*

$$\sigma(A) = \sigma(B) \Longrightarrow \sigma(\varphi(A)) \cap \sigma(\varphi(B)) \neq \emptyset \qquad (A, B \in \mathcal{M}_n). \tag{1.4}$$

*i) If $n = 2$, there exist $c, d \in \mathbb{F}$ with $c \neq 0$ and $c + 2d \neq 0$ and an invertible matrix $U \in \mathcal{M}_2$ such that $\varphi$ is either of the form (1.2) or (1.3).*
*ii) If $n = 3$, there exist a nonzero $c \in \mathbb{F}$, a scalar $d \in \{0, -c, c\}$ with $c + 3d \neq 0$ and an invertible $U \in \mathcal{M}_3$ such that $\varphi$ is either of the form (1.2) or (1.3).*
*iii) If $n = 4$ and char$(\mathbb{F}) \in \{2, 3\}$, there exist a nonzero $c \in \mathbb{F}$, a scalar $d \in \{0, c\}$ and an invertible $U \in \mathcal{M}_4$ such that $\varphi$ is either of the form (1.2) or (1.3).*
*iv) If either $n = 4$ and char$(\mathbb{F}) \notin \{2, 3\}$, or $n \geq 5$, then there exist $c \in \mathbb{F}\backslash\{0\}$ and $U \in \mathcal{M}_n$ invertible such that either*

$$\varphi(T) = cUTU^{-1} \qquad (T \in \mathcal{M}_n), \tag{1.5}$$

*or*

$$\varphi(T) = cUT^tU^{-1} \qquad (T \in \mathcal{M}_n). \tag{1.6}$$

*Conversely, any map $\varphi$ of the above forms, with the corresponding properties for $n$, $\mathbb{F}$, $c$, $d$, and $U$, is linear and bijective and satisfies (1.4).*

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As another corollary of Theorem 1.1, we obtain a characterization of linear bijective maps on $\mathcal{M}_n$ preserving the equality of the spectrum.

THEOREM 1.3. *Let $\varphi : \mathcal{M}_n \to \mathcal{M}_n$ be a bijective linear map such that*

$$\sigma(A) = \sigma(B) \Longrightarrow \sigma(\varphi(A)) = \sigma(\varphi(B)) \qquad (A, B \in \mathcal{M}_n). \tag{1.7}$$

*i) If $n = 2$, there exist $c, d \in \mathbb{F}$ with $c \neq 0$ and $c + 2d \neq 0$ and an invertible matrix $U \in \mathcal{M}_2$ such that $\varphi$ is either of the form (1.2) or (1.3).*
*ii) If $n \geq 3$, then there exist a nonzero $c \in \mathbb{F}$ and an invertible $U \in \mathcal{M}_n$ such that $\varphi$ is either of the form (1.5) or (1.6).*

*Conversely, any map $\varphi$ of the above forms, with the corresponding properties for $n$, $c$, $d$, and $U$, is linear and bijective and satisfies (1.7).*

In the case of linear maps $\varphi$ preserving pairs of matrices having at least one common element in their spectrum, if the image of $\varphi$ in $\mathcal{M}_n$ contains at least one invertible matrix, the bijectivity of $\varphi$ is automatic. (See also [2, Theorem 1].)

THEOREM 1.4. *Let $\varphi : \mathcal{M}_n \to \mathcal{M}_n$ be a linear map having the property that*

$$\sigma(A) \cap \sigma(B) \neq \emptyset \Longrightarrow \sigma(\varphi(A)) \cap \sigma(\varphi(B)) \neq \emptyset \qquad (A, B \in \mathcal{M}_n). \tag{1.8}$$

