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THE GRAHAM–HOFFMAN–HOSOYA-TYPE THEOREMS FOR THE EXPONENTIAL DISTANCE MATRIX*

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Abstract. Let G be a strongly connected digraph with vertex set $\{v_1, v_2, \dots, v_n\}$. Denote by D_{ij} the distance between vertices v_i and v_j in G. Two variant versions of the distance matrix were proposed by Yan and Yeh (Adv. Appl. Math.), and Bapat *et al.* (Linear Algebra Appl.) independently, one is the q-distance matrix, and the other is the exponential distance matrix. Given a nonzero indeterminate q, the q-distance matrix $\mathscr{D}_G = (\mathscr{D}_{ij})_{n\times n}$ of G is defined as

$$\mathscr{D}_{ij} = \begin{cases} 1 + q + \cdots + q^{D_{ij}-1} & \text{if } i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, when $q = 1$, it would be reduced to the distance matrix of G. The exponential distance matrix $\mathscr{F}_G = (\mathscr{F}_{ij})_{n\times n}$ of G is defined as

$$\mathscr{F}_{ij} = q^{D_{ij}}.$$

In 1977, Graham *et al.* (J. Graph Theory) established a classical formula connecting the determinants and cofactor sums of the distance matrices of strongly connected digraphs in terms of their blocks, which plays a powerful role in the subsequent researches on the determinants of distance matrices. Sivasubramanian (Electron. J. Combin.) and Li *et al.* (Discuss. Math. Graph Theory) independently extended it from the distance matrix to the q-distance matrix. In this note, three formulae of such types for the exponential distance matrices of strongly connected digraphs will be presented.

Key words. Exponential distance matrix, q-distance matrix, Distance matrix, Determinant, Cofactor sum.

AMS subject classifications. 05C50, 15A15.

1. Introduction and preliminaries.

Let G be a strongly connected digraph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. For $1 \leq i, j \leq n$, the distance between vertices v_i and v_j in G, denoted by $D_G(i, j)$, or simply D_{ij}, is the length of a shortest path connecting them in G. Based on the distances of vertex pairs, the well-studied distance matrix of G is defined as $D_G = (D_{ij})_{n\times n}$.

The earliest result about the distance matrix could date back to the classical work of Graham and Pollack [6] in 1971, in which they showed that for any n-vertex tree T, the determinant of D_T is expressed as $\det(D_T) = (-1)^{n-1}(n-1)2^{n-2}$. This interesting formula reveals that such determinant depends only on the order of trees, but is independent of the particular structure.

Let M be an $n \times n$ matrix. For $1 \leq i, j \leq n$, let $c_{ij}(M)$ represent the (i, j)-cofactor of M, which is equal to $(-1)^{i+j}$ times the determinant of the submatrix obtained from M by deleting its ith row and jth column. The cofactor sum of M, denoted by $\mathrm{cof}(M)$, is defined to be the sum of all cofactors of M, i.e.,

$$\mathrm{cof}(M) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}(M).$$

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For $1 \leq i \leq n$, let

$$r_i(M) = \sum_{j=1}^{n} c_{ij}(M).$$

That is, $\mathrm{cof}(M) = \sum_{i=1}^{n} r_i(M)$.

In 1977, Graham *et al.* [5] established the following two remarkable formulae on the determinant and cofactor sum of the distance matrix of a strongly connected digraph, by using the corresponding determinants and cofactor sums of the distance matrices of all its blocks.

THEOREM 1.1 ([5]). *Let G be a strongly connected digraph with blocks $G_1, G_2, \ldots, G_r$. Then*

$$\mathrm{cof}(D_G) = \prod_{i=1}^{r} \mathrm{cof}(D_{G_i}),$$

and

$$\det(D_G) = \sum_{i=1}^{r} \det(D_{G_i}) \prod_{j \neq i} \mathrm{cof}(D_{G_j}).$$

We call Theorem 1.1 the Graham–Hoffman–Hosoya theorem throughout this note.

