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ON THE SUM OF THE K LARGEST ABSOLUTE VALUES OF LAPLACIAN EIGENVALUES OF DIGRAPHS*

XIUWEN YANG†, XIAOGANG LIU†, AND LIGONG WANG†

Abstract. Let $L(G)$ be the Laplacian matrix of a digraph G and $S_k(G)$ be the sum of the k largest absolute values of Laplacian eigenvalues of G. Let C_n^+ be a digraph with $n+1$ vertices obtained from the directed cycle C_n by attaching a pendant arc whose tail is on C_n. A digraph is $\mathbb{C}_n^+$-free if it contains no C_ℓ^+ as a subdigraph for any $2 \leq \ell \leq n-1$. In this paper, we present lower bounds of $S_n(G)$ of digraphs of order n. We provide the exact values of $S_k(G)$ of directed cycles and $\mathbb{C}_n^+$-free unicyclic digraphs. Moreover, we obtain upper bounds of $S_k(G)$ of $\mathbb{C}_n^+$-free digraphs which have vertex-disjoint directed cycles.

Key words. Laplacian eigenvalues, Directed cycles, $\mathbb{C}_n^+$-free unicyclic digraphs.

AMS subject classifications. 05C20, 05C50.

1. Introduction. Let $G = (\mathcal{V}(G), \mathcal{A}(G))$ be a digraph with vertex set $\mathcal{V}(G) = \{v_1, v_2, \ldots, v_n\}$ and arc set $\mathcal{A}(G)$. We denote an arc from a vertex v_i to a vertex v_j by (v_i, v_j), and we call v_i the tail and v_j the head of the arc (v_i, v_j). Let $d_i^- = d_G^-(v_i)$ be the indegree of vertex v_i of G, and the indegree is the number of arcs whose head is vertex v_i. Let $d_i^+ = d_G^+(v_i)$ be the outdegree of vertex v_i of G, and the outdegree is the number of arcs whose tail is vertex v_i. A directed walk π of length ℓ from vertex u to vertex v is a sequence of vertices π: $u = v_0, v_1, \ldots, v_\ell = v$, where (v_{k-1}, v_k) is an arc of G for any $1 \leq k \leq \ell$. If $u = v$, then π is called a directed closed walk. We use $(c_2(v_1), c_2(v_2), \ldots, c_2(v_n))$ to denote the directed closed walk sequence of length 2, and let $c_2 = \sum_{i=1}^n c_2(v_i)$ denote the number of all directed closed walks of length 2 in G. If all vertices of the directed walk π of length n are distinct, then we call it a directed path and denote it by P_{n+1}; a directed closed walk of length n in which all except the end vertices are distinct is called a directed cycle and is denoted by C_n. Let C_n^+ be a digraph with $n+1$ vertices obtained from C_n by attaching a pendant arc whose tail is on C_n. A digraph is connected if its underlying graph is connected. If a connected digraph contains only a unique directed cycle, then it is a unicyclic digraph. Throughout this paper, we consider the digraphs without loops and multiple arcs.

Let $A(G) = (a_{ij})_{n\times n}$ be the adjacency matrix of a digraph G whose (i,j)-entry equals to 1 if (v_i, v_j) is an arc of G, and equals to 0 otherwise. Let $D^+(G) = diag(d_1^+, d_2^+, \ldots, d_n^+)$ be the diagonal outdegree matrix of G. Let $L(G) = D^+(G) - A(G)$ be the Laplacian matrix of G. The characteristic polynomial $\phi_{L(G)}(x)$ of Laplacian matrix is $\phi_{L(G)}(x) = |xI_n - L(G)|$. The roots of $\phi_{L(G)}(x)$ are the eigenvalues of $L(G)$, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $S_k(G)$ be the sum of the k largest absolute values of Laplacian eigenvalues of G.

It is very interesting in theoretical chemistry to study $S_k(G)$ of graphs or digraphs. Relevant literature can be found in [12, 13]. Up to now, there are a few results of $S_k(G)$ of digraphs. However, there are many scholars who study the sum of the k largest adjacency eigenvalues (or Laplacian eigenvalues) of graphs. In

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[17], Mohar proved that the sum of the k largest adjacency eigenvalues of a graph is at most $\frac{1}{2}(\sqrt{k}+1)n$. In [6], Das et al. presented a sharp upper bound on the sum of the k largest adjacency eigenvalues ν_i of a graph in terms of the order n and negative inertia θ: $\sum_{i=1}^{k}\nu_i \leq \frac{n}{2(\theta+1)}(\theta+\sqrt{\theta(k\theta+k-1)})$. In [1], Brouwer conjectured that the sum of the k largest Laplacian eigenvalues of a graph H is $S_k(H) \leq e(H)+\binom{k+1}{2}$, for all $k=1,2,\ldots,n$ and $e(H)$ is the number of edges of H. Progress on Brouwer's conjecture can be found in [4, 8, 11, 14, 23] and other works. Numerous research results have been obtained on the Laplacian matrix of graphs and digraphs, as detailed in [2, 10, 22, 24, 25, 28]. In this paper, we will consider the sum of the k largest absolute values of Laplacian eigenvalues of digraphs.

Let G be a digraph and let $\mathcal{H}$ be a set of digraphs. We say that G is $\mathcal{H}$-free if it does not contain any digraph in $\mathcal{H}$ as a subdigraph. In particular, if $\mathcal{H}=\{H\}$, then we also say that G is H-free. A digraph is $\mathbb{C}_n^+$-free if it contains no C_ℓ^+ as a subdigraph for any $2\leq \ell \leq n-1$. Then, $\mathbb{C}_n^+$-free digraph does not contain any digraph in $\{C_2^+, C_3^+, \ldots, C_{n-1}^+\}$ as a subdigraph.

In 2010, Nikiforov [20] proposed a spectral version of extremal graph theory problem (spectral Turán problem), that is, what is the maximal spectral radius of an H-free graph of order n? Recently, much attention has been paid to the spectral Turán problem. Nikiforov [18] solved the spectral Turán problem of K_{r+1}-free graph. Nosal [21] established the spectral version of Mantel's theorem: each C_3-free graph H satisfies $\lambda_1(H) \leq \sqrt{|E(H)|}$. More results about spectral Turán problem of graphs can be found in [5, 15, 16, 19, 26, 27]. But for digraphs, there are only a few results about spectral Turán problem. The maximal spectral radius of C_2-free digraphs has been proved in [3, 7]: if n is odd, the extremal tournaments are precisely the ones that are regular, that is have indegree and outdegree $\frac{n-1}{2}$ at each vertex; if n is even, the extremal tournaments are those which are isomorphic to the Brualdi-Li tournament. The aforementioned spectral Turán problems primarily focus on the adjacency matrix. However, our objective is to consider the upper bound of $S_k(G)$ of Laplacian eigenvalues of $\mathbb{C}_n^+$-free digraphs.