*Suppose also that there exists $A_0 \in \mathcal{M}_n$ such that $\varphi(A_0) \in \mathcal{M}_n$ is invertible.*
*i) If $n = 2$, there exist $c \neq 0$ and $d \in \{0, -c\}$ in $\mathbb{F}$ and $U \in \mathcal{M}_2$ invertible such that $\varphi$ is either of the form (1.2) or (1.3).*
*ii) If $n \geq 3$, then there exist a nonzero $c \in \mathbb{F}$ and an invertible $U \in \mathcal{M}_n$ such that $\varphi$ is either of the form (1.5) or (1.6).*

*Any map $\varphi$ of the above forms, with the corresponding properties for $n$, $c$, $d$, and $U$, is linear and bijective on $\mathcal{M}_n$ and satisfies (1.8).*

**2. Proofs.** Let us start by obtaining the characterization of linear bijective maps which preserve at least one element in the spectrum of images of similar matrices.

*Proof of Theorem 1.1.* Let $X \in \mathcal{M}_n$ be an invertible matrix, and define the linear map $\varphi_X : \mathcal{M}_n \to \mathcal{M}_n$ by putting

$$\varphi_X(T) = \varphi\left(X\varphi^{-1}(T)X^{-1}\right) \qquad (T \in \mathcal{M}_n).$$

Since $X\varphi^{-1}(T)X^{-1} \sim \varphi^{-1}(T)$, then (1.1) gives $\sigma(\varphi_X(T)) \cap \sigma(T) \neq \emptyset$ for each $T \in \mathcal{M}_n$. As a result of Akbari and Aryapoor (see [1, Theorem 2]), the map $\varphi_X$ preserves the whole spectrum of matrices in $\mathcal{M}_n$, and there exists an invertible matrix $V_X \in \mathcal{M}_n$ such that either $\varphi_X(T) = V_X T V_X^{-1}$ for each $T \in \mathcal{M}_n$, or $\varphi_X(T) = V_X T^t V_X^{-1}$ for each $T \in \mathcal{M}_n$. Thus, either $\varphi\left(XWX^{-1}\right) = V_X \varphi(W) V_X^{-1}$ for each $W \in \mathcal{M}_n$, or $\varphi\left(XWX^{-1}\right) = V_X(\varphi(W))^t V_X^{-1}$ for each $W \in \mathcal{M}_n$. In particular, for each pair of similar matrices $A, B \in \mathcal{M}_n$ we have that either $\varphi(A)$ is similar to $\varphi(B)$ or $\varphi(A)$ is similar to $\varphi(B)^t$. By a general result (see, for example, [8, Lemma 2.1]), any matrix in $\mathcal{M}_n$ is similar to its transpose. Therefore, $A \sim B$ in $\mathcal{M}_n$ implies $\varphi(A) \sim \varphi(B)$ in $\mathcal{M}_n$. Since $\varphi$ is bijective on $\mathcal{M}_n$, then [6, Theorem 2 and Theorem 3] imply the existence of scalars $c$ and $d$ and an invertible matrix $U \in \mathcal{M}_n$ such that $\varphi$ is either of the form (1.2) or (1.3). The bijectivity of $\varphi$ implies that $c \neq 0$. Also, since $\varphi(I_n) = (c + dn)I_n$, we also have that $c + dn \neq 0$.

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Conversely, given any $c, d \in \mathbb{F}$ with $c \neq 0$ and $c + dn \neq 0$ and an invertible $U \in \mathcal{M}_n$, for $\varphi$ on $\mathcal{M}_n$ given by either (1.2) or (1.3) we have that $\varphi$ is linear and bijective, and $A \sim B$ in $\mathcal{M}_n$ implies $\varphi(A) \sim \varphi(B)$ in $\mathcal{M}_n$. In particular, $A \sim B$ in $\mathcal{M}_n$ implies $\sigma(\varphi(A)) = \sigma(\varphi(B))$. Since the field $\mathbb{F}$ is algebraically closed, the spectrum cannot be empty, and therefore, $\sigma(\varphi(A)) \cap \sigma(\varphi(B)) \neq \emptyset$. Thus, (1.1) holds. $\square$

Since $\sigma(A) = \sigma(B)$ for every similar matrices $A$ and $B$, then (1.4) gives more information on the map $\varphi$ than (1.1). Thus, in this case we can deduce further properties on the scalar $d$.

