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AN IMPROVED ALGORITHM FOR SOLVING AN INVERSE EIGENVALUE PROBLEM FOR BAND MATRICES*

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Abstract. The construction of matrices with prescribed eigenvalues is a kind of inverse eigenvalue problems. The authors proposed an algorithm for constructing band oscillatory matrices with prescribed eigenvalues based on the extended discrete hungry Toda equation (*Numer. Algor.* 75:1079–1101, 2017). In this paper, we develop a new algorithm for constructing band matrices with prescribed eigenvalues based on a generalization of the extended discrete hungry Toda equation. The new algorithm improves the previous algorithm so that the new one can produce more generic band matrices than the previous one in a certain sense. We compare the new algorithm with the previous one by numerical examples. Especially, we show an example of band oscillatory matrices which the new algorithm can produce but the previous one cannot.

Key words. Inverse eigenvalue problem, Band matrix, Oscillatory matrix, Discrete integrable system.

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1. Introduction. Solving inverse eigenvalue problems (IEPs) is an important subject in numerical linear algebra. The problem to construct a matrix with prescribed eigenvalues is one of IEPs. Boley and Golub [7] presented a survey of IEPs for symmetric matrices, and Chu and Golob [8] gave a review for various kinds of IEPs.

In this paper, we consider an IEP for totally nonnegative (TN) and oscillatory matrices. Let us first remind their definitions and some properties.

Definition 1.1 (Totally nonnegative (positive) matrices [6, 21, 10, 23]). *A matrix A is totally nonnegative (resp. positive) if all the minors of A are nonnegative (resp. positive).*

Definition 1.2 (Oscillatory matrices [12, 10]). *A totally nonnegative matrix A is oscillatory if A^n is totally positive for some nonnegative integer n.*

A totally nonnegative matrix is oscillatory if it is invertible and irreducible [12, 10]. The following facts are fundamental for totally nonnegative matrices and oscillatory matrices.

Theorem 1.3 ([23]). *Any product of totally nonnegative (resp. positive) matrices is totally nonnegative (resp. positive).*

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Theorem 1.4 ([12]). *The eigenvalues of any oscillatory matrix are all positive and distinct.*

Totally nonnegative matrices and oscillatory matrices have been discussed in several papers. Adm and Garloff showed a matrix interval in which matrices are totally nonnegative (see, e.g., [1, 2]) from the viewpoint of matrix analysis. On eigenvalue problems, Koev [22] presented an algorithm for computing eigenvalues of symmetric totally nonnegative matrices.

IEPs for totally nonnegative matrices and oscillatory matrices are much discussed in symmetric case. Gladwell [14, 15] discussed an IEP for symmetric pentadiagonal oscillatory matrices appearing in a vibrating beam problem which is modeled as a mass-spring system. Ghanbari [13] showed how to construct symmetric pentadiagonal oscillatory matrices with two and three eigenvalues in common based on matrix decomposition. Rojo et al. [25] proposed an algorithm for constructing symmetric oscillatory matrices with prescribed eigenvalues based on the Householder transformation. However, IEPs for totally nonnegative matrices and oscillatory matrices which may be nonsymmetric have not been much discussed. For example, Fallat et al. [9] fully examined the Jordan structure of irreducible totally nonnegative matrices from a combinatorial viewpoint.

In this paper, we consider the following IEP for band matrices which are nonsymmetric.

Problem. *Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be nonzero and distinct. Then, find a band matrix A of the form*

$$A = \begin{pmatrix} \ast & \multicolumn{4}{c}{\overbrace{\ast \quad \cdots \quad \ast \quad 1}^{M}} & & \\ \ast & \ast & \ast & \cdots & \ast & \ddots & \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 1 \\ \ast & & \ddots & \ddots & \ddots & & \ast \\ & \ddots & & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & \ddots & \ast & \ast \\ & & & \multicolumn{3}{c}{\underbrace{\ast \quad \cdots \quad \ast}_{N}} & \ast \end{pmatrix}, \tag{1.1}$$

whose eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_m$, where positive integers M and N are the upper and lower bandwidths of A, respectively.

In this problem, all the entries of A on the uppermost band are fixed to one. We note that this normalizing condition is not essential because we can set arbitrary nonzero values to those entries by a similarity transformation with diagonal matrices.

The authors have proposed some algorithms for constructing such band matrices with prescribed eigenvalues based on discrete integrable systems [3, 4, 5]. Integrable systems are nonlinear dynamical systems which have explicit solutions. A time discretization of integrable systems which respects their integrability is called integrable discretization. In [3], some of the authors proposed an algorithm for constructing tridiagonal matrices with prescribed eigenvalues, of the form (1.1) with $M = N = 1$, based on the discrete Toda equation. Any tridiagonal matrix A obtained from the algorithm is a product of lower and upper bidiagonal matrices as $A = L_1 R_1$, where L_i and R_j here, and hereafter, represent lower and upper bidiagonal matrices, respectively. The discrete Toda equation [20] is a discrete integrable system and is known to be the same as

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the recursion formula of the quotient-difference algorithm (see, e.g., [26, 27, 18]) for computing eigenvalues of tridiagonal matrices.

When M is general and $N = 1$, some of authors in [4] proposed an algorithm for constructing upper Hessenberg oscillatory matrices with prescribed eigenvalues, of the form (1.1), based on the discrete hungry Toda equation. Any upper Hessenberg matrix obtained from the algorithm is a product of lower and upper bidiagonal matrices as $A = L_1 R_M \cdots R_2 R_1$. The discrete hungry Toda equation is a generalization of the discrete Toda equation [29]. Fukuda et al. [11] discussed eigenvalue computation of upper Hessenberg oscillatory matrices by using the discrete hungry Toda equation.

In [5], the authors introduced a discrete integrable system called the extended discrete hungry Toda equation and proposed an algorithm for constructing band oscillatory matrices with prescribed eigenvalues, of the form (1.1) with M, N general such that the prescribed data are their eigenvalues according to this equation. Any band matrix obtained from the algorithm is also a product of lower and upper bidiagonal matrices as $A = L_1 L_2 \cdots L_N R_M \cdots R_2 R_1$.

