

# RANK AND INERTIA OF SUBMATRICES OF THE MOORE–PENROSE INVERSE OF A HERMITIAN MATRIX\*

YONGGE TIAN<sup>†</sup>

**Abstract.** Closed-form formulas are derived for the rank and inertia of submatrices of the Moore–Penrose inverse of a Hermitian matrix. A variety of consequences on the nonsingularity, nullity and definiteness of the submatrices are also presented.

**Key words.** Hermitian matrix, Partitioned matrix, Moore–Penrose inverse, Rank, Inertia.

**AMS subject classifications.** 15A03, 15A09, 15A23, 15A57.

**1. Introduction.** Throughout this paper,  $\mathbb{C}^{m \times n}$  and  $\mathbb{C}_h^{m \times m}$  stand for the sets of all  $m \times n$  complex matrices and all  $m \times m$  Hermitian complex matrices, respectively. The symbols  $A^*$ ,  $r(A)$  and  $\mathcal{R}(A)$  stand for the conjugate transpose, rank and range (column space) of a matrix  $A \in \mathbb{C}^{m \times n}$ , respectively.  $[A, B]$  denotes a row block matrix consisting of  $A$  and  $B$ . The inertia of a Hermitian matrix  $A$  is defined to be the triplet  $\text{In}(A) = \{i_+(A), i_-(A), i_0(A)\}$ , where  $i_+(A)$ ,  $i_-(A)$  and  $i_0(A)$  are the numbers of the positive, negative and zero eigenvalues of  $A$  counted with multiplicities, respectively. It is obvious that  $r(A) = i_+(A) + i_-(A)$ . We write  $A > 0$  ( $A \geq 0$ ) if  $A$  is Hermitian positive (nonnegative) definite. Two Hermitian matrices  $A$  and  $B$  of the same size are said to satisfy the inequality  $A > B$  ( $A \geq B$ ) in the Löwner partial ordering if  $A - B$  is positive (nonnegative) definite. The Moore–Penrose inverse of  $A \in \mathbb{C}^{m \times n}$ , denoted by  $A^\dagger$ , is defined to be the unique solution  $X$  of the four matrix equations

$$(i) AXA = A, \quad (ii) XAX = X, \quad (iii) (AX)^* = AX, \quad (iv) (XA)^* = XA.$$

A matrix  $X \in \mathbb{C}_h^{m \times m}$  is called a Hermitian  $g$ -inverse of  $A$ , denoted by  $A_h^-$ , if it satisfies  $AXA = A$ . Further, the symbols  $E_A$  and  $F_A$  stand for the two orthogonal projectors (idempotent Hermitian matrices)  $E_A = I_m - AA^\dagger$  and  $F_A = I_n - A^\dagger A$ . A well-known property of the Moore–Penrose inverse is  $(A^\dagger)^* = (A^*)^\dagger$ . In particular,  $(A^\dagger)^* = A^\dagger$  and  $AA^\dagger = A^\dagger A$  if  $A = A^*$ . Results on the Moore–Penrose inverse can be found, e.g., in [1, 2, 7].

\*Received by the editors September 3, 2009. Accepted for publication April 19, 2010. Handling Editor: Moshe Goldberg.

<sup>†</sup>China Economics and Management Academy, Central University of Finance and Economics, Beijing 100081, China (yongge.tian@gmail.com)

One of the fundamental operations in matrix theory is to partition a matrix into block forms. Many properties of a matrix and its operations can be derived from partitions of the matrix and their operations. A typical partitioned Hermitian matrix is given by the following  $2 \times 2$  form

$$M = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}, \quad (1.1)$$

where  $A \in \mathbb{C}_h^{m \times m}$ ,  $B \in \mathbb{C}^{m \times n}$  and  $D \in \mathbb{C}_h^{n \times n}$ . Correspondingly, the Moore–Penrose inverse of  $M$  is Hermitian as well, and a partitioned expression of  $M^\dagger$  can be written as

$$M^\dagger = \begin{bmatrix} G_1 & G_2 \\ G_2^* & G_3 \end{bmatrix}, \quad (1.2)$$

where  $G_1 \in \mathbb{C}_h^{m \times m}$ ,  $G_2 \in \mathbb{C}^{m \times n}$  and  $G_3 \in \mathbb{C}_h^{n \times n}$ . When  $M$  in (1.1) is nonsingular, (1.2) reduces to the usual inverse of  $M$ .