*Proof of Theorem* 1.2. Since $A \sim B$ implies $\sigma(A) = \sigma(B)$, then (1.4) true implies that (1.1) is true. By Theorem 1.1, there exist scalars $c$ and $d$ with $c \neq 0$ and $c + dn \neq 0$ and an invertible matrix $U \in \mathcal{M}_n$ such that $\varphi$ is either of the form (1.2) or (1.3).

- Suppose that $n = 2$. Then, $\sigma(A) = \sigma(B)$ implies that $\operatorname{tr}(A) = \operatorname{tr}(B)$. This implies that if $\varphi$ is of the form (1.2) or (1.3), with $c \neq 0$, $c + 2d \neq 0$ and $U$ invertible, then $\varphi$ is a bijective linear map satisfying $\sigma(\varphi(A)) = \sigma(\varphi(B))$ for every such $A$ and $B$. In particular, $\sigma(\varphi(A)) \cap \sigma(\varphi(B)) \neq \emptyset$ for every $A$ and $B$ with $\sigma(A) = \sigma(B)$.

- Suppose that $n = 3$. If $d = 0$, then $\varphi$ is either of the form (1.5) or (1.6). This again implies that $\varphi$ is a bijective linear map such that $\sigma(\varphi(A)) = \sigma(\varphi(B))$ for every $A$ and $B$ having the same spectrum. Suppose now that $d$ is a nonzero scalar. Let $A \in \mathcal{M}_3$ be a diagonal matrix having two times 0 and one time 1 on the main diagonal, and let $B \in \mathcal{M}_3$ be a diagonal matrix having two times 1 and one time 0 on the main diagonal. Then, $\sigma(A) = \sigma(B) = \{0, 1\}$ and $\operatorname{tr}(A) = 1$, while $\operatorname{tr}(B) = 2$. By (1.4), we have that

$$\{d, c + d\} \cap \{2d, c + 2d\} \neq \emptyset.$$

Since $d \neq 0$, then $d \neq 2d$ and $c + d \neq c + 2d$, and therefore either $c + d = 2d$ or $c + 2d = d$. Thus, either $c = d$ or $c = -d$.

Let us prove now that if $d = c$, for $\varphi : \mathcal{M}_3 \to \mathcal{M}_3$ given by

$$\varphi(T) = c(UTU^{-1} + (\operatorname{tr}(T))I_3) \qquad (T \in \mathcal{M}_3),$$

with $4 \neq 0$ and $U \in \mathcal{M}_3$ invertible, we have that $\varphi$ is a bijective linear map on $\mathcal{M}_3$ satisfying (1.4). (The same reasoning works also in the case when $\varphi(T) = c(UT^tU^{-1} + (\operatorname{tr}(T))I_3)$ for each $T \in \mathcal{M}_3$.) So let $A, B \in \mathcal{M}_3$ such that $\sigma(A) = \sigma(B)$. If the common set has only one element, then $\operatorname{tr}(A) = \operatorname{tr}(B)$, and thus, $\sigma(\varphi(A)) = \sigma(\varphi(B))$. The same is true if the common set has exactly 3 elements. Suppose now that $\sigma(A) = \sigma(B) = \{\alpha, \beta\}$ for some $\alpha \neq \beta$ in $\mathbb{F}$. If $\operatorname{tr}(A) = \operatorname{tr}(B)$, then again $\sigma(\varphi(A)) = \sigma(\varphi(B))$. If $\operatorname{tr}(A) \neq \operatorname{tr}(B)$, then for example $\operatorname{tr}(A) = 2\alpha + \beta$ and $\operatorname{tr}(B) = \alpha + 2\beta$. This gives $\sigma(\varphi(A)) = \{(3\alpha + \beta)c, 2(\alpha + \beta)c\}$ and $\sigma(\varphi(B)) = \{2(\alpha + \beta)c, (\alpha + 3\beta)c\}$, and therefore, $\sigma(\varphi(A)) \cap \sigma(\varphi(B)) \neq \emptyset$.

