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# THE MOORE-PENROSE INVERSE OF THE DISTANCE MATRIX OF A HELM GRAPH*

I. JEYARAMAN†, T. DIVYADEVI, AND R. AZHAGENDRAN

**Abstract.** In this paper, we give necessary and sufficient conditions for a real symmetric matrix and, in particular, for the distance matrix $D(H_n)$ of a helm graph $H_n$ to have their Moore-Penrose inverses as the sum of a symmetric Laplacian-like matrix and a rank-one matrix. As a consequence, we present a short proof of the inverse formula, given by Goel (*Linear Algebra Appl.* 621:86–104, 2021), for $D(H_n)$ when $n$ is even. Further, we derive a formula for the Moore-Penrose inverse of singular $D(H_n)$ that is analogous to the formula for $D(H_n)^{-1}$. Precisely, if $n$ is odd, we find a symmetric positive semi-definite Laplacian-like matrix $L$ of order $2n-1$ and a vector $\mathbf{w} \in \mathbb{R}^{2n-1}$ such that

$$D(H_n)^{\dagger} = -\frac{1}{2}L + \frac{4}{3(n-1)}\mathbf{w}\mathbf{w}',$$

where the rank of $L$ is $2n-3$. We also investigate the inertia of $D(H_n)$.

**Key words.** Distance matrix, Helm graph, Inverse, Moore-Penrose inverse, Inertia.

**AMS subject classifications.** 05C12, 05C50, 15A09.

## 1. Introduction.

We consider a simple connected graph $G$ with the vertex set $\{v_1, v_2, \ldots, v_n\}$. In literature, several matrices have been associated with $G$ through which many structural properties of the graphs have been studied. A few classical graph matrices are the adjacency matrix, the Laplacian matrix, the distance matrix, the incidence matrix etc., see [5, 7]. Let us recall the distance matrix which is relevant to our discussion. The *distance matrix* $D(G) := (d_{ij})$ of $G$ is an $n \times n$ symmetric matrix with $d_{ij} = d(v_i, v_j)$ for all $i$ and $j$, where $d(v_i, v_j)$ denotes the length of a shortest path between the vertices $v_i$ and $v_j$. This matrix has been widely studied in the literature and has applications in chemistry, physics, computer science, etc., see [5, 7] and the references therein. For a brief introduction, we refer to the survey article [1].

A basic problem in graph matrices is to give a simple formula to compute the inverses of these matrices. This problem has been extensively studied in the literature, see [4, 5, 8, 10, 11, 18]. To state the famous inverse formula given in [10], let us recall the Laplacian matrix corresponding to $G$. The *Laplacian matrix* of $G$ is given by $L := (l_{ij})$ where $l_{ii}$ is the degree of the vertex $v_i$, $l_{ij} = -1$ if $v_i$ and $v_j$ are adjacent and zero elsewhere. Then, $L$ is a symmetric matrix of order $n$ whose row sums are zero, positive semi-definite and rank of $L$ is $n-1$ (see [5]). In [10], an expression for the inverse of the distance matrix $D(T)$ of a tree $T$ is obtained, which is given by

$$D(T)^{-1} = -\frac{1}{2}L + \frac{1}{2(n-1)}\mathbf{u}\mathbf{u}', \tag{1.1}$$

where $\mathbf{u}' = \big(2 - \deg(v_1), 2 - \deg(v_2), \ldots, 2 - \deg(v_n)\big)$ and $\deg(v_i)$ is the degree of the vertex $v_i$. Motivated by this, analogous inverse formulae for the distance matrices of various graphs have been established. For instance, the distance matrices of wheel graphs $W_n$ when $n$ is even [4], helm graphs with even number of

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†Department of Mathematics, National Institute of Technology, Tiruchirappalli-620 015, India (jeyaraman@nitt.edu, tdivyadevi@gmail.com, razhagu2014@gmail.com).

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vertices [8] and fan graphs [11]. Let us mention that for all these matrices, the inverse formulae are obtained by replacing the Laplacian matrix by Laplacian-like matrix which we recall next. A symmetric matrix $\bar{L}$ is said to be *Laplacian-like* if all its row sums are zero [18]. The results state that the inverse formulae for all the above matrices are expressed as the sum of a symmetric positive semi-definite Laplacian-like matrix and a rank-one matrix. Similar inverse formulae have been determined for the distance matrices of weighted trees, cycles, complete graphs, block graphs and bi-block graphs, see [18] and the references therein for more graphs.

Another basic problem in this topic is to find the Moore–Penrose inverses of singular and rectangular graph matrices, which have been well studied in the literature (see [2, 3, 5, 15, 16] and references therein). Let us recall that for an $m\times n$ real matrix $M$, an $n\times m$ matrix $X$ is said to be the *Moore-Penrose inverse* of $M$ if $MXM = M,\ XMX = X,\ (MX)' = MX$ and $(XM)' = XM$. It is known that the Moore–Penrose inverse always exists and is unique. It is denoted by $M^\dagger$. Furthermore, $M^\dagger$ coincides with the usual inverse $M^{-1}$ if $M$ is non-singular, see [6] for more details. Inspired by the inverse formula result of [10], the Moore-Penrose inverse of the distance matrix $D(W_n)$ of the wheel graph $W_n$, similar to (1.1), was obtained for the singular case [3]. More precisely, if $n$ is odd, then the Moore-Penrose inverse of $D(W_n)$ is given by

$$D(W_n)^\dagger = -\frac{1}{2}L + \frac{4}{n-1}\mathbf{t}\mathbf{t}', \tag{1.2}$$

where $L$ is a real symmetric positive semi-definite Laplacian-like matrix of order $n$ and $\mathbf{t}\in\mathbb{R}^n$.

This paper focuses on the distance matrix of a helm graph $H_n$, which is a generalization of star graph [14]. Several studies on helm graphs have been carried out in the literature. The resolving domination numbers of helm graphs were analysed in [12]. It was proved in [14] that the distance matrix $D(H_n)$ of a helm graph $H_n$ is a circum Euclidean distance matrix. In [8], it was shown that if $n$ is even then $D(H_n)$ is non-singular. Further, the problem of finding the inverse of $D(H_n)$ was considered and the inverse formula for $D(H_n)$, similar to (1.1), was provided using the inverse of the distance matrix $D(W_n)$ of the wheel graph $W_n$. That is,

$$D(H_n)^{-1} = -\frac{1}{2}\mathcal{L} + \frac{4}{3(n-1)}\mathbf{w}\mathbf{w}', \tag{1.3}$$

where $\mathcal{L}$ is a Laplacian-like matrix and $\mathbf{w}\in\mathbb{R}^{2n-1}$, see Theorem 3 in [8]. Motivated by the above-mentioned results on $H_n$, our aim herein is to explore more results on $D(H_n)$, which may be helpful in studying the distance matrices of generalizations of star graphs.

In Section 3, we show that $D(H_n)$ is singular if $n$ is odd and then derive the inertia of $D(H_n)$ by finding its rank. We study the Moore-Penrose inverse formula for $D(H_n)$ in Section 4. We first establish necessary and sufficient conditions under which the Moore-Penrose inverses of symmetric matrices generally and $D(H_n)$, in particular, is of the form similar to (1.3). Using these results, we give an alternative proof of the inverse formula given in (1.3). Further, we establish an analogous formula for $D(H_n)^\dagger$ for the singular case. That is, if $n$ is odd, we construct a symmetric Laplacian-like matrix $L$ of order $2n-1$ and a vector $\mathbf{w}\in\mathbb{R}^{2n-1}$ such that $D(H_n)^\dagger$ is expressed as the sum of a constant multiple of $L$ and a rank-one symmetric matrix defined by $\mathbf{w}$ (see Theorem 4.13). It is noteworthy to mention that our approach of finding $D(H_n)^\dagger$ is significantly different from those of [3, 4, 8, 11]. Unlike [8], the techniques employed to obtain a different proof for $D(H_n)^{-1}$ do not depend on $D(W_n)^{-1}$ and the proof given for $D(H_n)^\dagger$ formula is also independent of $D(W_n)^\dagger$. Finally, we prove that the constructed $L$ is a positive semi-definite matrix of rank $2n-3$ using the concept of simultaneous diagonalization.

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**2. Preliminaries.** In this section, we fix the notations and collect some basic results which will be needed in this paper. For a matrix $M$, we denote the $i$-th row of $M$, the $j$-th column of $M$, the transpose of $M$, the range of $M$, the null space of $M$ and the rank of $M$ by $M_{i*}$, $M_{*j}$, $M'$, $R(M)$, $N(M)$ and $\mathrm{rank}(M)$, respectively. We write the determinant of a square matrix $M$ as $\det(M)$ and the identity matrix of order $n$ as $I_n$. The symbol $\mathrm{diag}(\mu_1,\mu_2,\dots,\mu_n)$ represents the $n\times n$ diagonal matrix whose $j$-th diagonal entry is $\mu_j$. The notations $J_n$ and $O_n$ are used to denote the matrices with all elements equal to 1 and 0, respectively. The subscripts are omitted if the order of the matrix is clear from the context. All the vectors are assumed to be column vectors and are denoted by lowercase boldface letters. We use the notations $\mathbf{e}$ and $\mathbf{0}$ to represent the vectors in $\mathbb{R}^n$ whose coordinates are all one and zero, respectively. Let $\mathrm{Circ}(\mathbf{a}')$ denote the circulant matrix of order $n$ defined by the vector $\mathbf{a}=(a_1,a_2,\dots,a_n)'\in\mathbb{R}^n$, and the notation $T_n(2,1,1)$ stands for the tridiagonal matrix of order $n$ whose diagonal entries are all 2. That is,

$$\mathrm{Circ}(\mathbf{a}')=\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n\\ a_n & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1}\\ a_{n-1} & a_n & a_1 & \cdots & a_{n-3} & a_{n-2}\\ \vdots & \vdots & & \ddots & \vdots & \vdots\\ a_3 & a_4 & a_5 & \cdots & a_1 & a_2\\ a_2 & a_3 & a_4 & \cdots & a_n & a_1\end{bmatrix} \text{ and } T_n(2,1,1)=\begin{bmatrix} 2 & 1 & 0 & \cdots & 0 & 0 & 0\\ 1 & 2 & 1 & & 0 & 0 & 0\\ 0 & 1 & 2 & & 0 & 0 & 0\\ \vdots & & \ddots & \ddots & \ddots & & \\ 0 & & & & 2 & 1 & 0\\ 0 & \cdots & & & 1 & 2 & 1\\ 0 & \cdots & & & 0 & 1 & 2\end{bmatrix}.$$

Suppose that $A=\mathrm{Circ}(\mathbf{x}')$ and $B=\mathrm{Circ}(\mathbf{y}')$ where $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$. Then,

$$AB=BA,\quad AB=\mathrm{Circ}(\mathbf{x}'B)\quad\text{and}\quad \mathrm{Circ}(a\mathbf{x}'+b\mathbf{y}')=aA+bB. \tag{2.4}$$

The above interesting properties of the circulant matrix will be used frequently in this paper. For more results, we refer to the books [13, 17].

