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STARLIKE TREES WITH MAXIMUM DEGREE 4 ARE DETERMINED BY THEIR SIGNLESS LAPLACIAN SPECTRA*

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Abstract. A graph is said to be determined by its signless Laplacian spectrum if there is no other non-isomorphic graph with the same spectrum. In this paper, it is shown that each starlike tree with maximum degree 4 is determined by its signless Laplacian spectrum.

Key words. Starlike trees, Spectra of graphs, Cospectral graphs.

AMS subject classifications. 05C50.

1. Introduction. In this paper, we are only concerned with undirected simple graphs (loops and multiple edges are not allowed). Let G be a graph with n vertices, m edges and the adjacency matrix A. We denote the maximum degree of G by $\Delta(G)$. Let D be the diagonal matrix of vertex degrees. The matrices $L = D - A$ and $Q = D + A$ are called the *Laplacian matrix* and *signless Laplacian matrix* of G, respectively. Since A, L and Q are real symmetric matrices, their eigenvalues are real numbers. So we can assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are the adjacency and signless Laplacian eigenvalues of G, respectively.

Let M be an associated matrix of a graph G (the adjacency matrix, the Laplacian matrix and the signless Laplacian matrix). The multiset of eigenvalues of M is called the *M spectrum* of G. Two graphs are said to be *cospectral with respect to* M if they have the same M spectrum. A graph is said to be *determined* (DS for short) *by the M spectrum* if there is no other non-isomorphic graph with the same spectrum of M. A tree is called *starlike* if it has exactly one vertex of degree greater than two. We will denote by $S(l_1, l_2, \ldots, l_r)$ the unique starlike tree such that $S(l_1, l_2, \ldots, l_r) - v = P_{l_1} \cup P_{l_2} \cup \ldots \cup P_{l_r}$, where P_{l_i} is the path on l_i vertices ($i = 1, \ldots, r$) and v is the vertex of degree greater than two. A starlike with maximum degree 3 is called a T-*shape*

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and is denoted by $T(l_1, l_2, l_3)$.

Since the problem of characterization of DS graphs is very difficult, finding any new infinite family of DS graphs is interesting. In [7], it was shown that $T(1, 1, n-3)$ and some graphs related to it are determined by their adjacency spectra as well as their Laplacian spectra. In [9], Wang and Xu proved that $T(l_1, l_2, l_3)$ is determined by its adjacency spectrum if and only if $(l_1, l_2, l_3) \neq (l, l, 2l-2)$ for any integer $l \geq 2$. In [10] they moreover showed that T-shape trees are determined by their Laplacian spectra. Tajbakhsh and Omidi showed that starlike trees are determined by their Laplacian spectra (see [6]). In [5] it has been shown that $T(l_1, l_2, l_3)$ is determined by its signless Laplacian spectrum if and only if $(l_1, l_2, l_3) \neq (l, l, 2l-1)$ for any integer $l \geq 1$. In this paper, we show that each starlike tree with maximum degree 4 is determined by its signless Laplacian spectrum.

2. Preliminaries.

First we give some facts that are needed in the next section.

LEMMA 2.1. *[8](*Interlacing*) Suppose that A is a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then the eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ of a principal submatrix of A of size $m \times m$ satisfy $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for $i = 1, \ldots, m$.*

LEMMA 2.2. *([8])Let G be a graph. For the adjacency matrix, the Laplacian matrix and the signless Laplacian the following can be obtained from the spectrum.*

i) The number of vertices.
ii) The number of edges.
For the adjacency matrix the following follows from the spectrum.
iii) The number of closed walks of any length.
iv) Whether G is bipartite.

LEMMA 2.3. *[2] The least eigenvalue of the signless Laplacian of a connected graph is equal to 0 if and only if the graph is bipartite. In this case 0 is a simple eigenvalue.*

COROLLARY 2.4. *In any graph (possibly disconnected) the multiplicity of the eigenvalue* 0 *of the signless Laplacian is equal to the number of bipartite components.*
The line graph of a starlike tree $S(l_1, l_2, \ldots, l_r)$ is called the *sunlike* graph. We will denote this by $K(l_1, l_2, \ldots, l_r)$.

THEOREM 2.5. *[4] If $K(l_1, l_2, \ldots, l_r)$ and $K(l'_1, l'_2, \ldots, l'_m)$ are two cospectral sunlike graphs with respect to the adjacency matrix, then they are isomorphic.*

LEMMA 2.6. *[2] Let G be a connected graph and let H be a proper subgraph of G. Then $\lambda_1(H) < \lambda_1(G)$.*

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Theorem 2.7. *[2] Let G and H be connected graphs and $\{G, H\} \neq \{K_{1,3}, K_3\}$. Then G and H are isomorphic if and only if their line graphs $L(G)$ and $L(H)$ are isomorphic.*

Lemma 2.8. *[3] Let G be a connected graph that is not isomorphic to W_n, where W_n is a graph obtained from the path $P_{(n-2)}$ (indexed by the natural order of $1, 2, \ldots, n-2$) by adding two pendant edges at vertices 2 and $n-3$. Let G_{uv} be the graph obtained from G by subdividing the edge uv of G. If uv lies on an internal path of G, then $\lambda_1(G_{uv}) \leq \lambda_1(G)$.*

Let n, m, R be the number of vertices, the number of edges and the vertex-edge incidence matrix of a graph G, respectively. The following relations are well-known:

$$RR^T = A + D, \qquad R^T R = A_L + 2I, \tag{2.1}$$

where D is the diagonal matrix of vertex degrees and A_L is the adjacency matrix of the line graph $L(G)$ of G. Let $P_{L(G)}(\lambda)$ and $Q_G(\lambda)$ be characteristic polynomials of $L(G)$ and G with respect to the adjacency and signless Laplacian matrices, respectively. Since non-zero eigenvalues of RR^T and $R^T R$ are the same, by relations (2.1), we immediately obtain:

$$P_{L(G)}(\lambda) = (\lambda + 2)^{(m-n)} Q_G(\lambda + 2). \tag{2.2}$$

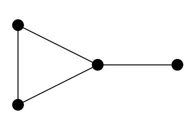
Remark 2.9. *If $m < n$, the matrix Q must have eigenvalue 0 with multiplicity at least $n - m$.*

Corollary 2.10. *If two graphs G and G' are cospectral with respect to the signless Laplacian matrix, then $L(G)$ and $L(G')$ are cospectral with respect to the adjacency matrix.*

The following useful Lemma provides some formulas for calculating the number of closed walks of small lengths.

Lemma 2.11. *[5] Let $N_G(H)$ be the number of subgraphs of a graph G which are isomorphic to H and let $N_G(i)$ be the number of closed walks of length i of G. Then:*

i) $N_G(2) = 2m$, $N_G(3) = 6N_G(K_3)$ *and* $N_G(4) = 2m + 4N_G(P_3) + 8N_G(C_4)$,
ii) $N_G(5) = 30N_G(K_3) + 10N_G(C_5) + 10N_G(T_0)$. *(see Fig.1)*



T_0

Fig.1

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LEMMA 2.12. *[1] Let G be a line graph. Then G does not have T_i for $i \in \{1,2,3\}$ as an induced subgraph (see Fig.2).*

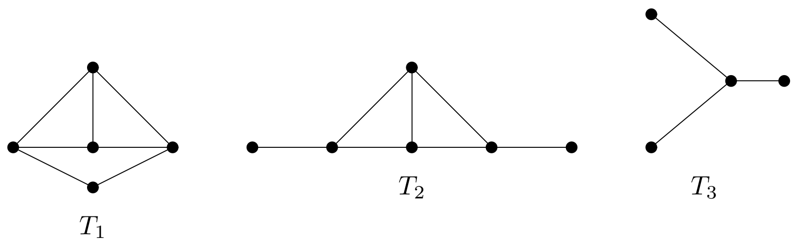

Fig.2

3. Main results. Using the previous facts, we show that each non-isomorphic graph Q-cospectral to a given starlike tree with maximum degree 4 is either of type T_{44} or a disjoint union of T_{45} with one path (see Fig.7). Finally we show that there is no such graph and so each starlike tree with maximum degree 4 is determined by its signless Laplacian spectrum.