Analogously, we obtain for $d = -c$ that if $2 \neq 0$, then $T \mapsto c(UTU^{-1} - (\operatorname{tr}(T))I_3)$ and $T \mapsto c(UT^tU^{-1} - (\operatorname{tr}(T))I_3)$ are both linear bijective maps satisfying (1.4).

- Suppose that $n \geq 4$. If $d = 0$, then again $\varphi$ of the form (1.5) or (1.6) is a bijective linear map such that $\sigma(\varphi(A)) = \sigma(\varphi(B))$ for every $A$ and $B$ having the same spectrum. Suppose for the remaining that $d \neq 0$. As above, there exist matrices in $\mathcal{M}_n$ having the spectrum equal to $\{0, 1\}$ and the trace any of the elements from $\{1, ..., n-1\} \subseteq \mathbb{F}$. Since (1.4) holds, then given any $j$ and $k$ in $\{1, ..., n-1\}$ we have that

$$\{dj, c + dj\} \cap \{dk, c + dk\} \neq \emptyset.$$

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For $j=1$ and $k=2$, since $d\neq 0$, then $dj\neq dk$ and $c+dj\neq c+dk$, and therefore either $c+dj=dk$ or $dj=c+dk$. Thus, either $d=c$ or $d=-c$. For $j=1$ and $k=3$, we have that $\{d,c+d\}\cap\{3d,c+3d\}\neq\emptyset$. If $d=c$, this means that $\{c,2c\}\cap\{3c,4c\}\neq\emptyset$, and since $c\neq 0$, this means that either $2=0$, or $3=0$ in $\mathbb{F}$. If $d=-c$, this means that $\{-c,0\}\cap\{-3c,-2c\}\neq\emptyset$, and we arrive to the same conclusion. Thus $d\in\{-c,c\}$ and $\operatorname{char}(\mathbb{F})\in\{2,3\}$.

Suppose now that $n=4$, $\operatorname{char}(\mathbb{F})=2$, and $d=c$. Then, $c+4d\neq 0$, and let us prove that maps $\varphi$ of form (1.2) or (1.3) satisfy (1.4). If $\sigma(A)=\sigma(B)$ and the common set has either one element or four elements, then $\operatorname{tr}(A)=\operatorname{tr}(B)$, and therefore, $\sigma(\varphi(A))=\sigma(\varphi(B))$. Suppose that $\sigma(A)=\sigma(B)=\{\alpha,\beta\}$, for some $\alpha\neq\beta$ in $\mathbb{F}$. Then, $\operatorname{tr}(A),\operatorname{tr}(B)\in\{\alpha+\beta,0\}$, and therefore, $\sigma(\varphi(A))=\sigma(\varphi(B))$. Suppose now that $\sigma(A)=\sigma(B)=\{\alpha,\beta,\gamma\}$, for some pairwise distinct $\alpha,\beta$, and $\gamma$ in $\mathbb{F}$. Then, $\operatorname{tr}(A)$ and $\operatorname{tr}(B)$ both belong to $\{\alpha+\beta,\alpha+\gamma,\beta+\gamma\}$, and therefore, $\sigma(\varphi(A))$ and $\sigma(\varphi(B))$ belong to $\{c\{\beta,\alpha,\alpha+\beta+\gamma\}$, $c\{\gamma,\alpha+\beta+\gamma,\alpha\},\ c\{\alpha+\beta+\gamma,\gamma,\beta\}\}$. Thus, $\sigma(\varphi(A))\cap\sigma(\varphi(B))\neq\emptyset$.

