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SOME RESULTS ON MATRIX PARTIAL ORDERINGS AND REVERSE ORDER LAW*

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Abstract. Some results relating different matrix partial orderings and the reverse order law for the Moore-Penrose inverse and the group inverse are given. Special attention is paid when at least one of the two involved matrices is EP.

Key words. Partial ordering, Moore-Penrose inverse, Group inverse, Reverse order law.

AMS subject classifications. 15A09, 15A57.

1. Introduction. The symbol $\mathbb{C}^{m\times n}$ will denote the set of $m \times n$ complex matrices. Let A^*, $\mathcal{R}(A)$, and $\mathrm{rk}(A)$ denote the conjugate transpose, column space, and rank of $A \in \mathbb{C}^{m\times n}$, respectively. In 1955 Penrose showed in [17] that, for every matrix $A \in \mathbb{C}^{m\times n}$, there is a unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the four equations

$$AXA = A \quad (1), \qquad XAX = X \quad (2), \qquad (AX)^* = AX \quad (3), \qquad (XA)^* = XA \quad (4).$$

This matrix X is commonly known as the Moore-Penrose inverse of A, and is denoted by $A^\dagger$. It is evident that if $A, U \in \mathbb{C}^{n\times n}$ and U is unitary, then $(UAU^*)^\dagger = UA^\dagger U^*$. Also, it is obvious that $(P \oplus Q)^\dagger = P^\dagger \oplus Q^\dagger$ holds for every pair of square matrices P and Q.

As we shall wish to deal with a number of different subsets of the set of equations (1)–(4), we need a convenient notation for a generalized inverse satisfying certain specified equations. For any $A \in \mathbb{C}^{m\times n}$, let $A\{i, j, \ldots, k\}$ denote the set of matrices $X \in \mathbb{C}^{n\times m}$ which satisfy equations (i), (j), ..., (k) from among equations (1)–(4).

For a matrix $A \in \mathbb{C}^{n\times n}$, the solution (unique if it exists) to the three equations

$$AXA = A, \qquad XAX = X, \qquad AX = XA$$

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with respect to X is called the group inverse of A denoted by $A^{\#}$. It is well known that $A^{\#}$ exists if and only if $\mathrm{rk}(A) = \mathrm{rk}(A^2)$ (see, for example, [6, Section 4.4]), in which case A is called a group matrix. The symbol $\mathbb{C}_n^{\mathsf{GP}}$ will stand for the subset of $\mathbb{C}^{n\times n}$ consisting of group matrices.

For a square matrix A, in general one has that $AA^\dagger \neq A^\dagger A$. A matrix $A \in \mathbb{C}^{n\times n}$ satisfying $AA^\dagger = A^\dagger A$ is said to be EP (the name comes from Equal Projection, because $AA^\dagger$ is the orthogonal projection onto $\mathcal{R}(A)$ and $A^\dagger A$ is the orthogonal projection onto $\mathcal{R}(A^*)$). The symbol $\mathbb{C}_n^{\mathsf{EP}}$ will denote the subset of $\mathbb{C}^{n\times n}$ consisting of EP matrices. Obviously one has that $\mathbb{C}_n^{\mathsf{EP}} \subset \mathbb{C}_n^{\mathsf{GP}}$.

A crucial role in subsequent considerations is played by the theorem given below, which constitutes part (i) $\Leftrightarrow$ (iv) of Theorem 4.3.1 in [7].

Theorem 1.1. *Let $A \in \mathbb{C}^{n\times n}$ have rank r. The following statements are equivalent:*

(i) *A is an EP matrix.*
(ii) *There exist a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a nonsingular $K \in \mathbb{C}^{r\times r}$ such that*

$$A = U(K \oplus 0)U^*. \tag{1.1}$$

It is easy to prove that if A is an EP matrix being represented as in (1.1), then

$$A^\dagger = U(K^{-1} \oplus 0)U^*. \tag{1.2}$$

Contrary to the usual inverse, it is not true in general that $(AB)^\dagger = B^\dagger A^\dagger$ (when $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$). This relationship is customarily known as the reverse order law. There does not seem to be a simple criterion for distinguishing the cases in which $(AB)^\dagger = B^\dagger A^\dagger$ holds. The following result is due to Greville [10]: For matrices A, B such that AB exists, $(AB)^\dagger = B^\dagger A^\dagger$ holds if and only if $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$ and $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$.

The reverse order law for the generalized inverses of the multiple-matrix products yields a class of interesting problems that are fundamental in the theory of generalized inverses of matrices and statistics. As suggested reading, we can cite (among others) [8, 18].

In this paper we shall give several results concerning the reverse order law for the Moore–Penrose inverse and group inverse and different orderings in the set $\mathbb{C}^{n\times n}$.

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Such orderings are defined in the sequel. The first of them is the star ordering introduced by Drazin [9] in 1978, which can be defined by

$$A \overset{*}{\leq} B \quad \Longleftrightarrow \quad A^*A = A^*B \text{ and } AA^* = BA^*. \tag{1.3}$$

In 1991, Baksalary and Mitra [5] defined the left-star and right-star orderings characterized as

$$A \ast\!\leq B \quad \Longleftrightarrow \quad A^*A = A^*B \text{ and } \mathcal{R}(A) \subset \mathcal{R}(B) \tag{1.4}$$

and

$$A \leq\!\ast\, B \quad \Longleftrightarrow \quad AA^* = BA^* \text{ and } \mathcal{R}(A^*) \subset \mathcal{R}(B^*) \tag{1.5}$$

respectively.

Another binary relation, the so-called sharp ordering, introduced by Mitra [15] in 1987, is defined in the set $\mathbb{C}_n^{\mathsf{GP}}$ by

$$A \overset{\#}{\leq} B \quad \Longleftrightarrow \quad A^{\#}A = A^{\#}B \text{ and } AA^{\#} = BA^{\#}.$$

In the following, $A \overset{\#}{\leq} B$ should entail the assumption that A and B are group matrices. It is easy to verify that

$$A \overset{\#}{\leq} B \quad \Longleftrightarrow \quad AB = A^2 = BA. \tag{1.6}$$

The last partial ordering we will deal with in this paper is the minus (or rank subtractivity) ordering defined by Hartwig [12] and Nambooripad [16] independently in 1980:

$$A \overset{-}{\leq} B \quad \Longleftrightarrow \quad \mathrm{rk}(B-A) = \mathrm{rk}(B) - \mathrm{rk}(A).$$

The interested reader can consult [1, 2, 4] and references therein in order to get a deep insight into the aforementioned orderings in $\mathbb{C}^{n\times n}$.

When we study matrices $A, B \in \mathbb{C}^{n\times n}$ satisfying $A \overset{?}{\leq} B$, where $\overset{?}{\leq}$ is any of the orderings defined above (except the sharp ordering), we will require that at least one of the involved matrices is EP. We shall use Theorem 1.1 to deal with these situations.

2. The star ordering and the reverse order law. In this section, we study the relation between $A \overset{*}{\leq} B$ and the reverse order law for the Moore-Penrose inverse of AB or BA when A is EP or B is EP.