We first define a vertex-deletion operation of digraphs used for later.

Vertex-deletion operation: Let G be a digraph with n vertices and e arcs. If G has a vertex u with $d_G^-(u)=0$, then we delete the vertex u.

If G has a vertex u with $d_G^-(u)=0$, then the elements of the u-th column of $xI_n - L(G)$ are all 0s, except for the diagonal element. By the properties of the determinant, we can deduce that $|xI_n - L(G)| = (x-d_G^+(u))|xI_{n-1}-L(G')|$, where G' is a new digraph by deleting the vertex u from G. Consequently, $d_G^+(u)$ is a Laplacian eigenvalue of G, and the other Laplacian eigenvalues of G remain the same as those of G'.

The order of this paper is as follows. In Section 2, we present tight lower bounds of $S_n(G)$ of digraphs. In Section 3, we provide the exact values of $S_k(C_n)$ of directed cycles C_n. In Section 4, we provide the exact values of $S_k(G)$ of $\mathbb{C}_n^+$-free unicyclic digraphs. We also obtain upper bounds of $S_k(G)$ of $\mathbb{C}_n^+$-free digraphs which have vertex-disjoint directed cycles.

2. Lower bounds of the sum of the absolute values of Laplacian eigenvalues of digraphs. In this section, we consider the lower bounds of the sum of the absolute values of Laplacian eigenvalues of digraphs. Before proceeding, we give definitions of acyclic digraphs and directed trees. A digraph is acyclic if it has no directed cycles. A directed tree is a connected digraph with n vertices and $n-1$ arcs whose underlying graph does not contain any cycles. An out-star $\overrightarrow{K}_{1,n-1}$ with n vertices is a directed tree which has one vertex with outdegree $n-1$ and other vertices with outdegree 0, as depicted in Fig. 1. A bidirected star $\overleftrightarrow{K}_{1,n-1}$ with n vertices is a digraph which has one vertex with outdegree (and indegree) $n-1$ and other

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vertices with outdegree (and indegree) 1, also shown in Fig. 1.

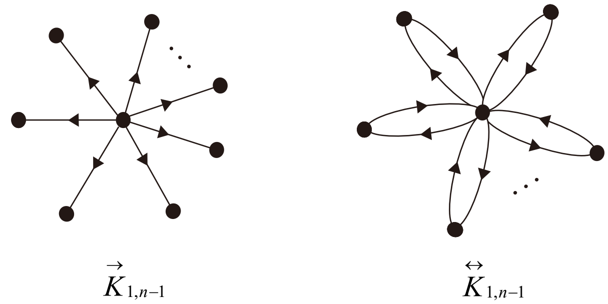

FIG. 1. *The out-star $\overrightarrow{K}_{1,n-1}$ and the bidirected star $\overleftrightarrow{K}_{1,n-1}$.*

LEMMA 2.1. *Let G be a connected digraph with n vertices and e arcs. Then, the rank of $L(G)$ is 1 if and only if G is an out-star $\overrightarrow{K}_{1,n-1}$ or G is a directed cycle C_2.*

Proof. Let $L(G) = D^+(G) - A(G)$ be the Laplacian matrix of G. Then, the Laplacian matrix is an $n \times n$ matrix $L(G) = (\ell_{ij})$, where

$$\ell_{ij} = \begin{cases} d_i^+, & \text{if } i = j, \\ -1, & \text{if } (v_i, v_j) \in \mathcal{A}(G), \\ 0, & \text{otherwise.} \end{cases}$$

We use $r(L(G))$ to denote the rank of $L(G)$. Obviously, $r(L(\overrightarrow{K}_{1,n-1})) = 1$ and $r(L(C_2)) = 1$. Next, we prove the necessity.

Let $\vec{\ell_i} = (\ell_{i1}, \ell_{i2}, \dots, \ell_{in})$, where $i = 1, 2, \dots, n$. Then, $r(L(G)) = 1$ means that any two vectors $\vec{\ell_u}, \vec{\ell_v}$ are linearly dependent. That is, there are some numbers a, b which are not all zero, such that $a\vec{\ell_u} + b\vec{\ell_v} = 0$. So we have

$$\ell_{uu} = d_u^+ = \begin{cases} \frac{b}{a}, & \text{if } \ell_{vu} = -1, \\ 0, & \text{if } \ell_{vu} = 0, \end{cases} \quad \textit{and} \quad \ell_{vv} = d_v^+ = \begin{cases} \frac{a}{b}, & \text{if } \ell_{uv} = -1, \\ 0, & \text{if } \ell_{uv} = 0. \end{cases}$$

For $L(G)$, we know $d_i^+ = -\sum_{j=1, j\neq i}^{n} \ell_{ij}$ and $\ell_{ij} = 0$ or -1 when $i \neq j$. Since $\vec{\ell_u}, \vec{\ell_v}$ are arbitrary, if $r(L(G)) = 1$, only two cases hold.

Case 1. There is one vector $\vec{\ell_u} \neq \vec{0}$, and other vectors are null vector. Without loss of generality, let $\vec{\ell_1} \neq \vec{0}$. Since G is connected, we get $\ell_{11} = d_1^+ = n - 1$, $\ell_{1j} = -1$ and $\ell_{ii} = d_i^+ = 0$, where $i, j = 2, 3, \dots, n$. That is, G is an out-star $\overrightarrow{K}_{1,n-1}$.

Case 2. There are two vectors $\vec{\ell_u} = -\vec{\ell_v} \neq \vec{0}$, and other vectors are null vector. When $n \geq 3$, it is impossible since G is connected. When $n = 2$, G is a directed cycle C_2.