Suppose now that $n=4$ and $\operatorname{char}(\mathbb{F})=3$. Since $c+4d\neq 0$, then $d$ cannot be $-c$. Then, $d=c$, in which case the condition $c+4d\neq 0$ clearly holds. Let us prove that maps $\varphi$ of form (1.2) or (1.3) satisfy (1.4). Again, if $\sigma(A)=\sigma(B)$ and the common set has either one element or four elements, then $\operatorname{tr}(A)=\operatorname{tr}(B)$, and therefore, $\sigma(\varphi(A))=\sigma(\varphi(B))$. If $\sigma(A)=\sigma(B)=\{\alpha,\beta\}$ for some $\alpha\neq\beta$ in $\mathbb{F}$, then $\operatorname{tr}(A)$ and $\operatorname{tr}(B)$ belong to the set $\{\alpha,\beta,2(\alpha+\beta)\}$, and therefore, $\sigma(\varphi(A))$ and $\sigma(\varphi(B))$ belong to $\{c\{2\alpha,\alpha+\beta\},\ c\{\alpha+\beta,2\beta\},\ c\{2\beta,2\alpha\}\}$. Thus, $\sigma(\varphi(A))\cap\sigma(\varphi(B))\neq\emptyset$. Suppose now that $\sigma(A)=\sigma(B)=\{\alpha,\beta,\gamma\}$, for some pairwise distinct $\alpha,\beta$, and $\gamma$ in $\mathbb{F}$. Then, $\operatorname{tr}(A)$ and $\operatorname{tr}(B)$ both belong to $\{\alpha+\beta+2\gamma,\alpha+2\beta+\gamma,2\alpha+\beta+\gamma\}$. Thus, $\sigma(\varphi(A))$ and $\sigma(\varphi(B))$ belong to $\{c\{2\alpha+\beta+2\gamma,\alpha+2\beta+2\gamma,\alpha+\beta\},\ c\{2\alpha+2\beta+\gamma,\alpha+\gamma,\alpha+2\beta+2\gamma\},\ c\{\beta+\gamma,2\alpha+2\beta+\gamma,2\alpha+\beta+2\gamma\}\}$. Again, in all possible cases, we have that $\sigma(\varphi(A))\cap\sigma(\varphi(B))\neq\emptyset$.

If $n=5$, $\operatorname{char}(\mathbb{F})=2$ and $d=c$, then $c+nd=0$. Then maps $\varphi$ of the form (1.2) or (1.3) are not bijective. Let us show now that if $n\geq 6$ and $\operatorname{char}(\mathbb{F})=2$, then no map of the form (1.2) or (1.3) with $c$ nonzero and $d=c$ can satisfy (1.4). Let $\mu_1\in\mathbb{F}\backslash\{0\}$, and pick inductively $\mu_j\in\mathbb{F}$ for $j=2,..,n-2$ such that $\mu_j\notin\{\sum_{s=1}^{j-1}\varepsilon_s\mu_s:\varepsilon_s\in\{0,1\},\ s=\overline{1,j-1}\}$ for $j=\overline{2,n-2}$. Consider the matrix $A=\operatorname{diag}(\mu_1,...,\mu_{n-2},\mu_{n-3},\mu_{n-4})$ and put $B=\operatorname{diag}(\mu_1,...,\mu_{n-2},\mu_{n-2},\mu_{n-5})$. Then, $\sigma(A)=\sigma(B)=\{\mu_1,...,\mu_{n-2}\}$ and $t:=\operatorname{tr}(A)-\operatorname{tr}(B)$ equals $\mu_{n-2}+\mu_{n-3}+\mu_{n-4}+\mu_{n-5}$. That $\sigma(\varphi(A))\cap\sigma(\varphi(B))\neq\emptyset$ means that $(c\{\sigma(A)+\operatorname{tr}(A)\})\cap(c\{\sigma(B)+\operatorname{tr}(B)\})\neq\emptyset$, and this implies the existence of $i,j\in\{1,...,n-2\}$ such that $t=\mu_i-\mu_j$. We deduce the existence of $\varepsilon_1,...,\varepsilon_{n-2}\in\{0,1\}$, not all zero, such that $\sum_{s=1}^{n-2}\varepsilon_s\mu_s=0$, and this contradicts our construction of the $\mu_j$'s.