LEMMA 3.1. *Let $G = K(a,b,c,d)$ with $\min\{a,b,c,d\} \geq 1$. Then:*

i) 2 can not be an adjacency eigenvalue of G,
ii) If $b = c = d = 1$ and $a > 1$, then 0 can not be an adjacency eigenvalue of G.

Proof.

i) Let 2 be an eigenvalue of G and let $Z \neq 0$ be the eigenvector corresponding to 2 of G. Suppose $V(G) = \{v_1, v_2, \cdots, v_n\}$ be the vertices of G and let $N_i = \{j | v_i v_j \in E(G)\}$ for $1 \leq i \leq n$. Let z_i be the i-th entry of Z. Since $AZ = 2Z$, for $1 \leq i \leq n$, we have :

$$\sum_{j \in N_i} z_j = 2z_i. \tag{3.1}$$

It is easy to see that if $z_1 z_{a+1} z_{a+b+1} z_{a+b+c+1} = 0$, then $Z = 0$. Which is not true. So $z_i \neq 0$, for $i \in \{1, a+1, a+b+1, a+b+c+1\}$. Using relation (3.1), we have $z_i = iz_1$ for $1 \leq i \leq a$, $z_{a+i} = iz_{a+1}$ for $1 \leq i \leq b$, $z_{a+b+i} = iz_{a+b+1}$ for $1 \leq i \leq c$ and $z_{a+b+c+i} = iz_{a+b+c+1}$ for $1 \leq i \leq d$. Again by relation (3.1), we have $2z_a = z_{a-1} + z_{a+b} + z_{a+b+c} + z_{a+b+c+d}$, $2z_{a+b} = z_{a+b-1} + z_a + z_{a+b+c} + z_{a+b+c+d}$, $2z_{a+b+c} = z_{a+b+c-1} + z_{a+b} + z_a + z_{a+b+c+d}$ and $2z_{a+b+c+d} = z_{a+b+c+d-1} + z_a + z_{a+b} + z_{a+b+c}$. So

$$(2a-1)z_1 + (2b-1)z_{a+1} + (2c-1)z_{a+b+1} + (2d-1)z_{a+b+c+1} = 0.$$

Moreover it is clear that

$$(2a+1)z_1 = (2b+1)z_{a+1} = (2c+1)z_{a+b+1} = (2d+1)z_{a+b+c+1}$$

$$= az_1 + bz_{a+1} + cz_{a+b+1} + dz_{a+b+c+1}.$$

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Since a, b, c and d are positive integers, we have $z_1 = z_{a+1} = z_{a+b+1} = z_{a+b+c+1} = 0$, which is not true (see Fig.3).

ii) In the similar way we can prove that 0 can not be the eigenvalue of G. ◻

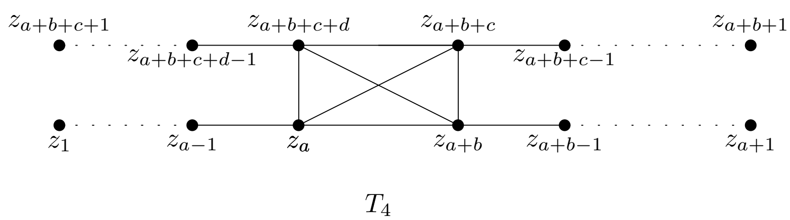

Fig.3

Lemma 3.2. *Let $G_1 = S(a, b, c, d)$ be the starlike tree where* $\min\{a, b, c, d\} \geq 1$ *and let G_2 be a cospectral to G_1 with respect to the signless Laplacian matrix. Let H_1 and H_2 be the line graphs of G_1 and G_2, respectively. Let y_i and x_i be the numbers of vertices of degree i of H_1 and H_2 respectively. Then:*

i) *The graph G_2 has exactly one bipartite component,*
ii) $x_0 \leq 1$,
iii) $\Delta(H_2) \in \{3, 4\}$,
iv) $x_1 = 2x_4 + x_3 - 2x_0 - 4$ *and* $x_0 + x_3 + 3x_4 + 2N_{H_2}(C_4) = 6 + 2y_4$,
v) $\lambda_3(H_1) < 2$.

Proof. i) Since G_1 is a connected bipartite graph, by Corollary 2.4, G_2 has exactly one bipartite component.

ii) Each vertex of degree 0 of H_2 is corresponding to the component P_2 of G_2, so by i), $x_0 \leq 1$.

iii) By Corollary 2.10, two graphs H_1 and H_2 are cospectral with respect to the adjacency matrix. By Lemma 2.11 and Lemma 2.2, $N_{H_1}(K_3) = N_{H_2}(K_3) = 4$. So $\Delta(H_2) \geq 2$. If $\Delta(H_2) = 2$, then each component of H_2 is either a path or a cycle. Since each cycle has 2 as an eigenvalue, by Lemma 3.1, H_2 contains no any cycle as a component. So each component of H_2 is a path. Hence $N_{H_2}(K_3) = 0$, which is a contradiction. Now let $\Delta(H_2) = t$ and let x be a vertex of degree t of H_2. Suppose $e = uv$ be the corresponding edge to x of G_2. Since x is a vertex of degree t, the edge $e = uv$ has t edges of G_2 as neighborhoods. Let $(deg(u), deg(v)) = (r, s)$, where $r + s - 2 = t$. Then $4 = N_{H_2}(K_3) \geq N_{K_r}(K_3) + N_{K_s}(K_3)$ and so $r + s \leq 6$. Hence $\Delta(H_2) = t \leq 4$.

iv) Since $H_1 = L(G_1) = K(a, b, c, d)$, it is clear that $y_1 = y_4$, $y_0 = 0$, $y_3 = 4 - y_4$ and $y_2 = n - y_4 - 4$. Then by ii) and iii) of Lemma 2.2, we have $\sum_{i=0}^{4} i^2 x_i + 4N_{H_2}(C_4) =$

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$\sum_{i=0}^{4} i^2 y_i + 4N_{H_1}(C_4)$. So

$$\sum_{i=0}^{4} i^2 x_i + 4N_{H_2}(C_4) = 4n + 4y_4 + 24. \tag{3.2}$$

By Lemma 2.2, the number of edges of H_2 is equal $n+2$, where n is the number of vertices of H_1. Hence by Lemma 2.2, we have $\sum_{i=0}^{4} x_i = n$ and $\sum_{i=1}^{4} ix_i = 2n+4$. By relation (3.2), we have $x_1 = 2x_4 + x_3 - 2x_0 - 4$ and $x_0 + x_3 + 3x_4 + 2N_{H_2}(C_4) = 6 + 2y_4$.

v) Let K be a graph obtain by deleting two vertices of degree at least 3 of H_1. Then each component of K is a path. Since the largest adjacency eigenvalue of each path is less than 2, by Lemma 2.1, we have $\lambda_3(H_1) \leq \lambda_1(K) < 2$. □

LEMMA 3.3. *Let $G_1 = S(a,b,c,d) \neq K_{1,4}$ and let G_2 be cospectral graphs with respect to the signless Laplacian matrix. Let H_1 and H_2 be the line graphs of G_1 and G_2 respectively. If x is the vertex of degree 4 in H_2, then the induced subgraph of x and its neighborhoods is of type T_5 or T_6 (see Fig.4).*