**3. The distance matrix of a Helm graph and its inertia.** We first define the distance matrix $D(H_n)$ of a helm graph $H_n$. Then, we show that $D(H_n)$ is singular when $n$ is odd. Finally, we derive the inertia of $D(H_n)$ after determining its rank.

We first recall the definition of a wheel graph. For $n\ge 4$, the notation $C_{n-1}$ denotes the cycle of length $n-1$ and the vertices in $C_{n-1}$ are labelled as $v_1,v_2,\dots,v_{n-1}$. The wheel graph $W_n$ on $n$ vertices is a graph containing the cycle $C_{n-1}$ and a vertex, say $v_0$, not in the cycle $C_{n-1}$ which is adjacent to every vertex $v_i$ in the cycle $C_{n-1}$. This paper is concerned with the helm graph which we define next. The helm graph on $2n-1$ vertices, denoted by $H_n$, is a supergraph of $W_n$ which is obtained from $W_n$ by attaching a pendant vertex $u_i$ to the vertex $v_i$ lying on the outer cycle for all $i=1,2,\dots,n-1$. The helm graph $H_7$ on 13 vertices is given in Figure 1.

Let $\mathbf{u}=(0,1,2,2,\dots,2,1)'\in\mathbb{R}^{n-1}$. Then, the distance matrix $D(H_n)$ of the helm graph $H_n$ is given by

$$D(H_n)=\begin{bmatrix} 0 & \mathbf{e}' & 2\mathbf{e}'\\ \mathbf{e} & \widetilde{D} & \widetilde{D}+J_{n-1}\\ 2\mathbf{e} & \widetilde{D}+J_{n-1} & \widetilde{D}+2(J_{n-1}-I_{n-1})\end{bmatrix},$$

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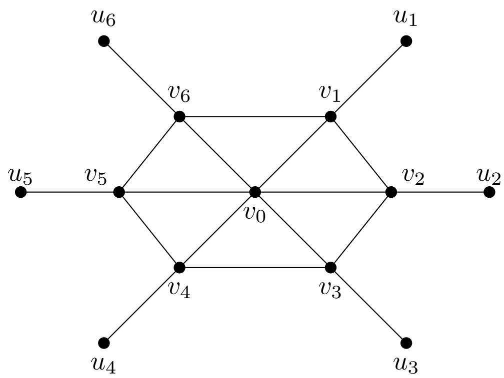


Figure 1: Helm graph $H_7$ on 13 vertices.

where $\widetilde{D} = \mathrm{Circ}(\mathbf{u}')$, see [8, 14]. We fix the symbol $\mathbf{s}$ to denote the vector $(2,1,0,\ldots,0,1)'$ in $\mathbb{R}^{n-1}$. By defining $S = \mathrm{Circ}(\mathbf{s}')$, the matrix $D(H_n)$ can be re-written as $D(H_n) = D_a + D_b$, where

$$D_a = \begin{bmatrix} 0 & \mathbf{e}' & 2\mathbf{e}' \\ \mathbf{e} & 2J_{n-1} & 3J_{n-1} \\ 2\mathbf{e} & 3J_{n-1} & 4J_{n-1} \end{bmatrix} \text{ and } D_b = \begin{bmatrix} 0 & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0} & -S & -S \\ \mathbf{0} & -S & -(S+2I_{n-1}) \end{bmatrix}.$$

It has been shown that $\det(D(H_n)) = 3(n-1)2^{n-1}$ when $n$ is even ([8], Theorem 2). From numerical computations, it has been observed in [8] that $D(H_n)$ is singular if $n$ is odd. In the following, we give a proof of this result. Precisely, we show that $\det(H_n) = 0$ if $n$ is odd.

THEOREM 3.1. *Let $n \geq 5$. If $n$ is an odd integer, then $D(H_n)$ is singular.*

*Proof.* Define $\mathbf{v} = (1,-1,1,-1,\ldots,1,-1)' \in \mathbb{R}^{n-1}$ and $\mathbf{p}_0 = (0,\mathbf{v}',\mathbf{0}')' \in \mathbb{R}^{2n-1}$. Then,

$$D(H_n)\mathbf{p}_0 = \begin{bmatrix} 0 & \mathbf{e}' & 2\mathbf{e}' \\ \mathbf{e} & 2J & 3J \\ 2\mathbf{e} & 3J & 4J \end{bmatrix}\begin{bmatrix} 0 \\ \mathbf{v} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} 0 & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0} & -S & -S \\ \mathbf{0} & -S & -(S+2I) \end{bmatrix}\begin{bmatrix} 0 \\ \mathbf{v} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{e}'\mathbf{v} \\ 2J\mathbf{v} \\ 3J\mathbf{v} \end{bmatrix} + \begin{bmatrix} 0 \\ -S\mathbf{v} \\ -S\mathbf{v} \end{bmatrix}.$$

Since $n-1$ is even, we have $\mathbf{e}'\mathbf{v} = 0$ which implies $J\mathbf{v} = \mathbf{0}$. It is easy to see that $S\mathbf{v} = \mathbf{0}$. Therefore, $D(H_n)\mathbf{p}_0 = \mathbf{0}$, and hence, $D(H_n)$ is singular. ☐

**3.1. The inertia of $D(H_n)$.** Recall that the *inertia* of a real symmetric matrix $M$ of order $n$, denoted by $\mathrm{In}(M)$, is the ordered triple $(i_+, i_-, i_0)$, where $i_+$, $i_-$ and $i_0$, respectively, denote the number of positive, negative, and zero eigenvalues of $M$ including the multiplicities. It is well known that $\mathrm{rank}(M) = i_+ + i_-$.

It is shown in [9] that the inertia of the distance matrix of any tree with $n \geq 2$ vertices is $(1, n-1, 0)$. The inertias of the distance matrices of wheel graphs and fan graphs are studied, see [1]. The objective of this section is to find the inertia of the distance matrix of the helm graph.

Next, we find the rank of $D(H_n)$ which will be used to compute the $\mathrm{In}(D(H_n))$.

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Theorem 3.2. *Let $n \geq 4$ be an integer. Then,*

$$\operatorname{rank}(D(H_n)) = \begin{cases} 2n-1 & \text{if } n \text{ is even}, \\ 2n-2 & \text{if } n \text{ is odd}. \end{cases}$$

*Proof.* If $n$ is even, then $\det(D(H_n)) = 3(n-1)2^{n-1} \neq 0$ by Theorem 2 in [8]. Hence, $\operatorname{rank}(D(H_n)) = 2n-1$. Assume that $n$ is odd. We claim that $\operatorname{rank}(D(H_n)) = 2n-2$. To prove the claim, it is enough to show $N(D(H_n)) = \operatorname{span}\{\mathbf{z_0}\}$ for some non-zero $\mathbf{z_0} \in \mathbb{R}^{2n-1}$. Let $\mathbf{z} = (\gamma, \mathbf{p}', \mathbf{q}')' \in \mathbb{R}^{2n-1}$ where $\mathbf{p}, \mathbf{q} \in \mathbb{R}^{n-1}$ and $\gamma \in \mathbb{R}$. Suppose $D(H_n)\mathbf{z} = \mathbf{0}$. Then, we have the following system of equations:

$$\mathbf{e}'\mathbf{p} + 2\mathbf{e}'\mathbf{q} = 0. \tag{3.5}$$

$$\gamma\mathbf{e} + 2J\mathbf{p} - S\mathbf{p} + 3J\mathbf{q} - S\mathbf{q} = \mathbf{0}. \tag{3.6}$$

$$2\gamma\mathbf{e} + 3J\mathbf{p} - S\mathbf{p} + 4J\mathbf{q} - S\mathbf{q} = 2\mathbf{q}. \tag{3.7}$$

Subtracting (3.6) from (3.7), we obtain $\gamma\mathbf{e} + J\mathbf{p} + J\mathbf{q} = 2\mathbf{q}$ which can be written as

$$(\gamma + \mathbf{e}'\mathbf{p} + \mathbf{e}'\mathbf{q})\mathbf{e} = 2\mathbf{q}. \tag{3.8}$$