To finish the proof, let us show now that if $n\geq 5$ and $\operatorname{char}(\mathbb{F})=3$, again there is no map of the form (1.2) or (1.3) with $c$ nonzero and $d\in\{-c,+c\}$ satisfying (1.4). Let $\mu_1\in\mathbb{F}\backslash\{0\}$ and consider $\mu_j\in\mathbb{F}$ for $j=2,..,n-2$ such that $\mu_j\notin\{\sum_{s=1}^{j-1}\varepsilon_s\mu_s:\varepsilon_s\in\{0,1,2\},\ s=\overline{1,j-1}\}$ for $j=\overline{2,n-2}$. Consider the matrices $A=\operatorname{diag}(\mu_1,...\mu_{n-2},\mu_{n-3},\mu_{n-4})$ and $B=\operatorname{diag}(\mu_1,...,\mu_{n-2},\mu_{n-2},\mu_{n-2})$. Then, $\sigma(A)=\sigma(B)=\{\mu_1,...,\mu_{n-2}\}$ and $t:=\operatorname{tr}(A)-\operatorname{tr}(B)$ equals $\mu_{n-2}+\mu_{n-3}+\mu_{n-4}$. Since $\sigma(\varphi(A))\cap\sigma(\varphi(B))\neq\emptyset$, for both cases $d=c$ and $d=-c$ we deduce the existence of $i,j\in\{1,...,n-2\}$ such that $t=\mu_i-\mu_j$. Thus, there exist $\varepsilon_1,...,\varepsilon_{n-2}\in\{0,1,2\}$, not all zero, such that $\sum_{s=1}^{n-2}\varepsilon_s\mu_s=0$, and we arrive again at a contradiction.$\square$

In the case of maps preserving the equality of the spectrum, there are less possible forms for the preserving map $\varphi$.

*Proof of Theorem* 1.3. Since $A\sim B$ implies $\sigma(A)=\sigma(B)$ and the spectrum is always non-empty, then (1.7) true implies that (1.1) is also true. Thus, we can use once again Theorem 1.1 to deduce the existence

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of two scalars $c$ and $d$ with $c \neq 0$ and $c + nd \neq 0$ and an invertible matrix $U \in \mathcal{M}_n$ such that $\varphi$ is either of the form (1.2) or (1.3).

- If $n = 2$, then $\sigma(A) = \sigma(B)$ implies that $\operatorname{tr}(A) = \operatorname{tr}(B)$. Thus, for $\varphi$ of the form (1.2) or (1.3), with $c \neq 0$ and $c + 2d \neq 0$ in $\mathbb{F}$ and $U$ invertible, we have that $\varphi$ is a bijective linear map satisfying $\sigma(\varphi(A)) = \sigma(\varphi(B))$ for every such $A$ and $B$. That is, (1.7) holds.

- Let now $n \geq 3$, and suppose that $d \neq 0$. There are matrices $A$ and $B$ such that $\sigma(A) = \sigma(B) = \{0, 1\}$ and $\operatorname{tr}(A) = 1$ while $\operatorname{tr}(B) = 2$. That $\sigma(\varphi(A)) = \sigma(\varphi(B))$ gives $\{d, d + c\} = \{2d, 2d + c\}$. Since $d \neq 2d$ and $d + c \neq 2d + c$, then $d + c = 2d$ and $d = 2d + c$. This gives $d = c$ and $d = -c$, and therefore, $2c = 0$. Thus, $d = c$ and $\operatorname{char}(\mathbb{F}) = 2$. If $n = 3$, since $c + 3d \neq 0$ we arrive to a contradiction. If $n \geq 4$, consider a scalar $\lambda \notin \{0, 1\}$ and let $A = \operatorname{diag}(0, 1, \lambda, \lambda, 0, ..., 0)$ and $B = \operatorname{diag}(0, 1, \lambda, 1, 0, ..., 0)$ such that $\sigma(A) = \sigma(B) = \{0, 1, \lambda\}$ and $\operatorname{tr}(A) = 1$ while $\operatorname{tr}(B) = \lambda$. We then have $\sigma(\varphi(A)) = c(\{0, 1, \lambda\} + 1)$ and $\sigma(\varphi(B)) = c(\{0, 1, \lambda\} + \lambda)$, and therefore, $\{1, 0, \lambda + 1\} = \{\lambda, \lambda + 1, 0\}$. Since $\lambda \notin \{0, 1\}$, we arrive again at a contradiction.