In the investigation of a partitioned matrix and its inverse or generalized inverse, attention is often given to expressions of submatrices of the inverse or generalized inverse, as well as their properties. If (1.1) is nonsingular, explicit expressions of the three submatrices  $G_1$ ,  $G_2$  and  $G_3$  of the inverse in (1.2) were given in a recent paper [19]. If (1.1) is singular, expressions of the three submatrices  $G_1$ ,  $G_2$  and  $G_3$  in (1.2) can also be derived from certain decompositions of  $M$ . Various formulas for  $G_1$ ,  $G_2$  and  $G_3$  in (1.2) were given in the literature; see, e.g., [9, 13, 16, 17]. These expressions, however, are quite complicated in general. In addition to the expressions of the submatrices in (1.2), another important task is to describe various algebraic properties of the submatrices in (1.2), such as, their rank, range, nullity, inertia, and definiteness. Some previous and recent work on these properties can be found, e.g., in [4, 10, 11, 14, 15, 18, 19, 20, 21]. Motivated by the work on nullity and inertia of submatrices in a nonsingular (Hermitian) matrix and its inverse, we derive in this paper closed-form formulas for the rank and inertia of the submatrices  $G_1$ ,  $G_2$  and  $G_3$  in (1.2) through some known and new results on ranks and inertias of (Hermitian) matrices. As applications, we use these formulas to characterize the nonsingularity, nullity and definiteness of the submatrices in (1.2).

Some well-known equalities and inequalities for ranks of partitioned matrices are given below.

LEMMA 1.1 ([12]). *Let  $A \in \mathbb{C}^{m \times n}$ ,  $B \in \mathbb{C}^{m \times k}$ ,  $C \in \mathbb{C}^{l \times n}$  and  $D \in \mathbb{C}^{l \times k}$ . Then,*

(a) *The following rank equalities hold*

$$r[A, B] = r(A) + r(E_A B) = r(B) + r(E_B A), \quad (1.3)$$

$$r \begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \quad (1.4)$$

$$r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B AF_C), \quad (1.5)$$

$$r \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} = r[A, B] + r(B) \quad \text{if } A \geq 0, \quad (1.6)$$

$$r \begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r \begin{bmatrix} 0 & E_A B \\ CF_A & D - CA^\dagger B \end{bmatrix}. \quad (1.7)$$

(b) *The following rank inequalities hold*

$$r(A) + r(B) + r(C) \geq r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \geq r(B) + r(C), \quad (1.8)$$

$$r(CA^\dagger B) \geq r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r \begin{bmatrix} A \\ C \end{bmatrix} - r[A, B] + r(A). \quad (1.9)$$

$$(c) \quad r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(A) + r(B) + r(C) \Leftrightarrow \mathcal{R}(A) \cap \mathcal{R}(B) = \{0\} \text{ and } \mathcal{R}(A^*) \cap \mathcal{R}(C^*) = \{0\}.$$

$$(d) \quad r \begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) \Leftrightarrow AA^\dagger B = B, \quad CA^\dagger A = C \text{ and } D = CA^\dagger B.$$

Note that the inertia of a Hermitian matrix divides the eigenvalues of the matrix into three sets on the real line. Hence the inertia of a Hermitian matrix can be used to characterize the definiteness of the matrix. The following result is obvious from the definitions of the rank and inertia of a matrix.