Hence, the rank of $L(G)$ is 1 if and only if G is an out-star $\overrightarrow{K}_{1,n-1}$ or G is a directed cycle C_2. □

THEOREM 2.2. *Let G be a connected digraph with n vertices and e arcs. Let $d_1^+, d_2^+, \dots, d_n^+$ be the outdegrees of vertices of G and c_2 be the number of all directed closed walks of length 2. Then,*

$$S_n(G) \geq \sqrt{\sum_{i=1}^{n} (d_i^+)^2 + c_2}.$$

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The equality holds if and only if G is an out-star $\overrightarrow{K}_{1,n-1}$ or G is a directed cycle C_2.

Proof. If $G = \overrightarrow{K}_{1,n-1}$ or $G = C_2$, then $S_n(G) = \sqrt{\sum_{i=1}^{n}(d_i^+)^2 + c_2}$ holds. Next, we prove the necessity.

In [22], the Laplacian energy $LE(G)$ of G is defined as $LE(G) = \sum_{i=1}^{n}\lambda_i^2$ by using second spectral moment. In [24], the formula of the Laplacian energy $LE(G)$ of G is

$$LE(G) = \sum_{i=1}^{n}(d_i^+)^2 + c_2.$$

Then, we get

$$S_n(G) = \sum_{i=1}^{n}|\lambda_i| \geq \sqrt{\sum_{i=1}^{n}\lambda_i^2} = \sqrt{\sum_{i=1}^{n}(d_i^+)^2 + c_2}.$$

The equality holds if and only if $\sum_{i=1}^{n}|\lambda_i| = \sqrt{\sum_{i=1}^{n}\lambda_i^2}$.

Let $\lambda_i = \mathrm{Re}(\lambda_i) + \mathrm{Im}(\lambda_i)\iota$, where $\mathrm{Re}(\lambda_i)$ is the real part of eigenvalue λ_i and $\mathrm{Im}(\lambda_i)$ is the imaginary part of eigenvalue λ_i, $\iota = \sqrt{-1}$. Then, $|\lambda_i| = \sqrt{(\mathrm{Re}(\lambda_i))^2 + (\mathrm{Im}(\lambda_i))^2}$ and $\lambda_i^2 = (\mathrm{Re}(\lambda_i))^2 - (\mathrm{Im}(\lambda_i))^2 + 2\mathrm{Re}(\lambda_i)\mathrm{Im}(\lambda_i)\iota$. If $\sum_{i=1}^{n}|\lambda_i| = \sqrt{\sum_{i=1}^{n}\lambda_i^2}$, we have

$$\begin{aligned}&\sum_{i=1}^{n}\left((\mathrm{Re}(\lambda_i))^2 + (\mathrm{Im}(\lambda_i))^2\right) + 2\sum_{i<j}\sqrt{(\mathrm{Re}(\lambda_i))^2 + (\mathrm{Im}(\lambda_i))^2}\sqrt{(\mathrm{Re}(\lambda_j))^2 + (\mathrm{Im}(\lambda_j))^2}\\ &= \sum_{i=1}^{n}\left((\mathrm{Re}(\lambda_i))^2 - (\mathrm{Im}(\lambda_i))^2 + 2\mathrm{Re}(\lambda_i)\mathrm{Im}(\lambda_i)\iota\right).\end{aligned}$$

That is,

$$\sum_{i=1}^{n}(\mathrm{Im}(\lambda_i))^2 + \sum_{i<j}\sqrt{(\mathrm{Re}(\lambda_i))^2 + (\mathrm{Im}(\lambda_i))^2}\sqrt{(\mathrm{Re}(\lambda_j))^2 + (\mathrm{Im}(\lambda_j))^2} = 0.$$

Since $(\mathrm{Im}(\lambda_i))^2 \geq 0$ and $\sqrt{(\mathrm{Re}(\lambda_i))^2 + (\mathrm{Im}(\lambda_i))^2} \geq 0$ for all $i = 1, 2, \ldots, n$, we get $\mathrm{Im}(\lambda_i) = 0$. Then,

$$\sum_{i<j}|\mathrm{Re}(\lambda_i)||\mathrm{Re}(\lambda_j)| = 0.$$

Without loss of generality, we obtain $\mathrm{Re}(\lambda_1) \neq 0$ and $\mathrm{Re}(\lambda_i) = 0$ for $i = 2, \ldots, n$ since G is connected. That is, $\lambda_1 = |\lambda_1| = \mathrm{Re}(\lambda_1) = \sum_{i=1}^{n} d_i^+ = e$ and $\lambda_i = 0$ for $i = 2, \ldots, n$. Hence, $L(G)$ only has one nonzero eigenvalue. Then, the rank of $L(G)$ is 1. From Lemma 2.1, the rank of $L(G)$ is 1 if and only if G is an out-star $\overrightarrow{K}_{1,n-1}$ or G is a directed cycle C_2.

This completes the proof. □

THEOREM 2.3. *Let G be an acyclic digraph with n vertices and e arcs. Then, $\lambda_i = d_i^+$ for all $i = 1, 2, \ldots, n$, where λ_i is the eigenvalue of $L(G)$ and d_i^+ is the outdegree of vertex of G.*

Proof. Any acyclic digraph admits a topological ordering, that is, an ordering of its vertices $\{v_1, v_2, \ldots, v_n\}$ such that for every arc (v_i, v_j), we have $i < j$. So the Laplacian matrix of acyclic digraph is an upper triangular matrix. Hence, $|xI_n - L(G)| = \Pi_{i=1}^{n}(x - d_i^+)$. Then, $\lambda_i = d_i^+$ for all $i = 1, 2, \ldots, n$. □

THEOREM 2.4. *Let G be a digraph with n vertices and e arcs. Then,*

$$S_n(G) \geq e.$$

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Proof. Note that

$$S_n(G) = \sum_{i=1}^{n} |\lambda_i| \geq \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} d_i^+ = e. \qquad \square$$

We present a class of digraphs that achieves the lower bound $S_n(G) = e$.

EXAMPLE 2.5. *Let G be a digraph containing m bidirected stars $\overleftrightarrow{K}_{1,n_i-1}$ which the outdegrees of vertices of $\overleftrightarrow{K}_{1,n_i-1}$ in G are $\{n_i-1,1,1,\ldots,1\}$, and every component of $G - \bigcup_{i=1}^{m} \overleftrightarrow{K}_{1,n_i-1}$ is acyclic. Then, $S_n(G) = e$.*

Proof. (i) If $m = 0$, then G is an acyclic digraph. From Theorem 2.3, we find the Laplacian eigenvalues of an acyclic digraph are equal to the outdegrees of vertices of acyclic digraph, so $S_n(G) = \sum_{i=1}^{n} d_i^+ = e$.