For a nonzero indeterminate q, Yan and Yeh [10] and Bapat *et al.* [2] independently proposed two variant versions of the distance matrix of G. One is the q-distance matrix $\mathscr{D}_G = (\mathscr{D}_{ij})_{n\times n}$, defined as

$$\mathscr{D}_{ij} = \begin{cases} 1 + q + \cdots + q^{D_{ij}-1} & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

In particular, when $q = 1$, the q-distance matrix would be reduced to the distance matrix. Hence, the distance matrix is a special case of the q-distance matrix. The other is the exponential distance matrix $\mathscr{F}_G = (\mathscr{F}_{ij})_{n\times n}$, defined as

$$\mathscr{F}_{ij} = q^{D_{ij}}.$$

Obviously, the diagonal entries of $\mathscr{F}_G$ are all equal to 1. When $q = \frac{1}{2}$, the exponential distance matrix $\mathscr{F}_G$ is almost identical to the so-called closeness matrix investigated in [11] (the only difference lies on the diagonal entries, which are all 1 in $\mathscr{F}_G$, but all 0 in the closeness matrix).

In the past several decades, the distance matrices of connected graphs have been studied extensively, e.g., see [1]. Besides that, the above-mentioned variant versions of distance matrix of graphs have also many interesting applications. In quantum chemistry, the Wiener polynomial of G is defined as [7]

$$W_G = \sum_{i<j} q^{D_{ij}} = \sum_{i<j} \mathscr{F}_{ij}.$$

Obviously, $\frac{\mathrm{d}W_G}{\mathrm{d}q}|_{q=1} = \sum_{i<j} D_{ij}$ is the Wiener index, one of the most studied molecular-graph-based structural descriptors (usually also called "topological indices"). The exponential distance matrix can also be used as a measure of some properties (e.g., centrality, the network resistance) of network structures under the context of closeness and residual closeness, see [3, 4].

Aiming to acquire a q-analogue of the Graham–Hoffman–Hosoya theorem, Sivasubramanian [9] defined a weighted cofactor sum of $\mathscr{D}_G$ as

$$\mathrm{R}_i^q(\mathscr{D}_G) = \sum_{j=1}^{n} q^{D_{ji}} r_j(\mathscr{D}_G),$$

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where $1 \leq i \leq n$. It is shown in [9] that $\mathrm{R}_i^q(\mathscr{D}_G)$ is independent of the index i, and thus could be uniformly denoted by $\mathrm{R}^q(\mathscr{D}_G)$, i.e.,

$$\mathrm{R}^q(\mathscr{D}_G) = \mathrm{R}_i^q(\mathscr{D}_G),$$

for any $1 \leq i \leq n$. It is clear that when $q = 1$,

$$\mathrm{R}^q(\mathscr{D}_G) = \mathrm{cof}(\mathscr{D}_G) = \mathrm{cof}(D_G).$$

A Graham–Hoffman–Hosoya-type theorem for the q-distance matrix $\mathscr{D}_G$ was obtained by Sivasubramanian in [9]. A similar result can be also found in [8].

THEOREM 1.2 ([9]). *Let G be a strongly connected digraph with blocks $G_1, G_2, \ldots, G_r$. Then*

$$\mathrm{R}^q(\mathscr{D}_G) = \prod_{i=1}^{r} \mathrm{R}^q\left(\mathscr{D}_{G_i}\right),$$

and

$$\det(\mathscr{D}_G) = \sum_{i=1}^{r} \det\left(\mathscr{D}_{G_i}\right) \prod_{j \neq i} \mathrm{R}^q\left(\mathscr{D}_{G_j}\right).$$

Combining the two formulae in Theorem 1.2 (or Theorem 1.1 for $q = 1$), one can get the following neat formula.