LEMMA 1.2. *Let  $A \in \mathbb{C}^{m \times m}$ ,  $B \in \mathbb{C}^{m \times n}$  and  $C \in \mathbb{C}_h^{m \times m}$ . Then,*

- (a)  *$A$  is nonsingular if and only if  $r(A) = m$ .*
- (b)  *$B = 0$  if and only if  $r(B) = 0$ .*
- (c)  *$C > 0$  ( $C < 0$ ) if and only if  $i_+(C) = m$  ( $i_-(C) = m$ ).*
- (d)  *$C \geq 0$  ( $C \leq 0$ ) if and only if  $i_-(C) = 0$  ( $i_+(C) = 0$ ).*

LEMMA 1.3. Let  $A \in \mathbb{C}_h^{m \times m}$ ,  $B \in \mathbb{C}^{m \times n}$  and  $P \in \mathbb{C}^{m \times m}$ . Then,

$$i_{\pm}(PAP^*) = i_{\pm}(A) \text{ if } P \text{ is nonsingular,} \quad (1.10)$$

$$i_{\pm}(PAP^*) \leq i_{\pm}(A) \text{ if } P \text{ is singular,} \quad (1.11)$$

$$A(A^3)^{\dagger}A = A^{\dagger}, \quad r(A^3) = r(A), \quad (1.12)$$

$$i_{\pm}(A^3) = i_{\pm}(A), \quad i_{\pm}(A^{\dagger}) = i_{\pm}(A), \quad (1.13)$$

$$i_{\pm}(\lambda A) = \begin{cases} i_{\pm}(A) & \text{if } \lambda > 0 \\ i_{\mp}(A) & \text{if } \lambda < 0 \end{cases}, \quad (1.14)$$

$$i_{\pm} \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix} = r(B). \quad (1.15)$$

Equation (1.10) is the well-known Sylvester's law of inertia (see, e.g., [8, Theorem 4.5.8]). Equation (1.11) was given in [18, Lemma 1.6]. Equations (1.12), (1.13) and (1.14) follow from the spectral decomposition of  $A$  and the definitions of the Moore–Penrose inverse, rank and inertia. Equation (1.15) is well known (see, e.g., [5, 6]).

The following result was given in [18, Theorem 2.3].

LEMMA 1.4. Let  $A \in \mathbb{C}_h^{m \times m}$ ,  $B \in \mathbb{C}^{m \times n}$ ,  $D \in \mathbb{C}_h^{n \times n}$ , and denote

$$U = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad V = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}, \quad S_A = D - B^*A^{\dagger}B.$$

Then,

$$i_{\pm}(U) = r(B) + i_{\pm}(E_B A E_B), \quad (1.16)$$

$$i_{\pm}(V) = i_{\pm}(A) + i_{\pm} \begin{bmatrix} 0 & E_A B \\ B^* E_A & S_A \end{bmatrix}, \quad (1.17)$$

$$r(B) \leq i_{\pm}(U) \leq r(B) + i_{\pm}(A), \quad (1.18)$$

$$r[A, B] - i_{\mp}(A) \leq i_{\pm}(V) \leq r[A, B] + i_{\pm}(S_A) - i_{\mp}(A), \quad (1.19)$$

$$i_{\pm}(B^*A^{\dagger}B) \geq i_{\mp}(U) - r[A, B] + i_{\pm}(A). \quad (1.20)$$

In particular,

- (a) If  $A \geq 0$ , then  $i_{+}(U) = r[A, B]$  and  $i_{-}(U) = r(B)$ .
- (b) If  $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ , then  $i_{\pm}(U) = r(B)$ .
- (c) If  $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ , then  $i_{\pm}(U) = i_{\pm}(A) + r(B)$ .
- (d) If  $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$  and  $\mathcal{R}(B^*) \cap \mathcal{R}(D) = \{0\}$ , then  $i_{\pm}(V) = i_{\pm}(A) + r(B) + i_{\pm}(D)$ .
- (e)  $i_{\pm}(V) = i_{\pm}(A)$  if and only if  $\mathcal{R}(B) \subseteq \mathcal{R}(A)$  and  $i_{\pm}(D - B^*A^{\dagger}B) = 0$ .