Theorem 2.1. *Let $A \in \mathbb{C}_n^{\mathsf{EP}}$ and $B \in \mathbb{C}^{n\times n}$. Then the following statements are equivalent:*

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(i) $A \overset{*}{\leq} B$.
(ii) $AB = BA = A^2$.
(iii) $(AB)^\dagger = B^\dagger A^\dagger = A^\dagger B^\dagger$ *and* $A = AA^\dagger B$.
(iv) $(AB)^\dagger = B^\dagger A^\dagger = A^\dagger B^\dagger$ *and* $A = BAA^\dagger$.

Proof. Since A is EP, by Theorem 1.1 there exist a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a nonsingular matrix $K \in \mathbb{C}^{r\times r}$ such that matrix A is represented as in (1.1). Let us remark that representation (1.2) also holds.

(i) $\Rightarrow$ (ii) Let us write matrix B as follows:

$$B = U \begin{pmatrix} P & Q \\ R & S \end{pmatrix} U^*, \qquad P \in \mathbb{C}^{r\times r},\ S \in \mathbb{C}^{(n-r)\times(n-r)}. \tag{2.1}$$

The first equality of the right side of the equivalence (1.3) implies that

$$\begin{pmatrix} K^* & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} K^* & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P & Q \\ R & S \end{pmatrix},$$

which with the nonsingularity of K^* leads to $K = P$ and $Q = 0$. Whereas the second equality of the right side of the equivalence (1.3) leads to

$$\begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} K^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}\begin{pmatrix} K^* & 0 \\ 0 & 0 \end{pmatrix},$$

which yields $R = 0$. Thus, we get $B = U(K \oplus S)U^*$, and therefore,

$$AB = U(K \oplus 0)(K \oplus S)U^* = U(K^2 \oplus 0)U^* = A^2 = U(K \oplus S)(K \oplus 0)U^* = BA.$$

(ii) $\Rightarrow$ (iii) As before, we can write B as in (2.1). From $AB = BA$ and the nonsingularity of K we get $Q = 0$ and $R = 0$, i.e., $B = U(P \oplus S)U^*$. From $AB = A^2$ we get $KP = K^2$, and the nonsingularity of K leads to $K = P$. Hence

$$B = U(K \oplus S)U^* \qquad \text{and} \qquad B^\dagger = U(K^{-1} \oplus S^\dagger)U^*.$$

Now, using (1.1) and (1.2) we have

$$(AB)^\dagger = [U(K \oplus 0)U^* U(K \oplus S)U^*]^\dagger = U(K^{-2} \oplus 0)U^* = A^\dagger B^\dagger = B^\dagger A^\dagger.$$

Moreover, it is satisfied that $AA^\dagger B = A$ because

$$AA^\dagger B = U(K \oplus 0)(K^{-1} \oplus 0)(K \oplus S)U^* = U(K \oplus 0)U^* = A.$$

(iii) $\Rightarrow$ (i) Let us write matrix $B^\dagger$ as follows:

$$B^\dagger = U \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} U^*, \qquad X \in \mathbb{C}^{r\times r},\ T \in \mathbb{C}^{(n-r)\times(n-r)}.$$

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The assumption $A^\dagger B^\dagger = B^\dagger A^\dagger$ and representation (1.2) entail

$$\begin{pmatrix} K^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X & Y \\ Z & T \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}\begin{pmatrix} K^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

from which we get $Y = 0$ and $Z = 0$. Hence $B^\dagger = U(X \oplus T)U^*$ holds, and therefore,

$$B = (B^\dagger)^\dagger = U(X^\dagger \oplus T^\dagger)U^*. \tag{2.2}$$

Now we use $A = AA^\dagger B$:

$$AA^\dagger B = U(K \oplus 0)(K^{-1} \oplus 0)(X^\dagger \oplus T^\dagger)U^* = U(X^\dagger \oplus 0)U^*.$$

This last computation, (1.1), and $A = AA^\dagger B$ lead to $X^\dagger = K$. From (1.1) and (2.2) we get

$$A^*A = U(K^*K \oplus 0)U^* = A^*B \quad \text{and} \quad AA^* = U(KK^* \oplus 0)U^* = BA^*.$$

Thus, we have that $A \overset{*}{\leq} B$.

(ii) ⇒ (iv) and (iv) ⇒ (i). These implications have the same proof as (ii) ⇒ (iii) and (iii) ⇒ (i), and thus, the theorem is demonstrated. □

Let us remark that part (i) ⇔ (ii) is known in the literature, see e.g., relationship (3.9) in [3]. We give an alternative approach based on block matrices. Observe that condition (ii) of Theorem 2.1 appears in the right side of the equivalence (1.6). In [11, Theorem 1], the author studied a simultaneous decomposition of matrices $A \in \mathbb{C}_n^{\mathsf{GP}}$ and $B \in \mathbb{C}^{n\times n}$ satisfying $AB = BA = A^2$.

To state the following theorem, let us permit to introduce the following notation: if $\mathcal{X}$ is a subspace of $\mathbb{C}^{n\times 1}$, the symbol $P_{\mathcal{X}}$ denotes the orthogonal projection onto $\mathcal{X}$. Let us recall that for every matrix A, one has that $P_{\mathcal{R}(A)} = AA^\dagger$ and $P_{\mathcal{R}(A^*)} = A^\dagger A$.

Theorem 2.2. *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}_n^{\mathsf{EP}}$.*

(i) *If $A \overset{*}{\leq} B$, then $ABB^\dagger = A = BB^\dagger A$.*
(ii) *If $ABB^\dagger = BB^\dagger A$, then $P_{\mathcal{R}(B)}$ commutes with $P_{\mathcal{R}(A)}$ and $P_{\mathcal{R}(A^*)}$, $B^\dagger A^\dagger \in AB\{1,2,3\}$, and $A^\dagger B^\dagger \in BA\{1,2,4\}$.*

Proof. Since B is EP, there exist a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a nonsingular matrix $K \in \mathbb{C}^{r\times r}$ such that

$$B = U(K \oplus 0)U^* \tag{2.3}$$

being $r = \mathrm{rk}(B)$. From the representation (2.3) we have

$$B^\dagger = U(K^{-1} \oplus 0)U^* \qquad \text{and} \qquad BB^\dagger = U(I_r \oplus 0)U^*. \tag{2.4}$$

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Let us write matrix A as follows:

$$A = U \begin{pmatrix} P & Q \\ R & S \end{pmatrix} U^*, \qquad P \in \mathbb{C}^{r\times r},\ S \in \mathbb{C}^{(n-r)\times(n-r)}. \tag{2.5}$$

(i) The first equality of the right side of the equivalence (1.3) implies that

$$\begin{pmatrix} P^* & R^* \\ Q^* & S^* \end{pmatrix}\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} P^* & R^* \\ Q^* & S^* \end{pmatrix}\begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix},$$

whose lower-right block gives $Q^*Q + S^*S = 0$, which yields $Q = 0$ and $S = 0$. The second equality of the right side of the equivalence (1.3) implies that

$$\begin{pmatrix} P & 0 \\ R & 0 \end{pmatrix}\begin{pmatrix} P^* & R^* \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P^* & R^* \\ 0 & 0 \end{pmatrix},$$

whose lower-right block gives $RR^* = 0$, which implies that $R = 0$. Thus, we have $A = U(P \oplus 0)U^*$. Hence

$$ABB^\dagger = U(P \oplus 0)(I_r \oplus 0)U^* = U(P \oplus 0)U^* = A = U(I_r \oplus 0)(P \oplus 0)U^* = BB^\dagger A$$

holds.

