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SOME CHARACTERIZATIONS OF TOTALLY NONPOSITIVE (TOTALLY NEGATIVE) MATRICES*

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Abstract. An $n \times m$ real matrix A is said to be totally nonpositive (totally negative) if every minor is nonpositive (negative). In this paper, we study characterizations of these classes of matrices by minors, by their full rank factorization and by their thin QR factorization.

Key words. Totally nonpositive and totally negative matrices, Full rank factorization, Thin QR factorization.

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1. Introduction. A matrix $A \in \mathbb{R}^{n \times m}$ is called *totally nonpositive (totally negative)* if all its minors are nonpositive (negative) and abbreviated as t.n.p. (t.n.). For square t.n. matrices, spectral properties and LDU factorization are analyzed in [9], and a characterization in terms of the parameters of the Neville elimination is obtained in [12]. For nonsingular t.n.p. matrices, an LDU factorization and some properties are presented in [3]. In addition, characterizations by minors for rectangular t.n.p. and t.n. matrices are obtained in [5] taking into account their full rank factorization in echelon form.

The t.n.p. (t.n.) matrices can be considered an extension of the partially nonpositive matrices, p.n.p. (partially negative matrices, p.n.), that is, matrices such that all its principals minors are nonpositive (negative). A p.n. matrix is called N-matrix in economic models [2]. Also, the p.n. matrices arise in conjunction with Lemke's algorithm for solving linear and convex quadratic programming problems [16, 17].

On the other hand, the totally positive matrices, TP (strictly totally positive matrices, STP), that is, matrices such that all its minors are nonnegative (positive) have a wide variety of applications in approximation theory, numerical mathematics, statistics, economics and others fields. Note that TP and STP matrices have been studied extensively [1, 2, 6, 7, 8, 10, 11, 12, 13, 15], but there exist few papers on t.n.p. and t.n. matrices [3, 5, 9]. A relation between the TP and t.n.p. matrices is given

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by the following result: "the product of two t.n.p. matrices is a TP matrix". But the question: *"Can any $n \times n$ TP matrix be written as the product of two t.n. matrices?"* is an open problem, which can be found in the initial paper for t.n. matrices [9].

To solve this open problem, we study and characterize the t.n. matrices. We extend the well-known results on TP matrices for t.n. and t.n.p. matrices. Hence, the main goal of this paper is to derive characterizations of rectangular t.n.p. (t.n.) matrices by minors and in terms of their thin QR factorization.

2. Notation and definitions. Following the notation of [1], given $k, n \in \mathbb{N}$, $1 \leq k \leq n$, the subcollection of all such sequences that consist of consecutive integers is denoted by $\mathcal{Q}_{k,n}$. When the natural numbers are consecutive the sequence is denoted by $\mathcal{Q}_{k,n}^0$. If $A \in \mathbb{R}^{n \times m}$, $\alpha \in \mathcal{Q}_{k,n}$ and $\beta \in \mathcal{Q}_{k,m}$ with $k = 1, 2, \ldots, \min\{n, m\}$, $A[\alpha|\beta]$ denotes the $k \times k$ submatrix of A lying in rows α and columns β. The principal submatrix $A[\gamma|\gamma]$ is abbreviated as $A[\gamma]$, where $\gamma \in \mathcal{Q}_{k,\min\{n,m\}}$. Therefore, $A \in \mathbb{R}^{n \times m}$ is t.n.p. (t.n.) if $\det A[\alpha|\beta] \leq 0$ (< 0), for all $\alpha \in \mathcal{Q}_{k,n}$ and $\beta \in \mathcal{Q}_{k,m}$ with $k = 1, 2 \ldots, \min\{n, m\}$.

We recall that a matrix is an *upper echelon matrix* if it satisfies the following conditions:

1. The first nonzero entry in each row is called *leading entry for that nonzero row*.
2. Each leading entry is to the right of all first leading entry in the row above it.
3. All zero rows are at the bottom.

If, in addition, the matrix satisfies the following condition is called *upper reduced echelon matrix*

4. Each leading entry is the only nonzero entry in its column.

A matrix is a lower (reduced) echelon matrix if its transpose is an upper (reduced) echelon matrix. Moreover, if each leading entry is equal to 1, we add the adjective *unit* to these definitions.

Given a matrix $A \in \mathbb{R}^{n \times m}$ with $\text{rank}(A) = r$, a decomposition $A = FG$ is called *full rank factorization of A* if $F \in \mathbb{R}^{n \times r}$, $G \in \mathbb{R}^{r \times m}$ and $\text{rank}(F) = \text{rank}(G) = r$. It is known that the full rank factorization of any nonzero matrix is not unique but if the matrix G is a unit upper reduced echelon matrix then this factorization is unique [4].

Recall that a characterization for square TP and STP matrices by their QR factorization is obtained in [11]. For the rectangular case, these characterizations are

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extended in [6] by using their thin QR factorization.

From now on, we suppose that $A = (a_{ij})$ has not zero rows and zero columns. In addition, as it can be proved applying [3, Proposition 3.2] and [5, Theorem 9], if A is a t.n.p. matrix with $a_{11} < 0$ then $a_{ij} < 0$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, except for $a_{nm} \leq 0$.

