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INEQUALITIES FOR THE MINIMUM EIGENVALUE OF M-MATRICES*

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Abstract. Let A be a nonsingular M-matrix, and $\tau(A)$ denote its minimum eigenvalue. Shivakumar et al. [*SIAM J. Matrix Anal. Appl.*, 17(2):298-312, 1996] presented some bounds of $\tau(A)$ when A is a weakly chained diagonally dominant M-matrix. The present paper establishes some new bounds of $\tau(A)$ for a general nonsingular M-matrix A. Numerical examples show that the results obtained are an improvement over some known results in certain cases.

Key words. M-matrix, Hadamard product, Minimum eigenvalue, Eigenvector.

AMS subject classifications. 15A06, 15A42, 15A48.

1. Introduction.

Let Z denote the class of all $n \times n$ real matrices all of whose off-diagonal entries are nonpositive. A matrix $A \in Z$ is called an M-matrix [1] if there exists an $n \times n$ nonnegative matrix B and some nonnegative real number λ such that $A = \lambda I_n - B$ and $\lambda \geq \rho(B)$, where $\rho(B)$ is the spectral radius of B, I_n is the identity matrix; if $\lambda > \rho(B)$, then A is called a *nonsingular M-matrix*; if $\lambda = \rho(B)$, we call A a *singular M-matrix*. If D is the diagonal matrix of A and $C = D - A$, then the spectral radius of the Jacobi iterative matrix $J_A = D^{-1}C$ of A denoted by $\rho(J_A)$ is less than 1 (see also [1]). Let $q = (q_1, q_2, \ldots, q_n)^T$ denote the eigenvector corresponding to $\rho(J_A)$.

For two real matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same size, the Hadamard product of A and B is defined as the matrix $A \circ B = (a_{ij}b_{ij})$. If A and B are two nonsingular M-matrices, then it is proved [2] that $A \circ B^{-1}$ is a nonsingular M-matrix.

If A is a nonsingular M-matrix, then there exists a positive eigenvalue of A equal to $\tau(A) = [\rho(A^{-1})]^{-1}$, where $\rho(A^{-1})$ is the spectral radius of the nonnegative matrix A^{-1}. $\tau(A)$ is called the *minimum eigenvalue* of A [3]. The Perron-Frobenius theorem [1] tells us that $\tau(A)$ is a eigenvalue of A corresponding to a nonnegative eigenvector,

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$q = (q_1, q_2, \ldots, q_n)^T \geq 0$. If, in addition, A is irreducible, then $\tau(A)$ is simple and $q > 0$.

For convenience, we shall employ the following notations throughout. Let $N = \{1, \ldots, n\}$. Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be nonsingular with $a_{ii} \neq 0$ for all $i \in N$, and $A^{-1} = (\alpha_{ij})$,

$$R_i(A) = \sum_{j=1}^{n} a_{ij}, \quad C_i(A) = \sum_{j=1}^{n} a_{ji},$$
$$\sigma_i = \frac{1}{|a_{ii}|} \sum_{j \neq i} |a_{ij}|, \quad \delta_i = \frac{1}{|a_{ii}|} \sum_{j \neq i} |a_{ji}|,$$
$$R(A) = \max_{i \in N} \sum_{j=1}^{n} a_{ij}, \quad r(A) = \min_{i \in N} \sum_{j=1}^{n} a_{ij},$$

and

$$M = \max_{i \in N} \sum_{j=1}^{n} \alpha_{ij}, \quad m = \min_{i \in N} \sum_{j=1}^{n} \alpha_{ij}.$$

We shall always assume $a_{ii} \neq 0$ for all $i \in N$. The following definitions can be found in [1, 7, 8]. Recall that A is called *diagonally dominant by rows* (*by columns*) if $\sigma_i \leq 1$ ($\delta_i \leq 1$, respectively) for all $i \in N$. If $\sigma_i < 1$ ($\delta_i < 1$), we say that A is *strictly diagonally dominant by rows* (*by columns*, respectively). A is called *weakly chained diagonally dominant* if $\sigma_i \leq 1$, $J(A) = \{i \in N : \sigma_i < 1\} \neq \phi$ and for all $i \in N \setminus J(A)$, there exist indices $i_1, i_2, \ldots, i_k$ in N with $a_{i_r, i_{r+1}} \neq 0$, $0 \leq r \leq k-1$, where $i_0 = i$ and $i_k \in J(A)$. Notice that a strictly diagonally dominant matrix is also weakly chained diagonally dominant.