LEMMA 1.5. Let  $A \in \mathbb{C}_h^{m \times m}$  and  $B \in \mathbb{C}^{m \times n}$ . Then,

$$i_{\pm}(B^* A^{\dagger} B) = i_{\mp} \begin{bmatrix} A^3 & AB \\ (AB)^* & 0 \end{bmatrix} - i_{\mp}(A), \quad (1.21)$$

$$r(B^* A^{\dagger} B) = r \begin{bmatrix} A^3 & AB \\ (AB)^* & 0 \end{bmatrix} - r(A). \quad (1.22)$$

*Proof.* Applying (1.12), (1.14) and (1.17) gives

$$i_{\pm} \begin{bmatrix} A^3 & AB \\ (AB)^* & 0 \end{bmatrix} = i_{\pm}(A^3) + i_{\pm}[-B^* A(A^3)^{\dagger} AB] = i_{\pm}(A) + i_{\mp}(B^* A^{\dagger} B),$$

as required for (1.21). Adding the two equalities in (1.21) gives (1.22).  $\square$

**2. Main results.** Note that the three submatrices  $G_1$ ,  $G_2$  and  $G_3$  in (1.2) can be represented as

$$G_1 = P_1 M^{\dagger} P_1^*, \quad G_2 = P_1 M^{\dagger} P_2^*, \quad G_3 = P_2 M^{\dagger} P_2^*, \quad (2.1)$$

where  $P_1 = [I_m, 0]$  and  $P_2 = [0, I_n]$ . Applying Lemma 1.5 to (2.1) gives the following result.

THEOREM 2.1. Let  $M$  and  $M^{\dagger}$  be given by (1.1) and (1.2), and denote

$$W_1 = [A, B] \quad \text{and} \quad W_2 = [B^*, D]. \quad (2.2)$$

Then,

$$r(G_1) = r \begin{bmatrix} W_2^* D W_2 & W_1^* \\ W_1 & 0 \end{bmatrix} - r(M), \quad (2.3)$$

$$r(G_2) = r \begin{bmatrix} W_1^* B W_2 & W_2^* \\ W_1 & 0 \end{bmatrix} - r(M), \quad (2.4)$$

$$r(G_3) = r \begin{bmatrix} W_1^* A W_1 & W_2^* \\ W_2 & 0 \end{bmatrix} - r(M), \quad (2.5)$$

and

$$i_{\pm}(G_1) = i_{\mp} \begin{bmatrix} W_2^* D W_2 & W_1^* \\ W_1 & 0 \end{bmatrix} - i_{\mp}(M), \quad (2.6)$$

$$i_{\pm}(G_3) = i_{\mp} \begin{bmatrix} W_1^* A W_1 & W_2^* \\ W_2 & 0 \end{bmatrix} - i_{\mp}(M). \quad (2.7)$$

Hence,

(a)  $G_1$  is nonsingular if and only if  $r \begin{bmatrix} W_2^*DW_2 & W_1^* \\ W_1 & 0 \end{bmatrix} = r(M) + m$ .

(b)  $G_1 = 0$  if and only if  $r \begin{bmatrix} W_2^*DW_2 & W_1^* \\ W_1 & 0 \end{bmatrix} = r(M)$ .

(c)  $G_1 > 0$  ( $G_1 < 0$ ) if and only if

$$i_- \begin{bmatrix} W_2^*DW_2 & W_1^* \\ W_1 & 0 \end{bmatrix} = i_-(M) + m \quad \left( i_+ \begin{bmatrix} W_2^*DW_2 & W_1^* \\ W_1 & 0 \end{bmatrix} = i_+(M) + m \right).$$

(d)  $G_1 \geq 0$  ( $G_1 \leq 0$ ) if and only if

$$i_+ \begin{bmatrix} W_2^*DW_2 & W_1^* \\ W_1 & 0 \end{bmatrix} = i_+(M) \quad \left( i_- \begin{bmatrix} W_2^*DW_2 & W_1^* \\ W_1 & 0 \end{bmatrix} = i_-(M) \right).$$

(e)  $G_2 = 0$  if and only if  $r \begin{bmatrix} W_1^*BW_2 & W_2^* \\ W_1 & 0 \end{bmatrix} = r(M)$ .