From (3.5) and (3.8), we get $(\gamma - \mathbf{e}'\mathbf{q})\mathbf{e} = 2\mathbf{q}$. This implies $(\gamma - \mathbf{e}'\mathbf{q})\mathbf{e}'\mathbf{e} = 2\mathbf{e}'\mathbf{q}$, and hence,

$$\gamma = \frac{n+1}{n-1}\mathbf{e}'\mathbf{q}. \tag{3.9}$$

Now we obtain another expression for $\gamma$ by premultiplying (3.6) by $\mathbf{e}'$. Using the fact $\mathbf{e}'S = 4\mathbf{e}'$, we get $\gamma(n-1) + 2(n-1)\mathbf{e}'\mathbf{p} - 4\mathbf{e}'\mathbf{p} + 3(n-1)\mathbf{e}'\mathbf{q} - 4\mathbf{e}'\mathbf{q} = 0$. By (3.5), the above equation reduces to

$$\gamma = \frac{n-5}{n-1}\mathbf{e}'\mathbf{q}. \tag{3.10}$$

Comparing (3.9) and (3.10) gives $\mathbf{e}'\mathbf{q} = 0$ which implies $\gamma = 0$. It follows from (3.5) that $\mathbf{e}'\mathbf{p} = 0$, and hence by (3.8), $\mathbf{q} = \mathbf{0}$. Thus, $\mathbf{z} = (0, \mathbf{p}', \mathbf{0}')'$ and the system $D(H_n)\mathbf{z} = \mathbf{0}$ reduces to $S\mathbf{p} = \mathbf{0}$. Note that $S$ can be written as

$$S = \begin{bmatrix} 2 & \mathbf{e_1}' + \mathbf{e_{n-2}}' \\ \mathbf{e_1} + \mathbf{e_{n-2}} & T_{n-2}(2,1,1) \end{bmatrix}.$$

By Theorem 5.5 in [17], $\det(T_{n-2}(2,1,1)) = n-1$ which gives $\operatorname{rank}(S) \geq n-2$. Note that the order of $S$ is $n-1$ and $S\mathbf{v} = \mathbf{0}$ where $\mathbf{v} = (1,-1,1,-1,\ldots,1,-1)' \in \mathbb{R}^{n-1}$. Hence, $\operatorname{rank}(S) = n-2$ and $N(S) = \operatorname{span}\{\mathbf{v}\}$. This implies $N(D(H_n)) = \operatorname{span}\{\mathbf{z_0}\}$ where $\mathbf{z_0} = (0, \mathbf{v}', \mathbf{0}')'$. Hence, the proof. □

Theorem 3.3. *Let $n \geq 4$. The inertia of $D(H_n)$ is*

$$\operatorname{In} D(H_n) = \begin{cases} (1, 2n-2, 0) & \text{if } n \text{ is even}, \\ (1, 2n-3, 1) & \text{if } n \text{ is odd}. \end{cases}$$

*Proof.* It is proved that $D(H_n)$ is a Euclidean distance matrix ([14], Theorem 14). Therefore, $D(H_n)$ has exactly one positive eigenvalue (see Theorem 2 in [14]), that is $n_+(D(H_n)) = 1$ for all $n \geq 4$. By Theorem 3.2 and using the fact that $\operatorname{rank}(D(H_n)) = n_+(D(H_n)) + n_-(D(H_n))$, we obtain the desired result. □

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**4. Formulae for the inverse and the Moore-Penrose inverse of $D(H_n)$.** It is shown in [8] that if $n \geq 4$ is an even integer, then $D(H_n)$ is non-singular and a formula for $D(H_n)^{-1}$ has been obtained using the inverse of the distance matrix of the wheel graph $W_n$. Also, the formula is expressed as a constant multiple of a symmetric Laplacian-like matrix $\mathcal{L}$ plus a symmetric rank-one matrix defined by a vector $\mathbf{w}$. To be more precise,

$$D(H_n)^{-1} = -\frac{1}{2}\mathcal{L} + \frac{4}{3(n-1)}\mathbf{w}\mathbf{w}', \tag{4.11}$$

where $\mathcal{L}$ is a symmetric positive semi-definite matrix of rank $2n-2$ and $\mathbf{w} = \frac{1}{4}(5-n, -\mathbf{e}', 2\mathbf{e}')' \in \mathbb{R}^{2n-1}$ ([8], Theorem 3). In Section 4.2, we give another proof of this result. We point out that our proof does not depend on $D(W_n)^{-1}$.

Motivated by the inverse formula given in (4.11), we study an analogous formula for the Moore-Penrose inverse of $D(H_n)$ when $D(H_n)$ is singular, see Section 4.3. If $n$ is odd, we proved that $D(H_n)$ is singular (Theorem 3.1). In this case, we establish a formula for $D(H_n)^\dagger$ by finding equivalent formulations for the general matrix case and in particular, $D(H_n)$ which will be discussed in Section 4.1.

**4.1. Characterizations for the Moore-Penrose inverse of a general symmetric matrix and $D(H_n)$.** In this subsection, we derive a necessary and sufficient condition for a symmetric matrix $D$ to have its Moore-Penrose inverse of the form (4.11). The conditions are given in terms of system of linear equations and matrix equations, where the precise statement is given below.

Theorem 4.1. *Let $D$ be a symmetric matrix of order $n$ with $\mathbf{e} \in R(D)$ and $\mathbf{w} \in \mathbb{R}^n$ such that $\mathbf{e}'\mathbf{w} = 1$. Suppose that $L$ is a symmetric Laplacian-like matrix and $\alpha$ is a non-zero real number. Then, $D^\dagger = -\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}'$ if and only if $D\mathbf{w} = \frac{1}{\alpha}\mathbf{e}$ and $LD + 2I = 2\mathbf{w}\mathbf{e}' + \widetilde{V}$ for some symmetric matrix $\widetilde{V}$ satisfying $D\widetilde{V} = O$ and $\widetilde{V}(-\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}') = O$.*

*Proof.* Assume that $D^\dagger = -\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}'$ with $L\mathbf{e} = \mathbf{0}$. Premultiplying $D^\dagger$ by $D$, we get $DD^\dagger = -\frac{1}{2}DL + \alpha D\mathbf{w}\mathbf{w}'$. Since $\mathbf{e} \in R(D)$, we have $DD^\dagger\mathbf{e} = \mathbf{e}$ (see [6]). Therefore, $DD^\dagger\mathbf{e} = -\frac{1}{2}DL\mathbf{e} + \alpha D\mathbf{w}\mathbf{w}'\mathbf{e} = \alpha D\mathbf{w}$ which gives $D\mathbf{w} = \frac{1}{\alpha}\mathbf{e}$. Now postmultiplying $D^\dagger$ by $D$, we obtain $D^\dagger D = -\frac{1}{2}LD + \alpha\mathbf{w}\mathbf{w}'D = -\frac{1}{2}LD + \mathbf{w}\mathbf{e}'$. This implies $LD + 2I = 2\mathbf{w}\mathbf{e}' + 2(I - D^\dagger D) = 2\mathbf{w}\mathbf{e}' + \widetilde{V}$, where $\widetilde{V} := 2(I - D^\dagger D)$. Note that $\widetilde{V}$ is a symmetric matrix as $D^\dagger D$ is symmetric. Clearly $D\widetilde{V} = O$. We claim that $\widetilde{V}(-\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}') = O$. We have $\widetilde{V}D^\dagger = (2I - 2D^\dagger D)D^\dagger = 2D^\dagger - 2D^\dagger DD^\dagger = O$. By the assumption on $D^\dagger$, the claim follows. Conversely, assume that

$$D\mathbf{w} = \frac{1}{\alpha}\mathbf{e} \text{ and } LD + 2I = 2\mathbf{w}\mathbf{e}' + \widetilde{V}, \tag{4.12}$$

where $L$ and $\widetilde{V}$ are symmetric matrices such that $L\mathbf{e} = \mathbf{0}$, $D\widetilde{V} = O$ and $\widetilde{V}(-\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}') = O$. Let $X = -\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}'$. We claim that $X$ is the Moore-Penrose inverse of $D$. First we show that $XD$ is symmetric. We have $XD = -\frac{1}{2}LD + \alpha\mathbf{w}\mathbf{w}'D$. Using (4.12), we can write $XD = I - \frac{1}{2}\widetilde{V}$. Therefore, $XD$ is symmetric. Also $DX$ is a symmetric matrix which follows from the fact that $X$, $D$ and $XD$ are symmetric. Since $D\widetilde{V} = O$, it is easy to see that $DXD = D(I - \frac{1}{2}\widetilde{V}) = D$. The assumption $\widetilde{V}X = O$ gives $XDX = X$. This completes the proof. ∎

The next theorem gives the uniqueness of the Laplacian-like matrix $L$ and the vector $\mathbf{w}$, which satisfy the identities in (4.12).

Theorem 4.2. *Let $D$ be a symmetric matrix of order $n$. If there exists a non-zero real number $\alpha$ and $\mathbf{w} \in \mathbb{R}^n$ with $\mathbf{e}'\mathbf{w} = 1$ such that $D^\dagger = -\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}'$ for some Laplacian-like matrix $L$, then the scalar $\alpha$, the vector $\mathbf{w}$ and the matrix $L$ are unique.*

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*Proof.* Suppose $D^{\dagger} = -\frac{1}{2}L_0 + \beta\mathbf{z}\mathbf{z}'$ for some non-zero $\beta \in \mathbb{R}$, $\mathbf{z} \in \mathbb{R}^n$ with $\mathbf{e}'\mathbf{z} = 1$ and $L_0\mathbf{e} = \mathbf{0}$. Then, $\mathbf{e}'D^{\dagger}\mathbf{e} = \mathbf{e}'(\alpha\mathbf{w}\mathbf{w}')\mathbf{e} = \mathbf{e}'(\beta\mathbf{z}\mathbf{z}')\mathbf{e}$. This implies $\alpha = \beta$ and hence $D^{\dagger} = -\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}' = -\frac{1}{2}L_0 + \alpha\mathbf{z}\mathbf{z}'$. By finding $D^{\dagger}\mathbf{e}$, we get $\mathbf{w} = \mathbf{z}$. Hence, $L = L_0$ which completes the proof. ∎

In the following remark, we mention certain facts about the system $D(H_n)\mathbf{w} = \frac{1}{\alpha}\mathbf{e}$ where $\alpha$ is a non-zero real number.