Finding bounds on $\tau(A)$ is a subject of interest on its own and various refined bounds can be found in [6, 8]. Shivakumar et al. [8] obtained the following bounds when A is a weakly chained diagonally dominant M-matrix.

THEOREM 1.1. *Let $A = (a_{ij})$ be a weakly chained diagonally dominant M-matrix, and $A^{-1} = (\alpha_{ij})$. Then*

$$r(A) \leq \tau(A) \leq R(A), \quad \tau(A) \leq \min_{i \in N} a_{ii} \quad \textit{and} \quad \frac{1}{M} \leq \tau(A) \leq \frac{1}{m}.$$

In Theorem 1.1, it is possible that $r(A)$ equals zero or that $\frac{1}{M}$ is very small. Moreover, whenever A is not (weakly chained) diagonally dominant, Theorem 1.1 can not be used to estimate some bounds of $\tau(A)$. In this paper, using the method of the optimally scaled matrices, we shall establish some new bounds of $\tau(A)$ for a general

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nonsingular M-matrix A. Numerical examples show that our results are better than some known results in some cases. Further, we also exhibit some new bounds of $\tau(A)$ that only depend on the entries of matrix A.

2. Some Lemmas. In this section, we will present some lemmas, which shall be useful in the following proofs. The following Lemma 2.1 comes from [9].

LEMMA 2.1. *(i) Let $A = (a_{ij})$ be a strictly diagonally dominant matrix by rows, that is, $\sigma_i < 1$ for all $i \in N$. Then $A^{-1} = (\alpha_{ij})$ exists, and for all $i \neq j$,*

$$|\alpha_{ji}| \leq \frac{\sum_{k\neq j} |a_{jk}|}{|a_{jj}|} |\alpha_{ii}| = \sigma_j |\alpha_{ii}|.$$

(ii) Let $A = (a_{ij})$ be a strictly diagonally dominant matrix by columns, that is, $\delta_i < 1$ for all $i \in N$. Then $A^{-1} = (\alpha_{ij})$ exists, and for all $i \neq j$

$$|\alpha_{ij}| \leq \frac{\sum_{k\neq j} |a_{kj}|}{|a_{jj}|} |\alpha_{ii}| = \delta_j |\alpha_{ii}|.$$

LEMMA 2.2. *(i) Let $A = (a_{ij})$ be a strictly diagonally dominant M-matrix by rows. Then $A^{-1} = (\alpha_{ij})$ satisfies*

$$\frac{1}{a_{ii}} \leq \alpha_{ii} \leq \frac{1}{a_{ii} + \sum_{j\neq i} a_{ij}\sigma_j} \leq \frac{1}{R_i(A)}.$$

(ii) Let $A = (a_{ij})$ be a strictly diagonally dominant M-matrix by columns. Then $A^{-1} = (\alpha_{ij})$ satisfies

$$\frac{1}{a_{ii}} \leq \alpha_{ii} \leq \frac{1}{a_{ii} + \sum_{j\neq i} a_{ji}\delta_j} \leq \frac{1}{C_i(A)}.$$

Proof. We prove only (i); the proof of (ii) is similar and is omitted. Since A is a strictly diagonally dominant M-matrix, $A^{-1} \geq 0$. By $A \cdot A^{-1} = I$, for all $i \in N$,

$$1 = a_{ii}\alpha_{ii} + \sum_{j\neq i} a_{ij}\alpha_{ji},$$

which implies

$$\frac{1}{a_{ii}} \leq \alpha_{ii}.$$

By Lemma 2.1, one has

$$\alpha_{ji} \leq \frac{\sum_{k\neq j} |a_{jk}|}{a_{jj}} \alpha_{ii} = \sigma_j \alpha_{ii}.$$