(ii) Writing matrix A as in (2.5) and using $A(BB^\dagger) = (BB^\dagger)A$ we get $Q = 0$ and $R = 0$, and therefore,

$$A = U(P \oplus S)U^*. \tag{2.6}$$

It follows that

$$AA^\dagger = U(PP^\dagger \oplus SS^\dagger)U^* \qquad \text{and} \qquad A^\dagger A = U(P^\dagger P \oplus S^\dagger S)U^*.$$

It is evident from (2.4) that $AA^\dagger$ and $A^\dagger A$ commute with $BB^\dagger$.

Next we shall prove from (2.3), (2.4), and (2.6) that $B^\dagger A^\dagger \in AB\{1,2,3\}$. Firstly we prove $(AB)(B^\dagger A^\dagger)(AB) = AB$:

$$\begin{aligned}&(AB)(B^\dagger A^\dagger)(AB)\\ &= U(P \oplus S)(K \oplus 0)(K^{-1} \oplus 0)(P^\dagger \oplus S^\dagger)(P \oplus S)(K \oplus 0)U^* = U(PK \oplus 0)U^* = AB.\end{aligned}$$

Secondly we prove $(B^\dagger A^\dagger)(AB)(B^\dagger A^\dagger) = B^\dagger A^\dagger$:

$$\begin{aligned}(B^\dagger A^\dagger)(AB)(B^\dagger A^\dagger) &= U(K^{-1} \oplus 0)(P^\dagger \oplus S^\dagger)(P \oplus S)(K \oplus 0)(K^{-1} \oplus 0)(P^\dagger \oplus S^\dagger)U^*\\ &= U(K^{-1}P^\dagger \oplus 0)U^* = B^\dagger A^\dagger.\end{aligned}$$

Lastly we prove that $(AB)(B^\dagger A^\dagger)$ is Hermitian:

$$(AB)(B^\dagger A^\dagger) = U(P \oplus S)(K \oplus 0)(K^{-1} \oplus 0)(P^\dagger \oplus S^\dagger)U^* = U(PP^\dagger \oplus 0)U^* \text{ is Hermitian.}$$

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The proof of $A^\dagger B^\dagger \in BA\{1,2,4\}$ is similar and we will not give it. ☐

THEOREM 2.3. *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}_n^{\mathsf{EP}}$. If $A \overset{*}{\leq} B$, then*

(i) $(AB)^\dagger = B^\dagger A^\dagger$ *if and only if* $BB^*P_{\mathcal{R}(A^*)} = P_{\mathcal{R}(A^*)}BB^*$.
(ii) $(BA)^\dagger = A^\dagger B^\dagger$ *if and only if* $B^*BP_{\mathcal{R}(A)} = P_{\mathcal{R}(A)}B^*B$.

Proof. We shall prove the first equivalence, and we will not give the proof of the other because its proof is similar. By Theorem 2.2, we have that $B^\dagger A^\dagger \in AB\{1,2,3\}$. Thus $(AB)^\dagger = B^\dagger A^\dagger$ if and only if $B^\dagger A^\dagger \in AB\{4\}$, or in another parlance, $(AB)^\dagger = B^\dagger A^\dagger$ if and only if $B^\dagger A^\dagger AB$ is Hermitian. In order to prove item (i), we will use the proof of Theorem 2.2. Since

$$B^\dagger A^\dagger AB = U(K^{-1}\oplus 0)(P^\dagger \oplus S^\dagger)(P\oplus S)(K\oplus 0)U^* = U(K^{-1}P^\dagger PK \oplus 0)U^*,$$

$$BB^*A^\dagger A = U(K\oplus 0)(K^*\oplus 0)(P^\dagger \oplus S^\dagger)(P\oplus S)U^* = U(KK^*P^\dagger P\oplus 0)U^*,$$

$$A^\dagger ABB^* = U(P^\dagger \oplus S^\dagger)(P\oplus S)(K\oplus 0)(K^*\oplus 0)U^* = U(P^\dagger PKK^*\oplus 0)U^*,$$

and $(P^\dagger P)^* = P^\dagger P$, we have that

$$\begin{aligned} B^\dagger A^\dagger AB \text{ is Hermitian} &\iff K^{-1}P^\dagger PK \text{ is Hermitian} \\ &\iff (K^{-1}P^\dagger PK)^* = K^{-1}P^\dagger PK \\ &\iff K^*P^\dagger P(K^{-1})^* = K^{-1}P^\dagger PK \\ &\iff K^*P^\dagger P(K^*)^{-1} = K^{-1}P^\dagger PK \\ &\iff KK^*P^\dagger P = P^\dagger PKK^* \\ &\iff BB^*A^\dagger A = A^\dagger ABB^*. \end{aligned}$$

This finishes the proof. ☐

COROLLARY 2.4. *Let $A, B \in \mathbb{C}^{n\times n}$. If B is EP and a partial isometry (i.e., $B^\dagger = B^*$), and $A \overset{*}{\leq} B$, then $(AB)^\dagger = B^\dagger A^\dagger$ and $(BA)^\dagger = A^\dagger B^\dagger$.*

Proof. Let us remark that since B is EP we have that $P_{\mathcal{R}(B)} = BB^\dagger = B^\dagger B = P_{\mathcal{R}(B^*)}$. Now, the proof of this corollary follows from Theorems 2.2 and 2.3. ☐

3. The left-star, right-star orderings and the reverse order law. Next results concern the left-star partial ordering. Before establishing them, let us write a useful representation for $A, B \in \mathbb{C}^{n\times n}$ when $A * \leq B$ and A is EP.

LEMMA 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be an EP matrix that is written as in* (1.1) *and let $B \in \mathbb{C}^{n\times n}$. Then $A * \leq B$ if and only if there exist $S \in \mathbb{C}^{(n-r)\times(n-r)}$ and*

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$Z \in \mathbb{C}^{(n-r)\times r}$ *such that*

$$B = U \begin{pmatrix} K & 0 \\ -SZ & S \end{pmatrix} U^*. \tag{3.1}$$

Proof. First of all, let us remark that the inclusion $\mathcal{R}(A) \subset \mathcal{R}(B)$ is equivalent to the existence of a matrix $C \in \mathbb{C}^{n\times n}$ such that $A = BC$.

Assume that $A \ast\!\leq B$. Since $\mathcal{R}(A) \subset \mathcal{R}(B)$, there exists $C \in \mathbb{C}^{n\times n}$ such that $A = BC$. Let us write matrices B and C as

$$B = U \begin{pmatrix} P & Q \\ R & S \end{pmatrix} U^*, \;\; C = U \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} U^*, \qquad P, X \in \mathbb{C}^{r\times r}, \; S, T \in \mathbb{C}^{(n-r)\times(n-r)}.$$

From the first equality of the right side of the equivalence of (1.4) and the nonsingularity of K we easily get $P = K$ and $Q = 0$. Now we use $A = BC$ and the invertibility of K to get $I_r = X$ and $0 = R + SZ$. Hence B can be written as in (3.1).