3. Characterization of rectangular t.n.p. matrices by minors. The main goal of this section is to obtain characterizations of rectangular t.n.p. matrices which reduce the number of minors to be checked to deduce if a matrix is t.n.p.

PROPOSITION 3.1. *Let* $A = (a_{ij}) \in \mathbb{R}^{n \times r}$ *be a matrix with* $\mathrm{rank}(A) = r$, $a_{nr} \leq 0$ *and the remaining entries less than zero. Consider the nonsingular matrix* $A_1 \in \mathbb{R}^{r \times r}$ *formed by the first r linearly independent rows of A. Then, A is t.n.p. if and only if for each $k = 1, 2, \ldots, r$, the following inequalities hold*

$$(3.1) \qquad \det A[\alpha | 1, 2, \ldots, k] \leq 0, \quad for\ all\ \ \alpha \in \mathcal{Q}_{k,n}$$

$$(3.2) \qquad \det A_1[1, 2, \ldots, k | \beta] \leq 0, \quad for\ all\ \ \beta \in \mathcal{Q}_{k,r}$$

$$(3.3) \qquad \det A_1[1, 2, \ldots, k] < 0.$$

Proof. If A is t.n.p. the inequality (3.1) holds. Moreover, since A_1 is a nonsingular t.n.p. matrix, (3.2) and (3.3) follow by [3].

Conversely, by (3.1) for all $\alpha \in \mathcal{Q}_{k,r}$ the following inequality holds

$$(3.4) \qquad \det A_1[\alpha | 1, 2, \ldots, k] \leq 0.$$

Then, by (3.2), (3.3), (3.4) and [3, Theorem 4.1] we have that A_1 is a nonsingular t.n.p. matrix. As a consequence A_1 admits a factorization $A_1 = L_{A_1} D_{A_1} U_{A_1}$, where $L_{A_1} \in \mathbb{R}^{r \times r}$ is a unit lower triangular TP matrix, $D_{A_1} = \mathrm{diag}(-d_1, d_2, \ldots, d_r) \in \mathbb{R}^{r \times r}$, with $d_i > 0$, $i = 1, 2, \ldots, r$, and $U_{A_1} \in \mathbb{R}^{r \times r}$ is a unit upper triangular TP matrix.

Since $A = F A_1$, where $F \in \mathbb{R}^{n \times r}$ is a unit lower reduced echelon matrix we have that

$$A = F A_1 = F(L_{A_1} D_{A_1} U_{A_1}) = (F L_{A_1}) D_{A_1} U_{A_1} = LDU,$$

where $D = D_{A_1}$, $U = U_{A_1}$ and $L = F L_{A_1}$ is a unit lower echelon matrix such that

$$\det A[\alpha | 1, 2, \ldots, k] = - \prod_{i=1}^{k} d_i \det L[\alpha | 1, 2, \ldots, k] \leq 0 \implies L[\alpha | 1, 2, \ldots, k] \geq 0.$$

Hence L is TP by [6, Proposition 2] and we have that A is a t.n.p. matrix by [5]. ◻

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The following proposition allows us to give a characterization by minors of rectangular t.n.p. matrices without full rank.

PROPOSITION 3.2. *Let $A = (a_{ij}) \in \mathbb{R}^{n \times m}$ be a matrix with $\mathrm{rank}(A) = r$, $a_{nm} \leq 0$ and the remaining entries less than zero. Consider the matrices $A_1 \in \mathbb{R}^{r \times m}$, $A_2 \in \mathbb{R}^{n \times r}$, $F \in \mathbb{R}^{n \times r}$ in unit lower reduced echelon form and $C \in \mathbb{R}^{r \times m}$ in unit upper reduced echelon form, such that $A = FA_1 = A_2C$. Then, A is t.n.p. if and only if A_1 and A_2 are t.n.p.*

Proof. Suppose that A is t.n.p. Since F and C are matrices in lower and upper echelon form, respectively, we have that A_1 and A_2 are the submatrices of A formed by its first r linearly independent rows or columns, respectively. Then, they are t.n.p. matrices.

Conversely, since A_1 is t.n.p. with full row rank by [5, Theorem 9], $A_1 = L_1D_1U_1$ is the unique full rank factorization in unit echelon form, where $L_1 \in \mathbb{R}^{r \times r}$ is a unit lower triangular TP matrix, $D_1 = \mathrm{diag}(-d_{11} \ d_{12} \ \ldots \ d_{1r}) \in \mathbb{R}^{r \times r}$ with $d_{1i} > 0$ for $i = 1, 2, \ldots, r$, $U_1 \in \mathbb{R}^{r \times m}$ is a unit upper echelon TP matrix and $\mathrm{rank}(L_1) = \mathrm{rank}(U_1) = r$. Then, A admits the following full rank factorization in unit echelon form

$$A = FA_1 = F(L_1D_1U_1) = (FL_1)D_1U_1.$$

Analogously, A_2 is a t.n.p. matrix with full column rank which admits the unique full rank factorization in unit echelon form $A_2 = L_2D_2U_2$ where $L_2 \in \mathbb{R}^{n \times r}$ is a unit lower echelon TP matrix, $D_2 = \mathrm{diag}(-d_{21} \ d_{22} \ \ldots \ d_{2r}) \in \mathbb{R}^{r \times r}$ with $d_{2i} > 0$ for $i = 1, 2, \ldots, r$, $U_2 \in \mathbb{R}^{r \times r}$ is a unit upper triangular TP matrix and $\mathrm{rank}(L_2) = \mathrm{rank}(U_2) = r$. In this case, A admits the full rank factorization in unit echelon form