(ii) If $m = 1$ and $G = \overleftrightarrow{K}_{1,n-1}$, by a careful calculation, the Laplacian eigenvalues of $\overleftrightarrow{K}_{1,n-1}$ are n of multiplicity 1, 1 of multiplicity $n-2$ and 0 of multiplicity 1. So $S_n(G) = 2n-2 = e$.

(iii) If $m \geq 1$ and $G \neq \overleftrightarrow{K}_{1,n-1}$, then G is a digraph containing m bidirected stars $\overleftrightarrow{K}_{1,n_i-1}$ which the outdegrees of vertices of $\overleftrightarrow{K}_{1,n_i-1}$ in G are $\{n_i-1,1,1,\ldots,1\}$, and every component of $G - \bigcup_{i=1}^{m} \overleftrightarrow{K}_{1,n_i-1}$ is acyclic. Obviously, G does not contain any directed cycles except C_2. And there must have a vertex u with $d_G^-(u) = 0$. By vertex-deletion operation, we use the following procedure:

$G^0 := G$;

$i := 0$;

while $\exists\ u \in \mathcal{V}(G^i)$ s.t. $d_{G^i}^-(u) = 0$ **do begin**

  $G^{i+1} := G^i - u$;

  $i := i+1$;

end.

Finally, we can get m bidirected stars with vertex-disjoint. By (i) and (ii), we have $S_n(G) = e$. $\square$

To better illustrate the aforementioned class of digraphs, we provide a visual representation of a specific digraph in Fig. 2.

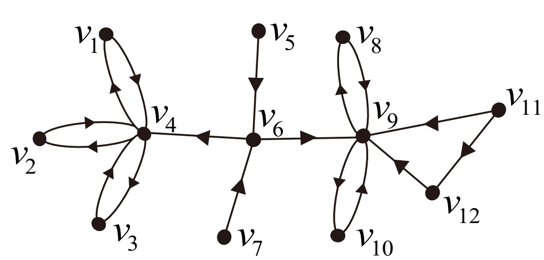

FIG. 2. *A digraph with the lower bound $S_n(G) = e$.*

LEMMA 2.6. *Let G be a digraph with n vertices and e arcs. Let $d_1^+, d_2^+, \ldots, d_n^+$ be the outdegrees of*

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vertices of G and c_2 be the number of all directed closed walks of length 2. Then,

$$\sum_{i=1}^{n} d_i^+ \geq \sqrt{\sum_{i=1}^{n}(d_i^+)^2 + c_2}.$$

Proof. Since $(\sum_{i=1}^n d_i^+)^2 = \sum_{i=1}^n (d_i^+)^2 + 2\sum_{i<j}(d_i^+ d_j^+)$ and $c_2 = 2\sum_{i<j}(a_{ij}a_{ji})$, we only need to prove $d_i^+ d_j^+ \geq a_{ij}a_{ji}$, where a_{ij} is the element of adjacency matrix of G for $i, j = 1, 2, \ldots, n$.

We know $d_i^+ = \sum_{j=1}^n a_{ij} \geq 0$. If $a_{ij} = 0$ or $a_{ji} = 0$, then $d_i^+ d_j^+ \geq 0 = a_{ij}a_{ji}$. If $a_{ij} = a_{ji} = 1$, then $d_i^+ \geq 1$ and $d_j^+ \geq 1$, so $d_i^+ d_j^+ \geq 1 = a_{ij}a_{ji}$. Hence $d_i^+ d_j^+ \geq a_{ij}a_{ji}$. ∎

Remark 2.7. From Theorems 2.2, 2.4 and Lemma 2.6,

$$S_n(G) \geq e = \sum_{i=1}^{n} d_i^+ \geq \sqrt{\sum_{i=1}^{n}(d_i^+)^2 + c_2}.$$

So the lower bound of Theorem 2.4 is stronger than the lower bound of Theorem 2.2.

3. The sum of the k largest absolute values of Laplacian eigenvalues of directed cycles. In this section, we determine the exact values of $S_k(G)$ of directed cycles C_n. Since the Laplacian matrix $L(C_n) = circ(1, -1, 0, \ldots, 0)$ is a circulant matrix, the Laplacian eigenvalues λ_i of C_n is

$$\lambda_i = \left(1 - \cos\frac{2\pi i}{n}\right) - \iota\sin\frac{2\pi i}{n},$$

where $i = 1, 2, \ldots, n$ and $\iota = \sqrt{-1}$.

Lemma 3.1. (see [9]) *For $x \in (0, \frac{\pi}{2}]$, the following inequality holds:*

$$\frac{1}{x} - 0.429x \leq \cot x \leq \frac{1}{x} - \frac{x}{3}.$$

Theorem 3.2. *Let C_n $(n \geq 2)$ be a directed cycle. Then,*

$$S_{n-1}(C_n) = S_n(C_n) \leq \frac{4n}{\pi} - \frac{\pi}{3n}.$$

Proof. Since $\lambda_i = \left(1 - \cos\frac{2\pi i}{n}\right) - \iota\sin\frac{2\pi i}{n}$, where $i = 1, 2, \ldots, n$, we have

$$|\lambda_i| = \sqrt{\left(1 - \cos\frac{2\pi i}{n}\right)^2 + \left(-\sin\frac{2\pi i}{n}\right)^2} = \sqrt{2\left(1 - \cos\frac{2\pi i}{n}\right)} = 2\sin\frac{\pi i}{n}.$$

From Lemma 3.1, since $\lambda_n = 0$,

$$S_{n-1}(C_n) = S_n(C_n) = \sum_{i=1}^{n} 2\sin\frac{\pi i}{n} = 2\cot\frac{\pi}{2n} \leq \frac{4n}{\pi} - \frac{\pi}{3n}.$$ ∎

Theorem 3.3. *Let C_n $(n \geq 2)$ be a directed cycle. Let*

$$f(x) = \frac{\sin\frac{\pi x}{n} + \sin\frac{\pi}{n} - \sin\frac{\pi(x+1)}{n}}{1 - \cos\frac{\pi}{n}}.$$