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Notice that $0 \le \sigma_j < 1$, one obtains

$$1 \ge a_{ii}\alpha_{ii} + \sum_{j\neq i} a_{ij}\sigma_j\alpha_{ii} = \left(a_{ii} + \sum_{j\neq i} a_{ij}\sigma_j\right)\alpha_{ii} \ge \left(a_{ii} + \sum_{j\neq i} a_{ij}\right)\alpha_{ii},$$

which implies

$$\alpha_{ii} \le \frac{1}{a_{ii} + \sum_{j\neq i} a_{ij}\sigma_j} \le \frac{1}{R_i(A)}.$$

This completes the proof. ∎

The following Lemmas 2.3 and 2.4 can be found in [4].

Lemma 2.3. *Let $M = (m_{ij})$ be a nonsingular M-matrix, and $N = (n_{ij})$ be a nonnegative matrix of same size. Then $\rho(M^{-1}N)$ satisfies:*

(i) If M is strictly diagonally dominant by rows, then

$$\min_{i\in N}\left\{\frac{\sum_{j=1}^n n_{ij}}{m_{ii} + \sum_{j\neq i} m_{ij}}\right\} \le \rho(M^{-1}N) \le \max_{i\in N}\left\{\frac{\sum_{j=1}^n n_{ij}}{m_{ii} + \sum_{j\neq i} m_{ij}}\right\}.$$

(ii) If M is strictly diagonally dominant by columns, then

$$\min_{i\in N}\left\{\frac{\sum_{j=1}^n n_{ji}}{m_{ii} + \sum_{j\neq i} m_{ji}}\right\} \le \rho(M^{-1}N) \le \max_{i\in N}\left\{\frac{\sum_{j=1}^n n_{ji}}{m_{ii} + \sum_{j\neq i} m_{ji}}\right\}.$$

Lemma 2.4. *Let $A = (a_{ij})$ be an irreducible matrix and $a_{ii} \neq 0$ for all $i \in N$. Then there exists a positive diagonal matrix $Q = \mathrm{diag}(q_1, q_2, \ldots, q_n)$ and $\tilde{A} = (\tilde{a_{ij}}) = AQ$ such that*

$$\sum_{j\neq i} \frac{|\tilde{a_{ij}}|}{|\tilde{a_{ii}}|} = \rho(J_A),$$

where $\rho(J_A)$ is the spectral radius of the Jacobi iterative matrix J_A of A and $q = (q_1, q_2, \ldots, q_n)^T$ is the eigenvector corresponding to $\rho(J_A)$. $\tilde{A}$ is called the optimally scaled matrix of A.

To proof Lemma 2.6, we also need the following Lemma 2.5 (see [1]).

Lemma 2.5. *Let A be a nonnegative matrix. If $Az \le kz$ for a positive nonzero vector z, then $\rho(A) \le k$.*

Lemma 2.6. *Suppose that $A = (a_{ij})$ is a nonnegative matrix and $B = (b_{ij})$ is a nonsingular M-matrix. Let $B^{-1} = (\beta_{ij})$. Then*

$$\rho(A \circ B^{-1}) \le \max_{i\in N}\{a_{ii}\beta_{ii} + \beta_{ii}\rho(J_B)(\rho(A) - a_{ii})\} \le \max_{i\in N} \beta_{ii}\rho(A). \tag{2.1}$$

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Proof. It is quite evident that (2.1) holds with equality for $n = 1$. In the following, we shall assume that $n \geq 2$, considering the following two cases:

Case 1: Both A and B are irreducible. Suppose that $v = (v_1, v_2, \ldots, v_n)^T$ is the eigenvector corresponding to the special radius of the Jacobi iterative matrix of B^T. Let $V = \mathrm{diag}(v_1, v_2, \ldots, v_n)$. Then, by Lemma 2.4, $B^T V$ is an optimally scaled matrix by rows. Thus $\tilde{B} = (\tilde{b_{ij}}) = VB$ is also an optimally scaled matrix by columns. Let $\tilde{B}^{-1} = (\tilde{\beta_{ij}})$. Notice that both B^{-1} and $\tilde{B}^{-1}$ are positive matrices, and by Lemma 2.1,

$$\tilde{\beta}_{ij} \leq \frac{\sum_{k\neq j} |\tilde{b}_{kj}|}{\tilde{b}_{jj}} \tilde{\beta}_{ii} = \frac{\sum_{k\neq j} |b_{kj}| v_k}{b_{jj} v_j} \frac{\beta_{ii}}{v_i} = \rho(J_{B^T}) \frac{\beta_{ii}}{v_i}.$$

Let D be the diagonal matrix of B and $C = D - B$. Then the Jacobi iterative matrix of B is $J_B = D^{-1}C$. Since $B^T = D - C^T$, then $J_{B^T} = D^{-1}C^T$. It is easy to verify that $\rho(J_B) = \rho(J_{B^T})$. Hence,

$$\beta_{ij} \leq \rho(J_B) \frac{\beta_{ii}}{v_i} v_j.$$

Now let $P = A \circ B^{-1}$ and $y = (y_1, y_1, \ldots, y_n)^T$ denote the eigenvector corresponding to $\rho(A)$, that is, $Ay = \rho(A)y$. Let $z = (z_1, z_2, \ldots, z_n)^T$, where $z_i = y_i/v_i$. Since $B^{-1} > 0$, it follows from both A and B are irreducible that P is irreducible as well, and for each $i \in N$,

$$\begin{aligned}
(Pz)_i = \sum_{j=1}^{n} a_{ij}\beta_{ij}z_j &= a_{ii}\beta_{ii}z_i + \sum_{j\neq i} a_{ij}\beta_{ij}z_j \\
&\leq a_{ii}\beta_{ii}z_i + \sum_{j\neq i} a_{ij}(\rho(J_B)\frac{\beta_{ii}}{v_i}v_j)z_j \\
&= a_{ii}\beta_{ii}z_i + \rho(J_B)\frac{\beta_{ii}}{v_i}\sum_{j\neq i} a_{ij}y_j \\
&= a_{ii}\beta_{ii}z_i + \rho(J_B)\frac{\beta_{ii}}{v_i}(\rho(A) - a_{ii})y_i \\
&= [a_{ii}\beta_{ii} + \beta_{ii}\rho(J_B)(\rho(A) - a_{ii})]z_i \\
&\leq \beta_{ii}\rho(A)z_i,
\end{aligned}$$

where the last inequality follows from $\rho(J_B) \leq 1$. Thus

$$(Pz)_i \leq \{\max_{i\in N}[a_{ii}\beta_{ii} + \beta_{ii}\rho(J_B)(\rho(A) - a_{ii})]\}z_i \leq \{\max_{i\in N}\beta_{ii}\rho(A)\}z_i.$$

By Lemma 2.5, this shows that Lemma 2.6 is valid for this case.

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Case 2: One of A and B is reducible. By replacing the zeros of A and B with ε and $-\varepsilon$, respectively, we obtain the nonnegative matrix $A(\varepsilon)$ and the Z-matrix $B(\varepsilon)$, both irreducible. $B(\varepsilon)$ is a nonsingular M-matrix if ε is a sufficiently small positive number. Now replace A and B with $A(\varepsilon)$ and $B(\varepsilon)$, respectively, in the previous case. Letting ε approach 0, the result follows by continuity. □

Remark 2.7. Under the hypotheses of Lemma 2.6, it follows that

$$\operatorname{diag}(\beta_{11}, \beta_{22}, \ldots, \beta_{nn}) \leq B^{-1},$$

and one may show that $\max_{i\in N} \beta_{ii} \leq \rho(B^{-1})$. Thus

$$\max_{i\in N} \beta_{ii}\rho(A) \leq \rho(A)\rho(B^{-1}).$$

This shows that Lemma 2.6 is an improvement on Theorem 5.7.4 of [3] when B^{-1} is an inverse M-matrix.