*Remark* 4.3. It was proved in ([14], Theorem 15) that the system $D(H_n)\mathbf{w} = \frac{3(n-1)}{4}\mathbf{e}$ has a solution $\mathbf{w_0} = \frac{1}{4}(5-n, -\mathbf{e}', 2\mathbf{e}')' \in \mathbb{R}^{2n-1}$. Note that $\mathbf{e}'\mathbf{w_0} = 1$ and $\mathbf{e} \in R(D(H_n))$. In fact, it can be shown that if the system $D(H_n)\mathbf{w} = \frac{1}{\alpha}\mathbf{e}$ has a solution $\mathbf{w} \in \mathbb{R}^{2n-1}$ with $\mathbf{e}'\mathbf{w} = 1$ then $\alpha = \frac{4}{3(n-1)}$. We omit the proof as it is similar to the case of $D(H_n)\mathbf{z} = \mathbf{0}$, (see Theorem 3.2). Moreover, if $D(H_n)$ is singular then the solution set of $D(H_n)\mathbf{w} = \frac{3(n-1)}{4}\mathbf{e}$ is $\mathbf{w_0} + N(D(H_n)) = \left\{\frac{1}{4}(5-n, \beta\mathbf{v}' - \mathbf{e}', 2\mathbf{e}')' \in \mathbb{R}^{2n-1} : \beta \in \mathbb{R}\right\}$ where $\mathbf{v} = (1, -1, 1, -1, \dots, 1, -1)' \in \mathbb{R}^{n-1}$ (see Theorem 3.2). Of course, the solution is unique if $D(H_n)$ is non-singular.

The next result gives a necessary condition on $\widetilde{V}$ which will be useful in obtaining the identities (4.12) in the case of $D(H_n)$.

Lemma 4.4. *Let $S = \mathrm{Circ}(\mathbf{s}')$ where $\mathbf{s} = (2, 1, 0, \dots, 0, 1)' \in \mathbb{R}^{n-1}$ and let $D$ be the distance matrix of $H_n$. Then, there exists a symmetric matrix $\widetilde{V}$ such that $D\widetilde{V} = O$ if and only if $\widetilde{V} = \begin{bmatrix} 0 & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0} & X & O \\ \mathbf{0} & O & O \end{bmatrix}$ for some symmetric matrix $X$ of order $n-1$ with $X\mathbf{e} = \mathbf{0}$ and $SX = O$.*

*Proof.* Assume that there exists a symmetric matrix $\widetilde{V}$ such that $D\widetilde{V} = O$. Suppose $\widetilde{V} = \begin{bmatrix} \gamma & \mathbf{p}' & \mathbf{q}' \\ \mathbf{p} & X & F \\ \mathbf{q} & F' & G \end{bmatrix}$ where $\gamma \in \mathbb{R}$, $\mathbf{p}, \mathbf{q} \in \mathbb{R}^{n-1}$ and $X$, $F$, $G$ are matrices of order $n-1$. Then, $(D\widetilde{V})_{*1} = D\widetilde{V}_{*1} = D\begin{bmatrix} \gamma \\ \mathbf{p} \\ \mathbf{q} \end{bmatrix} = \mathbf{0}$ gives that $\gamma = 0$ and $\mathbf{q} = \mathbf{0}$ (see the proof of Theorem 3.2). Similarly, we conclude that the first and the last $(n-1)$ coordinates of each column of $\widetilde{V}$ are zero. That is, $\mathbf{p} = \mathbf{0}$, $F' = O$ and $G = O$. Now $D\widetilde{V} = O$ reduces to $\mathbf{e}'X = \mathbf{0}'$ and $(2J - S)X = O$. Thus, $X\mathbf{e} = \mathbf{0}$ and $SX = O$. Conversely, assume that $\widetilde{V} = \begin{bmatrix} 0 & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0} & X & O \\ \mathbf{0} & O & O \end{bmatrix}$ where $X$ is an $(n-1) \times (n-1)$ symmetric matrix satisfying the conditions $X\mathbf{e} = \mathbf{0}$ and $SX = O$. By a direct verification, we see that $D\widetilde{V} = O$. ∎

We now derive a necessary and sufficient condition on the Laplacian-like matrix $L$ such that the identities in (4.12) are satisfied for the case of $D(H_n)$. Recall that for a real matrix $M$, $R(M') = R(M^{\dagger})$ (see [6]), which will be used in the next proof.

Theorem 4.5. *Suppose that $S = \mathrm{Circ}(\mathbf{s}')$ where $\mathbf{s} = (2, 1, 0, \dots, 0, 1)' \in \mathbb{R}^{n-1}$. Let $D$ be the distance matrix of $H_n$. Then, $D^{\dagger} = -\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}'$ for some Laplacian-like matrix $L$, non-zero $\alpha \in \mathbb{R}$ and $\mathbf{w} \in \mathbb{R}^{2n-1}$ with $\mathbf{e}'\mathbf{w} = 1$ if and only if $L = \begin{bmatrix} \frac{n-1}{2} & -\frac{1}{2}\mathbf{e}' & \mathbf{0}' \\ -\frac{1}{2}\mathbf{e} & A & B \\ \mathbf{0} & B' & I_{n-1} \end{bmatrix}$ where $A$ and $B$ are symmetric matrices of order $n-1$ satisfying the following conditions:*

| | | |
|---|---|---|
| *(i)* $A\mathbf{e} = \frac{3}{2}\mathbf{e}$ | *(iii)* $BS = -S$ | *(v)* $(B+I)B = O$ |
| *(ii)* $B\mathbf{e} = -\mathbf{e}$ | *(iv)* $(B+I)A = O$ | *(vi)* $(A+B)S + 2B = O$. |

*Proof.* Assume that $D^{\dagger} = -\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}'$ with $L\mathbf{e} = \mathbf{0}$ and $\mathbf{e}'\mathbf{w} = 1$. Note that $D^{\dagger}$ is symmetric as $D$ is symmetric, see [6]. This implies $L$ is symmetric. We first claim that $\mathbf{w} = \frac{1}{4}(5-n, -\mathbf{e}', 2\mathbf{e}')'$. By Theorem 4.1 and Remark 4.3, we have $D\mathbf{w} = \frac{1}{\alpha}\mathbf{e}$ where $\alpha = \frac{4}{3(n-1)}$. If $n$ is even then $D$ is non-singular and $\mathbf{w}$ is the unique solution of $D\mathbf{w} = \frac{1}{\alpha}\mathbf{e}$. Now assume that $n$ is odd. Then from Remark 4.3, $\mathbf{w} = \frac{1}{4}(5-n, \beta\mathbf{v}' - \mathbf{e}', 2\mathbf{e}')'$

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for some $\beta \in \mathbb{R}$ and $\mathbf{v} = (1,-1,1,-1,\dots,1,-1)' \in \mathbb{R}^{n-1}$. Note that $D^{\dagger}\mathbf{e} = -\frac{1}{2}L\mathbf{e} + \alpha\mathbf{w}\mathbf{w}'\mathbf{e} = \alpha\mathbf{w}$. Thus $\mathbf{w} \in R(D^{\dagger}) = R(D)$ because $D$ is symmetric. Since $(0,\mathbf{v}',\mathbf{0}')' \in N(D)$, we get $(0,\mathbf{v}',\mathbf{0}')\mathbf{w} = 0$. That is, $\frac{1}{4}\mathbf{v}'(\beta\mathbf{v}-\mathbf{e}) = \frac{1}{4}[\beta(n-1) - \mathbf{v}'\mathbf{e}] = \frac{1}{4}\beta(n-1) = 0$. Therefore, $\beta = 0$ and hence $\mathbf{w} = \frac{1}{4}(5-n,-\mathbf{e}',2\mathbf{e}')'$. Suppose $L = \begin{bmatrix} \gamma & \mathbf{p}' & \mathbf{q}' \\ \mathbf{p} & A & B \\ \mathbf{q} & B' & C \end{bmatrix}$ where $\gamma \in \mathbb{R}$, $\mathbf{p},\mathbf{q} \in \mathbb{R}^{n-1}$ and $A$ and $B$ are matrices of order $n-1$. Then, the condition $L\mathbf{e} = \mathbf{0}$ gives the following three equations:

$$\gamma + \mathbf{p}'\mathbf{e} + \mathbf{q}'\mathbf{e} = 0. \tag{4.13}$$

$$\mathbf{p} + A\mathbf{e} + B\mathbf{e} = \mathbf{0}. \tag{4.14}$$

$$\mathbf{q} + B'\mathbf{e} + C\mathbf{e} = \mathbf{0}. \tag{4.15}$$

By Theorem 4.1, $LD = 2\mathbf{w}\mathbf{e}' - 2I + \widetilde{V}$ with $D\widetilde{V} = O$ and $\widetilde{V}(-\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}') = O$ where $\widetilde{V}$ is a symmetric matrix. Since $D\widetilde{V} = O$, by Lemma 4.4, there exists an $(n-1)\times(n-1)$ symmetric matrix $X$ such that

$$\widetilde{V} = \begin{bmatrix} 0 & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0} & X & O \\ \mathbf{0} & O & O \end{bmatrix} \text{ with } X\mathbf{e} = \mathbf{0}. \tag{4.16}$$

Note that

$$2\mathbf{w}\mathbf{e}' - 2I + \widetilde{V} = \begin{bmatrix} \frac{1-n}{2} & \frac{5-n}{2}\mathbf{e}' & \frac{5-n}{2}\mathbf{e}' \\ -\frac{1}{2}\mathbf{e} & -\frac{1}{2}J - 2I + X & -\frac{1}{2}J \\ \mathbf{e} & J & J - 2I \end{bmatrix} = LD. \tag{4.17}$$

Suppose

$$LD = Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \tag{4.18}$$

where $Y$ is partitioned similarly to $L$. Equating the $(1,2)^{\text{th}}$ block of (4.17) and (4.18),

$$Y_{12} = \gamma\mathbf{e}' + 2\mathbf{p}'J - \mathbf{p}'S + 3\mathbf{q}'J - \mathbf{q}'S = \frac{5-n}{2}\mathbf{e}'. \tag{4.19}$$