To conclude the proof in this case too, let us remark that if $c$ is a nonzero scalar and $U$ an invertible matrix, then $\varphi$ of the form (1.5) or (1.6) has the property that $\sigma(\varphi(T)) = c\sigma(T)$ for each matrix $T$, and therefore (1.7) holds. ∎

Since any map $\varphi$ satisfying (1.8) clearly satisfies (1.1), the main part of the proof of Theorem 1.4 will be to obtain the bijectivity of $\varphi$.

*Proof of Theorem* 1.4. Let $T_0 \in \mathcal{M}_n$ such that $\varphi(T_0) = 0 \in \mathcal{M}_n$. Given an arbitrary $T \in \mathcal{M}_n$, we have that $\lambda \mapsto \det(\varphi(A_0) + \lambda\varphi(T))$ is a nonzero polynomial of degree at most $n$ and therefore has at most $n$ distinct roots in $\mathbb{F}$. Thus, there exists a finite subset $K_T \subseteq \mathbb{F}$ such that $\varphi(A_0) + \lambda\varphi(T) \in \mathcal{M}_n$ is invertible for each $\lambda \in \mathbb{F}\backslash K_T$. Thus, $\sigma(\varphi(A_0 + \mu T_0 + \lambda T)) \cap \sigma(\varphi(0)) = \emptyset$ for each $\mu \in \mathbb{F}$ and $\lambda \in \mathbb{F}\backslash K_T$, and then, (1.8) implies that $\sigma(A_0 + \mu T_0 + \lambda T) \cap \sigma(0) = \emptyset$ for each $\mu \in \mathbb{F}$ and $\lambda \in \mathbb{F}\backslash K_T$. Thus, $\det(A_0 + \mu T_0 + \lambda T) \neq 0$ for each $\mu \in \mathbb{F}$ and $\lambda \in \mathbb{F}\backslash K_T$. Fixing $\lambda \in \mathbb{F}\backslash K_T$, we have that the polynomial $\mu \mapsto \det(A_0 + \mu T_0 + \lambda T)$ has no roots in $\mathbb{F}$. This means that it is a constant nonzero polynomial with respect to $\mu$, and therefore, for any $\lambda \in \mathbb{F}\backslash K_T$ we have that

$$\det(A_0 + \mu T_0 + \lambda T) = \det(A_0 + \lambda T) \qquad (\mu \in \mathbb{F}),$$

with $\det(A_0 + \lambda T) \neq 0$. In particular,

$$\det(A_0 + \lambda(T_0 + T)) = \det(A_0 + \lambda T) \qquad (\lambda \in \mathbb{F}\backslash K_T).$$

Therefore, there exists an infinite subset $S_T \subseteq \mathbb{F}$ such that $\det(\mu A_0 + (T_0 + T)) = \det(\mu A_0 + T)$ for each $\mu \in S_T$. Since $\mu \mapsto \det(\mu A_0 + (T_0 + T))$ and $\mu \mapsto \det(\mu A_0 + T)$ are both polynomials with respect to $\mu$, we deduce that $\det(\mu A_0 + (T_0 + T)) = \det(\mu A_0 + T)$ for each $\mu \in \mathbb{F}$. In particular, $\det(T_0 + T) = \det T$. Since this holds for every $T \in \mathcal{M}_n$, by [4, Lemma 2.1] we deduce that $T_0 = 0$, as needed.