T_5 T_6

Fig.4

Proof. By Corollary 2.10, two graphs H_1 and H_2 are cospectral with respect to the adjacency matrix. So by Lemma 2.2, two graphs H_1 and H_2 have the same number of closed walks of length 3 and so by Lemma 2.11, $N_{H_2}(K_3) = 4$. Let $e = uv$ be corresponding edge of x of G_2. Since x is a vertex of degree 4, the edge $e = uv$ has 4 edges of G_2 as neighborhoods. We have the following cases:
Case1: If $(deg(u), deg(v)) \in \{(1,5),(5,1)\}$, then $N_{H_2}(K_3) > 4$. This is impossible.
Case2: If $(deg(u), deg(v)) \in \{(2,4),(4,2)\}$, since $N_{H_2}(K_3) = 4$, then the induced subgraph of x and its neighborhoods is of type T_5.
Case3: If $(deg(u), deg(v)) = (3,3)$, then the induced subgraph of x and its neighborhoods is of type T_6, T_7 and T_8 (see Fig.4 and Fig.5). If the induced subgraph of x and its neighborhoods is of type T_7, then $x_0 + x_3 + 3x_4 + 2N_{H_2}(C_4) > 14$. By Lemma 3.2 it is impossible. Now suppose the induced subgraph of x and its neighborhoods be of type T_8. First suppose $x_4 = 1$. By Lemma 3.1, H_2 does not have any component of type C_5. On the other hand by Lemma 3.2, $x_1 = x_3 - 2x_0 - 2$. Therefore $N_{H_2}(C_5) = 1$ and so $N_{H_2}(C_5) + N_{H_2}(T_0) \leq 16$. Moreover $N_{H_1}(C_5) = 0$ and $N_{H_1}(T_0) = 12 + 3y_4$. Since $N_{H_1}(5) = N_{H_2}(5)$ and $N_{H_1}(K_3) = N_{H_2}(K_3) = 4$, by Lemma 2.11, we have $y_4 \in \{0,1\}$. Since $G_1 = S(a,b,c,d) \neq K_{1,4}$, we have $y_4 = 1$ and so $b = c = d = 1$, $a > 1$. Therefore by Lemma 3.2, $8 = 6 + 2y_4 = x_0 + x_3 + 3x_4 + 2N_{H_2}(C_4) \geq 9$, which

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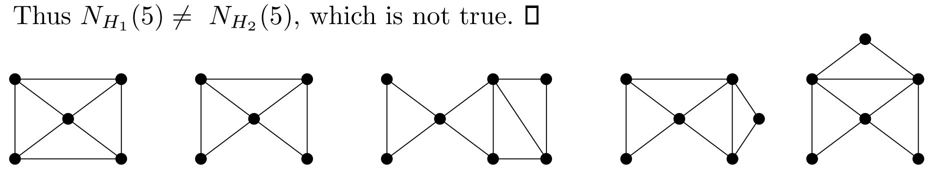
is impossible.
Now suppose H_2 has more than one vertex of degree greater than 4 , then H_2 has T_9, T_{10} or T_{11} as a subgraph (see Fig.5). It is easy to see that each graph on 6 vertices with T_{10} as a subgraph has either more than 4 triangles or is not the line graph of any graph. So if T_{10} is a subgraph of H_2, then it is an induced subgraph. Since $H_2 = L(G_2)$ and T_{10} is not the line graph of any graph, H_2 does not have T_{10} as a subgraph. If H_2 has a subgraph of type T_{11}, then by iv) of Lemma 3.2, $2y_4 + 6 = x_0 + x_3 + 3x_4 + 2N_{H_2}(C_4) \geq 15$ and so $y_4 \geq 9/2$, which is not true. Now let H_2 has T_9 as a subgraph. Since $N_{H_2}(C_4) \geq 2$, by iv) of Lemma 3.2, $x_4 \leq 3$ and so $x_4 \in \{2, 3\}$. If $x_4 = 3$, then by iv) of Lemma 3.2, $y_4 = 4$ and $x_0 + x_3 = 1$. Since T_9 has 4 vertices of degree greater than 2, H_2 has at least 4 vertices of degree greater than 2. Hence $x_3 = 1$ and $x_0 = 0$. Since H_2 does not have any cycle as a component, it is easy to see that H_2 has two components one of them is a path. Using Lemma 2.11, we have $N_{H_2}(5) = 280$ and $N_{H_1}(5) = 360$, which is not true. If $x_4 = 2$, since H_2 has at least 4 vertices of degree greater than 2, $x_3 \geq 2$. By iv) of Lemma 3.2, $x_0 + x_3 \leq 4$. If $x_3 = 2$, then H_2 has two components, one of them is T_9 and another is a path. By iv) of Lemma 3.2, $y_4 = 3$ and using Lemma 2.11, we have $N_{H_2}(5) = 270$ and $N_{H_1}(5) = 330$, which is not true. Now let H_2 has 3 or 4 vertices of degree 3. Using Lemma 2.11, we have $N_{H_2}(5) \in \{280, 290\}$ and $N_{H_1}(5) = 240 + 30y_4$ is a multiple of 30. Thus $N_{H_1}(5) \neq N_{H_2}(5)$, which is not true. ▯



T_7 T_8 T_9 T_{10} T_{11}

Fig.5

Let $H_1 = K(a, b, c, d)$ and H_2 be non-isomorphic cospectral graphs with respect to the adjacency matrix. Let N be the number of cycles of H_2 where the induced subgraph obtained by its vertices contains no any triangle as a subgraph. Again let x_i be the number of vertices of degree i of H_2. We have the following useful lemma.

Lemma 3.4. *Let H_2 does not have K_4 as a subgraph. Then:*

i) H_2 contains exactly one T_i as a subgraph for $12 \leq i \leq 43$ where different components of T_i lie in the different components of H_2 (see Fig.6).
Also if H_2 contains T_l as a subgraph for $12 \leq l \leq 40$, then
ii) If H_2 contains no path as a component or $x_0 = 1$, then $x_1 + 2N = x_3 - s$,
iii) If $x_0 = 0$ and H_2 contains exactly one path as a component, then $x_1 + 2N = x_3 - r$.

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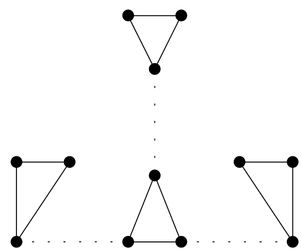
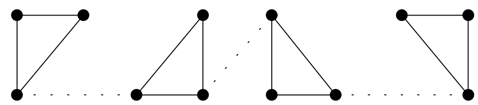
Where,

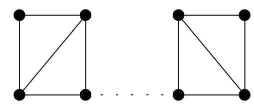
$$(s,r) = \begin{cases} (6,4) & 12 \le i \le 16. \\ (4,2) & 17 \le i \le 28. \\ (2,0) & 29 \le i \le 37. \\ (0,-2) & 38 \le i \le 40. \end{cases}$$