$$A = A_2C = (L_2D_2U_2)C = L_2D_2(U_2C).$$

Since the full rank factorization in unit echelon form is unique [4], we have that

$$A = (FL_1)D_1U_1 = L_2D_2(U_2C).$$

Then $FL_1 = L_2$, which implies that FL_1 is a unit lower echelon TP matrix. Therefore, by [5, Theorem 9] A is t.n.p. matrix. □

COROLLARY 3.3. *Let $A = (a_{ij}) \in \mathbb{R}^{n \times m}$ be a matrix with $\mathrm{rank}(A) = r$, $a_{nm} \leq 0$ and the remaining entries less than zero. Suppose that $A = F\bar{A}C$, with $F \in \mathbb{R}^{n \times r}$ unit lower reduced echelon matrix, $\bar{A} \in \mathbb{R}^{r \times r}$ nonsingular matrix and $C \in \mathbb{R}^{r \times m}$ unit upper reduced echelon matrix. Then, A is t.n.p. if and only if $\bar{A}C$ and $F\bar{A}$ are t.n.p.*

Proof. Consider the matrices $A_1 = \bar{A}C$ and $A_2 = F\bar{A}$. Then $A = FA_1$ with $F \in \mathbb{R}^{n \times r}$ unit lower reduced echelon matrix and $A = A_2C$ with $C \in \mathbb{R}^{r \times m}$ unit upper reduced echelon matrix. By Proposition 3.2 the result holds. □

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The Propositions 3.1 and 3.2 will be used to prove the following characterization by minors of rectangular t.n.p. matrices.

PROPOSITION 3.4. *Let $A = (a_{ij}) \in \mathbb{R}^{n \times m}$ be a matrix with $\mathrm{rank}(A) = r$, $a_{nm} \leq 0$ and the remaining entries less than zero. Suppose that $A_1 \in \mathbb{R}^{r \times m}$ is the submatrix of A formed by its first r linearly independent rows, $A_2 \in \mathbb{R}^{n \times r}$ is the submatrix of A formed by the first r linearly independent columns and $\bar{A} \in \mathbb{R}^{r \times r}$ is the nonsingular submatrix of A formed by its first r linearly independent rows and columns. Then, A is t.n.p. if and only if for each $k = 1, 2, \ldots, r$, the following inequalities hold*

$$(3.5) \qquad \det A_2[\alpha | 1, 2, \ldots, k] \leq 0 \qquad \forall \alpha \in \mathcal{Q}_{k,n},$$

$$(3.6) \qquad \det A_1[1, 2, \ldots, k | \beta] \leq 0 \qquad \forall \beta \in \mathcal{Q}_{k,m},$$

$$(3.7) \qquad \det \bar{A}[1, 2, \ldots, k] < 0.$$

Proof. Suppose that A is t.n.p. Since A_1 and A_2 are submatrices of A, the inequalities (3.5) and (3.6) hold. Moreover, note that $\bar{A}$ is the nonsingular submatrix of A_2 (A_1) formed by the first r linearly independent rows (columns), then $\bar{A}$ is a nonsingular t.n.p. matrix and by [3, Theorem 4.1] the inequality (3.7) holds.

Conversely, applying Proposition 3.1 to $A_2 \in \mathbb{R}^{n \times r}$ we know that A_2 is t.n.p. if and only if for each $k = 1, 2, \ldots, r$ the following inequalities hold

$$(3.8) \qquad \det A_2[\alpha | 1, 2, \ldots, k] \leq 0 \qquad \forall \alpha \in \mathcal{Q}_{k,n},$$

$$(3.9) \qquad \det \bar{A}[1, 2, \ldots, k | \beta] \leq 0 \qquad \forall \beta \in \mathcal{Q}_{k,r},$$

$$(3.10) \qquad \det \bar{A}[1, 2, \ldots, k] < 0.$$

Now, applying Proposition 3.1 to $A_1^T \in \mathbb{R}^{m \times r}$ we know that A_1 is t.n.p. if and only if for each $k = 1, 2, \ldots, r$ the following inequalities hold

$$(3.11) \qquad \det A_1[1, 2, \ldots, k | \beta] \leq 0 \qquad \forall \beta \in \mathcal{Q}_{k,m},$$

$$(3.12) \qquad \det \bar{A}[\alpha | 1, 2, \ldots, k] \leq 0 \qquad \forall \alpha \in \mathcal{Q}_{k,r},$$

$$(3.13) \qquad \det \bar{A}[1, 2, \ldots, k] < 0.$$

Since the inequalities (3.8) and (3.11) implies the inequalities (3.12) and (3.9) respectively, we can assure that A_1 and A_2 are t.n.p. matrices. Then, A is a t.n.p. matrix by Proposition 3.2. ∎

4. A characterization of rectangular t.n.p. matrices by their thin QR factorization. In [11] the authors introduce the concepts of lowerly TP and γ-matrix for nonsingular TP matrices, which allow to characterize this class of matrices by their QR factorizations. The extension of these concepts and characterization for rectangular TP matrices are given in [6]. In this section we give an equivalent characterization

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for rectangular t.n.p. matrices by using the concepts of *lowerly t.n.p. matrix* and *quasi γ-matrix*. From now on we denote by $I_{(-1)}$ the diagonal matrix $\operatorname{diag}(-1, 1, \ldots, 1)$.

