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THE INVERSE HORN PROBLEM*

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Abstract. Alfred Horn's conjecture on eigenvalues of sums of Hermitian matrices was proved more than 20 years ago. In this note, the problem is raised of, given an n-tuple γ in the solution polytope, constructing Hermitian matrices with the required spectra such that their sum has eigenvalues γ.

Key words. Eigenvalues, Hermitian matrices, Sums of matrices.

AMS subject classifications. 05E10, 15A18, 15A29.

1. Introduction. A classical problem in matrix theory is the following: given three n-tuples of real numbers $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, and $\gamma = (\gamma_1, \ldots, \gamma_n)$, ordered decreasingly, when is γ the spectrum of $A + B$, where A and B are Hermitian with spectra α and β, respectively? For fixed α and β, denote by $E(\alpha, \beta)$ the set of possible γ. Trivially, the set $E(\alpha, \beta)$ is compact and connected (as it is the image of the $n \times n$ unitary group under the continuous mapping taking U to the spectrum of $\mathrm{diag}(\alpha)+U\mathrm{diag}(\beta)U^*$), and it is contained in the hyperplane defined by the trace condition, which we abbreviate to $\Sigma\gamma = \Sigma\alpha + \Sigma\beta$.

The problem had a long history in the 20th century, starting with H. Weyl [19]. The theme that emerged gradually was that $E(\alpha, \beta)$ should be described by a family of inequalities of the type

$$\gamma_{k_1} + \cdots + \gamma_{k_r} \le \alpha_{i_1} + \cdots + \alpha_{i_r} + \beta_{j_1} + \cdots + \beta_{j_r}\,,$$

where $r \in \{1, \ldots, n\}$ and $i_1 < \cdots < i_r$, $j_1 < \cdots < j_r$, $k_1 < \cdots < k_r$.

In short,

$$\Sigma\,\gamma_K \;\le\; \Sigma\,\alpha_I \;+\; \Sigma\,\beta_J\,,$$

where $I = (i_1, \ldots, i_r), J = (j_1, \ldots, j_r), K = (k_1, \ldots, k_r)$. The question is to identify the right triples (I, J, K).

The big moment was a 1962 paper by A. Horn [11]. He presented a remarkable conjecture on the set $E(\alpha, \beta)$, which, in sightly changed form, reads as follows.

For a sequence of indices $I = (i_1, \ldots, i_r)$, with $1 \le i_1 < \cdots < i_r \le n$, write

$$\rho(I) = (i_r - r, \ldots, i_2 - 2, i_1 - 1)\,.$$

Then, Horn's conjecture is as follows: $\gamma \in E(\alpha, \beta)$ if and only if

$$\begin{cases} \Sigma\,\gamma \;=\; \Sigma\,\alpha \;+\; \Sigma\,\beta\,, \\ \Sigma\,\gamma_K \;\le\; \Sigma\,\alpha_I \;+\; \Sigma\,\beta_J \;\text{ whenever} \\ \qquad \rho(K) \;\in\; E(\rho(I), \rho(J)) \quad (\text{for all } r,\; 1 \le r < n). \end{cases}$$

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The conjecture is now a theorem. So $E(\alpha,\beta)$ is described recursively and is a convex polytope. We refer the reader to two excellent surveys on this story, by Fulton [9] and Bhatia [2].

2. An open problem. A natural question concerns the inverse problem, that is, construction of solutions: given α, β, and $\gamma \in E(\alpha,\beta)$, find Hermitian A with spectrum α and B with spectrum β such that $A+B$ has spectrum γ. In the remainder of this note, we make some comments on this open problem and give a few references.

Given the drop in dimension, it is to be expected that, for each γ, there may be many solutions. Since the proof of Horn's conjecture, several authors have studied a question related to the inverse problem: finding the probability distribution of γ, for given α and β, using the fact that, as mentioned before, γ is a continuous function on the unitary group, where we can take the Haar measure.

References on this, some very recent, are [5], [6], [7], [8], [17], [20], and [21].

3. Two particular cases. Only one paper – that we know of – addresses the actual construction problem. In [4], the authors use semidefinite programming and give an algorithm that works for $n = 3$. (The case $n = 2$ is trivial.)

In a different spirit, we can find an exact solution in a very particular case. Without loss of generality, we may assume the α's, the β's, and the γ's are ≥ 0. We proceed to address the case $\beta_2 = \cdots = \beta_n = 0$. So the second matrix to be constructed has rank 1. (By translation, this covers the case where β has $n-1$ coordinates equal.)

In this situation, the Horn inequalities reduce to

$$\gamma_1 + \cdots + \gamma_n = \alpha_1 + \cdots + \alpha_n + \beta_1\,,$$

$$\gamma_1 \geq \alpha_1 \geq \gamma_2 \geq \alpha_2 \geq \cdots \geq \gamma_n \geq \alpha_n\,.$$

Put $D_\alpha = \mathrm{diag}(\alpha)$. We are going to find a (real) column x such that $D_\alpha + xx^T$ has spectrum γ. Clearly, $\|x\|^2 = \beta_1$.