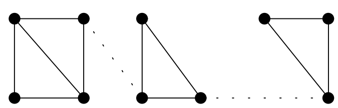
T_{12}



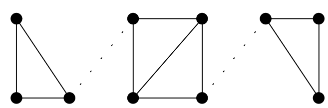
T_{13}



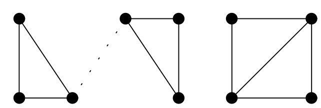
T_{14}



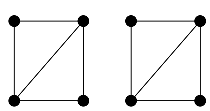
T_{15}



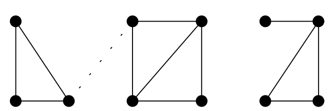
T_{16}



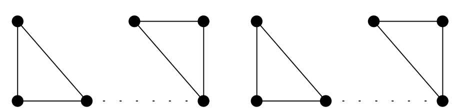
T_{17}



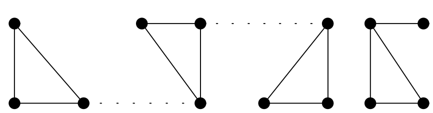
T_{18}



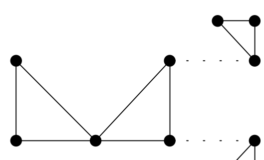
T_{19}



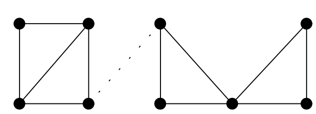
T_{20}



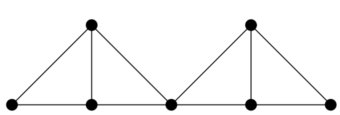
T_{21}



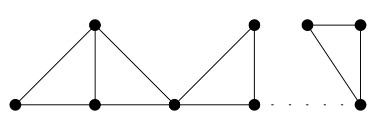
T_{22}



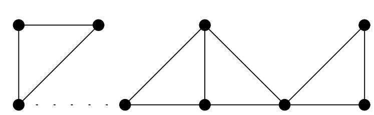
T_{23}



T_{24}



T_{25}



T_{26}

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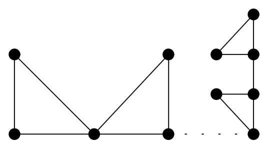
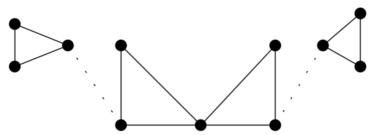


T_{27}



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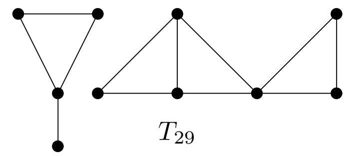
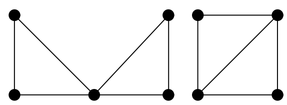
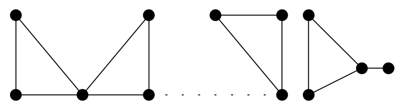


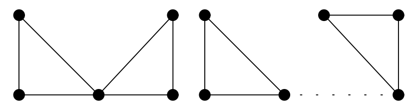
T_{29}



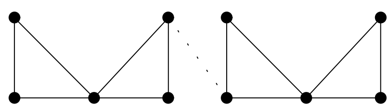
T_{30}



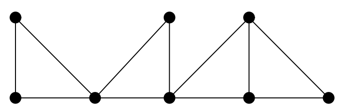
T_{31}



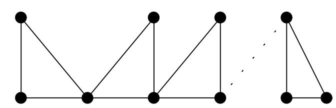
T_{32}



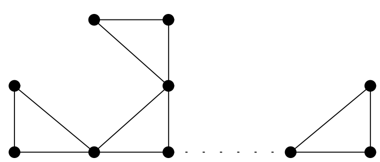
T_{33}



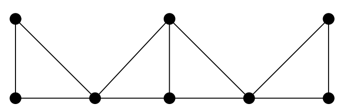
T_{34}



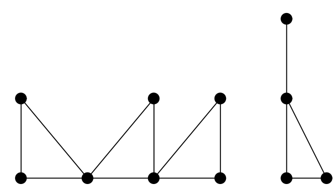
T_{35}



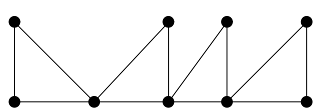
T_{36}



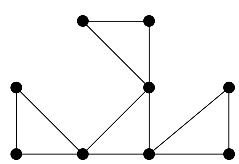
T_{37}



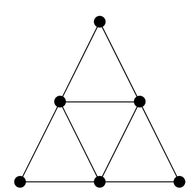
T_{38}



T_{39}



T_{40}



T_{41}

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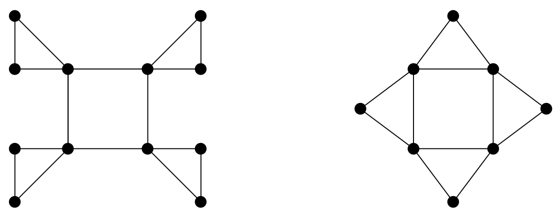

T_{42} T_{43}

Fig.6

Proof. i) Since H_1 and H_2 are cospectral with respect to the adjacency matrix, by Lemma 2.2, $N_{H_1}(3) = N_{H_2}(3)$ and so by Lemma 2.11, we have $N_{H_2}(K_3) = 4$. By Lemma 3.2, $\lambda_3(H_2) < 2$ and so the number of non-tree components of H_2 is at most 2. Moreover by Lemma 3.1, H_2 does not have any cycle as a component. On the other hand by Lemma 2.12, H_2 does not have $K_{1,3}$ as an induced subgraph and so each non-tree component of H_2 has K_3 as a subgraph. Let K be a disjoint union of non-tree components of H_2. By Lemma 3.2, $\Delta(H_2) \in \{3, 4\}$. Note that by Lemma 3.3, each vertex of degree 4 of H_2 is a vertex of K. If $\Delta(H_2) = 3$, then K has exactly one T_i for $12 \le i \le 21$ as a subgraph. Now let $\Delta(H_2) = 4$. If $x_4 = 1$, then K has exactly one T_i for $22 \le i \le 32$ as a subgraph. If $x_4 = 2$, then K has exactly one T_i for $33 \le i \le 38$ as a subgraph. If $x_4 = 3$, then K has exactly one T_i for $i \in \{39, 40, 41\}$ as a subgraph. If $x_4 = 4$, then K has exactly one T_i for $i \in \{42, 43\}$ as a subgraph.

ii, iii) Since H_2 does not have $K_{1,3}$ as an induced subgraph and $N_{H_2}(K_3) = 4$, each vertex of degree 3 of H_2 is a vertex of subgraph T_l. It is clear that the number of vertices with degree 2 of T_l where their degrees are 3 in H_2 is equal to $2N$ plus the number of vertices with degree 1 of $V(H_2) \setminus V(T_l)$. Let z_i be the number of vertices of degree i of T_l. So $x_3 - z_3 = 2N + x_1 - z_1$. If $s = z_3 - z_1$, where H_2 contains no any path as a component or $x_0 = 1$, then $x_1 + 2N = x_3 - s$. If $r = z_3 - z_1$, where $x_0 = 0$ and H_2 contains exactly one path as a component, then $x_1 + 2N = x_3 - r$. ∎

LEMMA 3.5. *Let $G_1 = S(a, b, c, d) \neq K_{1,4}$ and G_2 be cospectral graphs with respect to the signless Laplacian matrix. Let H_2 be the line graph of G_2. Then $\Delta(H_2) = 4$.*

Proof. By Corollary 2.10, the graphs $H_1 = L(G_1)$ and H_2 are cospectral with respect to the adjacency matrix. So by Lemma 2.2 and Lemma 2.11, $N_{H_2}(K_3) = 4$. Let x_i be the number of vertices of degree i of H_2 , by Lemma 3.2, it is sufficient to show that $\Delta(H_2) \neq 3$. Let $\Delta(H_2) = 3$. By Lemma 3.1, H_2 does not have any cycle as a component. If H_2 has K_4 as a subgraph, then since $\Delta(H_2) = 3$, K_4 is the component of H_2. Since $L(G_2) = H_2$ and $N_{H_2}(K_3) = N_{K_4}(K_3) = 4$, all other components of H_2 are trees. By Lemma 3.2, G_2 has exactly one bipartite component. Hence $H_2 = K_4$ and so $G_2 = K_{1,4}$ and by Theorem 2.2, $G_1 = K_{1,4}$. Which is impossible. So H_2 does

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not have K_4 as a subgraph. Since $N_{H_2}(K_3) = 4$ and $x_4 = 0$, H_2 has a subgraph of type T_i for $12 \leq i \leq 21$ (see Fig.6).