Assume that B is written as in (3.1). We have

$$A^*B = U \begin{pmatrix} K^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} K & 0 \\ -SZ & S \end{pmatrix} U^* = U \begin{pmatrix} K^*K & 0 \\ 0 & 0 \end{pmatrix} U^* = A^*A$$

and

$$B \left\{ U \begin{pmatrix} I_r & 0 \\ Z & 0 \end{pmatrix} U^* \right\} = U \begin{pmatrix} K & 0 \\ -SZ & S \end{pmatrix} \begin{pmatrix} I_r & 0 \\ Z & 0 \end{pmatrix} U^* = U \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} U^* = A.$$

Hence $A \ast\!\leq B$. The lemma is proved. □

THEOREM 3.2. *Let $A \in \mathbb{C}_n^{\mathsf{EP}}$ and $B \in \mathbb{C}^{n\times n}$. If $A \ast\!\leq B$, then*

(i) $AB = A^2$.
(ii) $A^\dagger B^\dagger = (AB)^\dagger$.
(iii) $B^\dagger A^\dagger \in AB\{1, 2, 3\}$.
(iv) $A^\dagger B^\dagger \in BA\{1, 2, 4\}$.

Proof. Let us write matrix A as in (1.1) and matrix B as in (3.1).

(i) It is evident.

(ii) We get $AB = U(K^2 \oplus 0)U^*$, hence

$$(AB)^\dagger = U(K^{-2} \oplus 0)U^*. \tag{3.2}$$

On the other hand, the expression for $B^\dagger$ is the following: (although a formula for the Moore-Penrose inverse of a block triangular matrix is hard to obtain, the fact that

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the lower blocks of representation (3.1) contain matrix S permits to find $B^\dagger$)

$$B^\dagger = U \begin{pmatrix} K^{-1} & 0 \\ S^\dagger S Z K^{-1} & S^\dagger \end{pmatrix} U^*. \tag{3.3}$$

The proof of the expression (3.3) is straightforward by checking the four equations of the definition of the Moore-Penrose inverse. Now, we have

$$A^\dagger B^\dagger = U \begin{pmatrix} K^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} K^{-1} & 0 \\ S^\dagger S Z K^{-1} & S^\dagger \end{pmatrix} U^* = U(K^{-2} \oplus 0)U^* = (AB)^\dagger.$$

(iii) We need to prove $(AB)(B^\dagger A^\dagger)(AB) = AB$, $(B^\dagger A^\dagger)(AB)(B^\dagger A^\dagger) = B^\dagger A^\dagger$, and $(AB)(B^\dagger A^\dagger)$ is Hermitian. To this end, we shall use the expressions (1.1), (1.2), (3.1), and (3.3). We easily get

$$ABB^\dagger A^\dagger = U \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} U^*,$$

which in particular implies the hermitancy of $ABB^\dagger A^\dagger$. Moreover we have

$$ABB^\dagger A^\dagger AB = U \begin{pmatrix} K^2 & 0 \\ 0 & 0 \end{pmatrix} U^* = AB$$

and

$$B^\dagger A^\dagger ABB^\dagger A^\dagger = U \begin{pmatrix} K^{-2} & 0 \\ S^\dagger S Z K^{-2} & 0 \end{pmatrix} U^* = B^\dagger A^\dagger.$$

(iv) The proof is similar as in (iii), and we will not give it. □

A problem which arises in the context of the different orderings defined in the introduction is to describe situations where all (or some of) the orderings become equivalent. Baksalary et al. proved in [1] that

$$A \overset{*}{\leq} B \iff A \ast\!\leq B \iff A \leq\!\ast\, B \iff A \overset{-}{\leq} B$$

hold for any partial isometries $A, B \in \mathbb{C}^{n\times n}$. Moreover, Baksalary et al. proved in [1] that

$$A \ast\!\leq B \iff A \overset{*}{\leq} B \qquad \text{and} \qquad A \leq\!\ast\, B \iff A \overset{*}{\leq} B$$

hold when A is EP and B is normal. The following result links the reverse order law for the Moore-Penrose inverse with the equivalence of the orderings.

Theorem 3.3. *Let $A \in \mathbb{C}_n^{\mathsf{EP}}$ and $B \in \mathbb{C}^{n\times n}$. If $A \ast\!\leq B$, then the following statements are equivalent:*

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(i) $B^\dagger A^\dagger = (AB)^\dagger$.
(ii) $AB = BA$.
(iii) $A \overset{*}{\leq} B$.
(iv) $A^\dagger B^\dagger = (BA)^\dagger$.

Proof. We will use Lemma 3.1 and the computations of Theorem 3.2 in order to prove that the four conditions of this theorem are equivalent to $SZ = 0$.

(i) $\Leftrightarrow$ ($SZ = 0$) From (1.2), (3.3), and (3.2) we have

$$B^\dagger A^\dagger - (AB)^\dagger = U \begin{pmatrix} 0 & 0 \\ S^\dagger SZK^{-2} & 0 \end{pmatrix} U^*.$$

The nonsingularity of K ensures that $B^\dagger A^\dagger = (AB)^\dagger$ if and only if $S^\dagger SZ = 0$. If $S^\dagger SZ = 0$, premultiplying by S and using $SS^\dagger S = S$ lead to $SZ = 0$, whereas if $SZ = 0$, then, obviously, $S^\dagger SZ = 0$.

(ii) $\Leftrightarrow$ ($SZ = 0$) From (1.1) and (3.1) we have

$$AB - BA = U \begin{pmatrix} 0 & 0 \\ SZK & 0 \end{pmatrix} U^*$$

and having in mind the nonsingularity of K, we trivially obtain that $AB = BA$ if and only if $SZ = 0$.

(iii) $\Leftrightarrow$ ($SZ = 0$) In view of the definitions of the star ordering and left-star ordering, see (1.3) and (1.4), we shall prove $AA^* = BA^*$ if and only if $SZ = 0$. From (1.1) and (3.1) we have

$$AA^* - BA^* = U \begin{pmatrix} 0 & 0 \\ SZK^* & 0 \end{pmatrix} U^*.$$

From the invertibility of K^*, $AA^* = BA^*$ if and only if $SZ = 0$.

(iv) $\Leftrightarrow$ ($SZ = 0$) Let $M = U^*BAU$ and $N = U^*A^\dagger B^\dagger U$. From (1.1), (1.2), (3.1), and (3.3) we have

$$M = \begin{pmatrix} K^2 & 0 \\ -SZK & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} K^{-2} & 0 \\ 0 & 0 \end{pmatrix}.$$

It is easy to verify that NM is Hermitian, $NMN = N$, $MNM = M$, and

$$MN = \begin{pmatrix} I_r & 0 \\ -SZK^{-1} & 0 \end{pmatrix}.$$

Thus,

$$\begin{aligned} (BA)^\dagger = A^\dagger B^\dagger &\iff M^\dagger = N \iff MN \text{ is Hermitian} \\ &\iff -SZK^{-1} = 0 \iff SZ = 0. \end{aligned}$$

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This completes the proof. $\square$

We can establish similar results as in Theorems 2.2 and 2.3.