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(i) *If n is even, then*

$$S_k(C_n) = \begin{cases} 2 + 2f\left(\frac{n-2}{2}\right) - f\left(\frac{n-k-2}{2}\right) - f\left(\frac{n-k}{2}\right), & \text{if } k \text{ is even,} \\ 2 + 2f\left(\frac{n-2}{2}\right) - 2f\left(\frac{n-k-1}{2}\right), & \text{if } k \text{ is odd.} \end{cases}$$

(ii) *If n is odd, then*

$$S_k(C_n) = \begin{cases} 2f\left(\frac{n-1}{2}\right) - 2f\left(\frac{n-k-1}{2}\right), & \text{if } k \text{ is even,} \\ 2f\left(\frac{n-1}{2}\right) - f\left(\frac{n-k-2}{2}\right) - f\left(\frac{n-k}{2}\right), & \text{if } k \text{ is odd.} \end{cases}$$

Proof. From the proof of Theorem 3.2, we have

$$|\lambda_i| = 2\sin\frac{\pi i}{n}.$$

Since $e^{\theta\iota} = \cos\theta + \iota\sin\theta$, we have

$$\sum_{i=0}^{x} 2\sin\frac{\pi i}{n} = 2\mathrm{Im}\left(\sum_{i=0}^{x} e^{\frac{\pi i}{n}\iota}\right),$$

where $\mathrm{Im}(e^{\theta\iota})$ is the imaginary part of $e^{\theta\iota}$. By geometric progression, we get

$$\sum_{i=0}^{x} e^{\frac{\pi i}{n}\iota} = \frac{\left(e^{\frac{\pi}{n}\iota}\right)^{x+1} - 1}{e^{\frac{\pi}{n}\iota} - 1}.$$

Hence,

$$\begin{aligned} \sum_{i=0}^{x} 2\sin\frac{\pi i}{n} &= 2\mathrm{Im}\left(\sum_{i=0}^{x} e^{\frac{\pi i}{n}\iota}\right) \\ &= \frac{1}{\iota}\left(\sum_{i=0}^{x} e^{\frac{\pi i}{n}\iota} - \sum_{i=0}^{x} e^{-\frac{\pi i}{n}\iota}\right) \\ &= \frac{1}{\iota}\left(\frac{\left(e^{\frac{\pi}{n}\iota}\right)^{x+1} - 1}{e^{\frac{\pi}{n}\iota} - 1} - \frac{\left(e^{-\frac{\pi}{n}\iota}\right)^{x+1} - 1}{e^{-\frac{\pi}{n}\iota} - 1}\right) \\ &= \frac{\sin\frac{\pi x}{n} + \sin\frac{\pi}{n} - \sin\frac{\pi(x+1)}{n}}{1 - \cos\frac{\pi}{n}} \\ &= f(x). \end{aligned}$$

See Table 1, since $\sin\frac{\pi i}{n}$ is increasing on $\left(0, \frac{n}{2}\right]$ and decreasing on $\left[\frac{n}{2}, n\right]$, we consider the decreasing sequence of $|\lambda_i| = 2\sin\frac{\pi i}{n}$ if n is even or odd.

TABLE 1
The value range of $\sin\frac{\pi i}{n}$

i	$\left(0, \frac{n}{6}\right]$	$\left[\frac{n}{6}, \frac{n}{2}\right]$	$\left[\frac{n}{2}, \frac{5n}{6}\right]$	$\left[\frac{5n}{6}, n\right]$
$\sin\frac{\pi i}{n}$	$0 \to \frac{1}{2}$	$\frac{1}{2} \to 1$	$1 \to \frac{1}{2}$	$\frac{1}{2} \to 0$

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(i) If n is even, since

$$\sin\left(\frac{\pi}{n}\cdot\frac{n}{2}\right)=1,\ \sin\left(\frac{\pi}{n}\cdot n\right)=0,\ \sin\left(\frac{\pi}{n}\cdot\left(\frac{n}{2}-i\right)\right)=\sin\left(\frac{\pi}{n}\cdot\left(\frac{n}{2}+i\right)\right),$$

where $i=1,2,\ldots,\frac{n}{2}-1$, we have

$$2=|\lambda_{\frac{n}{2}}|>|\lambda_{\frac{n}{2}-1}|=|\lambda_{\frac{n}{2}+1}|>|\lambda_{\frac{n}{2}-2}|=|\lambda_{\frac{n}{2}+2}|>\cdots>|\lambda_1|=|\lambda_{n-1}|>|\lambda_n|=0.$$

Hence, we obtain the following results.

If n is even and k is even, then

$$\begin{aligned}
S_k(C_n)&=|\lambda_{\frac{n}{2}}|+\sum_{i=1}^{\frac{k}{2}}|\lambda_{\frac{n}{2}-i}|+\sum_{i=1}^{\frac{k}{2}-1}|\lambda_{\frac{n}{2}-i}|\\
&=2+\sum_{i=0}^{\frac{k}{2}-1}2\sin\frac{\pi\left(\frac{n}{2}-1-i\right)}{n}+\sum_{i=0}^{\frac{k}{2}-2}2\sin\frac{\pi\left(\frac{n}{2}-1-i\right)}{n}\\
&=2+\left(\sum_{i=0}^{\frac{n}{2}-1}2\sin\frac{\pi i}{n}-\sum_{i=0}^{\frac{n}{2}-\frac{k}{2}-1}2\sin\frac{\pi i}{n}\right)+\left(\sum_{i=0}^{\frac{n}{2}-1}2\sin\frac{\pi i}{n}-\sum_{i=0}^{\frac{n}{2}-\frac{k}{2}}2\sin\frac{\pi i}{n}\right)\\
&=2+2f\left(\frac{n}{2}-1\right)-f\left(\frac{n}{2}-\frac{k}{2}-1\right)-f\left(\frac{n}{2}-\frac{k}{2}\right).
\end{aligned}$$

If n is even and k is odd, then

$$\begin{aligned}
S_k(C_n)&=|\lambda_{\frac{n}{2}}|+2\sum_{i=1}^{\frac{k-1}{2}}|\lambda_{\frac{n}{2}-i}|\\
&=2+2\sum_{i=0}^{\frac{k-1}{2}-1}2\sin\frac{\pi\left(\frac{n}{2}-1-i\right)}{n}\\
&=2+2\left(\sum_{i=0}^{\frac{n}{2}-1}2\sin\frac{\pi i}{n}-\sum_{i=0}^{\frac{n}{2}-\frac{k-1}{2}-1}2\sin\frac{\pi i}{n}\right)\\
&=2+2f\left(\frac{n}{2}-1\right)-2f\left(\frac{n}{2}-\frac{k-1}{2}-1\right).
\end{aligned}$$

(ii) If n is odd, since

$$\sin\left(\frac{\pi}{n}\cdot\left(\frac{n-1}{2}-i\right)\right)=\sin\left(\frac{\pi}{n}\cdot\left(\frac{n+1}{2}+i\right)\right),$$

where $i=0,1,\ldots,\frac{n-1}{2}-1$, we have

$$2>|\lambda_{\frac{n-1}{2}}|=|\lambda_{\frac{n+1}{2}}|>|\lambda_{\frac{n-1}{2}-1}|=|\lambda_{\frac{n+1}{2}+1}|>\cdots>|\lambda_1|=|\lambda_{n-1}|>|\lambda_n|=0.$$

Hence, we obtain the following results.