DEFINITION 4.1. Consider $A \in \mathbb{R}^{n \times r}$ with $\operatorname{rank}(A) = r$. Then, A is *lowerly t.n.p.* if and only if it can be decomposed in the form $A = LDU$, where $L \in \mathbb{R}^{n \times r}$ is a unit lower echelon matrix, $D = \operatorname{diag}(-d_1, d_2, \ldots, d_r)$ is a nonsingular matrix, $U \in \mathbb{R}^{r \times r}$ is a unit upper triangular matrix and $LDI_{(-1)}$ is a TP matrix.

Note that if $LDI_{(-1)}$ is a TP matrix then, L and $DI_{(-1)} = \operatorname{diag}(d_1, d_2, \ldots, d_r)$ are TP matrices. The following proposition presents the characterization of lowerly t.n.p. matrices.

PROPOSITION 4.2. *Let $A \in \mathbb{R}^{n \times r}$ be a matrix with $\operatorname{rank}(A) = r$. Consider the nonsingular matrix $A_1 \in \mathbb{R}^{r \times r}$ formed by the first r linearly independent rows of A. Then, A is lowerly t.n.p. if and only if for each $k = 1, 2, \ldots, r$, the following inequalities hold*

(4.1) $$\det A[\alpha | 1, 2, \ldots, k] \leq 0, \quad \text{for all} \ \ \alpha \in \mathcal{Q}_{k,n}$$

(4.2) $$\det A_1[1, 2, \ldots, k] < 0.$$

Proof. If A is lowerly t.n.p. it can be factorized as $A = LDU$, where $L \in \mathbb{R}^{n \times r}$ is a unit lower echelon matrix, $D = \operatorname{diag}(-d_1, d_2, \ldots, d_r)$ with $d_i > 0$ for $i = 1, 2, \ldots, r$, $U \in \mathbb{R}^{r \times r}$ is a unit upper triangular matrix and $LDI_{(-1)}$ is a TP matrix. Then

$$\det A[\alpha | 1, 2, \ldots, k] = -\left(\prod_{i=1}^{k} d_i\right) \det L[\alpha | 1, 2, \ldots, k] \leq 0.$$

Now, suppose that $s_1, s_2, \ldots, s_r$ are the indices of the first r linearly independent rows of A. Since $A = LDU$ the submatrix A_1 can be obtained as follow

$$A_1 = L[s_1, s_2, \ldots, s_r | 1, 2, \ldots, r] DU$$

and obviously the inequality (4.2) holds.

Conversely, since A_1 is the nonsingular submatrix of A formed by the first r linearly independent rows, by (4.2) we know that A admits the factorization $A = LDU$, where $L \in \mathbb{R}^{n \times r}$ is a unit lower echelon matrix, $D = \operatorname{diag}(-d_1, d_2, \ldots, d_r)$ with $d_i > 0$ for $i = 1, 2, \ldots, r$, and $U \in \mathbb{R}^{r \times r}$ is unit upper triangular matrix. Then, by (4.1) we have

$$\det A[\alpha | 1, 2, \ldots, k] = -\left(\prod_{i=1}^{k} d_i\right) \det L[\alpha | 1, 2, \ldots, k] \leq 0,$$

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which implies that

$$\det L[\alpha|1, 2, \ldots, k] \geq 0$$

for all $\alpha \in \mathcal{Q}_{k,n}$, $k = 1, 2, \ldots, r$. Then, L is TP by [6, Proposition 1]. As a consequence $LDI_{(-1)}$ is a TP matrix because $DI_{(-1)}$ is also TP. Therefore, A is lowerly t.n.p. $\square$

DEFINITION 4.3. A matrix $A \in \mathbb{R}^{n \times r}$ with $\text{rank}(A) = r$ is said to be a quasi γ-matrix if it is lowerly t.n.p. and in the factorization $A = LDU$, $(DUI_{(-1)})^{-1}$ is TP.

EXAMPLE 4.4. The rectangular matrix

$$A = \begin{bmatrix} -1 & -5 & 0 \\ -1 & 1 & -2 \\ -2 & 2 & 1 \\ -3 & 3 & -1 \end{bmatrix}$$

is a quasi γ-matrix since,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix} = LDU$$

with

$$LDI_{(-1)} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 12 & 5 \\ 3 & 18 & 5 \end{bmatrix} \quad \text{and} \quad (DUI_{(-1)})^{-1} = \begin{bmatrix} 1 & 5/6 & 1/3 \\ 0 & 1/6 & 1/15 \\ 0 & 0 & 1/5 \end{bmatrix}$$

lower and upper triangular TP matrices, respectively.