By Theorem 1.1, there exist scalars $c$ and $d$ with $c \neq 0$ and $c + nd \neq 0$ and $U \in \mathcal{M}_n$ invertible such that $\varphi$ is either of the form (1.2) or (1.3). If $d = 0$, then $\varphi$ is of the form (1.5) or (1.6). Suppose now that $d \neq 0$. For $A \in \mathcal{M}_n$ with $\sigma(A) = \{0, 1\}$ and $\operatorname{tr}(A) = 1$ and $B = 0 \in \mathcal{M}_n$, that $\sigma(A) \cap \sigma(B) \neq \emptyset$ implies $\{c\{0, 1\} + d\} \cap \{0\} \neq \emptyset$. That is, $d = -c$. This gives $\sigma(\varphi(T)) = c\{\sigma(T) - \operatorname{tr}(T)\}$ for each $T \in \mathcal{M}_n$. If $n \geq 3$, then let $\alpha, \beta \in \mathbb{F}$ such that $\alpha, \beta$, and $\alpha + \beta$ are all nonzero. For $A = \operatorname{diag}(\alpha, \beta, 0, ..., 0) \in \mathcal{M}_n$, since $\sigma(A) \cap \sigma(0) \neq \emptyset$ then $\sigma(\varphi(A))$ contains $0 \in \mathbb{F}$. That is, $0 \in \{-\beta, -\alpha, -(\alpha + \beta)\}$, and we arrive to a contradiction.

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To finish the proof, let us show that for $n = 2$, if $c \neq 0$ in $\mathbb{F}$ and $U \in \mathcal{M}_2$ is invertible, then for $\varphi$ of the form $T \mapsto c(UTU^{-1} - (\mathrm{tr}\,(T))I_2)$ and $T \mapsto c(UT^tU^{-1} - (\mathrm{tr}\,(T))I_2)$ we have that (1.8) holds. Indeed, let $A, B \in \mathcal{M}_2$ such that $\sigma(A) \cap \sigma(B) \neq \emptyset$. Say, $\sigma\,(A) = \{\alpha, \beta\}$ and $\sigma\,(B) = \{\alpha, \gamma\}$ for some scalars $\alpha, \beta$, and $\gamma$. Then, $\sigma\,(\varphi\,(A)) = c\{-\beta, -\alpha\}$ and $\sigma\,(\varphi\,(B)) = c\{-\gamma, -\alpha\}$, and therefore, $\sigma(\varphi\,(A)) \cap \sigma(\varphi\,(B)) \neq \emptyset$. One can also easily check that the maps $\varphi$ of the above two forms are also bijective. □

REFERENCES


[1] S. Akbari and M. Aryapoor. On linear transformations preserving at least one eigenvalue. *Proc. Amer. Math. Soc.,* 132:1621–1625, 2004.

[2] P. Botta. Linear maps that preserve singular and nonsingular matrices. *Linear Algebra Appl.,* 20:45–49, 1978.

[3] P. Botta, S. Pierce, and W. Watkins. Linear transformations that preserve the nilpotent matrices. *Pacific J. Math.,* 104: 39–46, 1983.

[4] G. Dolinar and P. Šemrl. Determinant preserving maps on matrix algebras. *Linear Algebra Appl.,* 348:189–192, 2002.

[5] F. Hiai. Similarity preserving linear maps on matrices. *Linear Algebra Appl.,* 97:127–139, 1987.

[6] M.H. Lim. A note on similarity preserving linear maps on matrices. *Linear Algebra Appl.,* 190:229–233, 1993.

[7] M. Marcus and B.N. Moyls. Transformations on tensor product spaces. *Pacific J. Math.,* 9:1215–1221, 1959.

[8] O. Taussky and H. Zassenhaus. On the similarity transformation between a matrix and its transpose. *Pacific J. Math.,* 9:893–896, 1959.