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If n is odd and k is even, then

$$\begin{aligned}
S_k(C_n) &= 2\sum_{i=0}^{\frac{k}{2}-1} |\lambda_{\frac{n-1}{2}-i}| \\
&= 2\sum_{i=0}^{\frac{k}{2}-1} 2\sin\frac{\pi\left(\frac{n-1}{2}-i\right)}{n} \\
&= 2\left(\sum_{i=0}^{\frac{n-1}{2}} 2\sin\frac{\pi i}{n} - \sum_{i=0}^{\frac{n-1}{2}-\frac{k}{2}} 2\sin\frac{\pi i}{n}\right) \\
&= 2f\left(\frac{n-1}{2}\right) - 2f\left(\frac{n-1}{2}-\frac{k}{2}\right).
\end{aligned}$$

If n is odd and k is odd, then

$$\begin{aligned}
S_k(C_n) &= \sum_{i=0}^{\frac{k-1}{2}} |\lambda_{\frac{n-1}{2}-i}| + \sum_{i=0}^{\frac{k-1}{2}-1} |\lambda_{\frac{n-1}{2}-i}| \\
&= \sum_{i=0}^{\frac{k-1}{2}} 2\sin\frac{\pi\left(\frac{n-1}{2}-i\right)}{n} + \sum_{i=0}^{\frac{k-1}{2}-1} 2\sin\frac{\pi\left(\frac{n-1}{2}-i\right)}{n} \\
&= \left(\sum_{i=0}^{\frac{n-1}{2}} 2\sin\frac{\pi i}{n} - \sum_{i=0}^{\frac{n-1}{2}-\frac{k-1}{2}-1} 2\sin\frac{\pi i}{n}\right) + \left(\sum_{i=0}^{\frac{n-1}{2}} 2\sin\frac{\pi i}{n} - \sum_{i=0}^{\frac{n-1}{2}-\frac{k-1}{2}} 2\sin\frac{\pi i}{n}\right) \\
&= 2f\left(\frac{n-1}{2}\right) - f\left(\frac{n}{2}-\frac{k}{2}-1\right) - f\left(\frac{n}{2}-\frac{k}{2}\right).
\end{aligned}$$

The proof is completed. ◻

4. The sum of the k largest absolute values of Laplacian eigenvalues of $\mathbb{C}_n^+$-free digraphs. In this section, firstly, we determine the exact values of $S_k(G)$ of $\mathbb{C}_n^+$-free unicyclic digraphs. A $\mathbb{C}_n^+$-free digraph does not contain any digraph in $\{C_2^+, C_3^+, \ldots, C_{n-1}^+\}$ as a subdigraph, but may contain C_2, C_3, ..., or C_n. Let G be a $\mathbb{C}_n^+$-free unicyclic digraph with n vertices and e arcs having only one directed cycle C_m, that is equivalent to G being a unicyclic digraph with n vertices and e arcs, having only one directed cycle C_m in which the outdegree of each vertex of C_m in G is 1.

LEMMA 4.1. *Let G be a $\mathbb{C}_n^+$-free unicyclic digraph with n vertices and e arcs having only one directed cycle C_m. Let $\mathcal{V}(G) = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{V}(C_m) = \{v_1, v_2, \ldots, v_m\}$. Then,*

$$\lambda_i = \begin{cases} \left(1-\cos\frac{2\pi i}{m}\right) - \iota\sin\frac{2\pi i}{m}, & \text{if } i = 1, 2, \ldots, m, \\ d_i^+, & \text{if } i = m+1, m+2, \ldots, n, \end{cases}$$

where λ_i is the eigenvalue of $L(G)$ and d_i^+ is the outdegree of vertex of G.

Proof. If $n = m$, then $G = C_n$. The result is obvious.

If $n > m$, we consider the characteristic polynomial $|xI_n - L(G)|$ of Laplacian matrix $L(G)$. For the $\mathbb{C}_n^+$-free unicyclic digraph, except C_m, it is impossible to find another directed cycle. So there is at least one

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vertex with indegree 0 among the vertices $v_{m+1}, v_{m+2}, \ldots, v_n$. Without loss of generality, assume that there is a vertex v_s with $d_s^- = 0$. By vertex-deletion operation, we use the procedure from the proof of Example 2.5. Note that the while loop stops when $i = n - m$. And we can get $G^{n-m} = C_m$. Then,

$$|xI_n - L(G)| = \prod_{i=1}^{n-m} (x - d_{m+i}^+)|xI_m - L(C_m)|.$$

Hence, we get

$$\lambda_i = \begin{cases} \left(1 - \cos\frac{2\pi i}{m}\right) - \iota \sin\frac{2\pi i}{m}, & \text{if } i = 1, 2, \ldots, m, \\ d_i^+, & \text{if } i = m+1, m+2, \ldots, n. \end{cases}$$ ∎