(f)  $G_3$  is nonsingular if and only if  $r \begin{bmatrix} W_1^*AW_1 & W_2^* \\ W_2 & 0 \end{bmatrix} = r(M) + m$ .

(g)  $G_3 = 0$  if and only if  $r \begin{bmatrix} W_1^*AW_1 & W_2^* \\ W_2 & 0 \end{bmatrix} = r(M)$ .

(h)  $G_3 > 0$  ( $G_3 < 0$ ) if and only if

$$i_- \begin{bmatrix} W_1^*AW_1 & W_2^* \\ W_2 & 0 \end{bmatrix} = i_-(M) + m \quad \left( i_+ \begin{bmatrix} W_1^*AW_1 & W_2^* \\ W_2 & 0 \end{bmatrix} = i_+(M) + m \right).$$

(i)  $G_3 \geq 0$  ( $G_3 \leq 0$ ) if and only if

$$i_+ \begin{bmatrix} W_1^*AW_1 & W_2^* \\ W_2 & 0 \end{bmatrix} = i_+(M) \quad \left( i_- \begin{bmatrix} W_1^*AW_1 & W_2^* \\ W_2 & 0 \end{bmatrix} = i_-(M) \right).$$

*Proof.* Applying (1.21) and (1.22) to (2.1) gives

$$r(G_1) = r \begin{bmatrix} M^3 & MP_1^* \\ P_1M & 0 \end{bmatrix} - r(M) = r \begin{bmatrix} M^3 & W_1^* \\ W_1 & 0 \end{bmatrix} - r(M), \quad (2.8)$$

$$r(G_2) = r \begin{bmatrix} M^3 & MP_2^* \\ P_1M & 0 \end{bmatrix} - r(M) = r \begin{bmatrix} M^3 & W_2^* \\ W_1 & 0 \end{bmatrix} - r(M), \quad (2.9)$$

$$r(G_3) = r \begin{bmatrix} M^3 & MP_2^* \\ P_2M & 0 \end{bmatrix} - r(M) = r \begin{bmatrix} M^3 & W_2^* \\ W_2 & 0 \end{bmatrix} - r(M), \quad (2.10)$$

$$i_{\pm}(G_1) = i_{\mp} \begin{bmatrix} M^3 & MP_1^* \\ P_1M & 0 \end{bmatrix} - i_{\mp}(M) = i_{\mp} \begin{bmatrix} M^3 & W_1^* \\ W_1 & 0 \end{bmatrix} - i_{\mp}(M), \quad (2.11)$$

$$i_{\pm}(G_3) = i_{\mp} \begin{bmatrix} M^3 & MP_2^* \\ P_2M & 0 \end{bmatrix} - i_{\mp}(M) = i_{\mp} \begin{bmatrix} M^3 & W_2^* \\ W_2 & 0 \end{bmatrix} - i_{\mp}(M). \quad (2.12)$$

Expanding  $M^3$  gives

$$M^3 = [W_1^*, W_2^*] \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = W_1^* A W_1 + W_1^* B W_2 + W_2^* B^* W_1 + W_2^* D W_2.$$

Hence,

$$\begin{aligned} & \begin{bmatrix} I_m & -\frac{1}{2}W_1^*A - W_2^*B^* \\ 0 & I_n \end{bmatrix} \begin{bmatrix} M^3 & W_1^* \\ W_1 & 0 \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -\frac{1}{2}A W_1 - B W_2 & I_n \end{bmatrix} \\ &= \begin{bmatrix} M^3 - W_1^* A W_1 - W_1^* B W_2 - W_2^* B^* W_1 & W_1^* \\ W_1 & 0 \end{bmatrix} = \begin{bmatrix} W_2^* D W_2 & W_1^* \\ W_1 & 0 \end{bmatrix}. \end{aligned}$$

Applying (1.10) to these equalities gives

$$r \begin{bmatrix} M^3 & W_1^* \\ W_1 & 0 \end{bmatrix} = r \begin{bmatrix} W_2^* D W_2 & W_1^* \\ W_1 & 0 \end{bmatrix} \quad \text{and} \quad i_{\mp} \begin{bmatrix} M^3 & W_1^* \\ W_1 & 0 \end{bmatrix} = i_{\mp} \begin{bmatrix} W_2^* D W_2 & W_1^* \\ W_1 & 0 \end{bmatrix}.$$

Substituting these equalities into (2.8) and (2.11) leads to (2.3) and (2.6). Equations (2.5) and (2.7) can be shown similarly.