REMARK 4.5. If $A \in \mathbb{R}^{n \times 2}$ with $\text{rank}(A) = 2$ is a t.n.p. matrix then it is easy to prove that A is a quasi γ-matrix. The converse is not true in general, as we can see with the following matrix, which is a quasi γ-matrix but not a t.n.p. matrix,

$$A = \begin{bmatrix} -2 & -2 \\ -1 & 3 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

From the quasi γ-matrices we give a characterization of rectangular t.n.p. matrices by their thin QR factorization.

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THEOREM 4.6. *Let $A = (a_{ij}) \in \mathbb{R}^{n \times r}$ be a matrix with* $\mathrm{rank}(A) = r$, $a_{nr} \leq 0$ *and the remaining entries less than zero. Then, A is t.n.p if and only if there exists a quasi γ-matrix $Q_1 \in \mathbb{R}^{n \times r}$ with orthonormal columns, an orthogonal quasi γ-matrix $Q_2 \in \mathbb{R}^{r \times r}$, a nonsingular upper triangular TP matrix $R_1 \in \mathbb{R}^{r \times r}$ and an upper echelon TP matrix $R_2 \in \mathbb{R}^{r \times n}$ such that $A = Q_1 R_1$ and $A^T = Q_2 R_2$.*

Proof. Suppose that A is t.n.p. with full column rank. By [14] A admits a unique thin QR factorization $A = Q_1 R_1$, where $Q_1 \in \mathbb{R}^{n \times r}$ has orthonormal columns and R_1 is nonsingular upper triangular. Since $A^T A$ is a TP matrix such that $A^T A = R_1^T R_1$, where R_1 is its upper triangular Cholesky factor, by [6] R_1 is a nonsingular upper triangular TP matrix. If we denote by $r_{1_i} > 0$, $i = 1, 2, \ldots, r$, the main diagonal entries of R_1, we have that

$$\det A[\alpha | 1, 2, \ldots, k] = \prod_{i=1}^{k} r_{1_i} \det Q_1[\alpha | 1, 2, \ldots, k] \leq 0.$$

Then, $\det Q_1[\alpha | 1, 2, \ldots, k] \leq 0$.

Moreover, since A is t.n.p., it admits the full rank factorization in unit echelon form $A = L_A D_A U_A$, where $L_A \in \mathbb{R}^{n \times r}$ is a unit lower echelon TP matrix, $D_A = \mathrm{diag}(-d_1, d_2, \ldots, d_r)$ with $d_i > 0$ for $i = 1, 2, \ldots, r$, and $U_A \in \mathbb{R}^{r \times r}$ is a unit upper triangular TP matrix. If we represent by $D_{R_1} = \mathrm{diag}(r_{1_1}, r_{1_2}, \ldots, r_{1_r})$, it follows that

$$\begin{aligned}
A = Q_1 R_1 = L_A D_A U_A &\Longrightarrow Q_1 = L_A D_A U_A R_1^{-1} \\
&= L_A \left(D_A D_{R_1}^{-1} \right) \left(D_{R_1} U_A R_1^{-1} \right) = L_{Q_1} D_{Q_1} U_{Q_1},
\end{aligned}$$

where $L_{Q_1} = L_A \in \mathbb{R}^{n \times r}$ is a unit lower echelon TP matrix, $D_{Q_1} = D_A D_{R_1}^{-1} = \mathrm{diag}(-d_{q_1}, d_{q_2}, \ldots, d_{q_r})$, with $d_{q_i} = d_i / r_{1_i} > 0$ for $i = 1, 2, \ldots, r$, and $U_{Q_1} = D_{R_1} U_A R_1^{-1} \in \mathbb{R}^{r \times r}$ is a unit upper triangular matrix.

Since L_{Q_1} and $D_{Q_1} I_{(-1)}$ are TP, their product is also TP, which implies that Q_1 is lowerly t.n.p.

On the other hand,

$$\begin{aligned}
Q_1^T = U_{Q_1}^T D_{Q_1} L_{Q_1}^T &\Longrightarrow \left(U_{Q_1}^T D_{Q_1} \right)^{-1} Q_1^T = L_{Q_1}^T \\
&\Longrightarrow \left(U_{Q_1}^T D_{Q_1} \right)^{-1} = L_{Q_1}^T Q_1 \\
&\Longrightarrow \left(U_{Q_1}^T D_{Q_1} \right)^{-1} I_{(-1)} = L_{Q_1}^T Q_1 I_{(-1)}.
\end{aligned}$$

Hence by the Binet-Cauchy identity [1],

$$\begin{aligned}
\det \left(U_{Q_1}^T D_{Q_1} \right)^{-1} I_{(-1)}[\alpha | 1, 2, \ldots, k] &= \det \left(L_{Q_1}^T Q_1 I_{(-1)} \right)[\alpha | 1, 2, \ldots, k] = \\
&= - \sum_{\forall \gamma \in \mathcal{Q}_{k,r}} \det \left(L_{Q_1}^T \right)[\alpha | \gamma] \det Q_1[\gamma | 1, 2, \ldots, k] \geq 0
\end{aligned}$$

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Therefore by [1, Corollary 2.2], the matrix

$$\left(U_{Q_1}^T D_{Q_1}\right)^{-1} I_{(-1)} = D_{Q_1}^{-T} U_{Q_1}^{-T} I_{(-1)}^{-T} = \left(I_{(-1)}^{-1} U_{Q_1}^{-1} D_{Q_1}^{-1}\right)^T$$

is lower triangular TP. Consequently, $\left(D_{Q_1} U_{Q_1} I_{(-1)}\right)^{-1}$ is an upper triangular TP matrix, which implies that Q_1 is a quasi γ-matrix.