Similarly $(1,3)^{\text{rd}}$ block gives

$$Y_{13} = 2\gamma\mathbf{e}' + 3\mathbf{p}'J - \mathbf{p}'S + 4\mathbf{q}'J - \mathbf{q}'S - 2\mathbf{q}' = \frac{5-n}{2}\mathbf{e}'. \tag{4.20}$$

Subtracting (4.19) from (4.20) gives $\gamma\mathbf{e}' + \mathbf{p}'J + \mathbf{q}'J = 2\mathbf{q}'$. This implies that $2q_i = \gamma + \mathbf{p}'\mathbf{e} + \mathbf{q}'\mathbf{e}$ for all $i = 1,2,\dots,n-1$ where $\mathbf{q} = (q_1,q_2,\dots,q_{n-1})' \in \mathbb{R}^{n-1}$. Using (4.13), we get $q_i = 0$, for all $i$ and hence $\mathbf{q} = \mathbf{0}$. We have $C\mathbf{e} = -B'\mathbf{e}$ from (4.15). Equating $(2,1)^{\text{th}}$ and $(3,1)^{\text{th}}$ blocks of (4.17) and (4.18), we get

$$Y_{21} = A\mathbf{e} + 2B\mathbf{e} = -\frac{1}{2}\mathbf{e} \quad \text{and} \quad Y_{31} = B'\mathbf{e} + 2C\mathbf{e} = \mathbf{e}. \tag{4.21}$$

Using $C\mathbf{e} = -B'\mathbf{e}$ in (4.21) gives $C\mathbf{e} = \mathbf{e}$ and hence $B'\mathbf{e} = -\mathbf{e}$. We prove the conditions $(i) - (vi)$ in the following order. Firstly, we show $(vi)$ followed by $(ii)$. Secondly, we prove $(i)$ and $(iii)$. Lastly, $(iv)$ and $(v)$

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will be verified. Substituting the first equation of (4.21) in (4.14), we get $\mathbf{p} - B\mathbf{e} = \frac{1}{2}\mathbf{e}$. Postmultiplying this equation by $\mathbf{e}'$ gives $\mathbf{p}\mathbf{e}' = BJ + \frac{1}{2}J$. Also, from the first equation of (4.21), we have $AJ = -2BJ - \frac{1}{2}J$. Note that $Y_{22} = \mathbf{p}\mathbf{e}' + A(2J - S) + B(3J - S)$ and $Y_{23} = 2\mathbf{p}\mathbf{e}' + A(3J - S) + B(4J - S - 2I)$. Comparing these with the corresponding blocks of (4.17) and using $\mathbf{p}\mathbf{e}'$ and $AJ$, we get $AS + BS = 2I - X$ and $AS + BS + 2B = O$. This implies $(vi)$ is proved and hence $2B = -(AS + BS) = X - 2I$. Therefore, $B$ is symmetric as $X$ is symmetric. Since $B'\mathbf{e} = -\mathbf{e}$, we have proved $(ii)$. Also, $A\mathbf{e} = -2B'\mathbf{e} - \frac{1}{2}\mathbf{e} = \frac{3}{2}\mathbf{e}$ and $\mathbf{p} = \frac{-1}{2}\mathbf{e}$ follows from (4.21) and (4.14), respectively. By (4.13), $\gamma = -\mathbf{p}'\mathbf{e} = \frac{n-1}{2}$. Substituting $BJ = -J$, $CJ = J$ and $\mathbf{q} = \mathbf{0}$ in (4.18), and then equating $(3,2)^{\text{th}}$ and $(3,3)^{\text{th}}$ blocks of (4.18) and (4.17), we have $BS + CS = O$ and $BS + CS + 2C = 2I$ which implies $C = I$ and $BS = -S$. This proves $(iii)$. To complete the only if part of the proof, it is remaining to show $(B + I)A = O$ and $(B + I)B = O$. It is easy to verify that $\widetilde{V}\mathbf{w} = \mathbf{0}$ where $\widetilde{V}$ is given in (4.16) with $X = 2(B + I)$.Then, the conditions $(B + I)A = O$ and $(B + I)B = O$ are easily derived from $\widetilde{V}(-\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}') = O$.

Conversely, assume that $L = \begin{bmatrix} \frac{n-1}{2} & \frac{-1}{2}\mathbf{e}' & \mathbf{0}' \\ \frac{-1}{2}\mathbf{e} & A & B \\ \mathbf{0} & B' & I \end{bmatrix}$ where the symmetric matrices $A$ and $B$ satisfy the conditions $(i) - (vi)$. Using the conditions $(i)$ and $(ii)$, it is easy to see that $L\mathbf{e} = \mathbf{0}$. Note that $D\mathbf{w} = \frac{1}{\alpha}\mathbf{e}$ where $\mathbf{w} = \frac{1}{4}(5 - n, -\mathbf{e}', 2\mathbf{e}')'$ and $\alpha = \frac{4}{3(n-1)}$. We claim that $D^{\dagger} = -\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}'$. By Theorem 4.1, it is enough to find a symmetric matrix $\widetilde{V}$ of order $2n - 1$ satisfying $LD + 2I = 2\mathbf{w}\mathbf{e}' + \widetilde{V}$ with $D\widetilde{V} = O$ and $\widetilde{V}(-\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}') = O$. Choose $\widetilde{V} = \begin{bmatrix} 0 & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0} & 2(B+I) & O \\ \mathbf{0} & O & O \end{bmatrix}$. Then it is easy to see that $LD + 2I = 2\mathbf{w}\mathbf{e}' + \widetilde{V}$. From the assumptions $(i)$ to $(v)$, it is clear that $D\widetilde{V} = O$, $\widetilde{V}L = O$ and $\widetilde{V}\mathbf{w} = \mathbf{0}$. Hence $\widetilde{V}(-\frac{1}{2}L + \alpha\mathbf{w}\mathbf{w}') = O$. This completes the proof. ☐

**4.2. A short proof of an inverse formula for $D(H_n)$.** If $n$ is even, an inverse formula for $D(H_n)$ is given as the sum of symmetric Laplacian-like matrix and a rank-one matrix ([8], Theorem 3). While deriving this, the inverse of the distance matrix $D(W_n)$ of a wheel graph $W_n$ is used. Using the inverse formula for $D(W_n)$, one may also get $D(H_n)^{-1}$ from [18]. In this section, we offer a different proof of this result without employing any results pertaining to $D(W_n)^{-1}$.

We use circulant matrices of order $n - 1$ while introducing the Laplacian-like matrices. The vectors defining the circulant matrices follow a certain type of symmetry in the last $n - 2$ coordinates, which we recall below.

Definition 4.6 ([3, 4]). *Let $n \geq 4$. A vector $\mathbf{z} = (z_1, z_2, \ldots, z_{n-1})' \in \mathbb{R}^{n-1}$ is said to follow symmetry in its last $n - 2$ coordinates if $z_i = z_{n+1-i}$ for all $i = 2, 3, \ldots, n - 1$.*

Let $\Delta$ be the collection of all vectors $\mathbf{z} \in \mathbb{R}^{n-1}$ that follow symmetry in its last $n - 2$ coordinates. That is, $\Delta := \{\mathbf{z} = (z_1, z_2, \ldots, z_{n-1})' : z_i = z_{n+1-i}$ for all $i = 2, 3, \ldots, n - 1\}$.

*Remark* 4.7. The above-defined $\Delta$ is a subspace of $\mathbb{R}^{n-1}$. It is observed in [14] (see Theorem 10) that if $\mathbf{c} \in \Delta$ and $C = \mathrm{Circ}(\mathbf{c}')$, then $C$ is a symmetric matrix.

The following lemma is useful in computing the vector which follows symmetry.

Lemma 4.8. *Let $\alpha$ and $\beta$ be real numbers and $\mathbf{g} = (\alpha, \beta, 0, \ldots, 0, \beta)' \in \mathbb{R}^{n-1}$. If $G = \mathrm{Circ}(\mathbf{g}')$, then $(\mathbf{z}'G)' \in \Delta$, for all $\mathbf{z} \in \Delta$.*

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*Proof.* Since $G = \mathrm{Circ}(\mathbf{g}')$, we have

$$G_{*k} = \begin{cases} \alpha\mathbf{e_1} + \beta(\mathbf{e_2} + \mathbf{e_{n-1}}) & \text{if } k = 1, \\ \beta(\mathbf{e_{k-1}} + \mathbf{e_{k+1}}) + \alpha\mathbf{e_k} & \text{if } 2 \le k \le n-2, \\ \beta(\mathbf{e_1} + \mathbf{e_{n-2}}) + \alpha\mathbf{e_{n-1}} & \text{if } k = n-1. \end{cases}$$

Let $\mathbf{z} = (z_1, z_2, \ldots, z_{n-1})' \in \Delta$. Then, $z_k = z_{n+1-k}$, for $k = 2, 3, \ldots, n-1$. Here we denote the $k$-th coordinate of $\mathbf{z}'G$ by $(\mathbf{z}'G)_k$. To show $(\mathbf{z}'G)' \in \Delta$, we first consider

$$\begin{aligned} (\mathbf{z}'G)_2 = \mathbf{z}'G_{*2} = \mathbf{z}'[\beta(\mathbf{e_1} + \mathbf{e_3}) + \alpha\mathbf{e_2}] &= \beta(z_1 + z_3) + \alpha z_2 \\ &= \beta(z_1 + z_{n-2}) + \alpha z_{n-1} \\ &= \mathbf{z}'[\beta(\mathbf{e_1} + \mathbf{e_{n-2}}) + \alpha\mathbf{e_{n-1}}] \\ &= (\mathbf{z}'G)_{n-1}. \end{aligned}$$

If $3 \le k \le n-2$, then $3 \le n-k+1 \le n-2$. The $k$-th coordinate of $\mathbf{z}'G$ is given by

$$\begin{aligned} (\mathbf{z}'G)_k = \mathbf{z}'G_{*k} &= \mathbf{z}'[\beta(\mathbf{e_{k-1}} + \mathbf{e_{k+1}}) + \alpha\mathbf{e_k}] \\ &= \beta(z_{k-1} + z_{k+1}) + \alpha z_k \\ &= \beta(z_{n+1-(k-1)} + z_{n+1-(k+1)}) + \alpha z_{n+1-k} \\ &= \mathbf{z}'[\beta(\mathbf{e_{n+1-(k-1)}} + \mathbf{e_{n+1-(k+1)}}) + \alpha\mathbf{e_{n+1-k}}] \\ &= (\mathbf{z}'G)_{(n+1-k)}. \end{aligned}$$

Hence the proof. ∎

To give an alternative proof of Theorem 3 in [8], let us recall the symmetric Laplacian-like matrix $\mathcal{L}$ associated with $D(H_n)^{-1}$.