THEOREM 3.4. *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}_n^{\mathsf{EP}}$. If $A \ast\!\leq B$, then*

(i) $A^\dagger B^\dagger \in BA\{1,2,4\}$ *and* $B^\dagger A^\dagger \in AB\{1,2,3\}$.
(ii) $A^\dagger B^\dagger = (BA)^\dagger$ *if and only if* B^*B *commutes with* $P_{\mathcal{R}(A)}$. *Moreover,* $B^\dagger A^\dagger = (AB)^\dagger$ *if and only if* BB^* *commutes with* $P_{\mathcal{R}(A^*)}$.

Proof. First of all, let us represent matrices A and B in a convenient form. Since B is EP, there exist a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a nonsingular $K \in \mathbb{C}^{r\times r}$, being $r = \mathrm{rk}(B)$, such that $B = U(K \oplus 0)U^*$. In view of $\mathcal{R}(A) \subset \mathcal{R}(B)$, there exists a matrix $C \in \mathbb{C}^{n\times n}$ such that $A = BC$. Let us write this matrix C as follows:

$$C = U \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} U^*, \qquad X \in \mathbb{C}^{r\times r},\ T \in \mathbb{C}^{(n-r)\times(n-r)}.$$

Now,

$$A = BC = U \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} U^* = U \begin{pmatrix} KX & KY \\ 0 & 0 \end{pmatrix} U^*.$$

Let us denote $M = KX$ and $L = KY$. From $A^*A = A^*B$ we get

$$\begin{pmatrix} M^* & 0 \\ L^* & 0 \end{pmatrix} \begin{pmatrix} M & L \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} M^* & 0 \\ L^* & 0 \end{pmatrix} \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}.$$

In view of the lower-right block of the former equality, we get $L^*L = 0$, which implies that $L = 0$. Therefore, matrix A can be represented as $A = U(M \oplus 0)U^*$. Now, the theorem should be easy to prove by mimicking the proofs of Theorems 2.2 and 2.3. $\square$

The above results in this section concern the left-star ordering. Having in mind that from (1.4) and (1.5), it is easy to see that $A \ast\!\leq B \iff A^* \leq\!\ast\, B^*$, we can obtain similar results for the right-star ordering. To prove Theorem 3.5 below, we recall the following three simple facts:

(1) For any matrix A, one has that $X \in A\{1,2,3\}$ if and only if $X^* \in A^*\{1,2,4\}$.
(2) $(A^\dagger)^* = (A^*)^\dagger$ holds for any matrix A.
(3) A is EP if and only if A^* is EP.

THEOREM 3.5. *Let $A \in \mathbb{C}_n^{\mathsf{EP}}$ and $B \in \mathbb{C}^{n\times n}$. If $A \leq\!\ast\, B$, then*

(i) $BA = A^2$.
(ii) $B^\dagger A^\dagger = (BA)^\dagger$.
(iii) $A^\dagger B^\dagger \in BA\{1,2,4\}$.
(iv) $B^\dagger A^\dagger \in AB\{1,2,3\}$.

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Proof. From the hypothesis we know that $A^* \ast\!\leq B^*$ and A^* is EP. Thus, applying Theorem 3.2 with matrices A^* and B^* we obtain $A^*B^* = (A^*)^2$, $(A^*)^\dagger(B^*)^\dagger = (A^*B^*)^\dagger$, $(B^*)^\dagger(A^*)^\dagger \in A^*B^*\{1,2,3\}$, and $(A^*)^\dagger(B^*)^\dagger \in B^*A^*\{1,2,4\}$. Now the theorem should be easy to prove in view of the simple facts stated before this theorem. □

THEOREM 3.6. *Let $A \in \mathbb{C}_n^{\mathsf{EP}}$ and $B \in \mathbb{C}^{n\times n}$. If $A \leq\!\ast\, B$, then the following statements are equivalent:*

(i) $A^\dagger B^\dagger = (BA)^\dagger$.
(ii) $AB = BA$.
(iii) $A \overset{*}{\leq} B$.
(iv) $B^\dagger A^\dagger = (AB)^\dagger$.

Proof. Let us remark that from (1.3) we have that $A \overset{*}{\leq} B \iff A^* \overset{*}{\leq} B^*$. Now the proof follows from the same argument as in the proof of Theorem 3.5. □

As before we can establish the following result, whose proof is omitted.

THEOREM 3.7. *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}_n^{\mathsf{EP}}$. If $A \leq\!\ast\, B$, then*

(i) $A^\dagger B^\dagger \in BA\{1,2,4\}$ *and* $B^\dagger A^\dagger \in AB\{1,2,3\}$.
(ii) $A^\dagger B^\dagger = (BA)^\dagger$ *if and only if* B^*B *commutes with* $P_{\mathcal{R}(A)}$. *Moreover,* $B^\dagger A^\dagger = (AB)^\dagger$ *if and only if* BB^* *commutes with* $P_{\mathcal{R}(A^*)}$.

4. The minus ordering and the reverse order law. In this section, we study the relation between the reverse order law and the matrix minus ordering. To this end, we need the following result developed by Hartwig and Styan [13].

LEMMA 4.1. *Let $A, B \in \mathbb{C}^{m\times n}$ and let $a = \mathrm{rk}(A) < \mathrm{rk}(B) = b$. Then $A \overset{-}{\leq} B$ if and only if*

$$A = U \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} V^* \quad \text{and} \quad B = U \begin{pmatrix} D_1 + RD_2S & RD_2 \\ D_2S & D_2 \end{pmatrix} V^*, \tag{4.1}$$

for some $U \in \mathbb{C}^{m\times b}$, $V \in \mathbb{C}^{n\times b}$ such that $U^*U = I_b = V^*V$, positive definite diagonal matrices D_1, D_2 of degree a, $b-a$, respectively, and arbitrary $R \in \mathbb{C}^{a\times(b-a)}$, $S \in \mathbb{C}^{(b-a)\times a}$.

Suppose that matrices $A, B \in \mathbb{C}^{n\times n}$ satisfy $A \overset{-}{\leq} B$ and let $U, V \in \mathbb{C}^{n\times b}$ be as specified in Lemma 4.1. Partition matrices U and V as follows:

$$U = \begin{pmatrix} U_1 & U_2 \end{pmatrix}, \qquad V = \begin{pmatrix} V_1 & V_2 \end{pmatrix},$$

where $U_1, V_1 \in \mathbb{C}^{n\times a}$ and $U_2, V_2 \in \mathbb{C}^{n\times(b-a)}$. The matrix $W = V^*U \in \mathbb{C}^{b\times b}$ can be

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partitioned in the following way:

$$W = V^*U = \begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix} \begin{pmatrix} U_1 & U_2 \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}, \tag{4.2}$$

where $W_{ij} = V_i^* U_j$, for $i, j = 1, 2$, and thus $W_{11} \in \mathbb{C}^{a\times a}$, $W_{12} \in \mathbb{C}^{a\times(b-a)}$, $W_{21} \in \mathbb{C}^{(b-a)\times a}$, and $W_{22} \in \mathbb{C}^{(b-a)\times(b-a)}$.