Since $A^T \in \mathbb{R}^{r \times n}$ is also a t.n.p. matrix, as in the previous case

$$A^T = Q_2 R_2,$$

where $Q_2 \in \mathbb{R}^{r \times r}$ is an orthogonal quasi γ-matrix and $R_2 \in \mathbb{R}^{r \times n}$ is an upper echelon TP matrix.

Conversely, since Q_1 is a lowerly t.n.p. matrix, by Proposition 4.2 for all $k = 1, 2, \ldots, r$, the following inequalities hold

(4.3) $$\det Q_1[\alpha|1, 2, \ldots, k] \le 0, \quad \text{for all} \ \ \alpha \in \mathcal{Q}_{k,n}$$

(4.4) $$\det Q_{1_1}[1, 2, \ldots, k] < 0.$$

where the columns of Q_{1_1} are the first r linearly independent rows of Q_1. Then, from (4.3) we have that

$$\det A[\alpha|1, 2, \ldots, k] \le 0, \quad \text{for all} \ \ \alpha \in \mathcal{Q}_{k,n}, \quad k = 1, 2, \ldots, r.$$

On the other hand, since the submatrix of A formed by its first r linearly independent rows can be obtained as $A_1 = Q_{1_1} R_1$, it follows by (4.4) that

$$\det A_1[1, 2, \ldots, k] < 0, \quad k = 1, 2, \ldots, r,$$

We also have that $A_1^T = Q_2 \tilde{R}_2$, where $\tilde{R}_2$ is the upper triangular submatrix of R_2 formed by its first r linearly independent columns. Moreover, since Q_2 is a lowerly t.n.p. matrix

$$\det A_1^T[\beta|1, 2, \ldots, k] = \left(\prod_{i=1}^{k} \tilde{r}_{2_i}\right) \det Q_2[\beta|1, 2, \ldots, k] \le 0,$$

where $\tilde{r}_{2_i}$, with $i = 1, 2, \ldots, r$, are main diagonal of $\tilde{R}_2$. Therefore,

$$\det A_1[1, 2, \ldots, k|\beta] \le 0, \quad \text{for all} \ \ \alpha \in \mathcal{Q}_{k,n}, \quad k = 1, 2, \ldots, r.$$

Finally, A is a t.n.p. matrix by Proposition 3.1. $\square$

Next theorem gives an extension of Theorem 4.6 to rectangular matrices without full column rank.

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THEOREM 4.7. *Let $A = (a_{ij}) \in \mathbb{R}^{n \times m}$ be a matrix with* $\mathrm{rank}(A) = r$, $a_{nm} \leq 0$ *and the remaining entries less than zero. Then, A is t.n.p. if and only if there exist two quasi γ-matrices $Q_1 \in \mathbb{R}^{n \times r}$ and $Q_2 \in \mathbb{R}^{m \times r}$, both with orthonormal columns and two upper echelon TP matrices $R_1 \in \mathbb{R}^{r \times m}$ and $R_2 \in \mathbb{R}^{r \times n}$, such that $A = Q_1 R_1$ and $A^T = Q_2 R_2$.*

Proof. Suppose that A is t.n.p. Since $\mathrm{rank}(A) = r$, then $A = \tilde{A}C_1$, where $\tilde{A} \in \mathbb{R}^{n \times r}$ is a t.n.p. matrix formed by the first r linearly independent columns of A and $C_1 \in \mathbb{R}^{r \times m}$ is a unit upper reduced matrix. Applying Theorem 4.6 to $\tilde{A}$ we have that

$$\tilde{A} = \tilde{Q}_1 \tilde{R}_1,$$

where $\tilde{Q}_1 \in \mathbb{R}^{n \times r}$ is a quasi γ-matrix with orthonormal columns and $\tilde{R}_1 \in \mathbb{R}^{r \times r}$ is a nonsingular upper triangular TP matrix. In consequence,

$$A = \tilde{A}C_1 = \tilde{Q}_1 \tilde{R}_1 C_1 = \tilde{Q}_1 \left(\tilde{R}_1 C_1 \right) = Q_1 R_1,$$

where $Q_1 = \tilde{Q}_1 \in \mathbb{R}^{n \times r}$ is a quasi γ-matrix with orthonormal columns and $R_1 = \tilde{R}_1 C_1 \in \mathbb{R}^{r \times m}$ is an upper echelon matrix. Since $R_1^T = A^T Q_1$,

$$\det R_1^T[\alpha | 1, 2, \ldots, k] = \sum_{\forall \gamma \in \mathcal{Q}_{k,r}} \det A^T[\alpha | \gamma] \det Q_1[\gamma | 1, 2, \ldots, k] \geq 0$$

and R_1^T is a lower echelon TP matrix by [6, Proposition 1]. Therefore R_1 is an upper echelon TP matrix.