Step1: Let $12 \leq i \leq 16$. If $x_0 = 1$, then by Lemma 3.2, $x_1 = x_3 - 6$ and by Lemma 3.4, $x_1 + 2N = x_3 - 6$. So $N = 0$. If $x_0 = 0$ and H_2 contains exactly one path as a component, then by Lemma 3.2, $x_1 = x_3 - 4$ and by Lemma 3.4, $x_1 + 2N = x_3 - 4$. Hence $N = 0$. So by Lemma 3.2, for $i \in \{12, 13\}$, we have $x_3 = 5+2y_4$ or $x_3 = 6+2y_4$. Hence by Lemma 2.11, $N_{H_2}(5) = 170 + 20y_4$ or $N_{H_2}(5) = 180 + 20y_4$. On the other hand $N_{H_1}(5) = 240 + 30y_4$. However $N_{H_1}(5) \neq N_{H_2}(5)$. Which is a contradiction to this fact H_1 and H_2 are cospectral with respect to adjacency matrix. For $i = 14$, we have $x_3 = 1 + 2y_4$ or $x_3 = 2 + 2y_4$. By Lemma 2.11, $N_{H_2}(5) = 170 + 20y_4$ or $N_{H_2}(5) = 180 + 20y_4$. On the other hand $N_{H_1}(5) = 240 + 30y_4$. So $N_{H_1}(5) \neq N_{H_2}(5)$. Which is a contradiction to this fact H_1 and H_2 are cospectral with respect to adjacency matrix. For $i = 15$, we have $x_3 = 3+2y_4$ or $x_3 = 4+2y_4$. By Lemma 2.11, $N_{H_2}(5) = 170+20y_4$ or $N_{H_2}(5) = 180+20y_4$. On the other hand $N_{H_1}(5) = 240+30y_4$. So $N_{H_1}(5) \neq N_{H_2}(5)$. Which is a contradiction to this fact H_1 and H_2 are cospectral with respect to adjacency matrix. If H_2 contains no any path as a component, then by Lemma 3.2, $x_1 = x_3 - 4$ and by Lemma 3.4, $x_1 + 2N = x_3 - 6$. Hence $N = -1$, which is impossible. It is easy to see that if H_2 has T_{16} as a subgraph, then it has either T_2 or T_3 as an induced subgraph. Since $H_2 = L(G_2)$ by Lemma 2.12, which is impossible.

Step2: Let $17 \leq i \leq 21$. First let $i \in \{17, 19\}$. If $x_0 = 1$, then by Lemma 3.2, $x_1 = x_3 - 6$ and by Lemma 3.4, $x_1 + 2N = x_3 - 4$. Also if $x_0 = 0$ and H_2 contains exactly one path as a component, then $x_1 + 2N = x_3 - 2$ and $x_1 = x_3 - 4$. Any way we have $N = 1$. Thus $N_{H_2}(C_4) = 1$ or 2. If $N_{H_2}(C_4) = 1$, then by Lemma 3.2, we have $x_3 = 3 + 2y_4$ or $x_3 = 4 + 2y_4$. By Lemma 2.11, $N_{H_2}(5) = 170 + 20y_4$ or $N_{H_2}(5) = 180 + 20y_4$. If $N_{H_2}(C_4) = 2$, then $x_3 = 1 + 2y_4$ or $x_3 = 2 + 2y_4$. So $N_{H_2}(5) = 150 + 20y_4$ or $N_{H_2}(5) = 160 + 20y_4$. Moreover $N_{H_1}(5) = 240 + 30y_4$. However, this is a contradiction to this fact H_1 and H_2 are cospectral with respect to adjacency matrix. If H_2 contains no any path as a component, then by Lemma 3.2, $x_1 = x_3 - 4$ and by Lemma 3.4, $x_1 + 2N = x_3 - 4$. Therefor $N = 0$ and so $N_{H_2}(C_4) = 1$. By Lemma 3.2, $x_3 = 4 + 2y_4$. So $N_{H_2}(5) = 180 + 20y_4 < 240 + 30y_4$. Which is impossible. Let $i = 18$. If $x_0 = 1$, then by Lemma 3.2, $x_1 = x_3 - 6$ and by Lemma 3.4, $x_1 + 2N = x_3 - 4$. If $x_0 = 0$ and H_2 contains exactly one path as a component, then $x_1 + 2N = x_3 - 2$ and $x_1 = x_3 - 4$, respectively. Any way we have $N = 1$. Since by Lemma 2.12, H_2 does not have T_1 as an induced subgraph, we have $N_{H_2}(C_4) = 2$. So by Lemma 3.2, $x_3 = 1 + 2y_4$ or $x_3 = 2 + 2y_4$. So $N_{H_2}(5) = 170 + 20y_4$ or $N_{H_2}(5) = 180 + 20y_4$. However, $N_{H_2}(5) < 240 + 30y_4$ this is not true. If H_2 contains no any path as a component, then $x_1 + 2N = x_3 - 4$ and $x_1 = x_3 - 4$. Hence $N = 0$ and so $N_{H_2}(C_4) = 2$. By Lemma 3.2, $x_3 = 2 + 2y_4$. So $N_{H_2}(5) = 180 + 20y_4 < N_{H_1}(5)$, which is a contradiction to this fact H_1 and H_2 are cospectral with respect to adjacency matrix.

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Now let $i \in \{20, 21\}$. If $x_0 = 1$, then by Lemma 3.2, $x_1 = x_3 - 6$ and by Lemma 3.4, $x_1 + 2N = x_3 - 4$. If $x_0 = 0$ and H_2 contains exactly one path as a component, then $x_1 + 2N = x_3 - 2$ and $x_1 = x_3 - 4$, respectively. Therefore $N = 1$ and so G_2 has more than one bipartite component which is impossible. If H_2 contains no any path as a component, then by Lemma 3.2, $x_1 = x_3 - 4$ and by Lemma 3.4, $x_1 + 2N = x_3 - 4$. Hence $N = 0$ and so G_2 has more than one bipartite component, which is false. ◻

In the following theorem by using the previous facts we show that only graphs of type T_{44} and disjoint union of T_{45} with one path can be cospectral to a given starlike tree with maximum degree 4 with respect to the signless Laplacian spectrum (see Fig.7).

THEOREM 3.6. *Let $G_1 = S(a, b, c, d)$ where $d \geq c \geq b \geq a \geq 1$ and let G_2 be cospectral to G_1 with respect to the signless Laplacian matrix. Then:*

i) If $a = b = 1$, then G_1 and G_2 are isomorphic,
ii) If $a = 1$, $b > 1$, then G_2 is either isomorphic to G_1 or is of type T_{44},
iii) If $a > 1$, then G_2 is either isomorphic to G_1 or it has two components, one of them is path and another is of type T_{45} (see Fig.7).