3. Upper and lower bounds for $\tau(A)$ and q.

In this section, we shall obtain some upper and lower bounds for $\tau(A)$.

Theorem 3.1. *Let $B = (b_{ij})$ be a nonsingular M-matrix and $B^{-1} = (\beta_{ij})$. Then*

$$\tau(B) \geq \frac{1}{1 + (n-1)\rho(J_B)} \cdot \frac{1}{\max_{i\in N} \beta_{ii}},$$

where $\rho(J_B)$ is the spectral radius of the Jacobi iterative matrix J_B of B.

Proof. Let $A = (a_{ij})$ be a nonnegative matrix. It follows from Lemma 2.6 that

$$\rho(A \circ B^{-1}) \leq \max_{i\in N}\{a_{ii}\beta_{ii} + \beta_{ii}\rho(J_B)(\rho(A) - a_{ii})\}. \tag{3.1}$$

Take $A = J$, where J denotes the matrix of all elements one. Notice that $\rho(A) = n$. The inequality (3.1) yields that

$$\tau(B) = \frac{1}{\rho(B^{-1})} \geq \frac{1}{1 + (n-1)\rho(J_B)} \cdot \frac{1}{\max_{i\in N} \beta_{ii}}.$$

This completes our proof. □

Let

$$A = \begin{bmatrix} 3 & -3 \\ -1 & 4 \end{bmatrix}.$$

Applying Theorem 3.1, one has

$$\tau(A) \geq \frac{2}{3} \cdot \frac{9}{4} = 1.5.$$

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However, applying Theorem 1.1, one has

$$\tau(A) \geq \frac{9}{7} \approx 1.286.$$

Hence the lower bound of Theorem 3.1 is better than that of Theorem 1.1 in some cases.

In Theorem 1.1, some bounds were given for $\tau(A)$ when A is weakly chained diagonally dominant M-matrix. Actually, we may obtain similar results for a general nonsingular M-matrix. The following Theorem 3.2 can be found in [7]. For the convenience of the readers, we provide its proof.

THEOREM 3.2. *Let $A = (a_{ij})$ be a nonsingular M-matrix and $A^{-1} = (\alpha_{ij})$. Then*

$$r(A) \leq \frac{1}{M} \leq \tau(A) \leq \frac{1}{m} \leq R(A).$$

Proof. Since A is a nonsingular M-matrix, then $A^{-1} \geq 0$. The Perron-Frobenius theorem [1] implies that

$$\frac{1}{M} \leq \frac{1}{\rho(A^{-1})} = \tau(A) \leq \frac{1}{m}.$$

In the following, we shall show that $r(A) \leq \frac{1}{M}$ and $\frac{1}{m} \leq R(A)$.

Expanding the determinant of A by column i,

$$\det A = \sum_{j=1}^{n} a_{ji}A_{ji} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}A_{ji} = \sum_{j=1}^{n} R_j(A)A_{ji} \geq r(A)\sum_{j=1}^{n} A_{ji}, \tag{3.2}$$

where A_{ji} denotes the (i,j)-th cofactor of A. Thus the inequality (3.2) implies that

$$r(A) \leq \frac{\det A}{\sum_{j=1}^{n} A_{ji}} = \frac{1}{R_i(A^{-1})}, \ \forall i \in N.$$

This shows that $r(A) \leq \frac{1}{M}$ as i is arbitrary.