DEFINITION 4.9 ([8]). *Let $n \ge 4$ be even. For $1 \le k \le \frac{n}{2} - 1$, $\beta_k = (-1)^k[(n-1) - 2k]$. Let*

$$\mathbf{z} = \frac{1}{2}(n+1, \beta_1, \beta_2, \ldots, \beta_{\frac{n}{2}-2}, \beta_{\frac{n}{2}-1}, \beta_{\frac{n}{2}-1}, \beta_{\frac{n}{2}-2}, \ldots, \beta_2, \beta_1)' \in \mathbb{R}^{n-1}.$$

*Define*

$$\mathcal{L} = \begin{bmatrix} \frac{n-1}{2} & \frac{-1}{2}\mathbf{e}' & \mathbf{0}' \\ \frac{-1}{2}\mathbf{e} & A & -I_{n-1} \\ \mathbf{0} & -I_{n-1} & I_{n-1} \end{bmatrix} \text{ where } A = \mathrm{Circ}(\mathbf{z}'). \tag{4.22}$$

Now we state Theorem 3 in [8] and present a short proof of this.

THEOREM 4.10. *Let $n \ge 4$ be an even integer. If $\mathcal{L}$ is the matrix defined in (4.22), then*

$$D(H_n)^{-1} = -\frac{1}{2}\mathcal{L} + \frac{4}{3(n-1)}\mathbf{w}\mathbf{w}'$$

*where* $\mathbf{w}' = \frac{1}{4}(5-n, -\mathbf{e}', 2\mathbf{e}') \in \mathbb{R}^{2n-1}$.

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*Proof.* Since $n$ is even, $D(H_n)$ is non-singular, see [8]. Therefore, $D(H_n)^{\dagger} = D(H_n)^{-1}$. It is straight forward that the blocks of $\mathcal{L}$ satisfy the conditions $(ii) - (v)$ given in Theorem 4.5. From [8], we have $\sum_{k=1}^{\frac{n}{2}-1}(-1)^k(n-1-2k) = \frac{2-n}{2}$. Since $A = \mathrm{Circ}(\mathbf{z}')$, we get

$$A\mathbf{e} = (\mathbf{z}'\mathbf{e})\mathbf{e} = \frac{1}{2}\left((n+1) + \sum_{k=1}^{\frac{n}{2}-1} 2\beta_k\right)\mathbf{e} = \left(\frac{n+1}{2} + \sum_{k=1}^{\frac{n}{2}-1}(-1)^k(n-1-2k)\right)\mathbf{e} = \frac{3}{2}\mathbf{e}.$$

If we prove $(A-I)S - 2I = O$, then the desired result follows by Theorem 4.5 together with Remark 4.3. By (2.4), it is enough to prove $\mathbf{z}'S = \mathbf{s}' + 2\mathbf{e_1}'$. We start with computing $\mathbf{z}'S$. Let $\mathbf{z}'S = (r_1, r_2, \ldots, r_{n-1})$. Then, $r_i = \mathbf{z}'S_{*i}$. Since $\mathbf{z} \in \Delta$, $(\mathbf{z}'S)' \in \Delta$ by Lemma 4.8. It is enough to compute the first $\frac{n}{2}$ coordinates of $\mathbf{z}'S$. We have $r_1 = \mathbf{z}'(2\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_{n-1}}) = \frac{1}{2}[2(n+1) + 2\beta_1] = (n+1) + (-1)(n-1-2) = 4$ and $r_2 = \mathbf{z}'(\mathbf{e_1} + 2\mathbf{e_2} + \mathbf{e_3}) = \frac{1}{2}[(n+1) + 2\beta_1 + \beta_2] = 1$. If $3 \le i \le \frac{n}{2} - 1$, then $r_i = \mathbf{z}'(\mathbf{e_{i-1}} + 2\mathbf{e_i} + \mathbf{e_{i+1}}) = \beta_{i-2} + 2\beta_{i-1} + \beta_i = 0$. Similarly, we see that $r_{\frac{n}{2}} = \beta_{\frac{n}{2}-2} + 3\beta_{\frac{n}{2}-1} = 0$. Thus, $\mathbf{z}'S = (4, 1, 0, 0, \ldots, 0, 1) = \mathbf{s}' + 2\mathbf{e_1}'$. ∎

**4.3. The Moore-Penrose inverse of $D(H_n)$.** In this section, we derive a formula for the Moore-Penrose Inverse of $D(H_n)$, when $n$ is odd. This is an analogous to the formula given for $D(H_n)^{-1}$. To obtain the desired formula, we introduce the Laplacian-like matrix $L$, similar to $\mathcal{L}$ in (4.22), involving two circulant matrices $A$ and $B$ which are defined by the vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^{n-1}$, respectively. The vectors $\mathbf{x}$ and $\mathbf{y}$ are identified from the numerical examples. Hereafter, it is assumed that $n \ge 5$ is an odd integer and $m = \frac{n-1}{2}$ is fixed.

Let $1 \le k \le m$. For each $k$, we define $\alpha_k \in \mathbb{R}$ and we fix the vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^{n-1}$ which are given by

$$\alpha_k := (-1)^{k+1}[2m^2 - 6(m-k)^2 + 7], \tag{4.23}$$

$$\mathbf{x} := \frac{1}{6(n-1)}(n^2 + 4n - 12, \alpha_1, \alpha_2, \ldots, \alpha_{m-1}, \alpha_m, \alpha_{m-1}, \alpha_{m-2}, \ldots, \alpha_2, \alpha_1)', \tag{4.24}$$

and

$$\mathbf{y} := \frac{1}{(n-1)}(2-n, -1, 1, -1, 1, \ldots, -1, 1, -1)'. \tag{4.25}$$

Using Theorem 4.5 and the above-defined vectors $\mathbf{x}$ and $\mathbf{y}$, we now construct the symmetric Laplacian-like matrix $L$ associated with the formula for $D(H_n)^{\dagger}$.

Definition 4.11. *Let $n \ge 5$ be an odd integer. Let $A = \mathrm{Circ}(\mathbf{x}')$ and $B = \mathrm{Circ}(\mathbf{y}')$ where $\mathbf{x}$ and $\mathbf{y}$ are given in (4.24) and (4.25), respectively. Define*

$$L = \begin{bmatrix} \frac{n-1}{2} & \frac{-1}{2}\mathbf{e}' & \mathbf{0}' \\ \frac{-1}{2}\mathbf{e} & A & B \\ \mathbf{0} & B & I_{n-1} \end{bmatrix}. \tag{4.26}$$

*Remark* 4.12. Note that the vectors $\mathbf{x}$ and $\mathbf{y}$ are in $\Delta$. By Remark 4.7, $A$ and $B$ are symmetric matrices of order $n-1$. Thus, the matrix $L$, of order $2n-1$, given in the above definition is symmetric.

We now state the result which gives the desired formula for $D(H_n)^{\dagger}$.

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THEOREM 4.13. *Let $n \geq 5$ be an odd integer and $L$ be the matrix given in Definition 4.11. Suppose $\mathbf{w} = \frac{1}{4}(5-n, -\mathbf{e}', 2\mathbf{e}')' \in \mathbb{R}^{2n-1}$ and $D$ is the distance matrix of $H_n$. Then,*

$$D^\dagger = -\frac{1}{2}L + \frac{4}{3(n-1)}\mathbf{w}\mathbf{w}'.$$

A proof of this result is based on the following three lemmas.

LEMMA 4.14. *Let $A$ and $B$ be the matrices given in Definition 4.11. Then,*

*(i)* $A\mathbf{e} = \frac{3}{2}\mathbf{e}$.
*(ii)* $B\mathbf{e} = -\mathbf{e}$.