The aim of this paper is to improve the algorithm in the preceding paper [5]. By using the algorithm in [5], we can construct band matrices of the form (1.1) for arbitrary bandwidths M and N. However, there remains a large defect in the algorithm that we cannot construct generic matrices of the form (1.1) when the bandwidths M and N are not co-prime. Actually, if $M = N \neq 1$, any band matrix constructed by using the algorithm must be a power of another tridiagonal matrix as $A = (L_1 R_1)^M$. Similarly, if $M = nN$ and $N = nM$, the algorithm always gives us matrices of the form $A = (L_1 R_n \cdots R_2 R_1)^N$ and $A = (L_1 L_2 \cdots L_n R_1)^M$, respectively [5, Proposition 8]. As these facts show, when the bandwidths M and N are not co-prime, the algorithm in [5] always produces a band matrix which is nothing but a power of another band matrix with narrower bandwidths. In this paper, we propose a new algorithm without this defect by employing another discrete integrable system which generalizes the extended discrete hungry Toda equation used in [5].

We mention the oscillatoriness of matrices. For the algorithm in [4], a criterion is given for upper Hessenberg matrices obtained from the algorithm to be oscillatory. For the algorithm in [5], a similar criterion is given for band matrices of the form (1.1). Unfortunately, for the new algorithm proposed in this paper, such an oscillatory criterion is not yet given. We only observe by numerical examples the capability of the new algorithm to produce a oscillatory matrix even when the bandwidths M and N are not co-prime.

This paper is organized as follows. In Section 2, we introduce some determinants with which we examine the eigenvalue problem of a band matrix of the form (1.1). In Section 3, we explain how to construct band matrices of the form (1.1) with prescribed eigenvalues and show a new algorithm (Algorithm 1). To do that, we utilize a generalization of the extended discrete hungry Toda equation. We also compare the new algorithm with the previous algorithm in [5]. In Section 4, we show numerical examples of band oscillatory matrices with prescribed eigenvalues constructed by using the new algorithm. We observe by an example that band matrices constructed with the new algorithm may be more generic than those with the algorithm in [5]. In Section 5, we give conclusions and discuss future works.

2. The eigenvalue problem for a band matrix. In this section, we introduce some determinants. Using the determinants, we examine the eigenvalue problem of a band matrix of the form (1.1).

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Let m, M and N be positive integers. Assume that $\lambda_1, \lambda_2, \dots, \lambda_m$ are nonzero and distinct. We define a sequence $f_{s,t}$ for $s, t = 0, 1, \dots$ by

$$f_{s,t} := \sum_{i=1}^{m} c_i^{(s \bmod M)} \lambda_i^{\frac{s}{M}} \hat{c}_i^{(t \bmod N)} \lambda_i^{\frac{t}{N}}, \tag{2.2}$$

where $c_1^{(s)}, c_2^{(s)}, \dots, c_m^{(s)}$ for $s = 0, 1, \dots, M-1$ and $\hat{c}_1^{(t)}, \hat{c}_2^{(t)}, \dots, \hat{c}_m^{(t)}$ for $t = 0, 1, \dots, N-1$ are arbitrary nonzero constants. Clearly,

$$f_{s+M,t} = f_{s,t+N}, \quad s, t = 0, 1, \dots. \tag{2.3}$$

Let us define

$$\xi_i^{(s)} := c_i^{(s \bmod M)} \lambda_i^{\frac{s}{M}}, \quad \hat{\xi}_i^{(t)} := \hat{c}_i^{(t \bmod N)} \lambda_i^{\frac{t}{N}}, \tag{2.4}$$

and

$$\boldsymbol{\xi}^{(s)} := \begin{pmatrix} \xi_1^{(s)} \\ \xi_2^{(s)} \\ \vdots \\ \xi_m^{(s)} \end{pmatrix}, \quad \hat{\boldsymbol{\xi}}^{(t)} := \begin{pmatrix} \hat{\xi}_1^{(t)} \\ \hat{\xi}_2^{(t)} \\ \vdots \\ \hat{\xi}_m^{(t)} \end{pmatrix}, \tag{2.5}$$

so that

$$f_{s,t} = \sum_{i=1}^{m} \xi_i^{(s)} \hat{\xi}_i^{(t)} = (\boldsymbol{\xi}^{(s)})^{\mathsf{T}} \hat{\boldsymbol{\xi}}^{(t)}. \tag{2.6}$$

2.1. Determinants. Let us define

$$\phi_{k,i}^{(s,t)} := \frac{\tau_{k,i}^{(s,t)}}{\tau_k^{(s,t)}}, \tag{2.7}$$

for $k = 0, 1, \dots, m$, $i = 1, 2, \dots, m$ and $s, t = 0, 1, \dots$, where $\tau_{k,i}^{(s,t)}$ and $\tau_k^{(s,t)}$ are determinants given by

$$\tau_{0,i}^{(s,t)} := \xi_i^{(s)}; \quad \tau_{k,i}^{(s,t)} := \begin{vmatrix} f_{s,t} & f_{s,t+1} & \cdots & f_{s,t+k-1} & \xi_i^{(s)} \\ f_{s+1,t} & f_{s+1,t+1} & \cdots & f_{s+1,t+k-1} & \xi_i^{(s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{s+k-1,t} & f_{s+k-1,t+1} & \cdots & f_{s+k-1,t+k-1} & \xi_i^{(s+k-1)} \\ f_{s+k,t} & f_{s+k,t+1} & \cdots & f_{s+k,t+k-1} & \xi_i^{(s+k)} \end{vmatrix}, \quad k = 1, 2, \dots, \tag{2.8}$$

and

$$\tau_0^{(s,t)} := 1; \quad \tau_k^{(s,t)} := \begin{vmatrix} f_{s,t} & f_{s,t+1} & \cdots & f_{s,t+k-1} \\ f_{s+1,t} & f_{s+1,t+1} & \cdots & f_{s+1,t+k-1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{s+k-1,t} & f_{s+k-1,t+1} & \cdots & f_{s+k-1,t+k-1} \end{vmatrix}, \quad k = 1, 2, \dots, \tag{2.9}$$

respectively. Hereafter, we assume that

$$\tau_k^{(s,t)} \neq 0, \quad k = 1, 2, \dots, m, \quad s, t = 0, 1, 2, \dots,$$

so that $\phi_{k,i}^{(s,t)}$ in (2.7) are well-defined.