Using the same method, one may show that $\frac{1}{m} \leq R(A)$. □

EXAMPLE 3.3. Let A be the following matrix (see [5])

$$A = \begin{bmatrix} 1 & -0.2 & -0.1 & -0.2 & -0.1 \\ -0.4 & 1 & -0.2 & -0.1 & -0.1 \\ -0.9 & -0.2 & 1 & -0.1 & -0.1 \\ -0.3 & -0.7 & -0.3 & 1 & -0.1 \\ -1 & -0.3 & -0.2 & -0.4 & 1 \end{bmatrix}.$$

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It is easy to verify that A is a nonsingular M-matrix, but it is not weakly chained diagonally dominant. Hence Theorem 1.1 may not be used to estimate the lower bounds of $\tau(A)$. However, applying Theorems 3.1 and 3.2, one has

$$\begin{aligned}\tau(A) &\geq \max\left\{\frac{1}{1+(n-1)\rho(J_A)}\cdot\frac{1}{\max_{i\in N}\alpha_{ii}},\ \frac{1}{M}\right\}\\ &\geq \max\left\{\frac{1}{1+4\times 0.9919}\cdot\frac{1}{33.6729},\ \frac{1}{215.2253}\right\}\\ &\approx \max\{0.0060,\ \ 0.0046\} = 0.0060.\end{aligned}$$

In fact, $\tau(A) \approx 0.0081$.

Combining Lemmas 2.2, 2.3 with Theorem 3.1, we may calculate lower bounds for $\tau(A)$ which depend on the entries of matrix A when A is a strictly diagonally dominant M-matrix.

Corollary 3.4. *(i) Let $A = (a_{ij})$ be a strictly diagonally dominant M-matrix by rows. Then*

$$\tau(A) \geq \frac{1}{1+(n-1)\max_{i\in N}\sigma_i}\min_{i\in N}\left\{a_{ii} + \sum\nolimits_{j\neq i} a_{ij}\sigma_j\right\}.$$

(ii) Let $A = (a_{ij})$ be a strictly diagonally dominant M-matrix by columns. Then

$$\tau(A) \geq \frac{1}{1+(n-1)\max_{i\in N}\delta_i}\min_{i\in N}\left\{a_{ii} + \sum\nolimits_{j\neq i} a_{ji}\delta_j\right\}.$$

Example 3.5. Let

$$A = \begin{bmatrix} 1.2 & -0.1 & -0.1\\ -0.3 & 1.0 & -0.1\\ -0.2 & -0.4 & 0.8 \end{bmatrix}.$$

It is easy to verify that A is a nonsingular M-matrix. Applying Lemmas 2.2 and 2.3, we have $0.8333 \leq \alpha_{11} \leq 0.9217$, $1.0000 \leq \alpha_{22} \leq 1.1429$, $1.2500 \leq \alpha_{33} \leq 1.6483$ and $0.25 \leq \rho(J_A) \leq 0.5$, respectively. Now we may use Theorem 3.1 to estimate the lower bound for $\tau(A)$

$$\tau(A) \geq \frac{1}{1+2\times 0.5}\cdot\frac{1}{1.6483} \approx 0.3033.$$

However, applying Theorem 3.3 in [8], one has $M \leq 4$. Then applying Theorem 1.1, we obtain

$$\tau(A) \geq \frac{1}{4} = 0.25.$$

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This shows that our results are better than Theorem 1.1 in some cases.

In Theorem 3.1, the spectral radius of the Jacobi iterative matrix may be estimated using Lemma 2.3, but it is difficult for us to estimate the upper bound of diagonal elements of A^{-1}. Lemma 2.2 provides an upper bound of diagonal elements of A^{-1} for a strictly diagonally dominant M-matrix A. Unfortunately, we are not able to give the corresponding upper bound when A is a general nonsingular M-matrix. It would be an interesting problem to be studied in future research.

Next, we shall exhibit some new bounds for $\tau(A)$ in terms of the spectral radius of the Jacobi iterative matrix and its corresponding eigenvector.

THEOREM 3.6. *Let $A = (a_{ij})$ be an irreducible nonsingular M-matrix. Then*

$$(1 - \rho(J_A))\frac{\min_{i\in N} a_{ii}q_i}{\max_{i\in N} q_i} \le \tau(A) \le (1 - \rho(J_A))\frac{\max_{i\in N} a_{ii}q_i}{\min_{i\in N} q_i}, \tag{3.3}$$

where $\rho(J_A)$ is the spectral radius of the Jacobi iterative matrix J_A of A and $q = (q_1, q_2, \ldots, q_n)^T$ is its eigenvector corresponding to $\rho(J_A)$.