COROLLARY 1.3. *Let G be a strongly connected digraph with blocks $G_1, G_2, \ldots, G_r$. If $\mathrm{R}^q\left(\mathscr{D}_{G_i}\right) \neq 0$ for each $1 \leq i \leq r$, then*

$$\frac{\det(\mathscr{D}_G)}{\mathrm{R}^q(\mathscr{D}_G)} = \sum_{i=1}^{r} \frac{\det\left(\mathscr{D}_{G_i}\right)}{\mathrm{R}^q\left(\mathscr{D}_{G_i}\right)}.$$

In this note, we will focus on the exponential distance matrix $\mathscr{F}_G$ and get the corresponding Graham–Hoffman–Hosoya-type theorems. Under the same assumptions as above, we obtain the following formulae:

$$\det(\mathscr{F}_G) = \prod_{i=1}^{r} \det\left(\mathscr{F}_{G_i}\right),$$

$$(q-1)^n \frac{\det(\mathscr{D}_G)}{\det(\mathscr{F}_G)} = \sum_{i=1}^{r} (q-1)^{n_i} \frac{\det\left(\mathscr{D}_{G_i}\right)}{\det\left(\mathscr{F}_{G_i}\right)},$$

and

$$1 - \frac{\mathrm{cof}(\mathscr{F}_G)}{\det(\mathscr{F}_G)} = \sum_{i=1}^{r} \left(1 - \frac{\mathrm{cof}\left(\mathscr{F}_{G_i}\right)}{\det\left(\mathscr{F}_{G_i}\right)}\right).$$

They appear in an analogous fashion as the classical Graham–Hoffman–Hosoya theorem as shown in Theorem 1.1 (see also Theorem 1.2) as well as Corollary 1.3. We will confirm them one by one in the subsequent sections.

2. The determinant of exponential distance matrix. First of all, let us consider the determinants of exponential distance matrices.

THEOREM 2.1. *Let G be a strongly connected digraph with blocks $G_1, G_2, \ldots, G_r$. Then*

$$\det(\mathscr{F}_G) = \prod_{i=1}^{r} \det\left(\mathscr{F}_{G_i}\right). \tag{2.1}$$

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Proof. We will prove (2.1) by the induction on the number of blocks of G, i.e., r. When $r = 1$, the result is trivial. Assume that (2.1) holds for any strongly connected digraph whose number of blocks is less than r, where $r \geq 2$.

Now suppose in the following that the strongly connected digraph G contains r blocks, say $G_1, G_2, \ldots, G_r$. Further, assume without loss of generality that G_1 is an end block of G (containing exactly one vertex adjacent to some vertex outside the block), and H is the subgraph of G induced by $V(G_2) \cup V(G_3) \cup \cdots \cup V(G_r)$. In particular, denote by v_1 the unique common vertex of G_1 and H and let $V(G_1) = \{v_1, v_2, \ldots, v_m\}$ and $V(H) = \{v_1, v_{m+1}, \ldots, v_n\}$.

For convenience, let $a_i = D_{1i}$ and $b_i = D_{i1}$ for any $2 \leq i \leq n$ (a_i and b_i are not necessarily equal since G is a digraph). For $2 \leq i \leq m$ and $m+1 \leq j \leq n$ ($v_i \in V(G_1)$ and $v_j \in V(H)$), it is easy to see that

$$D_{ij} = D_{i1} + D_{1j} = b_i + a_j,$$

and

$$D_{ji} = D_{j1} + D_{1i} = b_j + a_i,$$

and thus

$$\mathscr{F}_{ij} = q^{b_i+a_j} \quad \text{and} \quad \mathscr{F}_{ji} = q^{b_j+a_i}.$$

Based on these settings, we can write the exponential distance matrix of G as the following form:

$$\mathscr{F}_G = \left[\begin{array}{c|ccc|ccc}
1 & q^{a_2} & \cdots & q^{a_m} & q^{a_{m+1}} & \cdots & q^{a_n} \\ \hline
q^{b_2} & & & & q^{b_2+a_{m+1}} & \cdots & q^{b_2+a_n} \\
\vdots & & A & & \vdots & & \vdots \\
q^{b_m} & & & & q^{b_m+a_{m+1}} & \cdots & q^{b_m+a_n} \\ \hline
q^{b_{m+1}} & q^{b_{m+1}+a_2} & \cdots & q^{b_{m+1}+a_m} & & & \\
\vdots & \vdots & & \vdots & & B & \\
q^{b_n} & q^{b_n+a_2} & \cdots & q^{b_n+a_m} & & &
\end{array}\right].$$