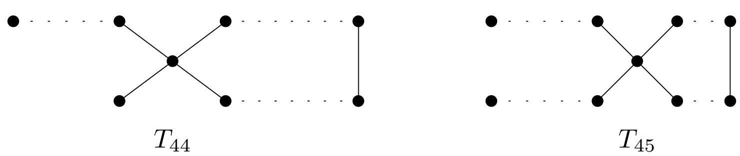

Fig.7

Proof. Let G_2 be a cospectral to G_1 with respect to the signless Laplacian matrix. Let H_1 and H_2 be the line graphs of G_1 and G_2, respectively. By Corollary 2.10, H_1 and H_2 are cospectral with respect to the adjacency matrix. If $G_1 = K_{1,4}$, then $H_1 = K_4$ and so $H_2 = K_4$. Hence G_1 and G_2 are isomorphic. Now let $G_1 \neq K_{1,4}$. By Lemmas 3.2 and 3.5, $0 < x_4 \leq 4$. Hence we have the following cases :
Case1: $x_4 = 4$. By Lemma 3.3 and the fact that $N_{H_2}(K_3) = 4$, H_2 has a subgraph of type T_i or K_4 for $i \in \{42, 43\}$ (see Fig.6). Since by Lemma 2.12, H_2 does not have T_3 as an induced subgraph and the fact that $N_{H_2}(K_3) = 4$, H_2 does not have T_{42} as a subgraph. If $i = 43$, then $N_{H_2}(C_4) \geq 1$ and by iv) of Lemma 3.2, we have $x_3 = x_0 = 0$ and $x_1 = 4$. Hence H_2 contains two path as a component and so G_2 has more than one bipartite component. Which is a contradiction. Therefor H_2 has K_4 as a subgraph. Again by iv) of Lemma 3.2, we have $x_3 = x_0 = 0$ and $x_1 = 4$. First let H_2 is a connected graph, then H_2 is the line graph of a starlike tree with maximum degree 4. Hence by Theorem 2.5, H_1 is isomorphic to H_2 and so by Theorem 2.7,

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G_1 is isomorphic to G_2. Now let H_2 is not a connected graph. Since $x_3 = x_0 = 0$ and $x_1 = 4$, using the fact that G_2 has exactly one bipartite component, H_2 has two components, one of them is path and another is of type T_{47} (see Fig.8).

Case2: $x_4 = 3$. Then by Lemma 3.3 and this fact that $N_{H_2}(K_3) = 4$, H_2 has a subgraph of type T_i or K_4 for $i \in \{39, 40, 41\}$ (see Fig.6). If $i = 41$, then $N_{H_2}(C_4) > 2$ and by Lemma 3.2, we have $x_3 < 0$. Which is impossible. Let $i \in \{39, 40\}$. If $x_0 = 1$, then by Lemma 3.2, $x_1 = x_3$ and by Lemma 3.4, $x_1 + 2N = x_3$. If $x_0 = 0$ and H_2 contains exactly one path as a component, then $x_1 + 2N = x_3 + 2$ and $x_1 = x_3 + 2$. Hence $N = 0$. However G_2 has more than one bipartite component which is a contradiction. If H_2 contains no any path as a component, then by Lemma 3.2, $x_1 = x_3 + 2$ and by Lemma 3.4, $x_1 + 2N = x_3$. So $N = -1$, which is impossible. So H_2 has K_4 as a subgraph. Since H_2 does not have $K_{1,3}$ as an induced subgraph and $N_{H_2}(K_3) = 4$, each vertex of degree at least 3 of H_2 is the vertex of subgraph K_4 of H_2. Therefore $x_3 + x_4 = 4$ and so $x_3 = 1$. On the other hand by Lemma 3.2, $N_{H_2}(C_4) \in \{1, 2\}$. First let $N_{H_2}(C_4) = 2$. By Lemma 3.2, $y_4 = 4$, $x_0 = 0$ and $x_1 = 3$ and so H_2 has 2 components, one of them is a path and another is of type T_{48} (see Fig.8). By Lemma 2.11, $N_{H_2}(5) = 350 < N_{H_1}(5) = 360$, which is not true. Now let $N_{H_2}(C_4) = 1$. By Lemma 3.2, $y_4 = 3$, $x_0 = 0$ and $x_1 = 3$. First let H_2 is a connected graph, then it is the line graph of a starlike tree with maximum degree 4. Hence by Theorem 2.5, H_1 is isomorphic to H_2 and so by Theorem 2.7, G_1 is isomorphic to G_2. Now let H_2 is not a connected graph. Since $x_0 = 0$ and $x_1 = 3$, using the fact that G_2 has exactly one bipartite component, H_2 has two components, one of them is path and another is of type T_{48}.

Case3: Let $x_4 = 2$. Then H_2 has a subgraph of type T_i or K_4 for $33 \leq i \leq 38$ (see Fig.6). Since $x_4 = 2$ and $N_{H_2}(K_3) = 4$, if H_2 has T_{37} as a subgraph, then H_2 has T_2 as an induced subgraph. By Lemma 2.12, which is impossible.

Let $i = 38$, if $x_0 = 1$, then by Lemma 3.2, $x_1 = x_3 - 2$ and by Lemma 3.4, $x_1 + 2N = x_3$. If $x_0 = 0$ and H_2 contains exactly one path as a component, then $x_1 + 2N = x_3 + 2$, $x_1 = x_3$. Hence $N = 1$, which is a contradiction to this fact that G_2 has exactly one bipartite component. If H_2 contains no path as a component, then $x_1 + 2N = x_3$ and $x_1 = x_3$ so $N = 0$. By i) of Lemma 3.2, it is a contradiction.

Let $i \in \{33, 35, 36\}$. If $x_0 = 1$, then by Lemma 3.2, $x_1 = x_3 - 2$ and by Lemma 3.4, $x_1 + 2N = x_3 - 2$. If $x_0 = 0$ and H_2 contains exactly one path as a component, then $x_1 + 2N = x_3$ and $x_1 = x_3$. Hence $N = 0$, by i) of Lemma 3.2, is a contradiction. If H_2 contains no path as a component, then $x_1 + 2N = x_3 - 2$ and $x_1 = x_3$ so $N = -1$, which is impossible.

Let $i = 34$. If $x_0 = 1$, then by Lemma 3.2, $x_1 = x_3 - 2$ and by Lemma 3.4, $x_1 + 2N = x_3 - 2$. So $N = 0$. By Lemma 3.2, we have $x_3 = 2y_4 - 3$. By Lemma 2.11, $N_{H_2}(5) = 190 + 20y_4 < N_{H_1}(5) = 240 + 30y_4$. Which is a contradiction to this fact H_1 and H_2 are cospectral with respect to adjacency matrix. If H_2 contains no path as a component, then $x_1 + 2N = x_3 - 2$ and $x_1 = x_3$. So $N = -1$, which is impossible.

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If $x_0 = 0$ and H_2 contains exactly one path as a component, then $x_1 + 2N = x_3$ and $x_1 = x_3$. So $N = 0$. By Lemma 3.2, we have $x_3 = 2y_4 - 2$. By Lemma 2.11, $N_{H_2}(5) = 200 + 20y_4 < N_{H_1}(5)$. Which is a contradiction to this fact H_1 and H_2 are cospectral with respect to adjacency matrix. So H_2 has K_4 as a subgraph. Since H_2 does not have $K_{1,3}$ as an induced subgraph and $N_{H_2}(K_3) = 4$, each vertex of degree at least 3 of H_2 is the vertex of subgraph K_4 of H_2. Therefore $x_3 + x_4 = 4$ and so $x_3 = 2$. Moreover H_2 does not have any cycle as a component. So H_2 has exactly one non-tree component. If H_2 is a connected graph, then by Lemma 3.2, $x_1 = 2$ and so $N_{H_2}(C_4) = 1$ and $y_4 = 2$. Therefore H_2 is a line graph of a starlike graph with maximum degree 4. Using Theorem 2.5, H_1 and H_2 are isomorphic. So by Theorem 2.7, G_1 and G_2 are isomorphic. If H_2 is not a connected graph, then by Lemma 3.2, we have $x_0 + 2N_{H_2}(C_4) = 2(y_4 - 1)$. Since $x_0 \leq 1$, we have $x_0 = 0$ and so $x_1 = 2$ and $N_{H_2}(C_4) = y_4 - 1$. Hence H_2 has 2 components one of them is path and another is of type T_{46}. It is easy to see that $N_{H_2}(C_4) \leq 2$. If $N_{H_2}(C_4) = 2$, then $y_4 = 3$ and so $N_{H_1}(5) = 330 > N_{H_2}(5) = 320$. That is false. Hence $N_{H_2}(C_4) = 1$ and so $y_4 = 2$. One can successively subdivide certain edges of the H_2 in an appropriate way, to obtain graph $\tilde{H}$, such that H_1 can be embedded in $\tilde{H}$ as a proper subgraph. So by Lemma 2.8, $\lambda_1(H_2) \geq \lambda_1(\tilde{H})$ and by Lemma 2.6, $\lambda_1(\tilde{H}) > \lambda_1(H_1)$. Hence $\lambda_1(H_2) > \lambda_1(H_1)$ which is a contradiction to the fact that H_2 and H_1 are cospectral with respect to the adjacency matrix.