The following lemma developed by Baksalary et al. in [1] gives some properties of the matrix W:

Lemma 4.2. *For any $A \in \mathbb{C}^{n\times n}$ of the form as in* (4.1) *and $W \in \mathbb{C}^{b\times b}$ of the form* (4.2),

(i) *A is EP if and only if W_{11} is unitary, $W_{12} = 0$, and $W_{21} = 0$.*
(ii) *A is normal if and only if W_{11} is unitary, $W_{12} = 0$, $W_{21} = 0$, and $W_{11}D_1^2 = D_1^2 W_{11}$.*
(iii) *A is Hermitian if and only if W_{11} is unitary, $W_{12} = 0$, $W_{21} = 0$, and $W_{11}D_1 = D_1 W_{11}^*$.*

Moreover, the last condition in item (iii) *may be replaced by $W_{11}D_1 = D_1W_{11}$.*

Now, based on the above lemmas, we can get the following result.

Theorem 4.3. *Let $A \in \mathbb{C}_n^{\mathsf{EP}}$ and $B \in \mathbb{C}^{n\times n}$. If $A \overset{-}{\leq} B$, then $B^\dagger A^\dagger \in (AB)\{1, 2, 3\}$ and $A^\dagger B^\dagger \in (BA)\{1, 2, 4\}$.*

Proof. In view of the assumption $A \overset{-}{\leq} B$, we can write A and B as in (4.1). It is easy to check that $A^\dagger = V(D_1^{-1} \oplus 0)U^*$. Moreover,

$$\begin{pmatrix} D_1 + RD_2S & RD_2 \\ D_2S & D_2 \end{pmatrix}^{-1} = \begin{pmatrix} D_1^{-1} & -D_1^{-1}R \\ -SD_1^{-1} & D_2^{-1} + SD_1^{-1}R \end{pmatrix}$$

and the second equality of (4.1) imply that

$$B^\dagger = V \begin{pmatrix} D_1^{-1} & -D_1^{-1}R \\ -SD_1^{-1} & D_2^{-1} + SD_1^{-1}R \end{pmatrix} U^*.$$

Since A is EP, Lemma 4.2 permits to write $V^*U = W = W_{11} \oplus W_{22}$, and therefore,

$$AB = U \begin{pmatrix} D_1W_{11}D_1 + D_1W_{11}RD_2S & D_1W_{11}RD_2 \\ 0 & 0 \end{pmatrix} V^*$$

and

$$B^\dagger A^\dagger = V \begin{pmatrix} D_1^{-1}W_{11}^*D_1^{-1} & 0 \\ -SD_1^{-1}W_{11}^*D_1^{-1} & 0 \end{pmatrix} U^*.$$

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Also, Lemma 4.2 assures that W_{11} is unitary. Thus

$$ABB^\dagger A^\dagger = U \begin{pmatrix} I_a & 0 \\ 0 & 0 \end{pmatrix} U^*,$$

which implies the hermitancy of $ABB^\dagger A^\dagger$. Moreover,

$$ABB^\dagger A^\dagger AB = U \begin{pmatrix} D_1 W_{11} D_1 + D_1 W_{11} R D_2 S & D_1 W_{11} R D_2 \\ 0 & 0 \end{pmatrix} V^* = AB$$

and

$$B^\dagger A^\dagger ABB^\dagger A^\dagger = V \begin{pmatrix} D_1^{-1} W_{11}^* D_1^{-1} & 0 \\ -S D_1^{-1} W_{11}^* D_1^{-1} & 0 \end{pmatrix} U^* = BA.$$

Hence, $B^\dagger A^\dagger \in (AB)\{1,2,3\}$. In a similar way, we get $A^\dagger B^\dagger \in (BA)\{1,2,4\}$. ◻

5. The sharp ordering and the reverse order law. Groß remarked the following fact in [11, Remark 1]: Let $A \in \mathbb{C}_n^{\mathsf{EP}}$ and $B \in \mathbb{C}_n^{\mathsf{GP}}$. Then $A \overset{*}{\leq} B$ if and only if $A \overset{\#}{\leq} B$. However, this equivalence does not hold for the case $A \in \mathbb{C}_n^{\mathsf{GP}}$ and $B \in \mathbb{C}_n^{\mathsf{EP}}$. As an example one may consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is easy to see that $A^2 = A$ and thus $A = A^\#$. Having in mind the obvious equalities $A^\# A = A^\# B$ and $AA^\# = BA^\#$, we get that $A \overset{\#}{\leq} B$. However, $A^*A \neq A^*B$ and $AA^* \neq BA^*$, which implies that $A \overset{*}{\leq} B$ does not hold.

Next result concerns the situation $A \overset{\#}{\leq} B$ when $A \in \mathbb{C}_n^{\mathsf{GP}}$ and $B \in \mathbb{C}_n^{\mathsf{EP}}$.

Theorem 5.1. *Let $A \in \mathbb{C}_n^{\mathsf{GP}}$ and $B \in \mathbb{C}_n^{\mathsf{EP}}$. If $A \overset{\#}{\leq} B$, then*

(i) *$A^\dagger B^\dagger \in BA\{1,2,4\}$ and $B^\dagger A^\dagger \in AB\{1,2,3\}$.*
(ii) *$A^\dagger B^\dagger = (BA)^\dagger$ if and only if $B^*BP_{\mathcal{R}(A)} = P_{\mathcal{R}(A)}B^*B$. Moreover, $B^\dagger A^\dagger = (AB)^\dagger$ if and only if $BB^*P_{\mathcal{R}(A^*)} = P_{\mathcal{R}(A^*)}BB^*$.*

Proof. Since $B \in \mathbb{C}_n^{\mathsf{EP}}$, by Theorem 1.1, there exist a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a nonsingular matrix $K \in \mathbb{C}^{r\times r}$ such that

$$B = U(K \oplus 0)U^*. \tag{5.1}$$

Let us write matrix A as follows

$$A = U \begin{pmatrix} P & Q \\ R & S \end{pmatrix} U^*, \qquad P \in \mathbb{C}^{r\times r},\ S \in \mathbb{C}^{(n-r)\times(n-r)}.$$

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From $AB = BA$ (obtained from the right side of the equivalence (1.6)) we get $KQ = 0$ and $RK = 0$. The nonsingularity of K entails $Q = 0$ and $R = 0$. Hence

$$A = U(P \oplus S)U^*. \tag{5.2}$$

From (5.2) and (5.1) we get $A^\dagger = U(P^\dagger \oplus S^\dagger)U^*$ and $B^\dagger = U(K^{-1} \oplus 0)U^*$.

(i) We shall prove $B^\dagger A^\dagger \in AB\{1,2,3\}$. We have

$$ABB^\dagger A^\dagger AB = U(PKK^{-1}P^\dagger PK \oplus 0)U^* = U(PK \oplus 0)U^* = AB,$$

$$B^\dagger A^\dagger ABB^\dagger A^\dagger = U(K^{-1}P^\dagger PKK^{-1}P^\dagger \oplus 0)U^* = U(K^{-1}P^\dagger \oplus 0)U^* = B^\dagger A^\dagger,$$

and

$$ABB^\dagger A^\dagger = U(PKK^{-1}P^\dagger \oplus 0)U^* = U(PP^\dagger \oplus 0)U^* \text{ is Hermitian.}$$

Thus, we conclude that $B^\dagger A^\dagger \in AB\{1,2,3\}$. The proof of $A^\dagger B^\dagger \in BA\{1,2,4\}$ is similar and we omit it.