THEOREM 4.2. *Let G be a $\mathbb{C}_n^+$-free unicyclic digraph with n vertices and e arcs having only one directed cycle C_m. Let $d_1^+ = d_2^+ = \cdots = d_m^+ = 1,\ d_{m+1}^+ \geq \cdots \geq d_n^+$ be the outdegrees of vertices of G. Let t and s be the number of vertices with $d_i^+ \geq 2$ and $d_i^+ = 0$, respectively, where $i = 1, 2, \ldots, n$. Let $p = m - 1 - 2\left\lfloor\frac{m}{6}\right\rfloor$ and*

$$F(y) = \begin{cases} 2 + 2f\left(\frac{m-2}{2}\right) - f\left(\frac{m-y-2}{2}\right) - f\left(\frac{m-y}{2}\right), & \text{if } m \text{ is even and } y \text{ is even,} \\ 2 + 2f\left(\frac{m-2}{2}\right) - 2f\left(\frac{m-y-1}{2}\right), & \text{if } m \text{ is even and } y \text{ is odd,} \\ 2f\left(\frac{m-1}{2}\right) - 2f\left(\frac{m-y-1}{2}\right), & \text{if } m \text{ is odd and } y \text{ is even,} \\ 2f\left(\frac{m-1}{2}\right) - f\left(\frac{m-y-2}{2}\right) - f\left(\frac{m-y}{2}\right), & \text{if } m \text{ is odd and } y \text{ is odd.} \end{cases}$$

where

$$f(x) = \frac{\sin\frac{\pi x}{m} + \sin\frac{\pi}{m} - \sin\frac{\pi(x+1)}{m}}{1 - \cos\frac{\pi}{m}}.$$

Then,

$$S_k(G) = \begin{cases} \sum_{i=m+1}^{m+k} d_i^+, & \text{if } k \leq t, \\ \sum_{i=m+1}^{m+t} d_i^+ + F(k-t), & \text{if } t < k \leq t+p, \\ \sum_{i=m+1}^{m+t} d_i^+ + F(p) + k - t - p, & \text{if } t+p < k \leq n-m-s+p, \\ \sum_{i=m+1}^{n} d_i^+ + F(k-n+m+s), & \text{if } n-m-s+p < k \leq n-s-1, \\ \sum_{i=m+1}^{n} d_i^+ + F(m-1), & \text{if } n-s-1 < k \leq n. \end{cases}$$

Proof. Let $d_1^+ = d_2^+ = \cdots = d_m^+ = 1,\ d_{m+1}^+ \geq \cdots \geq d_n^+$ be the outdegrees of vertices of G, from Lemma 4.1, we get

$$\lambda_i = \begin{cases} \left(1 - \cos\frac{2\pi i}{m}\right) - \iota \sin\frac{2\pi i}{m}, & \text{if } i = 1, 2, \ldots, m, \\ d_i^+, & \text{if } i = m+1, m+2, \ldots, n. \end{cases}$$

Let t and s be the number of vertices with $d_i^+ \geq 2$ and $d_i^+ = 0$, respectively, where $i = 1, 2, \ldots, n$. Then, there are $n - t - s$ vertices with $d_i^+ = 1$.

For $G - C_m$, we know $\lambda_i = d_i^+$ for $i = m+1, m+2, \ldots, n$. Then, there are t eigenvalues with $|\lambda_i| \geq 2$, $n - m - t - s$ eigenvalues with $|\lambda_i| = 1$ and s eigenvalues with $|\lambda_i| = 0$.

For C_m, we know $|\lambda_i| = 2\sin\frac{\pi i}{m}$ for $i = 1, 2, \ldots, m$. From Table 1, we get $1 < |\lambda_i| \leq 2$ when $i \in \left(\frac{m}{6}, \frac{5m}{6}\right)$, $0 < |\lambda_i| \leq 1$ when $i \in \left(0, \frac{m}{6}\right]$ and $i \in \left[\frac{5m}{6}, m-1\right]$, $|\lambda_i| = 0$ when $i = m$. So there are $2\left\lfloor\frac{m}{6}\right\rfloor$ eigenvalues with

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$0 < |\lambda_i| \leq 1$, $p = m - 1 - 2\left\lfloor \frac{m}{6} \right\rfloor$ eigenvalues with $1 < |\lambda_i| \leq 2$ and 1 eigenvalue with $|\lambda_i| = 0$. From Theorem 3.3, we also get $S_k(C_m) = F(k)$.

Hence, for $S_k(G)$, we have the following cases.

Case 1. $k \leq t$.

Since there are at least t eigenvalues with $|\lambda_i| \geq 2$, that is, $\lambda_i = d_i^+ \geq 2$ for $i = m+1, m+2, \ldots, m+t$ and $d_{m+1}^+ \geq \cdots \geq d_{m+t}^+ \geq 2$, we obtain $S_k(G) = \sum_{i=m+1}^{m+k} d_i^+$.

Case 2. $t < k \leq t + p$.

Since there are p eigenvalues with $1 < |\lambda_i| \leq 2$ for C_m, there are $t + p$ eigenvalues with $|\lambda_i| > 1$. So we obtain

$$S_k(G) = \sum_{i=m+1}^{m+t} d_i^+ + S_{k-t}(C_m) = \sum_{i=m+1}^{m+t} d_i^+ + F(k-t).$$

Case 3. $t + p < k \leq n - m - s + p$.

Since there are at least $n - m - t - s$ eigenvalues with $|\lambda_i| = 1$, there are at least $n - m - t - s + t + p = n - m - s + p$ eigenvalues with $|\lambda_i| \geq 1$. So we obtain

$$S_k(G) = \sum_{i=m+1}^{m+t} d_i^+ + S_p(C_m) + k - t - p = \sum_{i=m+1}^{m+t} d_i^+ + F(p) + k - t - p.$$

Case 4. $n - m - s + p < k \leq n - s - 1$.

Since there are $2\left\lfloor \frac{m}{6} \right\rfloor$ eigenvalues with $0 < |\lambda_i| \leq 1$ for C_m, there are $2\left\lfloor \frac{m}{6} \right\rfloor + n - m - s + p = n - s - 1$ eigenvalues with $|\lambda_i| > 0$. So we obtain

$$S_k(G) = \sum_{i=m+1}^{n} d_i^+ + S_{k-t-(n-m-t-s)}(C_m) = \sum_{i=m+1}^{n} d_i^+ + F(k - n + m + s).$$

Case 5. $n - s - 1 < k \leq n$.

Since there are $s + 1$ eigenvalues with $|\lambda_i| = 0$, we obtain

$$S_k(G) = \sum_{i=m+1}^{n} d_i^+ + S_{m-1}(C_m) = \sum_{i=m+1}^{n} d_i^+ + F(m-1).$$

The proof is completed. ∎

As an illustration of the above theorem, we give an example.