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Proposition 2.1. *$\phi_{m,i}^{(s,t)} = 0$ for every i and s,t.*

Proof. From the definition of $\phi_{k,i}^{(s,t)}$ in (2.7), the proof amounts to showing that $\tau_{m,i}^{(s,t)} = 0$. Since the $m+1$ vectors $\boldsymbol{\xi}^{(s)}, \boldsymbol{\xi}^{(s+1)}, \dots, \boldsymbol{\xi}^{(s+m)}$ of size m are linearly dependent, we have

$$\sum_{j=0}^{m} a_j \boldsymbol{\xi}^{(s+j)} = a_0 \boldsymbol{\xi}^{(s)} + a_1 \boldsymbol{\xi}^{(s+1)} + \cdots + a_m \boldsymbol{\xi}^{(s+m)} = \mathbf{0},$$

for some $a_0, a_1, \dots, a_m$ one or more of which are nonzero. Hence, the row vectors

$$\begin{pmatrix} f_{s+j,t} & f_{s+j,t+1} & \cdots & f_{s+j,t+m-1} & \xi_i^{(s+j)} \end{pmatrix}, \quad j = 0,1,\dots,m,$$

of the determinant $\tau_{m,i}^{(s,t)}$, defined in (2.8), are also linearly dependent because $\sum_{j=0}^{m} a_j \xi_i^{(s+j)} = 0$ from (2.5) and for $\ell = 0,1,\dots,m-1$

$$\sum_{j=0}^{m} a_j f_{s+j,t+\ell} = \sum_{j=0}^{m} a_j (\boldsymbol{\xi}^{(s+j)})^{\mathsf{T}} \hat{\boldsymbol{\xi}}^{(t+\ell)} = \mathbf{0}^{\mathsf{T}} \hat{\boldsymbol{\xi}}^{(t+\ell)} = 0,$$

from (2.6). ◻

Proposition 2.2. *$\tau_{m+1}^{(s,t)} = 0$ for every s,t.*

Proof. From (2.6) we have

$$\tau_{m+1}^{(s,t)} = \left| \begin{pmatrix} \boldsymbol{\xi}^{(s)} & \boldsymbol{\xi}^{(s+1)} & \cdots & \boldsymbol{\xi}^{(s+m)} \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \hat{\boldsymbol{\xi}}^{(t)} & \hat{\boldsymbol{\xi}}^{(t+1)} & \cdots & \hat{\boldsymbol{\xi}}^{(t+m)} \end{pmatrix} \right| = 0,$$

since each of $\boldsymbol{\xi}^{(s)}$ and $\hat{\boldsymbol{\xi}}^{(t)}$ is a column vector of size m. ◻

We show two linear relations among $\phi_{k,i}^{(s,t)}$.

Proposition 2.3. *The $\phi_{k,i}^{(s,t)}$ satisfy the linear relation that*

$$\phi_{k-1,i}^{(s+1,t)} = q_k^{(s,t)} \phi_{k-1,i}^{(s,t)} + \phi_{k,i}^{(s,t)}, \quad k = 1,2,\dots,m, \tag{2.10}$$

$$q_k^{(s,t)} := \frac{\tau_k^{(s+1,t)} \tau_{k-1}^{(s,t)}}{\tau_k^{(s,t)} \tau_{k-1}^{(s+1,t)}}. \tag{2.11}$$

Proof. From the Jacobi identity for determinants [19, 18], we have

$$\tau_{k,i}^{(s,t)} \begin{bmatrix} k+1 \\ k+1 \end{bmatrix} \cdot \tau_{k,i}^{(s,t)} \begin{bmatrix} 1 \\ k \end{bmatrix} = \tau_{k,i}^{(s,t)} \begin{bmatrix} k+1 \\ k \end{bmatrix} \cdot \tau_{k,i}^{(s,t)} \begin{bmatrix} 1 \\ k+1 \end{bmatrix} + \tau_{k,i}^{(s,t)} \cdot \tau_{k,i}^{(s,t)} \begin{bmatrix} k+1,1 \\ k,k+1 \end{bmatrix},$$

where $\tau_{k,i}^{(s,t)} \left[\begin{smallmatrix} i_1,\dots,i_\ell \\ j_1,\dots,j_\ell \end{smallmatrix}\right]$ denotes the minor of $\tau_{k,i}^{(s,t)}$ obtained by deleting the $i_1,\dots,i_\ell$th rows and the $j_1,\dots,j_\ell$th columns. From the definitions (2.8)–(2.9) of $\tau_{k,i}^{(s,t)}$ and $\tau_k^{(s,t)}$, the last identity is the same as

$$\tau_k^{(s,t)} \tau_{k-1,i}^{(s+1,t)} = \tau_{k-1,i}^{(s,t)} \tau_k^{(s+1,t)} + \tau_{k,i}^{(s,t)} \tau_k^{(s+1,t)}.$$

Dividing the both sides by $\tau_k^{(s,t)} \tau_{k-1}^{(s+1,t)}$, we obtain

$$\frac{\tau_{k-1,i}^{(s+1,t)}}{\tau_{k-1}^{(s+1,t)}} = \frac{\tau_k^{(s+1,t)} \tau_{k-1}^{(s,t)}}{\tau_k^{(s,t)} \tau_{k-1}^{(s+1,t)}} \cdot \frac{\tau_{k-1,i}^{(s,t)}}{\tau_{k-1}^{(s,t)}} + \frac{\tau_{k,i}^{(s,t)}}{\tau_k^{(s,t)}}.$$

From the definition (2.7) of $\phi_{k,i}^{(s,t)}$, this is equivalent to (2.10) with (2.11). ◻

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PROPOSITION 2.4. *The $\phi_{k,i}^{(s,t)}$ satisfy the linear relation that*

$$\phi_{k,i}^{(s,t)} = e_k^{(s,t)} \phi_{k-1,i}^{(s,t+1)} + \phi_{k,i}^{(s,t+1)}, \quad k = 0, 1, \ldots, m, \tag{2.12}$$

$$e_k^{(s,t)} := \frac{\tau_{k+1}^{(s,t)} \tau_{k-1}^{(s,t+1)}}{\tau_k^{(s,t+1)} \tau_k^{(s,t)}}, \tag{2.13}$$

where $e_0^{(s,t)} \phi_{-1,i}^{(s,t+1)} = 0$.