Remark that in Theorem 3.6, A must be irreducible to ensure that $q_i \ne 0$.

Proof. It is quite evident that (3.3) holds with equality for $n = 1$. In the following, suppose that $n \ge 2$. Since A is an irreducible nonsingular M-matrix, by Lemma 2.4, there exists a positive diagonal matrix $Q = \mathrm{diag}(q_1, q_2, \ldots, q_n)$ such that AQ satisfies

$$\sum_{j\ne i} \frac{|a_{ij}q_j|}{a_{ii}q_i} = \rho(J_A).$$

Since AQ is also a nonsingular M-matrix and $Q^{-1}A^{-1} \ge \min\limits_{i\in N} \frac{1}{q_i} A^{-1}$, then

$$\tau(AQ) = \frac{1}{\rho(Q^{-1}A^{-1})} \le \frac{1}{\rho(A^{-1}) \min_{i\in N} \frac{1}{q_i}} = \tau(A) \max_{i\in N} q_i.$$

Similarly,

$$\tau(AQ) \ge \tau(A) \min_{i\in N} q_i.$$

Thus, we have

$$\frac{\tau(AQ)}{\max_{i\in N} q_i} \le \tau(A) \le \frac{\tau(AQ)}{\min_{i\in N} q_i}. \tag{3.4}$$

Notice that AQ is strictly diagonally dominant matrix by rows and its row sums equal to $(1 - \rho(J_A))a_{ii}q_i$ for all $i \in N$. By Theorem 3.2,

$$(1 - \rho(J_A)) \min_{i\in N} a_{ii}q_i \le \tau(AQ) \le (1 - \rho(J_A)) \max_{i\in N} a_{ii}q_i. \tag{3.5}$$

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From (3.4) and (3.5), we get that the inequality (3.3) holds. □

Corollary 3.7. *Let A be an irreducible nonsingular M-matrix. Then there exist two positive diagonal matrices $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ and $E = \operatorname{diag}(e_1, e_2, \ldots, e_n)$ such that $DA^{-1}E$ is a doubly stochastic matrix and*

$$\min_{i\in N} d_i \min_{i\in N} e_i \le \tau(A) \le \max_{i\in N} d_i \max_{i\in N} e_i. \tag{3.6}$$

Proof. Since A is an irreducible nonsingular M-matrix, then A^{-1} is positive. By Theorem 2-6.34 in [1], there exist two positive diagonal matrices D and E such that $DA^{-1}E$ is a doubly stochastic matrix. This implies that $E^{-1}AD^{-1}$ is also a nonsingular M-matrix and $\tau(E^{-1}AD^{-1}) = 1$.

Let $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ and $E = \operatorname{diag}(e_1, e_2, \ldots, e_n)$. From the proof of Theorem 3.6, one obtains

$$\tau(E^{-1}AD^{-1}) \ge \min_{i\in N} \frac{1}{e_i}\tau(AD^{-1}) \ge \min_{i\in N} \frac{1}{d_i} \min_{i\in N} \frac{1}{e_i}\tau(A),$$

which implies

$$\tau(A) \le \max_{i\in N} d_i \max_{i\in N} e_i.$$

Similarly, we have

$$\tau(A) \ge \min_{i\in N} d_i \min_{i\in N} e_i.$$

This completes our proof. □

The following matrix A in Example 3.8 comes from [8].

Example 3.8. Let

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

It is easy to verify that A is a nonsingular M-matrix. Applying Theorem 1.1, one has

$$0.03704 \approx \frac{1}{27} \le \tau(A) \le 1.$$

Now, applying Theorem 3.6, we obtain

$$0.06458 \approx \frac{4-\sqrt{14}}{4} \le \tau(A) \le \frac{4\sqrt{14}-14}{2} \approx 0.4833.$$

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This shows that Theorem 3.6 provides tighter bounds than Theorem 1.1 in some cases.