Note that

$$\begin{aligned} & \begin{bmatrix} I_m & -W_1^*A - W_2^*B^* \\ 0 & I_n \end{bmatrix} \begin{bmatrix} M^3 & W_2^* \\ W_1 & 0 \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -D W_2 & I_n \end{bmatrix} \\ &= \begin{bmatrix} M^3 - W_1^* A W_1 - W_2^* B^* W_1 - W_2^* D W_2 & W_1^* \\ W_1 & 0 \end{bmatrix} = \begin{bmatrix} W_1^* B W_2 & W_2^* \\ W_1 & 0 \end{bmatrix}. \end{aligned}$$

Hence,

$$r \begin{bmatrix} M^3 & W_2^* \\ W_1 & 0 \end{bmatrix} = r \begin{bmatrix} W_1^* B W_2 & W_2^* \\ W_1 & 0 \end{bmatrix}.$$

Substituting this equality into (2.9) leads to (2.4). Results (a)–(i) follow from (2.3)–(2.7) and Lemma 1.2.  $\square$

We next obtain some consequences of Theorem 2.1 under various assumptions for  $M$  in (1.1).

**COROLLARY 2.2.** *Let  $M$  and  $M^\dagger$  be given by (1.1) and (1.2), and assume that  $M$  satisfies the rank additivity condition*

$$r(M) = r[A, B] + r[B^*, D], \quad \text{i.e.,} \quad \mathcal{R}([A, B]^*) \cap \mathcal{R}([B^*, D]^*) = \{0\}. \quad (2.13)$$

Then,

$$r(G_1) = r(D) + r[A, B] - r[B^*, D], \quad (2.14)$$

$$r(G_2) = r(B), \quad (2.15)$$

$$r(G_3) = r(A) + r[B^*, D] - r[A, B], \quad (2.16)$$

$$r(G_1) + r(G_3) = r(A) + r(D), \quad (2.17)$$

$$i_{\pm}(G_1) = i_{\mp}(D) + r[A, B] - i_{\mp}(M), \quad (2.18)$$

$$i_{\pm}(G_3) = i_{\mp}(A) + r[B^*, D] - i_{\mp}(M), \quad (2.19)$$

$$i_{\pm}(G_1) + i_{\mp}(G_3) = i_{\pm}(A) + i_{\mp}(D). \quad (2.20)$$

In particular, if  $M$  is nonsingular, then

$$r(G_1) = r(D) + m - n, \quad r(G_2) = r(B), \quad r(G_3) = r(A) + n - m, \quad (2.21)$$

$$i_{\pm}(G_1) = i_{\mp}(D) + m - i_{\mp}(M), \quad i_{\pm}(G_3) = i_{\mp}(A) + n - i_{\mp}(M). \quad (2.22)$$

*Proof.* Under (2.13), it follows from Lemmas 1.1(c) and 1.4(c) that

$$r \begin{bmatrix} W_2^* D W_2 & W_1^* \\ W_1 & 0 \end{bmatrix} = r(W_2 D W_2) + 2r(W_1), \quad (2.23)$$

$$i_{\mp} \begin{bmatrix} W_2^* D W_2 & W_1^* \\ W_1 & 0 \end{bmatrix} = i_{\mp}(W_2^* D W_2) + r(W_1), \quad (2.24)$$

where the matrix  $W_2^* D W_2 = \begin{bmatrix} B D B^* & B D^2 \\ D^2 B^* & D^3 \end{bmatrix}$  satisfies

$$\begin{bmatrix} I_m & -B D^{\dagger} \\ 0 & I_n \end{bmatrix} \begin{bmatrix} B D B^* & B D^2 \\ D^2 B^* & D^3 \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -D^{\dagger} B^* & I_n \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & D^3 \end{bmatrix}.$$