Proof. For $k = 0$, the relation (2.12) holds because $\phi_{0,i}^{(s,t)} = \phi_{0,i}^{(s,t+1)} = \xi_i^{(s)}$ from the definition (2.7)–(2.8) of $\phi_{k,i}^{(s,t)}$. For $k \geq 1$, we use the Plücker relation [19, 18] for a $(k+1) \times (k+3)$ matrix

$$F = \begin{pmatrix} f_{s,t+1} & \cdots & f_{s,t+k-1} & f_{s,t+k} & f_{s,t} & \xi_i^{(s)} & 0 \\ f_{s+1,t+1} & \cdots & f_{s+1,t+k-1} & f_{s+1,t+k} & f_{s+1,t} & \xi_i^{(s+1)} & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{s+k-1,t+1} & \cdots & f_{s+k-1,t+k-1} & f_{s+k-1,t+k} & f_{s+k-1,t} & \xi_i^{(s+k-1)} & 0 \\ f_{s+k,t+1} & \cdots & f_{s+k,t+k-1} & f_{s+k,t+k} & f_{s+k,t} & \xi_i^{(s+k)} & 1 \end{pmatrix},$$

with respect to the last four columns to obtain

$$F[k+1, k+3] \cdot F[k, k+2] - F[k+1, k+3] \cdot F[k, k+2] + F[k+1, k+2] \cdot F[k, k+3] = 0, \tag{2.14}$$

where $F[i, j]$ denotes the determinant of the submatrix of F obtained by deleting the ith and jth columns. From the definitions (2.8)–(2.9) of $\tau_{k,i}^{(s,t)}$ and $\tau_k^{(s,t)}$, the last identity is the same as

$$(-1)^k \tau_{k+1}^{(s,t)} \tau_{k-1,i}^{(s,t+1)} - (-1)^{k-1} \tau_{k,i}^{(s,t+1)} \tau_k^{(s,t)} + (-1)^{k-1} \tau_{k,i}^{(s,t)} \tau_k^{(s,t+1)} = 0.$$

Dividing the both sides by $(-1)^k \tau_k^{(s,t+1)} \tau_k^{(s,t)}$, we have

$$\frac{\tau_{k,i}^{(s,t)}}{\tau_k^{(s,t)}} = \frac{\tau_{k+1}^{(s,t)} \tau_{k-1}^{(s,t+1)}}{\tau_k^{(s,t+1)} \tau_k^{(s,t)}} \cdot \frac{\tau_{k-1,i}^{(s,t+1)}}{\tau_{k-1}^{(s,t+1)}} + \frac{\tau_{k,i}^{(s,t+1)}}{\tau_k^{(s,t+1)}}.$$

From the definition (2.7) of $\phi_{k,i}^{(s,t)}$, this is equivalent to (2.12) with (2.13). ◻

2.2. A band matrix and its eigenvalue problem.

Let us introduce a vector of size m

$$\Phi_i^{(s,t)} := \begin{pmatrix} \phi_{0,i}^{(s,t)} \\ \phi_{1,i}^{(s,t)} \\ \vdots \\ \phi_{m-1,i}^{(s,t)} \end{pmatrix}. \tag{2.15}$$

We rewrite the linear relations among $\phi_{k,i}^{(s,t)}$ in Propositions 2.3–2.4 in terms of matrices and vectors. Propositions 2.3 and 2.1 imply the following.

PROPOSITION 2.5. *The vectors $\Phi_i^{(s,t)}$ satisfy*

$$\Phi_i^{(s+1,t)} = R^{(s,t)} \Phi_i^{(s,t)}, \quad i = 1, 2, \ldots, m, \tag{2.16}$$

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where $R^{(s,t)}$ is an upper bidiagonal matrix given by

$$R^{(s,t)} := \begin{pmatrix} q_1^{(s,t)} & 1 & & \\ & q_2^{(s,t)} & \ddots & \\ & & \ddots & 1 \\ & & & q_m^{(s,t)} \end{pmatrix}. \tag{2.17}$$

Proposition 2.4 implies the following.

PROPOSITION 2.6. *The vectors $\Phi_i^{(s,t)}$ satisfy*

$$\Phi_i^{(s,t)} = L^{(s,t)} \Phi_i^{(s,t+1)}, \quad i = 1, 2, \ldots, m, \tag{2.18}$$

where $L^{(s,t)}$ is a lower bidiagonal matrix given by

$$L^{(s,t)} := \begin{pmatrix} 1 & & & \\ e_1^{(s,t)} & 1 & & \\ & \ddots & \ddots & \\ & & e_{m-1}^{(s,t)} & 1 \end{pmatrix}. \tag{2.19}$$

The vectors $\Phi_i^{(s,t)}$ also satisfy the following relation due to the definition (2.2) of $f_{s,t}$.

PROPOSITION 2.7. $\Phi_i^{(s+M,t)} = \lambda_i \Phi_i^{(s,t+N)}$ *for every* $s, t = 0, 1, 2, \ldots$.

Proof. From (2.4), we have

$$\xi_i^{(s+M)} = \lambda_i \xi_i^{(s)},$$

and then, from (2.8) with the help of (2.3),

$$\begin{aligned} \tau_{k,i}^{(s+M,t)} &= \begin{vmatrix} f_{s+M,t} & f_{s+M,t+1} & \cdots & f_{s+M,t+k-1} & \xi_i^{(s+M)} \\ f_{s+M+1,t} & f_{s+M+1,t+1} & \cdots & f_{s+M+1,t+k-1} & \xi_i^{(s+M+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{s+M+k-1,t} & f_{s+M+k-1,t+1} & \cdots & f_{s+M+k-1,t+k-1} & \xi_i^{(s+M+k-1)} \\ f_{s+M+k,t} & f_{s+M+k,t+1} & \cdots & f_{s+M+k,t+k-1} & \xi_i^{(s+M+k)} \end{vmatrix} \\ &= \lambda_i \begin{vmatrix} f_{s,t+N} & f_{s,t+N+1} & \cdots & f_{s,t+N+k-1} & \xi_i^{(s)} \\ f_{s+1,t+N} & f_{s+1,t+N+1} & \cdots & f_{s+1,t+N+k-1} & \xi_i^{(s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{s+k-1,t+N} & f_{s+k-1,t+N+1} & \cdots & f_{s+k-1,t+N+k-1} & \xi_i^{(s+k-1)} \\ f_{s+k,t+N} & f_{s+k,t+N+1} & \cdots & f_{s+k,t+N+k-1} & \xi_i^{(s+k)} \end{vmatrix} = \lambda_i \tau_{k,i}^{(s,t+N)}. \end{aligned}$$

Similarly, $\tau_k^{(s+M,t)} = \tau_k^{(s,t+N)}$ from (2.9), and hence, $\phi_{k,i}^{(s+M,t)} = \lambda_i \phi_{k,i}^{(s,t+N)}$ from (2.7). From the definition (2.15) of $\Phi_i^{(s,t)}$, this proves the proposition. ∎

By using the bidiagonal matrices $L^{(s,t)}$ and $R^{(s,t)}$, let us define a matrix

$$A^{(s,t)} := L^{(s,t)} L^{(s,t+1)} \cdots L^{(s,t+N-1)} R^{(s+M-1,t)} \cdots R^{(s+1,t)} R^{(s,t)}. \tag{2.20}$$

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Obviously, from (2.17) and (2.19), this $A^{(s,t)}$ is a band matrix of the form (1.1). In addition, we have the following theorem concerning the eigenvalue problem for $A^{(s,t)}$.