Case4: Let $x_4 = 1$. Then H_2 has a subgraph of type T_i or K_4 for $22 \leq i \leq 32$ (see Fig.6). If H_2 has a subgraph of type K_4, then by Lemma 3.2, H_2 is a line graph of a starlike graph with maximum degree 4. Using Theorem 2.5, H_1 and H_2 are isomorphic. So by Theorem 2.7, G_1 and G_2 are isomorphic. Now let H_2 has a subgraph of type T_i for $22 \leq i \leq 32$.

Let $i \in \{22, 27, 28\}$. If $x_0 = 1$, then by Lemma 3.2, $x_1 = x_3 - 4$ and by Lemma 3.4, $x_1 + 2N = x_3 - 4$. If $x_0 = 0$ and H_2 contains exactly one path as a component, then $x_1 + 2N = x_3 - 2$ and $x_1 = x_3 - 2$. Any way, $N = 0$. So G_2 has more than one bipartite component, which is impossible. If H_2 contains no path as a component, then $x_1 + 2N = x_3 - 4$ and $x_1 = x_3 - 2$. So $N = -1$, which is impossible.

Let $i = 23$. If $x_0 = 1$, then by Lemma 3.2, $x_1 = x_3 - 4$ and by Lemma 3.4, $x_1 + 2N = x_3 - 4$. Also if $x_0 = 0$ and H_2 contains exactly one path as a component, then $x_1 + 2N = x_3 - 2$ and $x_1 = x_3 - 2$. Any way we have $N = 0$. Thus $N_{H_2}(C_4) = 1$. By Lemma 3.2, we have $x_3 = 2y_4$ or $x_3 = 2y_4 + 1$. By Lemma 2.11, $N_{H_2}(5) = 180 + 20y_4$ or $N_{H_2}(5) = 190 + 20y_4$. However $N_{H_2}(5) < N_{H_1}(5)$, that is impossible. If H_2 contains no path as a component, then by Lemma 3.2, $x_1 = x_3 - 2$ and by Lemma 3.4, $x_1 + 2N = x_3 - 4$. So $N = -1$, which is impossible.

Let $i = 30$. If $x_0 = 1$, then by Lemma 3.2, $x_1 = x_3 - 4$ and by Lemma 3.4, $x_1 + 2N = x_3 - 2$. Also if $x_0 = 0$ and H_2 contains exactly one path as a component, then $x_1 + 2N = x_3$ and $x_1 = x_3 - 2$. Any way we have $N = 1$. Thus $N_{H_2}(C_4) = 1$ or 2. If $N_{H_2}(C_4) = 1$, then by Lemma 3.2, we have $x_3 = 2y_4$ or $x_3 = 2y_4 + 1$.

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By Lemma 2.11, $N_{H_2}(5) = 180 + 20y_4$ or $N_{H_2}(5) = 190 + 20y_4$. If $N_{H_2}(C_4) = 2$, then $x_3 = 2y_4 - 2$ or $x_3 = 2y_4 - 1$. By Lemma 2.11, $N_{H_2}(5) = 160 + 20y_4$ or $N_{H_2}(5) = 170 + 20y_4$. However $N_{H_2}(5) < N_{H_1}(5)$, this is a contradiction to this fact H_1 and H_2 are cospectral with respect to adjacency matrix. If H_2 contains no any path as a component, then by Lemma 3.2, $x_1 = x_3 - 2$ and by Lemma 3.4, $x_1 + 2N = x_3 - 2$. So $N = 0$. Hence by Lemma 3.2, $x_3 = 2y_4 + 1$. So $N_{H_2}(5) = 190 + 20y_4 < N_{H_1}(5)$. This is impossible.

Let $i \in \{31, 32\}$. If $x_0 = 1$, then by Lemma 3.2, $x_1 = x_3 - 4$ and by Lemma 3.4, $x_1 + 2N = x_3 - 2$. If $x_0 = 0$ and H_2 contains exactly one path as a component, then $x_1 + 2N = x_3$ and $x_1 = x_3 - 2$. So $N = 1$. However G_2 has more than one bipartite component and this is a contradiction. If H_2 contains no path as a component, then $x_1 + 2N = x_3 - 2$ and $x_1 = x_3 - 2$. So $N = 0$, which is a contradiction to this fact that G_2 has exactly one bipartite component.

Let $24 \leq i \leq 26$. If $x_0 = 1$, then by Lemma 3.2, $x_1 = x_3 - 4$ and by Lemma 3.4, $x_1 + 2N = x_3 - 4$. If $x_0 = 0$ and H_2 contains exactly one path as a component, then $x_1 + 2N = x_3 - 2$ and $x_1 = x_3 - 2$. So $N = 0$. By Lemma 3.2, for $i = 24$, we have $x_3 = 2y_4 - 2$ or $x_3 = 2y_4 - 1$. By Lemma 2.11, $N_{H_2}(5) = 180 + 20y_4$ or $N_{H_2}(5) = 190 + 20y_4$. By Lemma 3.2, for $i = 25$, we have $x_3 = 2y_4$ or $x_3 = 2y_4 + 1$. By Lemma 2.11, $N_{H_2}(5) = 180 + 20y_4$ or $N_{H_2}(5) = 190 + 20y_4$. However $N_{H_2}(5) < N_{H_1}(5)$, that is not true. Now if $x_0 = 0$ and H_2 contains no path as a component, then by Lemma 3.2, $x_1 = x_3 - 2$ and by Lemma 3.4, $x_1 + 2N = x_3 - 4$. So $N = -1$, which is impossible. Since $x_4 = 1$ and $N_{H_2}(K_3) = 4$, if H_2 has T_{26} as a subgraph, then H_2 has T_2 as an induced subgraph. By Lemma 2.12, which is impossible.