Hence, it follows from (1.10), (1.12) and (1.13) that

$$r(W_2^* D W_2) = r(D^3) = r(D), \quad i_{\mp}(W_2^* D W_2) = i_{\mp}(D^3) = i_{\mp}(D). \quad (2.25)$$

Substituting (2.25) into (2.23) and (2.24), and (2.23) and (2.24) into (2.3) and (2.6), leads to (2.14) and (2.18). Equations (2.15), (2.16) and (2.19) can be shown similarly. Adding (2.14) and (2.16) yields (2.17). Adding (2.18) and (2.19) yields (2.20). If  $M$  is nonsingular, then  $r[A, B] = m$  and  $r[B^*, D] = n$ . Hence, (2.14)–(2.20) reduce to (2.21) and (2.22).  $\square$

The three formulas in (2.21) were given in [4] in the form of nullity of matrices, and the corresponding results are usually called the nullity theorem; see also [3, 15, 20]. Note that

$$i_{\mp}(A) = m - i_{\pm}(A) - i_0(A), \quad i_{\mp}(D) = n - i_{\pm}(D) - i_0(D), \quad i_{\pm}(M) + i_{\mp}(M) = m + n.$$

Hence, (2.22) can alternatively be written as

$$i_{\pm}(M) = i_{\pm}(D) + i_0(D) + i_{\pm}(G_1) \quad \text{and} \quad i_{\pm}(M) = i_{\pm}(A) + i_0(A) + i_{\pm}(G_3).$$

These formulas were given in [10, 11].

**COROLLARY 2.3.** *Let  $M$  and  $M^{\dagger}$  be given by (1.1) and (1.2), and assume that  $M$  satisfies*

$$\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\} \quad \text{and} \quad \mathcal{R}(B^*) \cap \mathcal{R}(D) = \{0\}. \quad (2.26)$$

*Then,*

$$r(G_1) = r(A), \quad r(G_2) = r(B), \quad r(G_3) = r(D), \quad (2.27)$$

$$i_{\pm}(G_1) = i_{\pm}(A), \quad i_{\pm}(G_3) = i_{\pm}(D). \quad (2.28)$$

*Proof.* Equation (2.26) is equivalent to

$$r(W_1) = r(A) + r(B), \quad r(W_2) = r(B) + r(D), \quad r(M) = r(A) + 2r(B) + r(D) \quad (2.29)$$

by (1.3), (1.4) and (1.5). In this case, (2.14)–(2.16) reduce to (2.27). Also, substituting Lemma 1.4(d) and (2.29) into (2.18) and (2.19) yields (2.28).  $\square$

**COROLLARY 2.4.** *Let  $M$  and  $M^{\dagger}$  be given by (1.1) and (1.2), and assume that*

$$r(M) = r(A). \quad (2.30)$$

*Then,*

$$r(G_1) = r(A), \quad r(G_2) = r(B), \quad r(G_3) = r \begin{bmatrix} A^3 & B \\ B^* & 0 \end{bmatrix} - r(A), \quad (2.31)$$

$$i_{\pm}(G_1) = i_{\pm}(A), \quad i_{\pm}(G_3) = i_{\mp} \begin{bmatrix} A^3 & B \\ B^* & 0 \end{bmatrix} - i_{\mp}(A). \quad (2.32)$$

*Proof.* By Lemma 1.1(d), (2.30) is equivalent to  $E_A B = 0$  and  $D = B^* A^{\dagger} B$ , which imply  $r(W_1) = r(A)$ ,  $r(W_2) = r(B)$ , and  $\mathcal{R}(W_2^*) \subseteq \mathcal{R}(W_1^*)$  by (1.3). Applying