In particular,

$$\mathscr{F}_{G_1} = \left[\begin{array}{c|ccc}
1 & q^{a_2} & \cdots & q^{a_m} \\ \hline
q^{b_2} & & & \\
\vdots & & A & \\
q^{b_m} & & &
\end{array}\right] \quad \text{and} \quad \mathscr{F}_H = \left[\begin{array}{c|ccc}
1 & q^{a_{m+1}} & \cdots & q^{a_n} \\ \hline
q^{b_{m+1}} & & & \\
\vdots & & B & \\
q^{b_n} & & &
\end{array}\right].$$

Define two auxiliary matrices R_G and C_G as

$$R_G = \left[\begin{array}{c|ccc|ccc}
1 & & & & & & \\ \hline
-q^{b_2} & 1 & & & & & \\
\vdots & & \ddots & & & & \\
-q^{b_m} & & & 1 & & & \\ \hline
-q^{b_{m+1}} & & & & 1 & & \\
\vdots & & & & & \ddots & \\
-q^{b_n} & & & & & & 1
\end{array}\right], \quad C_G = \left[\begin{array}{c|ccc|ccc}
1 & & & & & & \\ \hline
-q^{a_2} & 1 & & & & & \\
\vdots & & \ddots & & & & \\
-q^{a_m} & & & 1 & & & \\ \hline
-q^{a_{m+1}} & & & & 1 & & \\
\vdots & & & & & \ddots & \\
-q^{a_n} & & & & & & 1
\end{array}\right].$$

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Note that $\det(R_G) = \det\left(C_G^\top\right) = 1$. It is standard (but somewhat tedious) to get that

$$\det(\mathscr{F}_G) = \det(R_G \mathscr{F}_G C_G^\top) = \det\left(\begin{bmatrix} 1 & & \\ & A' & \\ & & B' \end{bmatrix}\right) = \det(A')\det(B'), \tag{2.2}$$

where the (i,j)-entry of $\begin{bmatrix} 1 & & \\ & A' & \\ & & B' \end{bmatrix}$ is equal to $q^{D_{ij}} - q^{b_i + a_j}$, for $2 \le i,j \le m$ (in A') or $m+1 \le i,j \le n$ (in B').

For the matrices considered in (2.2), if we just retain the rows and columns with indices $\{1, 2, \ldots, m\}$ (leading to the corresponding principal submatrix for G_1), we have

$$\det(\mathscr{F}_{G_1}) \quad = \quad \det\left(\begin{bmatrix} 1 & \\ & A' \end{bmatrix}\right) = \det(A'), \tag{2.3}$$

while if the indices of rows and columns are confined to $\{1, m+1, \ldots, n\}$ (aiming to H this time), we have

$$\det(\mathscr{F}_H) \quad = \quad \det\left(\begin{bmatrix} 1 & \\ & B' \end{bmatrix}\right) = \det(B'). \tag{2.4}$$

Combining (2.2), (2.3) and (2.4), we are able to draw the conclusion

$$\det(\mathscr{F}_G) = \det(\mathscr{F}_{G_1})\det(\mathscr{F}_H). \tag{2.5}$$

Recall that the number of blocks of H is $r-1$ $(G_2, \ldots, G_r)$. Applying the inductive hypothesis to H, we have

$$\det(\mathscr{F}_H) = \prod_{i=2}^{r} \det\left(\mathscr{F}_{G_i}\right).$$

Finally, together with (2.5), we get

$$\det(\mathscr{F}_G) = \prod_{i=1}^{r} \det\left(\mathscr{F}_{G_i}\right),$$

as desired. ∎

3. The determinants and cofactor sums between the q-distance and exponential distance matrices. This section is devoted to revealing the relationship on the determinants and cofactor sums between $\mathscr{D}_G$ (the q-distance matrix) and $\mathscr{F}_G$ (the exponential distance matrix). Before proceeding, let us present two auxiliary lemmas.