Analogously, since $A^T \in \mathbb{R}^{m \times n}$ is a t.n.p. matrix with $\mathrm{rank}(A^T) = r$,

$$A^T = \tilde{A}^T C_2,$$

where $\tilde{A}^T \in \mathbb{R}^{m \times r}$ is a t.n.p. matrix formed by the first r linearly independent columns of A^T and $C_2 \in \mathbb{R}^{r \times n}$ is a unit upper reduced matrix. Reasoning in the same way we obtain that $A^T = Q_2 R_2$, where $Q_2 \in \mathbb{R}^{m \times r}$ is a quasi γ-matrix with orthonormal columns and $R_2 \in \mathbb{R}^{r \times n}$ is an upper echelon TP matrix.

Conversely, suppose that $A = Q_1 R_1$. Since $Q_1 \in \mathbb{R}^{n \times r}$ is a quasi γ-matrix with $\mathrm{rank}(Q_1) = r$, it admits the full rank factorization

$$Q_1 = L_{Q_1} D_{Q_1} U_{Q_1},$$

where $L_{Q_1} \in \mathbb{R}^{n \times r}$ is a unit lower echelon matrix, $D_{Q_1} = \mathrm{diag}(-d_{q_{1_1}}, d_{q_{1_2}}, \ldots, d_{q_{1_r}})$ with $d_{q_{1_i}} > 0$ for $i = 1, 2, \ldots, r$, $U_{Q_1} \in \mathbb{R}^{r \times r}$ is a unit upper triangular matrix, such that $\mathrm{rank}(L_{Q_1}) = \mathrm{rank}(U_{Q_1}) = r$ and $L_{Q_1} D_{Q_1} I_{(-1)}$ ia a TP matrix (with L_{Q_1} also a TP matrix). Then,

$$A = (L_{Q_1} D_{Q_1} U_{Q_1}) R_1 = (L_{Q_1} D_{Q_1}) (U_{Q_1} R_1)$$
$$= (L_{Q_1} D_{Q_1}) (D_1 U_1) = L_{Q_1} (D_{Q_1} D_1) U_1 = L_A D_A U_A$$

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is a full rank factorization of A, with $L_A = L_{Q_1} \in \mathbb{R}^{n \times r}$ a unit lower echelon TP matrix, $D_A = D_{Q_1} D_1 = \mathrm{diag}(-d_{1_1}, d_{1_2}, \ldots, d_{1_r})$ with $d_{1_i} > 0$ for $i = 1, 2, \ldots, r$, $U_A = U_1 \in \mathbb{R}^{r \times m}$ unit upper echelon matrix and $\mathrm{rank}(L_A) = \mathrm{rank}(U_A) = r$.

Analogously, $A^T = Q_2 R_2$ where $Q_2 \in \mathbb{R}^{m \times r}$ is a γ-matrix with $\mathrm{rank}(Q_2) = r$. Then, Q_2 admits the following full rank factorization

$$Q_2 = L_{Q_2} D_{Q_2} U_{Q_2},$$

where $L_{Q_2} \in \mathbb{R}^{m \times r}$ is a unit lower echelon matrix, $D_{Q_2} = \mathrm{diag}(-d_{q_{2_1}}, d_{q_{2_2}}, \ldots, d_{q_{2_r}})$ with $d_{q_{2_i}} > 0$ for $i = 1, 2, \ldots, r$, $U_{Q_2} \in \mathbb{R}^{r \times r}$ is a unit upper triangular matrix, such that $\mathrm{rank}(L_{Q_2}) = \mathrm{rank}(U_{Q_2}) = r$, and $L_{Q_1} D_{Q_1} I_{(-1)}$ is a TP matrix (with L_{Q_2} also a TP matrix). As a consequence,

$$
\begin{aligned}
A^T &= \left(L_{Q_2} D_{Q_2} U_{Q_2} \right) R_2 = \left(L_{Q_2} D_{Q_2} \right) \left(U_{Q_2} R_2 \right) \\
&= \left(L_{Q_2} D_{Q_2} \right) \left(D_2 U_2 \right) = L_{Q_2} \left(D_{Q_2} D_2 \right) U_2 = L_{A^T} D_{A^T} U_{A^T}
\end{aligned}
$$

with $L_{A^T} = L_{Q_2} \in \mathbb{R}^{m \times r}$ unit lower echelon TP matrix, $U_{A^T} = U_2 \in \mathbb{R}^{r \times n}$ unit upper echelon matrix with $\mathrm{rank}(L_{A^T}) = \mathrm{rank}(U_{A^T}) = r$ and $D_{A^T} = D_{Q_2} D_2 = \mathrm{diag}(-d_{2_1}, d_{2_2}, \ldots, d_{2_r})$ with $d_{2_i} > 0$, for $i = 1, 2, \ldots, r$.