Put $C = \left[\sqrt{D_\alpha}\ \ x\right]$. We have

$$D_\alpha + xx^T \;=\; \left[\sqrt{D_\alpha}\ \ x\right]\begin{bmatrix}\sqrt{D_\alpha}\\ x^T\end{bmatrix} \;=\; CC^T.$$

Therefore, we are looking for a column x such that C has singular values $\sqrt{\gamma_1},\ldots,\sqrt{\gamma_n}$. We may assume the α's to be all distinct (if $\alpha_i = \alpha_{i+1}$ just take $x_i = 0$). Denote by x^2 the column $[x_1^2\ x_2^2\ \cdots\ x_n^2]^T$.

Denote by $\sigma_k(\alpha)$ the k-th elementary symmetric function of $\alpha_1,\ldots,\alpha_n$ and write $\sigma(\alpha)$ for the column $[\sigma_1(\alpha)\ \sigma_2(\alpha)\ \cdots\ \sigma_n(\alpha)]^T$.

In [16], it was proved that

$$J(\alpha)\cdot x^2 = \sigma(\gamma) - \sigma(\alpha)\,,$$

where J is the Jacobian matrix of the elementary symmetric functions.

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We have $\det J(\alpha) = \prod_{i<j}(\alpha_i - \alpha_j)$. Since $\alpha_1 > \cdots > \alpha_n$, $J(\alpha)$ is nonsingular. In [16], the inverse of $J(\alpha)$ is found and related to the Vandermonde matrix with parameters α.

So the column x satisfies $x^2 = J(\alpha)^{-1} \cdot [\sigma(\gamma) - \sigma(\alpha)]$, and there is a nice simple expression for x.

An example: take the triples $\alpha = (6, 4, 2)$, $\beta = (3, 0, 0)$, $\gamma = (7, 5, 3)$. We get $x = \begin{bmatrix} 0.6124 \\ 0.8660 \\ 1.3693 \end{bmatrix}$, so $A = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0.3750 & 0.5303 & 0.8385 \\ 0.5303 & 0.7500 & 1.1859 \\ 0.8385 & 1.1859 & 1.8750 \end{bmatrix}$ solve the problem.

4. A possible general approach. A somewhat speculative idea to approach the inverse Horn problem uses the well-known Littlewood-Richardson rule, an object appearing in many settings, starting from representation theory.

Suppose the n-tuples α and β are integral and nonnegative. Denote by $LR(\alpha, \beta)$ the set of all n-tuples γ that can be obtained from α and β according to the Littlewood-Richardson rule (see *e.g.* [10, App. A.1]).

In [18], it was proved that $E(\alpha, \beta) \cap \mathbb{Z}^n \supseteq LR(\alpha, \beta)$. Shortly afterward, Knutson and Tao [15] proved a combinatorial theorem that, together with earlier work by Klyachko [13] (which also implies the result in [18], obtained independently), shows that there is in fact equality: $E(\alpha, \beta) \cap \mathbb{Z}^n = LR(\alpha, \beta)$, and this leads to a proof of Horn's conjecture. Without going into details (see [9]), the equality gives an idea of why the conjecture should be true, because nonempty intersections of Schubert varieties (which produce inequalities) are governed by the LR rule: using the above notations, we have

$$\rho(K) \in LR(\rho(I), \rho(J)) \Longrightarrow \Sigma\,\gamma_K \le \Sigma\,\alpha_I + \Sigma\,\beta_J .$$

Second, the equality suggests a connection to another problem: invariant factors of a product of two matrices over a principal ideal domain. (For more general rings see [3].) Let R be a p.i.d. Consider three n-tuples of nonzero elements of R

$$a = (a_n, \ldots, a_2, a_1)\ , \quad b = (b_n, \ldots, b_2, b_1)\ , \quad c = (c_n, \ldots, c_2, c_1),$$

ordered so that

$$a_n \mid \cdots \mid a_2 \mid a_1\ , \qquad b_n \mid \cdots \mid b_2 \mid b_1\ , \qquad c_n \mid \cdots \mid c_2 \mid c_1\,.$$

The invariant factor problem is: when is c the n-tuple of invariant factors of AB, where A and B are R-matrices with invariant factors a and b, respectively?

The problem was solved, not in the same exact language, by Klein in 1968 [12]. First, localize the situation: fix a prime $p \in R$ and work over the local ring R_p, that is, work with powers of p:

$$a_i \to p^{\alpha_i},\ b_i \to p^{\beta_i},\ c_i \to p^{\gamma_i}\ ,$$

where $\alpha_1 \ge \cdots \ge \alpha_n$, $\beta_1 \ge \cdots \ge \beta_n$, $\gamma_1 \ge \cdots \ge \gamma_n$ are nonnegative integers. Denote by $IF(\alpha, \beta)$ the set of possible γ in the invariant factor product problem. Then, Klein's result states that

$$IF(\alpha, \beta) = LR(\alpha, \beta)\,.$$

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So $E(\alpha,\beta)\cap\mathbb{Z}^n = IF(\alpha,\beta)$. But in [1], there is a constructive version of Klein's theorem. Our speculative question is then the following: is there a way of "transporting" this construction from the invariant factor setting to Hermitian matrices for the case of integral α, β, γ? In this context, it is relevant to note that the equality $E(\alpha,\beta)\cap\mathbb{Z}^n = LR(\alpha,\beta)$ reflects a deep result, the Kirwan-Ness theorem, relating symplectic geometry to geometric invariant theory. (See [14].)

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