Denote by $\mathbf{J}_n$ the $n \times n$ all-one matrix.

LEMMA 3.1. *Let G be an n-vertex strongly connected digraph. For any nonzero indeterminate q, we have*

$$(1-q)\mathscr{D}_G = \boldsymbol{J}_n - \mathscr{F}_G. \tag{3.6}$$

Proof. For any $1 \le i,j \le n$, if $i \ne j$, then

$$(1-q)\mathscr{D}_{ij} = (1-q)\left(1 + q + \cdots + q^{D_{ij}-1}\right) = 1 - q^{D_{ij}} = 1 - \mathscr{F}_{ij},$$

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and if $i = j$, then

$$(1-q)\mathscr{D}_{ij} = 0 = 1 - \mathscr{F}_{ij}.$$

It leads to (3.6) directly. □

The following identity involving the determinants and cofactor sum of matrices is well known, which frequently appears as an exercise about the properties of determinants in the textbooks on linear algebra.

Lemma 3.2. *Let M be an $n \times n$ matrix, and x a real number. Then*

$$\det(M + x\boldsymbol{J}_n) = \det(M) + x \cdot \operatorname{cof}(M).$$

In [9], $\mathrm{R}^q(\mathscr{D}_G)$ was viewed as a q-analog of $\operatorname{cof}(D_G)$, which can be reflected in the following expression.

Lemma 3.3 ([9]). *Let G be a strongly connected digraph. Then*

$$\mathrm{R}^q(\mathscr{D}_G) = (q-1)\det(\mathscr{D}_G) + \operatorname{cof}(\mathscr{D}_G). \tag{3.7}$$

Now we are ready to derive the relation on the determinants and cofactor sums between $\mathscr{D}_G$ and $\mathscr{F}_G$.

Theorem 3.4. *Let G be an n-vertex strongly connected digraph. Then*

$$\det(\mathscr{F}_G) = (q-1)^{n-1}\,\mathrm{R}^q(\mathscr{D}_G).$$

Proof. From (3.6), we have

$$\mathscr{F}_G = (q-1)\mathscr{D}_G + \mathbf{J}_n.$$

Applying Lemma 3.2 by setting $M = (q-1)\mathscr{D}_G$ and $x = 1$, one can get

$$\begin{aligned} \det(\mathscr{F}_G) &= \det((q-1)\mathscr{D}_G) + \operatorname{cof}((q-1)\mathscr{D}_G) \\ &= (q-1)^n \det(\mathscr{D}_G) + (q-1)^{n-1}\operatorname{cof}(\mathscr{D}_G) \\ &= (q-1)^{n-1}\left((q-1)\det(\mathscr{D}_G) + \operatorname{cof}(\mathscr{D}_G)\right) \\ &\overset{(3.7)}{=} (q-1)^{n-1}\,\mathrm{R}^q(\mathscr{D}_G). \end{aligned} \tag{3.8}$$

The result follows. □

Theorem 3.5. *Let G be an n-vertex strongly connected digraph, where $n \geq 2$. Then*

$$\operatorname{cof}(\mathscr{F}_G) = (q-1)^{n-1}\operatorname{cof}(\mathscr{D}_G).$$

Proof. If $q = 1$, then $\mathscr{F}_G = \mathbf{J}_n$, and thus

$$\operatorname{cof}(\mathscr{F}_G) = 0 = (q-1)^{n-1}\operatorname{cof}(\mathscr{D}_G)$$

follows directly. Now suppose that $q \neq 1$. From (3.6), we have an equivalent form:

$$\mathscr{D}_G = \frac{1}{q-1}\mathscr{F}_G + \frac{1}{1-q}\mathbf{J}_n,$$

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and thus from Lemma 3.2 by setting $M = \frac{1}{q-1}\mathscr{F}_G$ and $x = \frac{1}{1-q}$,

$$\begin{aligned}
\det(\mathscr{D}_G) &= \det\left(\frac{1}{q-1}\mathscr{F}_G\right) + \frac{1}{1-q}\operatorname{cof}\left(\frac{1}{q-1}\mathscr{F}_G\right) \\
&= \frac{1}{(q-1)^n}\det(\mathscr{F}_G) - \frac{1}{(q-1)^n}\operatorname{cof}(\mathscr{F}_G) \\
&= \frac{\det(\mathscr{F}_G) - \operatorname{cof}(\mathscr{F}_G)}{(q-1)^n}, \qquad (3.9)
\end{aligned}$$

or equivalently,

$$\operatorname{cof}(\mathscr{F}_G) = \det(\mathscr{F}_G) - (q-1)^n \det(\mathscr{D}_G).$$