Theorem 2.1. *The band matrix $A^{(s,t)}$ and the vector $\Phi_i^{(s,t)}$ satisfy*

$$A^{(s,t)}\Phi_i^{(s,t)} = \lambda_i \Phi_i^{(s,t)}, \quad \Phi_i^{(s,t)} \neq \mathbf{0}, \quad i = 1, 2, \ldots, m. \tag{2.21}$$

That is, $A^{(s,t)}$ has eigenpairs $(\lambda_i, \Phi_i^{(s,t)})$ for $i = 1, 2, \ldots, m$.

Proof. From the definition (2.15) of $\Phi_i^{(s,t)}$ with (2.7)–(2.8), the first entry of $\Phi_i^{(s,t)}$ is $\xi_i^{(s)} = c_i^{(s \bmod M)} \lambda_i^{\frac{s}{M}}$, that is nonzero since we assumed that each of λ_i and $c_i^{(s)}$ is nonzero. Hence, $\Phi_i^{(s,t)}$ is nonzero. Using (2.16) repeatedly, we derive

$$\begin{aligned}
A^{(s,t)}\Phi_i^{(s,t)} &= L^{(s,t)} \cdots L^{(s,t+N-2)} L^{(s,t+N-1)} R^{(s+M-1)} \cdots R^{(s+1,t)} (R^{(s,t)} \Phi_i^{(s,t)}) \\
&= L^{(s,t)} \cdots L^{(s,t+N-2)} L^{(s,t+N-1)} R^{(s+M-1,t)} \cdots (R^{(s+1,t)} \Phi_i^{(s+1,t)}) \\
&\ \vdots \\
&= L^{(s,t)} \cdots L^{(s,t+N-2)} L^{(s,t+N-1)} \Phi_i^{(s+M,t)}.
\end{aligned}$$

From Proposition 2.7, we here have $\Phi_i^{(s+M,t)} = \lambda_i \Phi_i^{(s,t+N)}$. Then, using (2.18) repeatedly, we obtain

$$\begin{aligned}
A^{(s,t)}\Phi_i^{(s,t)} &= \lambda_i L^{(s,t)} \cdots L^{(s,t+N-2)} (L^{(s,t+N-1)} \Phi_i^{(s,t+N)}) \\
&= \lambda_i L^{(s,t)} \cdots (L^{(s,t+N-2)} \Phi_i^{(s,t+N-1)}) \\
&\ \vdots \\
&= \lambda_i \Phi_i^{(s,t)}.
\end{aligned}$$

∎

3. Construction of band matrices with prescribed eigenvalues. In this section, we explain how to construct band matrices of the form (1.1) with prescribed eigenvalues. We owe the construction to a generalization of the extended discrete hungry Toda equation which naturally arises from the linear relations among $\phi_{k,i}^{(s,t)}$ discussed in Section 2.

3.1. Construction of band matrices by a generalization of the extended hungry Toda equation.

Theorem 3.1. *The bidiagonal matrices $L^{(s,t)}$ and $R^{(s,t)}$ satisfy*

$$L^{(s+1,t)} R^{(s,t+1)} = R^{(s,t)} L^{(s,t)}, \tag{3.22}$$

$$L^{(s+M,t)} = L^{(s,t+N)}, \quad R^{(s+M,t)} = R^{(s,t+N)}. \tag{3.23}$$

for $s, t = 0, 1, 2, \ldots$.

Proof. From Propositions 2.5–2.6, we have

$$\Phi_i^{(s+1,t)} = R^{(s,t)} \Phi_i^{(s,t)} = R^{(s,t)} L^{(s,t)} \Phi_i^{(s,t+1)},$$

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and

$$\Phi_i^{(s+1,t)} = L^{(s+1,t)}\Phi_i^{(s+1,t+1)} = L^{(s+1,t)}R^{(s,t+1)}\Phi_i^{(s,t+1)}.$$

Hence, $L^{(s+1,t)}R^{(s,t+1)}\Phi_i^{(s,t+1)} = R^{(s,t)}L^{(s,t)}\Phi_i^{(s,t+1)}$ for $i = 1, 2, \dots, m$. Here, the m vectors $\Phi_1^{(s,t+1)}$, $\Phi_2^{(s,t+1)}$, $\dots$, $\Phi_m^{(s,t+1)}$ of size m are linearly independent because, from Theorem 2.1, they are the eigenvectors of a common matrix $A^{(s,t)}$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$. Therefore, we have (3.22). Equation (3.23) is a direct consequence of (2.3). $\square$

The last theorem is useful to construct a band matrix $A^{(s,t)}$ with prescribed eigenvalues. To see that, we rewrite the relations (3.22)–(3.23) for matrices into those for entries as follows:

$$q_k^{(s,t+1)} + e_{k-1}^{(s+1,t)} = q_k^{(s,t)} + e_k^{(s,t)}, \quad q_k^{(s,t+1)} e_k^{(s+1,t)} = q_{k+1}^{(s,t)} e_k^{(s,t)}, \tag{3.24a}$$

$$q_k^{(s+M,t)} = q_k^{(s,t+N)}, \quad e_k^{(s+M,t)} = e_k^{(s,t+N)}, \tag{3.24b}$$

$$e_0^{(s,t)} \equiv 0, \quad e_m^{(s,t)} \equiv 0 \tag{3.24c}$$

for $s, t = 0, 1, 2, \dots$ and $k = 1, 2, \dots, m$. Then, in generic, we may construct a band matrix $A^{(s,t)}$ of (2.20) with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$ by the following procedure:

1. Take the nonzero constants $c_k^{(s)}$ and $\hat{c}_k^{(t)}$ for $k = 1, 2, \dots, m$, $s = 0, 1, \dots, M-1$ and $t = 0, 1, \dots, N-1$ in (2.2) arbitrarily. Then, determine the sequence $f_{s,t}$ by (2.2).
2. Compute $q_k^{(s,t)}$ in $R^{(s,t)}$ and $e_k^{(s,t)}$ in $L^{(s,t)}$ by using relations (3.24) as a recurrence from the initial values $q_1^{(s,t)} = \tau_1^{(s+1,t)}/\tau_1^{(s,t)} = f_{s+1,t}/f_{s,t}$. (Remember (2.11).)
3. Construct $A^{(s,t)}$ by (2.20) where the entries $q_k^{(s,t)}$ of $R^{(s,t)}$ and $e_k^{(s,t)}$ of $L^{(s,t)}$ are those already computed in the last step.