Let $i = 29$. If $x_0 = 1$, then by Lemma 3.2, $x_1 = x_3 - 4$ and by Lemma 3.4, $x_1 + 2N = x_3 - 2$. If $x_0 = 0$ and H_2 contains exactly one path as a component, then $x_1 + 2N = x_3$ and $x_1 = x_3 - 2$. Hence $N = 1$. Thus $N_{H_2}(C_4) = 1$ or 2. If $N_{H_2}(C_4) = 1$, then by Lemma 3.2, we have $x_3 = 2y_4$ or $x_3 = 2y_4 + 1$. By Lemma 2.11, $N_{H_2}(5) = 180 + 20y_4$ or $N_{H_2}(5) = 190 + 20y_4$. If $N_{H_2}(C_4) = 2$, then by Lemma 3.2, we have $x_3 = 2y_4 - 2$ or $x_3 = 2y_4 - 1$. By Lemma 2.11, $N_{H_2}(5) = 160 + 20y_4$ or $N_{H_2}(5) = 170 + 20y_4$, a contradiction. If $x_0 = 0$ and H_2 contains no path as a component, then by Lemma 3.2, $x_1 = x_3 - 2$ and by Lemma 3.4, $x_1 + 2N = x_3 - 2$. So $N = 0$. Thus by Lemma 3.2, $x_3 = 2y_4 + 1$. By Lemma 2.11, $N_{H_2}(5) = 190 + 20y_4 < N_{H_1}(5)$. This is impossible. $\square$

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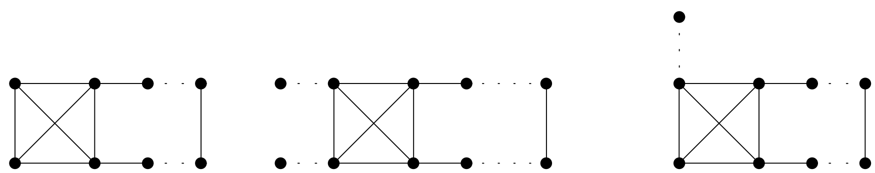

T_{46} T_{47} T_{48}

Fig.8

Let $N_G(H)$ be the number of subgraphs of a graph G which are isomorphic to H and let $N_G(i)$ be the number of closed walks of length i in G. Let $N'_H(i)$ be the number of closed walks of length i of H which contains all edges and let $S_i(G)$ be the set of all connected graphs such H with $N'_H(i) \neq 0$ where G has at least one subgraph isomorphic to H. Then:

$$N_G(i) = \sum_{H \in S_i(G)} N_G(H) N'_H(i). \tag{3.3}$$

THEOREM 3.7. *Let $G = S(a,b,c,d)$ where $d \geq c \geq b \geq a \geq 1$. Then G is determined by its signless Laplacian spectrum.*

Proof. Let $G_1 = G$ and let G_2 be cospectral to G_1 with respect to the signless Laplacian matrix. If G_2 is not isomorphic to G_1, then by using Theorem 3.6, we have the following cases:

Case1: Let $a = 1$, $b > 1$ and let $G_2 = A$ (see Fig.9). If $\overline{b} \geq b$, then we can subdivide certain edges of the cycle C_l of $L(G_2)$ in an appropriate way, to obtain graph $\tilde{H}$, such that $L(G_1)$ can be embedded in $\tilde{H}$ as a proper subgraph. So by Lemma 2.8, $\lambda_1(L(G_2)) \geq \lambda_1(\tilde{H})$ and by Lemma 2.6, $\lambda_1(\tilde{H}) > \lambda_1(L(G_1))$. Hence $\lambda_1(L(G_2)) > \lambda_1(L(G_1))$ which contradicts to the fact that $L(G_2)$ and $L(G_1)$ are cospectral with respect to the adjacency matrix. So $\overline{b} < b$. If $\overline{b} \geq (l-1)/2$, then $S_l(L(G_2)) = S_l(L(G_1)) \cup \{C_l\}$ and for each $K \in S_l(L(G_1))$, $N_{L(G_2)}(K) \geq N_{L(G_1)}(K)$. So by the equation (3.3), $N_{L(G_2)}(l) > N_{L(G_1)}(l)$, contradicting to the fact that $L(G_2)$ and $L(G_1)$ have the same number of closed walks of any length. If $\overline{b} < (l-3)/2$, then $S_{(2\overline{b}+3)}(L(G_2)) = S_{(2\overline{b}+3)}(L(G_1))$, $N_{L(G_1)}(K(1,1,\overline{b}+1)) > N_{L(G_2)}(K(1,1,\overline{b}+1))$ and $N_{L(G_1)}(K) = N_{L(G_2)}(K)$ for each $K \neq K(1,1,\overline{b}+1)$ in $S_{(2\overline{b}+3)}(L(G_2))$. Hence by the equation (3.3), we have $N_{L(G_1)}(2\overline{b}+3) > N_{L(G_2)}(2\overline{b}+3)$, which is again a contradiction. Hence $\overline{b} \in \{(l-3)/2, (l-2)/2\}$. On the other hand G_1 and G_2 have the same number of vertices and so $l > c + d \geq 2b \geq 2\overline{b} + 2$. Therefore $\overline{b} = (l-3)/2$ and so G_2 is not a bipartite graph, contradicting to the fact that G_1 is bipartite.

Case2: Let $a > 1$ and let G_2 has two components, one of them is path and another is

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B (see Fig.9) where $\overline{a} \leq \overline{b}$. Since G_2 has exactly one bipartite component, l is an odd number. If $\overline{a} > a$, then by easy task we can see that $N_{L(G_2)}(2a+3) > N_{L(G_1)}(2a+3)$, contradicting to the fact that $L(G_2)$ and $L(G_1)$ have the same number of closed walks of any length. If $\overline{a} = a$, then as a similar to case1, we have $\overline{b} = (l-3)/2 < b$. Since for each natural x we have $N'_{K(1,1,x)}(2x+1) = N'_{C_{(2x+1)}}(2x+1) = 4x+2$, it is easy to see that $N_{L(G_2)}(l) < N_{L(G_1)}(l)$, that is impossible. If $\overline{a} < a$, then again as a similar to case1, we have $\overline{a} = (l-3)/2$. Again we can see that $N_{L(G_2)}(l) < N_{L(G_1)}(l)$, which is impossible. $\square$

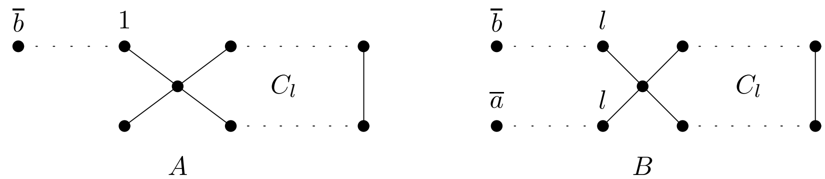

Fig.9

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REFERENCES

[1] J. A. Bondy and U. S. R. Murty. *Graph Theory with Applications*. The MacMillan Press Ltd, 1976.
[2] D. M. Cvetkovic, M. Doob, and H. Sachs. *Spectra of Graphs*. Third edition, Johann Abrosius Barth Verlag, 1995.
[3] A. J. Hoffman and J. H. Smith. On the spectral radii of topologically equivalent graphs. In *Recent Advances in Graph Theory, Academia Praha*, M. Fiedler, Editor, (1975) pp. 273–281.
[4] M. Lepović. Some results on starlike trees and sunlike graphs. *Journal of Applied Mathematics and Computing*, 11:109–123, 2003.
[5] G. R. Omidi. On a signless Laplacian spectral characterization of T-shape trees. *Linear Algebra and its Applications*, 431:1607–1615, 2009.
[6] G. R. Omidi and K. Tajbakhsh. Starlike trees are determined by their Laplacian spectrum. *Linear Algebra and its Applications*, 422:654–658, 2007.
[7] X. L. Shen, Y. P. Hou, and Y. P. Zhang. Graph Z_n and some graphs related to Z_n are determined by their spectrum. *Linear Algebra and its Applications*, 404: 58–68, 2005.
[8] E. R. van Dam and W. H. Haemers. Which graphs are determined by their spectrum? *Linear Algebra and its Applications*, 373:241–272, 2003.
[9] W. Wang and C. X. Xu. On the spectral charactrization of T-shape trees. *Linear Algebra and its Applications*, 414:492–501, 2006.
[10] W. Wang and C. X. Xu. Note: The T-shape tree is determined by its Laplacian spectrum. *Linear Algebra and its Applications*, 419:78–81, 2006.