EXAMPLE 4.3. *Let G be a $\mathbb{C}_{10}^+$-free unicyclic digraph shown in Fig. 3. Then, the $\mathbb{C}_{10}^+$-free unicyclic digraph G has 10 vertices, 12 arcs, and a unique directed cycle C_6. By the proof of Lemma 4.1,*

$$|xI_{10} - L(G)| = x(x-1)(x-2)(x-3)|xI_6 - L(C_6)| = 0.$$

So the Laplacian eigenvalues of G are $\{3, 2, 2, \sqrt{3}, \sqrt{3}, 1, 1, 1, 0, 0\}$.

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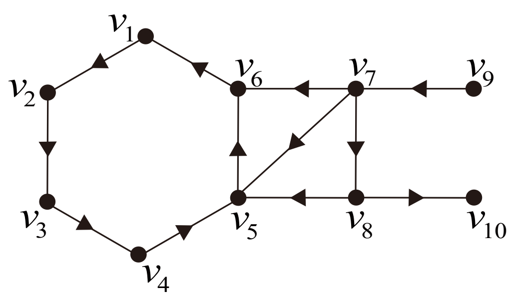

FIG. 3. *A $\mathbb{C}_{10}^{+}$-free unicyclic digraph*

From Theorem 4.2, if $m=6$, $t=2$, $s=1$ and $p=3$, then

$$S_k(G)=\begin{cases}3, & \text{if } k=1,\\ 5, & \text{if } k=2,\\ 5+F(k-2), & \text{if } 2<k\le 5,\\ 5+F(3)+k-5, & \text{if } 5<k\le 6,\\ 6+F(k-3), & \text{if } 6<k\le 8,\\ 6+F(5), & \text{if } 8<k\le 10.\end{cases}$$

Secondly, we obtain the upper bounds of $S_k(G)$ of $\mathbb{C}_n^+$-free digraphs which have vertex-disjoint directed cycles.

THEOREM 4.4. *Let G be a $\mathbb{C}_n^+$-free digraph with n vertices and e arcs having vertex-disjoint directed cycles $C_{m_1}, C_{m_2}, \ldots, C_{m_q}$, where $\sum_{i=1}^{q} m_i = m$. Let $d_1^+ = d_2^+ = \cdots = d_m^+ = 1$, $d_{m+1}^+ \ge \cdots \ge d_n^+$ be the outdegrees of vertices of G. Let t and s be the number of vertices with $d_i^+ \ge 2$ and $d_i^+ = 0$, respectively, where $i = 1, 2, \ldots, n$. Let $p = \sum_{i=1}^{q}\left(m_i - 1 - 2\left\lfloor \frac{m_i}{6}\right\rfloor\right)$. Then,*

$$S_k(G)\le\begin{cases}\sum_{i=m+1}^{m+k} d_i^+, & \text{if } k\le t,\\ \sum_{i=m+1}^{m+t} d_i^+ + 2\left(k-t\right), & \text{if } t<k\le t+p,\\ \sum_{i=m+1}^{m+t} d_i^+ + p + k - t, & \text{if } t+p<k\le n-s-q,\\ e+\left(\frac{4}{\pi}-1\right)m-\sum_{i=1}^{q}\frac{\pi}{3m_i}, & \text{if } n-s-q<k\le n.\end{cases}$$

Proof. Similar to the proof of Theorem 4.2, we get there are t eigenvalues with $|\lambda_i|\ge 2$, $n-m-t-s$ eigenvalues with $|\lambda_i|=1$ and s eigenvalues with $|\lambda_i|=0$ for $G-\bigcup_{i=1}^{q} C_{m_i}$. And there are $\sum_{i=1}^{q} 2\left\lfloor\frac{m_i}{6}\right\rfloor$ eigenvalues with $0<|\lambda_i|\le 1$, $p=\sum_{i=1}^{q}\left(m_i-1-2\left\lfloor\frac{m_i}{6}\right\rfloor\right)$ eigenvalues with $1<|\lambda_i|\le 2$ and q eigenvalues with $|\lambda_i|=0$ for $\bigcup_{i=1}^{q} C_{m_i}$. Hence, for $S_k(G)$, we have the following cases.

Case 1. $k\le t$.

Similar to the proof of Lemma 4.1, by vertex-deletion operation, we get $\lambda_i = d_i^+$ for $i=m+1, m+2, \ldots, n$. Since there are at least t eigenvalues with $|\lambda_i|\ge 2$, that is, $d_{m+1}^+\ge\cdots\ge d_{m+t}^+\ge 2$, we obtain $S_k(G)=\sum_{i=m+1}^{m+k} d_i^+$.

Case 2. $t<k\le t+p$.

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Since there are p eigenvalues with $1 < |\lambda_i| \le 2$ for $\bigcup_{i=1}^q C_{m_i}$, we obtain

$$S_k(G) \le \sum_{i=m+1}^{m+t} d_i^+ + 2(k-t).$$

Case 3. $t + p < k \le n - s - q$.

Since there are at least $n - m - t - s$ eigenvalues with $|\lambda_i| = 1$, and $\sum_{i=1}^q 2\left\lfloor \frac{m_i}{6} \right\rfloor$ eigenvalues with $0 < |\lambda_i| \le 1$ for $\bigcup_{i=1}^q C_{m_i}$, we obtain

$$S_k(G) \le \sum_{i=m+1}^{m+t} d_i^+ + 2p + k - (t+p) = \sum_{i=m+1}^{m+t} d_i^+ + p + k - t.$$

Case 4. $n - s - q < k \le n$.

Since there are $s + q$ eigenvalues with $|\lambda_i| = 0$, from Theorem 3.2, we obtain

$$S_k(G) \le \sum_{i=m+1}^{n} d_i^+ + \sum_{i=1}^{q} \left(\frac{4m_i}{\pi} - \frac{\pi}{3m_i} \right) = e + \left(\frac{4}{\pi} - 1 \right) m - \sum_{i=1}^{q} \frac{\pi}{3m_i}.$$

This completes the proof. ∎

5. Conclusion. In this paper, we obtain the lower bounds of $S_n(G)$ and consider the upper bounds of $S_k(G)$ of digraphs. But we have only found the upper bounds of $S_k(G)$ of $\mathbb{C}_n^+$-free digraphs which have vertex-disjoint directed cycles. It is challenging to study the upper bounds of $S_k(G)$ of all $\mathbb{C}_n^+$-free digraphs.

Declaration of competing interest. The authors declare that they have no conflict of interest.

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