Owing to Theorems 2.1 and 3.1, this procedure does make a band matrix $A^{(s,t)}$ of the form (1.1) whose eigenvalues are $\lambda_1, \lambda_2, \dots, \lambda_m$ *unless* the second step fails by division-by-zero. Such a failure only occurs when some $e_k^{(s,t)}$, $1 \le k < m$, becomes zero in the second step. From (2.13), that is the case where some $\tau_k^{(s,t)}$, $1 \le k \le m$, is zero. Thus, if and only if every $\tau_k^{(s,t)}$, $1 \le k \le m$, is nonzero, the above procedure works well.

The relations (3.24) can be viewed as a generalization of a discrete integrable system. In fact, (3.24) reduce into the extended hungry Toda equation

$$q_k^{(n+M)} + e_{k-1}^{(n+N)} = q_k^{(n)} + e_k^{(n)}, \quad q_k^{(n+M)} e_k^{(n+N)} = q_{k+1}^{(n)} e_k^{(n)}, \quad e_0^{(n)} \equiv 0, \quad e_m^{(n)} \equiv 0, \tag{3.25}$$

discussed in [5] by the authors when $q_k^{(s,t)} = q_k^{(sN+tM)}$, $e_k^{(s,t)} = e_k^{(sN+tM)}$ and $n = sN + tM$. This generalization of the extended discrete hungry Toda equation already appears in the literature concerning eigenvalue problems of band oscillatory matrices and is called the multiple qd formula in [30] and the hungry-type discrete 2-dimensional Toda equation in [28].

We discuss some criteria for the band matrix $A^{(s,t)}$ to be oscillatory.

THEOREM 3.2. *The band matrix $A^{(s,t)}$ in (2.20) is oscillatory provided that the determinant $\tau_k^{(s,t)}$ is positive for every $k = 0, 1, 2, \dots, m$ and $s, t = 0, 1, \dots$.*

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Proof. From (2.20) with (2.17) and (2.19), $A^{(s,t)}$ is oscillatory if every $q_k^{(s,t)}$ and $e_k^{(s,t)}$ are positive. From (2.11) and (2.13), every $q_k^{(s,t)}$ and $e_k^{(s,t)}$ are positive if every $\tau_k^{(s,t)}$ therein is positive. ∎

3.2. An algorithm. We summarize the procedure shown in Section 3.1 to construct band matrices in Algorithm 1. Algorithm 1 is written for computing $A^{(0,0)}$ only. We notice that Algorithm 1 includes a "going-back" operation from the second step to the first to prevent division-by-zero failure.

We compare Algorithm 1 with the previous algorithm proposed in [5] by the authors. Take the constants $c_i^{(s)}$ and $\hat{c}_i^{(t)}$ as

$$c_i^{(0)} = c_i^{(1)} = \cdots = c_i^{(M-1)} = \hat{c}_i^{(0)} = \hat{c}_i^{(1)} = \cdots = \hat{c}_i^{(N-1)}, \tag{3.26}$$

for $i = 1, 2, \ldots, m$. Then, the sequence $f_{s,t}$ in (2.2) becomes

$$f_{s,t} = \sum_{i=1}^{m} c_i \lambda_i^{\frac{s}{M}+\frac{t}{N}},$$

with some constants $c_1, c_2, \ldots, c_m$. This setting for $f_{s,t}$ is the same as in the previous algorithm in [5]. Thus, Algorithm 1 reduces into the previous algorithm in [5] just by taking the constants $c_i^{(s)}$ and $\hat{c}_i^{(t)}$ as in (3.26).

Algorithm 1 Construction of band matrices with prescribed eigenvalues

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Input: A matrix size m, positive integers M and N which determine upper and lower bandwidths, respectively, and distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$

Output: A band matrix $A = A^{(0,0)}$ of the form (1.1) whose eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_m$

1: Take $c_1^{(s)}, c_2^{(s)}, \ldots, c_m^{(s)}$ for $s = 0, 1, \ldots, M-1$ and $\hat{c}_1^{(t)}, \hat{c}_2^{(t)}, \ldots, \hat{c}_m^{(t)}$ for $t = 0, 1, \ldots, N-1$ arbitrarily.

2: In order to find $q_k^{(i,0)}$ ($k = 1, 2, \ldots, m$, $i = 0, 1, \ldots, M-1$) and $e_k^{(0,j)}$ ($k = 1, 2, \ldots, m-1$, $j = 0, 1, \ldots, N-1$), compute $q_k^{(s,t)}$ and $e_k^{(s,t)}$ by the recurrence

$$q_k^{(s,t)} = \begin{cases} q_k^{(s+M,t-N)} & (t \geq N), \\ \dfrac{f_{s+1,t}}{f_{s,t}} & (k = 1), \\ q_{k-1}^{(s,t+1)} \dfrac{e_{k-1}^{(s+1,t)}}{e_{k-1}^{(s,t)}} & (otherwise), \end{cases} \qquad e_k^{(s,t)} = \begin{cases} e_k^{(s+M,t-N)} & (t \geq N), \\ 0 & (k = 0), \\ q_k^{(s,t+1)} + e_{k-1}^{(s+1,t)} - q_k^{(s,t)} & (otherwise) \end{cases}$$

with $f_{s,t} = \sum_{i=1}^{m} c_i^{(s \bmod M)} \lambda_i^{\frac{s}{M}} \hat{c}_i^{(t \bmod N)} \lambda_i^{\frac{t}{N}}$. If $e_k^{(s,t)} = 0$ for some $k > 0$ then go back to the first step to take another set of $c_i^{(s)}$ and $\hat{c}_i^{(t)}$.

3: Construct

$$R^{(i,0)} = \begin{pmatrix} q_1^{(i,0)} & 1 & & \\ & q_2^{(i,0)} & \ddots & \\ & & \ddots & 1 \\ & & & q_m^{(i,0)} \end{pmatrix}, \quad L^{(0,j)} = \begin{pmatrix} 1 & & & \\ e_1^{(0,j)} & 1 & & \\ & \ddots & \ddots & \\ & & e_{m-1}^{(0,j)} & 1 \end{pmatrix}$$

for $i = 0, 1, \ldots, M-1$ and $j = 0, 1, \ldots, N-1$.

4: Compute $A = A^{(0,0)} = L^{(0,0)} L^{(0,1)} \cdots L^{(0,N-1)} R^{(M-1,0)} \cdots R^{(1,0)} R^{(0,0)}$.

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For the previous algorithm in [5], we have the following.