Finally, together with (3.8), it leads to

$$\begin{aligned}
\operatorname{cof}(\mathscr{F}_G) &= (q-1)^n \det(\mathscr{D}_G) + (q-1)^{n-1}\operatorname{cof}(\mathscr{D}_G) - (q-1)^n \det(\mathscr{D}_G) \\
&= (q-1)^{n-1}\operatorname{cof}(\mathscr{D}_G),
\end{aligned}$$

completing the proof. □

REMARK 3.6. *When $n = 1$, $\operatorname{cof}(\mathscr{F}_G) = (q-1)^{n-1}\operatorname{cof}(\mathscr{D}_G)$ is still valid for $q \neq 1$ (both sides are equal to 1), but becomes invalid for $q = 1$, since at this time $(q-1)^{n-1} = 0^0$ is undefined.*

THEOREM 3.7. *Let G be a strongly connected digraph. If $\operatorname{cof}(\mathscr{F}_G), \operatorname{cof}(\mathscr{D}_G) \neq 0$, then*

$$\frac{\det(\mathscr{F}_G)}{\operatorname{cof}(\mathscr{F}_G)} = 1 + (q-1)\cdot\frac{\det(\mathscr{D}_G)}{\operatorname{cof}(\mathscr{D}_G)}.$$

Proof. Let n denote the order of G. If $n = 1$, then the result follows trivially. Assume that $n \geq 2$. Observe that the hypothesis $\operatorname{cof}(\mathscr{F}_G) \neq 0$ implies $q \neq 1$. By (3.9), we have

$$\det(\mathscr{F}_G) - \operatorname{cof}(\mathscr{F}_G) = (q-1)^n \det(\mathscr{D}_G).$$

Furthermore, by Theorem 3.5,

$$\frac{\det(\mathscr{F}_G)}{\operatorname{cof}(\mathscr{F}_G)} - 1 = (q-1)^n\cdot\frac{\det(\mathscr{D}_G)}{\operatorname{cof}(\mathscr{F}_G)} = (q-1)^n\cdot\frac{\det(\mathscr{D}_G)}{(q-1)^{n-1}\operatorname{cof}(\mathscr{D}_G)} = (q-1)\frac{\det(\mathscr{D}_G)}{\operatorname{cof}(\mathscr{D}_G)},$$

implying the desired result. □

4. The Graham–Hoffman–Hosoya-type theorems. The following lemma acts as an auxiliary tool in our subsequent deduction.

LEMMA 4.1. *Let G be an n-vertex strongly connected digraph with blocks $G_1, G_2, \ldots, G_r$, where $n_i = |V(G_i)|$ for $1 \leq i \leq r$. Then*

$$n = 1 + \sum_{i=1}^{r}(n_i - 1). \qquad (4.10)$$

Proof. We will confirm (4.10) by virtue of the induction on r. The case when $r = 1$ is trivial. Suppose that $r \geq 2$, and (4.10) holds for any strongly connected digraph with $r - 1$ blocks.

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Let H be the subgraph of G induced by $V(G_1)\cup V(G_2)\cup\cdots\cup V(G_{r-1})$. Then H is a strongly connected digraph with $r-1$ blocks. Applying the inductive hypothesis to H, we have

$$|V(H)| = 1 + \sum_{i=1}^{r-1}(n_i - 1).$$

Further,

$$n = |V(H)| + n_r - 1 = 1 + \sum_{i=1}^{r-1}(n_i - 1) + n_r - 1 = 1 + \sum_{i=1}^{r}(n_i - 1),$$

confirming the validity of (4.10). □

We present two more Graham–Hoffman–Hosoya-type theorems for the exponential distance matrix as below.