*Proof.* Since $A = \mathrm{Circ}(\mathbf{x}')$, all the row sums of $A$ are equal to $\mathbf{e}'\mathbf{x}$. Therefore, $A\mathbf{e} = (\mathbf{e}'\mathbf{x})\mathbf{e}$. To determine $\mathbf{e}'\mathbf{x}$, we consider

$$\begin{aligned}\sum_{k=1}^{m-1}\alpha_k &= \sum_{k=1}^{m-1}(-1)^{k+1}(2m^2-6(m-k)^2+7)\\ &= (7-4m^2)\sum_{k=1}^{m-1}(-1)^{k+1} + 12m\sum_{k=1}^{m-1}(-1)^{k+1}k - 6\sum_{k=1}^{m-1}(-1)^{k+1}k^2. \end{aligned}\tag{4.27}$$

By a simple verification, it is easy to see that

$$\sum_{k=1}^{m-1}(-1)^{k+1} = \begin{cases}1 & \text{if } m \text{ is even,}\\ 0 & \text{if } m \text{ is odd,}\end{cases}\tag{4.28}$$

$$\sum_{k=1}^{m-1}(-1)^{k+1}k = \frac{1}{2}\begin{cases}m & \text{if } m \text{ is even,}\\ 1-m & \text{if } m \text{ is odd,}\end{cases}\tag{4.29}$$

and

$$\sum_{k=1}^{m-1}(-1)^{k+1}k^2 = \frac{1}{2}\begin{cases}m(m-1) & \text{if } m \text{ is even,}\\ -m(m-1) & \text{if } m \text{ is odd.}\end{cases}\tag{4.30}$$

Using (4.27)-(4.30) and $\alpha_m = (-1)^{m+1}(2m^2+7)$, we get

$$\begin{aligned}2\sum_{k=1}^{m-1}\alpha_k + \alpha_m &= \begin{cases}2(-m^2+3m+7)-(2m^2+7) & \text{if } m \text{ is even,}\\ 2(-3m^2+3m)+(2m^2+7) & \text{if } m \text{ is odd.}\end{cases}\\ &= -4m^2+6m+7\\ &= \left[-4\left(\frac{n-1}{2}\right)^2 + 6\left(\frac{n-1}{2}\right)+7\right]\\ &= -n^2+5n+3.\end{aligned}$$

Note that $\mathbf{e}'\mathbf{x} = \frac{1}{6(n-1)}\left[(n^2+4n-12) + 2\sum_{k=1}^{m-1}\alpha_k + \alpha_m\right]$. Thus, $\mathbf{e}'\mathbf{x} = \frac{3}{2}$ which implies $A\mathbf{e} = \frac{3}{2}\mathbf{e}$. This proves (i). Since $n-1$ is even, we have $\mathbf{e}'\mathbf{y} = -1$ and hence $B\mathbf{e} = -\mathbf{e}$. □

A square matrix $M$ is said to be *diagonalizable* if there exists a non-singular matrix $P$ such that $P^{-1}MP = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $M$. We now recall a result on simultaneous diagonalization which will be frequently used in the proofs.

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Theorem 4.15 ([13]). *Let $M_1, M_2, \ldots, M_k$ be diagonalizable matrices. Then, $M_1, M_2, \ldots, M_k$ are simultaneously diagonalizable if and only if $M_iM_j = M_jM_i$ for all $i, j \in \{1, 2, \ldots, k\}$. Moreover, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $M_1$, then there exists a non-singular matrix $P$ such that $P^{-1}M_1P = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $P^{-1}M_jP$ is diagonal for all $j = 2, 3, \ldots, k$.*

If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of a real symmetric matrix $M$ with respective multiplicities $m_1, m_2, \ldots, m_k$, then the set of all eigenvalues of $M$ is given by $\sigma(M) = \left\{\lambda_1^{[m_1]}, \lambda_2^{[m_2]}, \ldots, \lambda_k^{[m_k]}\right\}$.

Lemma 4.16. *Let $n \geq 5$ be an odd integer. Let $B = \operatorname{Circ}(\mathbf{y}')$ where the vector $\mathbf{y}$ is given in (4.25). Then, $\sigma(B) = \left\{0^{[1]}, -1^{[n-2]}\right\}$ and $\sigma\left(B - \frac{1}{2(n-1)}J_{n-1}\right) = \left\{0^{[1]}, -1^{[n-3]}, -\frac{3}{2}^{[1]}\right\}$.*

*Proof.* Let $\mathbf{v} = (1, -1, 1, -1, \ldots, 1, -1)' \in \mathbb{R}^{n-1}$. As rows of $\operatorname{Circ}(\mathbf{v}')$ are either $\mathbf{v}'$ or $-\mathbf{v}'$, we have $\operatorname{rank}(\operatorname{Circ}(\mathbf{v}')) = 1$. Hence, 0 is an eigenvalue of $\operatorname{Circ}(\mathbf{v}')$ with multiplicity $n-2$. Also, $\operatorname{Circ}(\mathbf{v}')\mathbf{v} = (n-1)\mathbf{v}$. Thus, $\sigma(\frac{1}{n-1}\operatorname{Circ}(\mathbf{v}')) = \left\{1^{[1]}, 0^{[n-2]}\right\}$. Since $\mathbf{y} = \frac{1}{n-1}\mathbf{v} - \mathbf{e_1}$, we write $B = \operatorname{Circ}(\mathbf{y}') = \frac{1}{n-1}\operatorname{Circ}(\mathbf{v}') - I$. Hence, $\sigma(B) = \left\{0^{[1]}, -1^{[n-2]}\right\}$. It is easy to verify that $B\mathbf{v} = \mathbf{0}$ and $J_{n-1}\mathbf{v} = (\mathbf{e}'\mathbf{v})\mathbf{e} = \mathbf{0}$. This gives $B - \frac{1}{2(n-1)}J_{n-1}$ is singular because $\left(B - \frac{1}{2(n-1)}J_{n-1}\right)\mathbf{v} = \mathbf{0}$. Note that $B$ and $\frac{1}{2(n-1)}J_{n-1}$ commute by (2.4). Hence, by Theorem 4.15, there exists a non-singular matrix $P$ such that $P^{-1}BP = \operatorname{diag}(0, -1, -1, \ldots, -1)$ and $P^{-1}\left(\frac{1}{2(n-1)}J_{n-1}\right)P = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$, where $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are the eigenvalues of $\frac{1}{2(n-1)}J_{n-1}$. Therefore, $P^{-1}\left(B - \frac{1}{2(n-1)}J_{n-1}\right)P = \operatorname{diag}(-\lambda_1, -1-\lambda_2, \ldots, -1-\lambda_{n-1})$. We claim that $\lambda_1 = 0$. On the contrary, $\lambda_1 \neq 0$. Then, $\lambda_1 = \frac{1}{2}$ and $\lambda_j = 0$ for $2 \leq j \leq n-1$ because the eigenvalues of $\frac{1}{2(n-1)}J_{n-1}$ are $\frac{1}{2}$ and 0 with multiplicities 1 and $n-2$, respectively. This implies that $\lambda_1 \neq 0$ and $-1-\lambda_j \neq 0$ for all $j = 2, 3, \ldots, n-1$, and hence, $B - \frac{1}{2(n-1)}J_{n-1}$ is non-singular, which is a contradiction. Therefore, $\lambda_1 = 0$, the result follows. ∎

Lemma 4.17. *Let $n \geq 5$ be an odd integer. Suppose $S = \operatorname{Circ}(\mathbf{s}')$ where $\mathbf{s} = (2, 1, 0, \ldots, 0, 1)'$ in $\mathbb{R}^{n-1}$. If $A$ and $B$ are the matrices given in the Definition 4.11, then the following conditions hold.*

*(i)* $A(B + I) = O$.
*(ii)* $B(B + I) = O$.
*(iii)* $BS = -S$.
*(iv)* $(A + B)S + 2B = O$.

*Proof.* Since $A$ and $B$ are circulant symmetric matrices, they commute by (2.4) and are diagonalizable. Therefore, by Theorem 4.15, they are simultaneously diagonalizable. That is, there exists an invertible matrix $P$ (whose columns are the eigenvectors of $A$ as well $B$) such that

$$P^{-1}AP = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_{n-1}) \text{ and } P^{-1}BP = \operatorname{diag}(\beta_1, \beta_2, \ldots, \beta_{n-1}).$$

By Lemma 4.16, the eigenvalues of $B$ are 0 and $-1$ with multiplicities 1 and $n-2$, respectively. We assume $P$ such that $\beta_1 = 0$ and $\beta_i = -1$ for $2 \leq i \leq n-1$.

(i) To show $A(B + I) = O$, it is enough to prove that all the eigenvalues of $A(B+I)$ are zero. Note that the eigenvalues of $A(B + I)$ are $\mu_i(\beta_i + 1)$, $i = 1, 2, \ldots, n-1$. We claim that $\mu_i(\beta_i + 1) = 0$ for all $i$. As $\beta_1 = 0$ and $\beta_i = -1$ for $i = 2, 3, \ldots, n-1$, it remains to prove $\mu_1 = 0$. Since $B\mathbf{v} = \mathbf{0} = \beta_1\mathbf{v}$ where $\mathbf{v} = (1, -1, 1, -1, \ldots, 1, -1)' \in \mathbb{R}^{n-1}$ and the multiplicity of the eigenvalue 0 of $B$ is 1, the first column of $P$ is a scalar multiple of $\mathbf{v}$. Therefore, $A\mathbf{v} = \mu_1\mathbf{v}$. It is easy to see that $A\mathbf{v} = (\mathbf{x}'\mathbf{v})\mathbf{v}$. Note that $\mathbf{x}'\mathbf{v} = \frac{1}{6(n-1)}\left[(n^2 + 4n - 12) + 2\left(\sum_{k=1}^{m-1}(-1)^k\alpha_k\right) + (-1)^m\alpha_m\right]$. Substituting $\alpha_k$'s and simplifying, we get

$$\mathbf{x}'\mathbf{v} = \frac{1}{6(n-1)}\left[(n^2 + 4n - 12) - 2\left((7 - 4m^2)\sum_{k=1}^{m-1}1 + 12m\sum_{k=1}^{m-1}k - 6\sum_{k=1}^{m-1}k^2\right) - (2m^2 + 7)\right].$$

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Using the formulae for the sum of first $m-1$ natural numbers and the sum of squares of first $m-1$ natural numbers, we have $\mathbf{x'v} = \frac{1}{6(n-1)}\left[(n^2+4n-12) - 2(m^2+6m-7) - (2m^2+7)\right] = 0$. Thus, $\mu_1 = 0$.

(ii) It is clear that all the eigenvalues of $B(B+I)$ are zero and hence $B(B+I) = O$.

(iii) Let $\mathbf{v} = (1,-1,1,-1,\dots,1,-1)' \in \mathbb{R}^{n-1}$. It is easy to see that $\mathbf{v}'S = \mathbf{0}'$. Note that $\mathbf{y} = \frac{1}{n-1}\mathbf{v} - \mathbf{e_1}$. Since $B = \mathrm{Circ}(\mathbf{y}')$, we have $BS = \mathrm{Circ}(\mathbf{y}'S) = \mathrm{Circ}(\frac{1}{n-1}\mathbf{v}'S - \mathbf{e_1}'S) = \mathrm{Circ}(-\mathbf{s}') = -S$.