THEOREM 3.9. *Let $A = (a_{ij})$ be an irreducible nonsingular M-matrix and the eigenvector $q = (q_1, q_2, \dots, q_n)^T$ corresponding to $\rho(J_A)$ satisfy $\| q \|_1 = 1$. Then*

$$\frac{\min_{j\neq i} |a_{ij}|}{a_{ii}\rho(J_A) + \min_{j\neq i} |a_{ij}|} \leq q_i \leq \frac{\max_{j\neq i} |a_{ij}|}{a_{ii}\rho(J_A) + \max_{j\neq i} |a_{ij}|}, \tag{3.7}$$

where $q_i > 0$ if $\min_{j\neq i} |a_{ij}| = 0$.

Proof. Let D be the diagonal matrix of A and $C = D - A$. Then the Jacobi iterative matrix of A is $J_A = D^{-1}C$, that is, $D^{-1}Cq = \rho q$, where $\rho = \rho(J_A)$. It follows from Lemma 2.4 that, for all $i \in N$,

$$a_{ii}\rho q_i = \sum_{j\neq i} |a_{ij}| q_j.$$

Hence,

$$a_{ii}\rho q_i \leq \max_{j\neq i} |a_{ij}| \sum_{j\neq i} q_j = \max_{j\neq i} |a_{ij}|(1 - q_i),$$

which implies

$$q_i \leq \frac{\max_{j\neq i} |a_{ij}|}{a_{ii}\rho + \max_{j\neq i} |a_{ij}|}.$$

Similarly, we get, for all $i \in N$,

$$q_i \geq \frac{\min_{j\neq i} |a_{ij}|}{a_{ii}\rho + \min_{j\neq i} |a_{ij}|}.$$

This completes our proof. □

Theorem 3.9, together with Theorem 3.6, may be used to estimate some bounds of $\tau(A)$ for an irreducible nonsingular M-matrix A. For example, let A be the matrix in Example 3.3. Applying Theorem 3.9, we obtain that

$$\min_i q_i \geq 0.09158, \ \max_i q_i \leq 0.50203.$$

Also applying Theorem 3.6, one has

$$0.001476 \leq \tau(A) \leq 0.04440.$$

When A is an irreducible nonsingular M-matrix and $a_{ij} \neq 0$ for all $i \neq j$, using Theorem 3.9, we may obtain positive bounds of q. Then Theorem 3.6 can be used

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to estimate some bounds of $\tau(A)$. However, if there exists some entry $a_{ij} = 0$ in all elements of A, we may only obtain $q > 0$. This show that Theorem 3.6 is invalid in this case. To determine some positive bounds of eigenvector corresponding to $\rho(J_A)$ in this case would be an interesting problem for future research.

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REFERENCES

[1] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. SIAM Press, Philadephia, 1994.
[2] M. Fiedler and T.L. Markham. An inequality for the Hadamard product of an M-matrix and inverse M-matrix. *Linear Algebra Appl.*, 101:1–8, 1988.
[3] R.A. Horn and C.R. Johnson. *Topics in Matrix Analysis*. Cambridge Press, New York, 1991.
[4] J. Hu. The estimation of $||M^{-1}N||_\infty$ and the optimally scaled matrix. *J. Comput. Math.*, 2:122–129, 1984.
[5] L. Li. On the iterative criterion for generalized diagonally dominant matrices. *SIAM J. Matrix Anal. Appl.*, 24(1):17–24, 2002.
[6] H. Minc. *Nonnegative Matrices*. John Wiley and Sons, New York, 1987.
[7] M. Pang. *Spectral Theory of Matrices*. Jilin University Press, Changchun, China, 1990.
[8] P.N. Shivakumar, J.J. Williams, Q. Ye, and C.A. Marinov. On two-sided bounds related to weakly diagonally dominant M-matrices with application to digital circuit dynamics. *SIAM J. Matrix Anal. Appl.*, 17(2):298–312, 1996.
[9] X.R. Yong and Z. Wang. On a conjecture of Fiedler and Markham. *Linear Algebra Appl.*, 288:259–267, 1999.