Therefore,

$$A = L_A D_A U_A = \left(L_{A^T} D_{A^T} U_{A^T} \right) = U_{A^T}^T D_{A^T} L_{A^T}^T,$$

where $U_A = L_{A^T}^T \in \mathbb{R}^{r \times m}$ is a unit upper echelon TP matrix because the full rank factorization in echelon form of any matrix is unique [4]. Finally, A is a t.n.p. matrix by [5, Theorem 8]. ∎

5. A characterization of rectangular t.n. matrices by their thin QR factorization. In this section we characterize the rectangular t.n. matrices by their thin QR factorization. In [6, Theorem 6] the authors give the following characterization of rectangular t.n. matrices

THEOREM 5.1. *Consider $A = (a_{ij}) \in \mathbb{R}^{n \times m}$ with $a_{nm} < 0$ and $n \le m$. A is t.n. if and only if for each $k = 1, 2, \ldots, n$, the following inequalities hold*

(5.1) $\qquad \det A[\alpha | \{1, 2, \ldots, k\}] < 0, \quad$ *for all $\alpha \in \mathcal{Q}_{k,n}^0$*

(5.2) $\qquad \det A[\{1, 2, \ldots, k\} | \beta] < 0, \quad$ *for all $\beta \in \mathcal{Q}_{k,m}^0$*

In analogous way to the previous section we extend to rectangular matrices the concept of lowerly t.n. matrix and give the characterization of this class of matrices by minors.

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DEFINITION 5.2. Consider $A \in \mathbb{R}^{n \times r}$ with $\operatorname{rank}(A) = r$. Then, A is *lowerly t.n.* if and only if it can be decomposed in the form $A = LDU$, where $L \in \mathbb{R}^{n \times r}$ is a unit lower echelon matrix, $D = \operatorname{diag}(-d_1, d_2, \ldots, d_r)$ is a nonsingular matrix, $U \in \mathbb{R}^{r \times r}$ is a unit upper triangular matrix and $LDI_{(-1)}$ is a $\triangle$STP matrix.

Note that if $LDI_{(-1)}$ is a $\triangle$STP matrix then, L is a $\triangle$STP matrix and $DI_{(-1)} = \operatorname{diag}(d_1, d_2, \ldots, d_r)$ is a TP matrix. The following proposition presents the characterization of lowerly quasi STP matrices.

PROPOSITION 5.3. *A matrix* $A = (a_{ij}) \in \mathbb{R}^{n \times m}$ *with* $\operatorname{rank}(A) = m$ *and* $a_{nm} < 0$, *is lowerly t.n. if and only if for each* $k = 1, 2, \ldots, m$ *the following inequalities hold*

$$(5.3) \qquad \det A[\alpha | 1, 2, \ldots, k] < 0 \qquad for\ all\ \ \alpha \in \mathcal{Q}_{k,n}^0.$$

Proof. If A is lowerly t.n. then $A = LDU$, with $D = \operatorname{diag}(-d_1, d_2, \ldots, d_m)$, $d_i > 0$, $i = 1, 2, \ldots, m$. By [6, Remark 3] we have that

$$\det A[\alpha | 1, 2, \ldots, k] = \det(LDU)[\alpha | 1, 2, \ldots, k] = \det(LD)[\alpha | 1, 2, \ldots, k]$$
$$= \left(-\prod_{i=1}^{k} d_i \right) \det L[\alpha | 1, 2, \ldots, k] < 0 \qquad \forall \alpha \in \mathcal{Q}_{k,n}^0.$$

Conversely, from (5.3) we have that A admits a factorization $A = LDU$ with no pivoting such that $L \in \mathbb{R}^{n \times m}$ is a unit lower echelon matrix, $D = \operatorname{diag}(-d_1, d_2, \ldots, d_m)$ with $d_i > 0$, for $i = 1, 2, \ldots, m$, and $U \in \mathbb{R}^{m \times m}$ is a unit upper triangular matrix. Since for all $\alpha \in \mathcal{Q}_{k,n}^0$ it is satisfied that

$$\det A[\alpha | 1, 2, \ldots, k] = \left(-\prod_{i=1}^{k} d_i \right) \det L[\alpha | 1, 2, \ldots, k] < 0,$$

then L is $\triangle$STP by [6, Remark 3], therefore LD is also $\triangle$STP and A is lowerly t.n. $\square$

Now, the following definition extends to rectangular matrices with full column rank the concept of quasi strict γ-matrix.

DEFINITION 5.4. A matrix $A \in \mathbb{R}^{n \times r}$, with $\operatorname{rank}(A) = r$, is said to be a quasi strict γ-matrix if it is lowerly t.n. and in the factorization $A = LDU$, $(DUI_{(-1)})^{-1}$ is $\triangle$STP.

Next result gives a characterization of the t.n. matrices by their thin QR factorization using quasi strict γ-matrices. The proof is similar to Theorem 4.6.

THEOREM 5.5. *Let* $A = (a_{ij}) \in \mathbb{R}^{n \times m}$ *be a matrix with* $\operatorname{rank}(A) = m$ *and all its entries less than zero. Then, A is t.n. if and only if there exist a quasi strict*

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γ-matrix $Q_1 \in \mathbb{R}^{n \times m}$ with orthonormal columns and an orthogonal quasi strict γ-matrix $Q_2 \in \mathbb{R}^{m \times m}$, a nonsingular upper triangular ΔSTP matrix $R_1 \in \mathbb{R}^{m \times m}$ and an upper echelon ΔSTP matrix $R_2 \in \mathbb{R}^{m \times n}$, such that $A = Q_1 R_1$ and $A^T = Q_2 R_2$.

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