(iv) Let $\mathbf{x}'S = (p_1, p_2, \dots, p_{n-1})$. Then, $p_i = \mathbf{x}'S_{*i}$. From (4.23) and (4.24), we have

$$\begin{aligned}
p_1 &= \mathbf{x}'(2\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_{n-1}}) \\
&= \frac{1}{6(n-1)}[2(n^2+4n-12) + 2\alpha_1] \\
&= \frac{2}{6(n-1)}[(n^2+4n-12) + (-1)^2(2m^2 - 6(m-1)^2 + 7)] \\
&= \frac{4n-6}{n-1}.
\end{aligned}$$

Note that $p_2 = \mathbf{x}'(\mathbf{e_1} + 2\mathbf{e_2} + \mathbf{e_3}) = \frac{1}{6(n-1)}\left[(n^2+4n-12) + 2\alpha_1 + \alpha_2\right] = \frac{n+1}{n-1}$. Let $3 \le i \le m$. Then, $p_i = \mathbf{x}'(\mathbf{e_{i-1}} + 2\mathbf{e_i} + \mathbf{e_{i+1}}) = \alpha_{i-2} + 2\alpha_{i-1} + \alpha_i = (-1)^i \frac{2}{n-1}$. Similarly, we see that $p_{m+1} = 2(\alpha_{m-1} + \alpha_m) = (-1)^{m+1}\frac{2}{n-1}$. Since $\mathbf{x} \in \Delta$, we have $(\mathbf{x}'S)' \in \Delta$ by Lemma 4.8. Thus,

$$\mathbf{x}'S = \frac{1}{(n-1)}(4n-6, n+1, -2, 2, -2, 2, \dots, 2, -2, n+1).$$

Now it is clear that $\mathbf{x}'S + 2\mathbf{y}' = \mathbf{s}'$. Also, from (iii), $\mathbf{y}'S = -\mathbf{s}'$. Hence $(A+B)S + 2B = \mathrm{Circ}(\mathbf{x}'S + \mathbf{y}'S + 2\mathbf{y}') = O$. □

*Proof of Theorem* 4.13. The result follows from Theorem 4.5 and Lemmas 4.14 and 4.17. □

In the following, we study two properties of the Laplacian-like matrix $L$ defined in (4.26). First, we show that $L$ is a positive semi-definite matrix. Let us recall that an $n \times n$ real symmetric matrix $M$ is said to be *positive semi-definite (positive definite)* if $\mathbf{z}'M\mathbf{z} \ge 0$ (respectively, $\mathbf{z}'M\mathbf{z} > 0$) for all non-zero $\mathbf{z} \in \mathbb{R}^n$. We abbreviate the positive semi-definite (positive definite) as psd (respectively, pd).

To prove $L$ is psd, we need to find the eigenvalues of $\mathrm{Circ}(\mathbf{s}')$, where $\mathbf{s} = (2,1,0,0,\dots,0,1)'$, which follows from the next theorem.

THEOREM 4.18 ([17]). *Let $\mathbf{m} = (m_0, m_1, \dots, m_{n-1})' \in \mathbb{R}^n$ and $f(x) = m_0 + m_1 x + m_2 x^2 + \cdots + m_{n-1}x^{n-1}$. Then the eigenvalues of $M = \mathrm{Circ}(\mathbf{m}')$ are $f(\omega^j)$, $j = 0,1,2,\dots,n-1$ where $\omega = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$ is an $n^{th}$ primitive root of unity.*

*Remark* 4.19. Consider the matrix $S = \mathrm{Circ}(\mathbf{s}')$ where $\mathbf{s} = (2,1,0,\dots,0,1)' \in \mathbb{R}^{n-1}$. In this case, $f(x) = 2 + x + x^{n-2}$. Therefore, $f(\omega_j) = 2 + \omega_j + \omega_j^{-1}$ because $\omega_j^{n-1} = 1$. From Theorem 4.18, the eigenvalues of $S$ are $4\cos^2(\frac{\pi j}{n-1})$, $j = 0,1,2,\dots,n-2$.

Using the same arguments as given in the proof of Theorem 7.2$(iv)$ in [3], it can be shown that $L$ is psd. However, we wish to give a different proof based on the following result involving the Schur complement for the sake of completeness.

THEOREM 4.20 ([13]). *Let $M_1$ and $M_3$ be square matrices. If $M = \begin{bmatrix} M_1 & M_2 \\ M_2' & M_3 \end{bmatrix}$ is a symmetric matrix, then the following statements are true.*

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*(i) Let $M_1$ be pd. Then, $M$ is psd if and only if $M_3 - M_2' M_1^{-1} M_2$ is psd.*
*(ii) Let $M_3$ be pd. Then, $M$ is psd if and only if $M_1 - M_2 M_3^{-1} M_2'$ is psd.*

THEOREM 4.21. *Let $L$ be the matrix given in Definition 4.11. Then $L$ is a positive semi-definite matrix.*

*Proof.* By $(i)$ of Theorem 4.20, it suffices to show that $X = \begin{bmatrix} A-\frac{1}{2(n-1)}J_{n-1} & B \\ B & I_{n-1}\end{bmatrix}$ is psd. Note that $B^2 = -B$ by $(ii)$ of Lemma 4.17. Now applying part $(ii)$ of Theorem 4.20 to $X$, we get $L$ is psd if and only if $A + B - \frac{1}{2(n-1)}J_{n-1}$ is psd. If we prove all the eigenvalues of $A + B - \frac{1}{2(n-1)}J_{n-1}$ are non-negative, then the desired result follows. Since $A$, $B$, $S$ and $B - \frac{1}{2(n-1)}J_{n-1}$ are symmetric circulant matrices, by (2.4) and Theorem 4.15, there exists an invertible matrix $P$ such that $P^{-1}AP =: \Lambda_1 = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_{n-1})$, $P^{-1}BP =: \Lambda_2 = \mathrm{diag}(0, -1, -1, \ldots, -1)$, $P^{-1}SP =: \Lambda_3 = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_{n-1})$, and $P^{-1}\left(B - \frac{1}{2(n-1)}J_{n-1}\right)P = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_{n-1})$ where the diagonal entries of $\Lambda_2$ are obtained from Lemma 4.16. Then, $\mu_1 = 0$ follows from $\Lambda_1(\Lambda_2 + I) = O$ by $(i)$ of Lemma 4.17. We have $(\Lambda_2 + I)\Lambda_3 = O$ by the identity $BS = -S$ from Lemma 4.17. This implies $\gamma_1 = 0$. We now claim that $\delta_1 = 0$. The multiplicities of the eigenvalue 0 with respect to $B$ and $B - \frac{1}{2(n-1)}J_{n-1}$ are equal to 1, see Lemma 4.16. Also, $B\mathbf{v} = \left(B - \frac{1}{2(n-1)}J_{n-1}\right)\mathbf{v} = \mathbf{0}$ where $\mathbf{v} = (1, -1, 1, -1, \ldots, 1, -1)' \in \mathbb{R}^{n-1}$. Therefore, the first column of $P$ must be a scalar multiple of $\mathbf{v}$. Hence, $\delta_1 = 0$. To complete the proof, we need to show that $\mu_j + \delta_j \geq 0$ for all $j = 2, 3, \ldots, n-1$. Let $2 \leq j \leq n-1$. From Lemma 4.16, it is clear that $\delta_j$ is either $-1$ or $-\frac{3}{2}$. In view of this, it is enough to prove that $\mu_j \geq \frac{3}{2}$. Using the fact that $\mathrm{rank}(S) = n-2$ (see Theorem 3.2) and by Remark 4.19, we get $\gamma_j > 0$. By $(iii)$ and $(iv)$ of Lemma 4.17, we write $(A - I)S = -2B$ which gives $(\Lambda_1 - I)\Lambda_3 = -2\Lambda_2$. This yields that $(\mu_j - 1)\gamma_j = 2$. This implies $\mu_j = \frac{2}{\gamma_j} + 1 = \frac{2}{4\cos^2\left(\frac{\pi j}{n-1}\right)} + 1 \geq \frac{3}{2}$. This completes the proof. ∎

In the next result, we find the rank of $L$. We will make use of the following theorem.

THEOREM 4.22 ([13]). *Let $A$ and $B$ be symmetric matrices of same order. Then,* $\mathrm{rank}(A + B) \leq \mathrm{rank}(A) + \mathrm{rank}(B)$. *Furthermore, equality holds if and only if $R(A) \cap R(B) = \{\mathbf{0}\}$.*

THEOREM 4.23. *Let $L$ be the matrix given in (4.26). Then, the rank of $L$ is $2n-3$.*

*Proof.* For $\mathbf{w} = \frac{1}{4}(5-n, -\mathbf{e}', 2\mathbf{e}')' \in \mathbb{R}^{2n-1}$, we have $\mathbf{e}'\mathbf{w} = 1$. We claim that $\mathbf{w} \notin R(L)$. On the contrary, suppose $L\mathbf{z} = \mathbf{w}$ for some $\mathbf{z} \in \mathbb{R}^{2n-1}$. Since $\mathbf{e}'L = \mathbf{0}'$, $\mathbf{e}'L\mathbf{z} = \mathbf{e}'\mathbf{w} = 0$ which is impossible. Hence, $\mathbf{w} \notin R(L)$. Thus, $R(L) \cap R(\mathbf{w}\mathbf{w}') = \{\mathbf{0}\}$. By Theorems 4.13 and 4.22, $\mathrm{rank}(L) = \mathrm{rank}(D(H_n)^\dagger) - 1$. Since $\mathrm{rank}(D(H_n)) = \mathrm{rank}(D(H_n)^\dagger)$, the result follows from Theorem 3.2. ∎

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