THEOREM 3.3 (cf. [5, Theorem 1, Proposition 8]). *Assume that the sequence $f_{s,t}$ is given by (2.2) with $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$ and $c_i^{(0)} = c_i^{(1)} = \cdots = c_i^{(M-1)} = \hat{c}_i^{(0)} = \hat{c}_i^{(1)} = \cdots = \hat{c}_i^{(N-1)} > 0$ for $i = 1, 2, \ldots, m$. Then, (i) $\tau_k^{(s,t)} > 0$ for every $k = 0, 1, 2, \ldots, m$ and $s, t = 0, 1, 2, \ldots,$ and hence, the band matrix $A^{(s,t)}$ is oscillatory; (ii) when the bandwidths M and N are not co-prime, the $A^{(s,t)}$ is a power of another band oscillatory matrix whose upper and lower bandwidths are smaller than M and N, respectively.*

Proof. The statement (i) is a direct consequence of [5, Theorem 1]. The statement (ii) is proved in [5, Proposition 8] in the case where one of the bandwidths M and N is a multiple of the other. Based on the same idea, we can prove the statement also when M and N are not co-prime. ◻

As Theorem 3.3 (ii) says, when the bandwidths M and N are not co-prime, the previous algorithm in [5], that is equivalent to Algorithm 1 with (3.26), always results in a band matrix $A^{(0,0)}$ which is a power of another band matrix X with narrower bandwidth: $A = X^p$, $p > 1$. This is a large defect in the previous algorithm because such a band matrix $A^{(0,0)}$ is non-generic. That is, almost every matrix of the form (1.1) does not have such power structure. Thus, the previous algorithm in [5] cannot produce generic band matrices of the form (1.1) with upper and lower bandwidths M and N. Actually, this is not the case with Algorithm 1. In the next section, we show by numerical examples that Algorithm 1 can produce a generic band matrix of the form (1.1) which is *not* a power of another band matrix with narrower bandwidth, and improves the previous algorithm in this defect.

4. Numerical examples.

We give numerical examples of Algorithm 1 mentioned in Section 3.2. Numerical construction of band oscillatory matrices was carried out on a computer, OS: Mac OS X 10.11.6, Software: Maple 17. We can arbitrarily prescribe nonzero and distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ for band oscillatory matrices. In the following examples, we choose $\lambda_i = (m - i + 1)^{\mathrm{lcm}(M,N)}$ so that the entries of the constructed matrices are rational.

In the first example, we demonstrate the defect in the previous algorithm proposed by the authors in [5] that, when the upper and lower bandwidths M and N are not co-prime, the algorithm cannot produce a generic band matrix, namely a band matrix which is not a power of another band matrix with narrower bandwidth. Note that, as is discussed in Section 3.2, we can perform computation with the previous algorithm by using Algorithm 1 with constants $c_i^{(s)}$ and $\hat{c}_i^{(t)}$ satisfying (3.26).

EXAMPLE 1. *Let the matrix size $m = 4$, the upper and lower bandwidths $M = N = 2$, and the eigenvalues $\lambda_1 = 16$, $\lambda_2 = 9$, $\lambda_3 = 4$ and $\lambda_4 = 1$. Suppose that $c_i^{(j)} = \hat{c}_i^{(j)} = 1$ for all i and j as satisfying (3.26). Then, the constructed matrix is*

$$A = A^{(0,0)} = L^{(0,0)} L^{(0,1)} R^{(1,0)} R^{(0,0)} = \begin{pmatrix} \frac{15}{2} & 5 & 1 & 0 \\ \frac{25}{4} & \frac{83}{10} & 5 & 1 \\ 1 & 4 & \frac{15}{2} & 5 \\ 0 & \frac{9}{25} & \frac{9}{4} & \frac{67}{10} \end{pmatrix},$$

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where

$$L^{(0,0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{2}{5} & 1 & 0 \\ 0 & 0 & \frac{3}{14} & 1 \end{pmatrix}, \quad L^{(0,1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{63}{155} & 1 & 0 \\ 0 & 0 & \frac{124}{483} & 1 \end{pmatrix},$$

$$R^{(0,0)} = \begin{pmatrix} \frac{5}{2} & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & \frac{21}{10} & 1 \\ 0 & 0 & 0 & \frac{16}{7} \end{pmatrix}, \quad R^{(1,0)} = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & \frac{31}{15} & 1 & 0 \\ 0 & 0 & \frac{414}{217} & 1 \\ 0 & 0 & 0 & \frac{140}{69} \end{pmatrix}.$$

This pentadiagonal matrix A with eigenvalues $\lambda_1 = 16$, $\lambda_2 = 9$, $\lambda_3 = 4$, $\lambda_4 = 1$, is non-generic, namely, is a square of a tridiagonal matrix, $A = (L^{(0,0)}R^{(0,0)})^2$. The tridiagonal matrix $L^{(0,0)}R^{(0,0)}$ has eigenvalues $4, 3, 2, 1$ which are the square roots of the eigenvalues $16, 9, 4, 1$ of A. Note that, as Theorem 3.3 assures, the pentadiagonal matrix A is oscillatory, for the irreducible A is totally nonnegative and invertible since each of $L^{(0,0)}, L^{(0,1)}, R^{(0,0)}$ and $R^{(1,0)}$ is so.

In the second example, we show that we may overcome the defect in the previous algorithm in [5] by employing Algorithm 1. Taking the constants $c_i^{(s)}$ and $\hat{c}_i^{(t)}$ without (3.26) in Algorithm 1, we may construct a generic band matrix which is not a power of another band matrix with narrower bandwidth even when the upper and lower bandwidths M and N are not co-prime.

Example 2. *Let the matrix size $m = 4$, the upper and lower bandwidths $M = N = 2$, and the eigenvalues $\lambda_1 = 16$, $\lambda_2 = 9$, $\lambda_3 = 4$ and $\lambda_4 = 1$. Suppose that $c_i^{(0)} = \hat{c}_i^{(0)} = \hat{c}_i^{(1)} = 1$ and $c_i^{(1)} = 2$ for all i. Then, the constructed matrix is*

$$A = A^{(0,0)} = L^{(0,0)}L^{(0,1)}R^{(1,0)}R^{(0,0)} = \begin{pmatrix} \frac{15}{2} & \frac{5}{2} & 1 & 0 \\ \frac{25}{2} & \frac{83}{10} & 10 & 1 \\ 1 & 2 & \frac{15}{2} & \frac{5}{2} \\ 0 & \frac{9}{25} & \frac{9}{2} & \frac{67}{10} \end{pmatrix},$$

where

$$L^{(0,0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & \frac{1}{5} & 1 & 0 \\ 0 & 0 & \frac{3}{7} & 1 \end{pmatrix}, \quad L^{(0,1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ 0 & \frac{63}{310} & 1 & 0 \\ 0 & 0 & \frac{248}{483} & 1 \end{pmatrix},$$