THEOREM 4.2. *Let G be an n-vertex strongly connected digraph with blocks $G_1, G_2, \ldots, G_r$, where $n_i = |V(G_i)|$ for $1 \le i \le r$. If $\det(\mathscr{F}_{G_i}) \neq 0$ for each $1 \le i \le r$, then*

$$(q-1)^n \frac{\det(\mathscr{D}_G)}{\det(\mathscr{F}_G)} = \sum_{i=1}^{r}(q-1)^{n_i}\frac{\det(\mathscr{D}_{G_i})}{\det(\mathscr{F}_{G_i})}.$$

Proof. If $q = 1$, then the result is trivial. Now suppose that $q \neq 1$. From Theorem 3.4,

$$\mathrm{R}^q(\mathscr{D}_{G_i}) = \frac{\det(\mathscr{F}_{G_i})}{(q-1)^{n_i-1}}$$

for $1 \le i \le r$, which, together with Theorems 1.2 and 2.1, implies that

$$\begin{aligned}
\frac{\det(\mathscr{D}_G)}{\det(\mathscr{F}_G)} &= \frac{\sum_{i=1}^{r}\det(\mathscr{D}_{G_i})\prod_{j\neq i}\mathrm{R}^q(\mathscr{D}_{G_j})}{\prod_{i=1}^{r}\det(\mathscr{F}_{G_i})}\\
&= \frac{\sum_{i=1}^{r}\det(\mathscr{D}_{G_i})\prod_{j\neq i}\frac{\det(\mathscr{F}_{G_j})}{(q-1)^{n_j-1}}}{\prod_{i=1}^{r}\det(\mathscr{F}_{G_i})}\\
&= \sum_{i=1}^{r}\frac{\det(\mathscr{D}_{G_i})}{\det(\mathscr{F}_{G_i})}\cdot\frac{1}{(q-1)^{\sum_{j\neq i}(n_j-1)}}\\
&\overset{(4.10)}{=} \sum_{i=1}^{r}\frac{1}{(q-1)^{n-n_i}}\cdot\frac{\det(\mathscr{D}_{G_i})}{\det(\mathscr{F}_{G_i})}.
\end{aligned}$$

After multiplying by $(q-1)^n$ on both sides, the result follows. □

THEOREM 4.3. *Let G be a strongly connected digraph with blocks $G_1, G_2, \ldots, G_r$. If $\det(\mathscr{F}_{G_i}) \neq 0$ for each $1 \le i \le r$, then*

$$1 - \frac{\mathrm{cof}(\mathscr{F}_G)}{\det(\mathscr{F}_G)} = \sum_{i=1}^{r}\left(1 - \frac{\mathrm{cof}(\mathscr{F}_{G_i})}{\det(\mathscr{F}_{G_i})}\right).$$

Proof. Let $n = |V(G)|$, and $n_i = |V(G_i)|$ for $1 \le i \le r$. If $n \ge 2$ and $q \neq 1$, then by (3.9), we have

$$(q-1)^n \det(\mathscr{D}_G) = \det(\mathscr{F}_G) - \mathrm{cof}(\mathscr{F}_G),$$

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and when $n = 1$ or $q = 1$, then it can be verified directly. Equivalently,

$$(q-1)^n \frac{\det(\mathscr{D}_G)}{\det(\mathscr{F}_G)} = 1 - \frac{\mathrm{cof}(\mathscr{F}_G)}{\det(\mathscr{F}_G)}.$$

Analogously, we can get

$$(q-1)^{n_i} \frac{\det(\mathscr{D}_{G_i})}{\det(\mathscr{F}_{G_i})} = 1 - \frac{\mathrm{cof}(\mathscr{F}_{G_i})}{\det(\mathscr{F}_{G_i})}$$

for each $1 \leq i \leq r$. Then the result follows directly from Theorem 4.2. □

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