(ii) Now, we shall prove that $B^\dagger A^\dagger = (AB)^\dagger \iff BB^* P_{\mathcal{R}(A^*)} = P_{\mathcal{R}(A^*)} BB^*$. By item (i) of this theorem, $B^\dagger A^\dagger = (AB)^\dagger$ if and only if $B^\dagger A^\dagger \in (AB)\{4\}$, i.e., $B^\dagger A^\dagger = (AB)^\dagger$ if and only if $B^\dagger A^\dagger AB$ is Hermitian. Since

$$B^\dagger A^\dagger AB = U(K^{-1}P^\dagger PK \oplus 0)U^*,$$

the hermitancy of $B^\dagger A^\dagger AB$ is equivalent to the hermitancy of $K^{-1}P^\dagger PK$. Let us recall that $P^\dagger P$ is Hermitian. Now

$$\begin{aligned}(K^{-1}P^\dagger PK)^* = K^{-1}P^\dagger PK &\iff K^*P^\dagger P(K^{-1})^* = K^{-1}P^\dagger PK \\ &\iff K^*P^\dagger P(K^*)^{-1} = K^{-1}P^\dagger PK \\ &\iff KK^*P^\dagger P = P^\dagger PKK^* \\ &\iff BB^*A^\dagger A = A^\dagger ABB^*,\end{aligned}$$

which proves item $B^\dagger A^\dagger = (AB)^\dagger \iff BB^* P_{\mathcal{R}(A^*)} = P_{\mathcal{R}(A^*)} BB^*$. The proof of the remaining part of (ii) follows from the same argument. □

Let us observe that Theorems 3.4 and 3.7 are quite similar: in fact, Theorem 3.7 differs from Theorem 3.4 only with the fact that assumption "$A \ast\!\leq B$" replaces "$A \leq\!\ast\, B$". Moreover, these theorems also are similar to Theorem 5.1 since the conditions "$A \ast\!\leq B$" or "$A \leq\!\ast\, B$" replace "$A \in \mathbb{C}_n^{\mathsf{GP}}$" and $A \overset{\#}{\leq} B$". One can expect that some (or all) of the following equivalences hold.

(i) If $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}_n^{\mathsf{EP}}$, then $A \ast\!\leq B \iff A \leq\!\ast\, B$.

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(ii) If $A \in \mathbb{C}_n^{\mathsf{GP}}$ and $B \in \mathbb{C}_n^{\mathsf{EP}}$, then $A \ast\!\leq B \iff A \overset{\#}{\leq} B$.
(iii) If $A \in \mathbb{C}_n^{\mathsf{GP}}$ and $B \in \mathbb{C}_n^{\mathsf{EP}}$, then $A \leq\!\ast\, B \iff A \overset{\#}{\leq} B$.

However the following example shows that none of these equivalences holds. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} a & b \\ c & 1 \end{pmatrix},$$

being $a, b, c \in \mathbb{C}$ such that $a \neq bc$. The idempotency of A leads to $A \in \mathbb{C}_n^{\mathsf{GP}}$. The nonsingularity of B entails $B \in \mathbb{C}_n^{\mathsf{EP}}$, $\mathcal{R}(A) \subset \mathcal{R}(B)$, and $\mathcal{R}(A^*) \subset \mathcal{R}(B^*)$. However, we have that

(i) $A^*A = A^*B \iff a = b = 1$,
(ii) $AA^* = BA^* \iff a + b = 2$ and $c = -1$,
(iii) $AB = A^2 = BA \iff a = 1, b = 0, c = 0$,

which proves that the conditions $A \ast\!\leq B$, $A \leq\!\ast\, B$, and $A \ast\!\leq B$ are independent, even if we assume that B is EP.

Let us remark that for $A \in \mathbb{C}^{n\times n}$ and a nonsingular $S \in \mathbb{C}^{n\times n}$ we have the equivalence

$$A \in \mathbb{C}_n^{\mathsf{GP}} \qquad \iff \qquad SAS^{-1} \in \mathbb{C}_n^{\mathsf{GP}},$$

and when $A \in \mathbb{C}_n^{\mathsf{GP}}$ one has that $(SAS^{-1})^\# = SA^\# S^{-1}$.

Also, for an arbitrary nonsingular $S \in \mathbb{C}^{n\times n}$ and $A, B \in \mathbb{C}_n^{\mathsf{GP}}$ we have the equivalence

$$A \overset{\#}{\leq} B \qquad \iff \qquad SAS^{-1} \overset{\#}{\leq} SBS^{-1}.$$

On the other hand, the relationship $(SAS^{-1})^\dagger = SA^\dagger S^{-1}$ is not true in general, for example, take

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad S = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

The above comments suggest that for $A, B \in \mathbb{C}_n^{\mathsf{GP}}$, the relation $A \overset{\#}{\leq} B$ is more related to the expression $(AB)^\# = B^\# A^\#$ than to the expression $(AB)^\dagger = B^\dagger A^\dagger$. The precise result is stated in Theorem 5.3 below. Before doing this, we provide a useful result to prove Theorem 5.3.

Lemma 5.2. *Let $A \in \mathbb{C}^{n\times n}$ be written as*

$$A = W(A_1 \oplus A_2)W^{-1}, \tag{5.3}$$

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where $W \in \mathbb{C}^{n\times n}$ is nonsingular, $A_1 \in \mathbb{C}^{r\times r}$, and $A_2 \in \mathbb{C}^{(n-r)\times(n-r)}$. The following statements are equivalent:

(i) $A \in \mathbb{C}_n^{\mathsf{GP}}$.
(ii) $A_1 \in \mathbb{C}_r^{\mathsf{GP}}$ *and* $A_2 \in \mathbb{C}_{n-r}^{\mathsf{GP}}$.

Under this equivalence, one has $A^{\#} = W(A_1^{\#} \oplus A_2^{\#})W^{-1}$.

Proof. (i) $\Rightarrow$ (ii) Let us recall that every square matrix X satisfies $\mathrm{rk}(X^2) \leq \mathrm{rk}(X)$. If $A_1 \notin \mathbb{C}_r^{\mathsf{GP}}$, then $\mathrm{rk}(A_1^2) < \mathrm{rk}(A_1)$. The assumption $A \in \mathbb{C}_n^{\mathsf{GP}}$ leads to $\mathrm{rk}(A^2) = \mathrm{rk}(A)$. From the representation (5.3) we have that $\mathrm{rk}(A) = \mathrm{rk}(A_1) + \mathrm{rk}(A_2)$ and $A^2 = W(A_1^2 \oplus A_2^2)W^{-1}$, hence $\mathrm{rk}(A^2) = \mathrm{rk}(A_1^2) + \mathrm{rk}(A_2^2)$ holds. Therefore

$$\mathrm{rk}(A_2^2) = \mathrm{rk}(A^2) - \mathrm{rk}(A_1^2) > \mathrm{rk}(A) - \mathrm{rk}(A_1) = \mathrm{rk}(A_2),$$

which is clearly unfeasible. Thus $A_1 \in \mathbb{C}_r^{\mathsf{GP}}$, and in a similar manner, we can prove that $A_2 \in \mathbb{C}_{n-r}^{\mathsf{GP}}$.