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$$R^{(0,0)} = \begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \frac{21}{5} & 1 \\ 0 & 0 & 0 & \frac{8}{7} \end{pmatrix}, \quad R^{(1,0)} = \begin{pmatrix} \frac{3}{2} & 1 & 0 & 0 \\ 0 & \frac{62}{15} & 1 & 0 \\ 0 & 0 & \frac{207}{217} & 1 \\ 0 & 0 & 0 & \frac{280}{69} \end{pmatrix}.$$

This pentadiagonal matrix A with eigenvalues $\lambda_1 = 16$, $\lambda_2 = 9$, $\lambda_3 = 4$, $\lambda_4 = 1$ is not *a square of any tridiagonal matrix. Note that the pentadiagonal matrix A is oscillatory since the irreducible A is totally nonnegative and invertible since each of $L^{(0,0)}, L^{(0,1)}, R^{(0,0)}$ and $R^{(1,0)}$ is so.*

In the final example, we examine the numerical stability of Algorithm 1 by using the floating point arithmetic. Numerical construction was carried out on a computer, OS: Mac OS X (ver. 10.11.5), CPU: 2.7GHz 12-Core Intel Xeon E5, and Compiler: Apple LLVM version 7.3.0. We employed multi-precision floating point libraries, GNU GMP Library [16] (Ver. 3.1.4) and GNU MPFR Library [17] (Ver. 6.1.0).

Example 3. *We tried to construct the same matrix $A^{(0,0)}$ as Example 1 from the same parameters by using 53-bit, 64-bit and 128-bit precision arithmetic. As a result from Algorithm 1 with 53-bit, 64-bit and 128-bit precision arithmetic, we obtained the matrices*

$$\hat{A}_{53}^{(0,0)} = \begin{pmatrix} 7.50000000000000 & 5.00000000000000 & 1.00000000000000 & 0.00000000000000 \\ 6.25000000000000 & 8.30000000000000 & 5.00000000000003 & 1.00000000000000 \\ 0.99999999999999 & 4.00000000000000 & 7.49999999999996 & 5.00000000000281 \\ 0.00000000000000 & 0.35999999999982 & 2.25000000000016 & 6.69999999999923 \end{pmatrix},$$

$$\hat{A}_{64}^{(0,0)} = \begin{pmatrix} 7.5000000000000000 & 4.5000000000000000 & 1.0000000000000000 & 0.0000000000000000 \\ 6.2500000000000000 & 8.3000000000000000 & 4.5000000000000000 & 1.0000000000000000 \\ 1.0000000000000000 & 4.0000000000000000 & 7.4999999999999999 & 5.0000000000000007 \\ 0.0000000000000000 & 0.3500000000000000 & 2.2500000000000001 & 6.7000000000000012 \end{pmatrix},$$

$$\hat{A}_{128}^{(0,0)} = \begin{pmatrix} 7.5000000000000000 & 4.5000000000000000 & 1.0000000000000000 & 0.0000000000000000 \\ 6.2500000000000000 & 8.3000000000000000 & 4.5000000000000000 & 1.0000000000000000 \\ 1.0000000000000000 & 4.0000000000000000 & 7.5000000000000000 & 5.0000000000000000 \\ 0.0000000000000000 & 0.3500000000000000 & 2.2500000000000000 & 6.7000000000000000 \end{pmatrix},$$

respectively, where all of the entries are rounded to 16-digit numbers. The matrix $\hat{A}_{128}^{(0,0)}$ obtained with 128-bit precision arithmetic is totally equal to the decimal expression of $A^{(0,0)}$ obtained in Example 1 with symbolic computation. We show in Table 1 the eigenvalues $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\lambda}_3$ and $\hat{\lambda}_4$ of $\hat{A}_{53}^{(0,0)}$, $\hat{A}_{64}^{(0,0)}$ and $\hat{A}_{128}^{(0,0)}$ computed with the QR algorithm. The program code of the QR algorithm was prepared according to [24] with 4096-bit precision arithmetic, and the results are rounded to 16-digit numbers. We observe from Table 1 that the matrix $\hat{A}_{*}^{(0,0)}$ constructed with floating point arithmetic and its eigenvalues $\hat{\lambda}_i$ get close to the exact $A^{(0,0)}$ and λ_i by using an enough number of bits. Actually, the authors observed in other examples that $\hat{\lambda}_i$ get linearly close to λ_i as the number of bits increases.

5. Concluding remarks. In this paper, we proposed an algorithm for solving an inverse eigenvalue problem for band matrices by using a generalization of a discrete integrable system called the extended discrete hungry Toda equation. When the upper and lower bandwidths of band matrices are not co-prime, the previous algorithm in [5] by the authors must produce a non-generic band matrix which is a power of another band matrix with narrower bandwidth. The proposed algorithm, Algorithm 1, improves the

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TABLE 1
The eigenvalues $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3$ and $\hat{\lambda}_4$ of $\hat{A}_{53}^{(0,0)}$, $\hat{A}_{64}^{(0,0)}$ and $\hat{A}_{128}^{(0,0)}$ that are computed with 53-bit, 64-bit and 128-bit precision arithmetic, respectively.

	53-bit	64-bit	128-bit
$\hat{\lambda}_1$	16.0000000000002	16.0000000000000	16.0000000000000
$\hat{\lambda}_2$	9.00000000000040	9.00000000000000	9.00000000000000
$\hat{\lambda}_3$	3.99999999999898	4.00000000000000	4.00000000000000
$\hat{\lambda}_4$	0.99999999999965	1.00000000000000	1.00000000000000

previous algorithm in this defect of genericity; Algorithm 1 enables us to construct a band matrix which is *not* a power of another band matrix with narrower bandwidth even when the upper and lower bandwidths are not co-prime. We demonstrated this improvement by numerical examples of constructing band matrices with prescribed eigenvalues by using Algorithm 1. The numerical examples shows that Algorithm 1 might be useful to construct band matrices which is generic (in the above sense) as well as oscillatory.

In the future, we should find a criterion, which would be more convenient than Theorem 3.2, for band matrices obtained from Algorithm 1 to be oscillatory. This problem is much concerned with positivity of solutions to the generalization of the extended discrete hungry Toda equation discussed in this paper. It is also expected that we may develop some algorithms with finite steps for solving inverse eigenvalue problems for other structured matrices from other discrete integrable systems.

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