(ii) $\Rightarrow$ (i) Let us define $B = W(A_1^{\#} \oplus A_2^{\#})W^{-1}$. It is simple to prove that $ABA = A$, $BAB = B$, and $AB = BA$. Hence $A \in \mathbb{C}_n^{\mathsf{GP}}$ and $A^{\#} = B$. $\square$

THEOREM 5.3. *Let $A, B \in \mathbb{C}_n^{\mathsf{GP}}$. If $A \overset{\#}{\leq} B$, then $AB \in \mathbb{C}_n^{\mathsf{GP}}$ and $(AB)^{\#} = B^{\#}A^{\#} = A^{\#}B^{\#}$.*

Proof. Since $B \in \mathbb{C}_n^{\mathsf{GP}}$, the Core-Nilpotent decomposition of B (see [14, Exercise 5.10.12]) assures that there exist nonsingular matrices $W \in \mathbb{C}^{n\times n}$ and $K \in \mathbb{C}^{r\times r}$ such that

$$B = W(K \oplus 0)W^{-1}. \tag{5.4}$$

Let us write matrix A as follows

$$A = W \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} W^{-1}, \qquad X \in \mathbb{C}^{r\times r},\ T \in \mathbb{C}^{(n-r)\times(n-r)}.$$

From $AB = BA$ (obtained from the right side of the equivalence (1.6)) we get $XK = KX$, $KY = 0$, and $ZK = 0$. Having in mind that K is nonsingular, one gets $Y = 0$ and $Z = 0$. Thus

$$A = W(X \oplus T)W^{-1}. \tag{5.5}$$

Moreover, the combination of $AB = A^2$ and representations (5.4), (5.5) leads to

$$XK = KX = X^2 \qquad \text{and} \qquad T^2 = 0.$$

From equality (5.5) and $A \in \mathbb{C}_n^{\mathsf{GP}}$, by Lemma 5.2, we get $X \in \mathbb{C}_r^{\mathsf{GP}}$ and $T \in \mathbb{C}_{n-r}^{\mathsf{GP}}$, and thus, from $T^2 = 0$, we get $0 = \mathrm{rk}(T^2) = \mathrm{rk}(T)$, which yields $T = 0$. Now, we

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have

$$A^{\#}B^{\#} = W(X^{\#}K^{-1} \oplus 0)W^{-1} \quad \text{and} \quad B^{\#}A^{\#} = W(K^{-1}X^{\#} \oplus 0)W^{-1}. \tag{5.6}$$

Premultiplying the equality $XK = X^2$ by $(X^{\#})^2$ leads to

$$X^{\#}K = X^{\#}X. \tag{5.7}$$

Similarly, postmultiplying $KX = X^2$ by $(X^{\#})^2$ entails

$$KX^{\#} = XX^{\#}. \tag{5.8}$$

It follows from (5.7) and (5.8) that $X^{\#}K = KX^{\#}$, and the nonsingularity of K leads to $K^{-1}X^{\#} = X^{\#}K^{-1}$. Recalling the expressions given in (5.6), one gets that $A^{\#}B^{\#} = B^{\#}A^{\#}$. Thus, it remains to prove that $(AB)^{\#} = B^{\#}A^{\#}$. To this end, we shall check that $B^{\#}A^{\#} \in AB\{1,2\}$ and $B^{\#}A^{\#}$ commutes with AB. Firstly, we will prove $B^{\#}A^{\#} \in AB\{1\}$:

$$ABB^{\#}A^{\#}AB = W(XKK^{-1}X^{\#}XK \oplus 0)W^{-1} = W(XK \oplus 0)W^{-1} = AB.$$

Secondly, we will demonstrate $B^{\#}A^{\#} \in AB\{2\}$:

$$\begin{aligned} B^{\#}A^{\#}ABB^{\#}A^{\#} &= W(K^{-1}X^{\#}XKK^{-1}X^{\#} \oplus 0)W^{-1} \\ &= W(K^{-1}X^{\#} \oplus 0)W^{-1} = B^{\#}A^{\#}. \end{aligned}$$

Finally, we will prove $ABB^{\#}A^{\#} = B^{\#}A^{\#}AB$. Observe that (5.7) and (5.8) imply

$$K^{-1}X^{\#}XK = K^{-1}(XX^{\#})K = K^{-1}(KX^{\#})K = X^{\#}K = X^{\#}X. \tag{5.9}$$

Since

$$ABB^{\#}A^{\#} = W(XKK^{-1}X^{\#} \oplus 0)W^{-1} = W(XX^{\#} \oplus 0)W^{-1}$$

and

$$B^{\#}A^{\#}AB = W(K^{-1}X^{\#}XK \oplus 0)W^{-1},$$

the computations made in (5.9) show that $ABB^{\#}A^{\#} = B^{\#}A^{\#}AB$. $\square$

6. Concluding remark. In this paper, we present some results relating different matrix partial orderings and the reverse order law for the Moore-Penrose inverse and group inverse. Special attention is paid when at least one of the two involved matrices is EP. The expression (1.1) of an EP matrix given in Theorem 1.1 plays a crucial role in the calculations throughout this paper. Let us remark that if we remove the EPness condition in Sections 2, 3, 4 and 5, many results are not valid. A simple example is provided by the matrices

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$$A = B = \begin{bmatrix} 1 & \mathrm{i} \\ 0 & 0 \end{bmatrix}.$$

By considering that $X \in \mathbb{C}_n^{\mathsf{EP}} \iff \mathcal{R}(X) = \mathcal{R}(X^*)$, it is evident that $A, B \notin \mathbb{C}_2^{\mathsf{EP}}$. By using the well-known formula $X^\dagger = X^*(XX^*)^\dagger$ for $X \in \mathbb{C}^{n\times n}$ we easily get

$$A^\dagger = B^\dagger = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -\mathrm{i} & 0 \end{bmatrix}.$$

Evidently, we have that $A \overset{*}{\leq} B$. Since $AB = A$ we get $(AB)^\dagger = A^\dagger$. An obvious computation shows that $B^\dagger A^\dagger \neq (AB)^\dagger$ and $A^\dagger B^\dagger \neq (AB)^\dagger$. This example shows that Theorem 2.1 and Theorem 3.2 do not hold if we delete the assumption $A \in \mathbb{C}_n^{\mathsf{EP}}$.

As easy to see, $ABB^\dagger \neq A$, which shows that Theorem 2.2 is not true if we remove the assumption $B \in \mathbb{C}_n^{\mathsf{EP}}$.

The fact $BAA^\dagger B^\dagger BA \neq BA$ shows again that the EPness condition for B and A is essential in Theorems 3.4 and Theorem 4.3, respectively.

From the obvious fact $A * \leq B \iff A^* \leq * \; B^*$, we can easily construct an example showing that Theorems 3.5 and 3.7 are false when we delete $A \in \mathbb{C}_n^{\mathsf{EP}}$ and $B \in \mathbb{C}_n^{\mathsf{EP}}$, respectively.

Finally, let us remark that A and B are idempotent matrices, hence $A, B \in \mathbb{C}_2^{\mathsf{GP}}$. From $BAA^\dagger B^\dagger BA \neq BA$ we obtain that Theorem 5.1 does not hold if we remove the condition $B \in \mathbb{C}_n^{\mathsf{EP}}$.

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