(1.5) to (2.3)–(2.5), and simplifying by elementary block matrix operations, produces

$$\begin{aligned}
 r(G_1) &= r \begin{bmatrix} W_2^* D W_2 & W_1^* \\ W_1 & 0 \end{bmatrix} - r(M) = r \begin{bmatrix} 0 & W_1^* \\ W_1 & 0 \end{bmatrix} - r(A) = r(A), \\
 r(G_2) &= r \begin{bmatrix} W_1^* B W_2 & W_2^* \\ W_1 & 0 \end{bmatrix} - r(M) = r \begin{bmatrix} 0 & W_2^* \\ W_1 & 0 \end{bmatrix} - r(A) = r(B), \\
 r(G_3) &= r \begin{bmatrix} W_1^* A W_1 & W_2^* \\ W_2 & 0 \end{bmatrix} - r(M) = r \begin{bmatrix} A^3 & A^2 B & B \\ B^* A^2 & B^* A B & D \\ B^* & D & 0 \end{bmatrix} - r(A) \\
 &= r \begin{bmatrix} A^3 & A^2 B & B \\ 0 & 0 & 0 \\ B^* & D & 0 \end{bmatrix} - r(A) \\
 &= r \begin{bmatrix} A^3 & 0 & B \\ 0 & 0 & 0 \\ B^* & 0 & 0 \end{bmatrix} - r(A) \\
 &= r \begin{bmatrix} A^3 & B \\ B^* & 0 \end{bmatrix} - r(A),
 \end{aligned}$$

as required for (2.31). Applying Lemma 1.4(b) and (e) to (2.6), and simplifying by elementary block congruence matrix operations and by (1.10), produces

$$\begin{aligned}
 i_{\pm}(G_1) &= i_{\mp} \begin{bmatrix} W_2^* D W_2 & W_1^* \\ W_1 & 0 \end{bmatrix} - i_{\mp}(M) = r(W_1) - i_{\mp}(A) = i_{\pm}(A), \\
 i_{\pm}(G_3) &= i_{\mp} \begin{bmatrix} W_1^* A W_1 & W_2^* \\ W_2 & 0 \end{bmatrix} - i_{\mp}(M) = i_{\mp} \begin{bmatrix} A^3 & A^2 B & B \\ B^* A^2 & B^* A B & D \\ B^* & D & 0 \end{bmatrix} - i_{\mp}(A) \\
 &= i_{\mp} \begin{bmatrix} A^3 & 0 & B \\ 0 & 0 & 0 \\ B^* & 0 & 0 \end{bmatrix} - i_{\mp}(A) \\
 &= i_{\mp} \begin{bmatrix} A^3 & B \\ B^* & 0 \end{bmatrix} - i_{\mp}(A),
 \end{aligned}$$

as required for (2.32).  $\square$

COROLLARY 2.5. *Let  $M$  and  $M^\dagger$  be given by (1.1) and (1.2), and assume that*

both  $A \geq 0$  and  $D \geq 0$ . Then,

$$r(G_1) = r \begin{bmatrix} A & BD \\ B^* & D^2 \end{bmatrix} + r[A, B] - r(M), \quad (2.33)$$

$$r(G_3) = r \begin{bmatrix} A^2 & AB \\ B^* & D \end{bmatrix} + r[B^*, D] - r(M), \quad (2.34)$$

$$i_+(G_1) = r[A, B] - i_-(M), \quad i_-(G_1) = r \begin{bmatrix} A & BD \\ B^* & D^2 \end{bmatrix} - i_+(M), \quad (2.35)$$

$$i_+(G_3) = r[B^*, D] - i_-(M), \quad i_-(G_3) = r \begin{bmatrix} A^2 & AB \\ B^* & D \end{bmatrix} - i_+(M). \quad (2.36)$$

Under the condition  $M \geq 0$ ,

$$r(G_1) = i_+(G_1) = r(A) \quad \text{and} \quad r(G_3) = i_+(G_3) = r(D). \quad (2.37)$$

*Proof.* If  $D \geq 0$ , then  $W_2^*DW_2 \geq 0$  and  $\mathcal{R}(W_2^*DW_2) = \mathcal{R}(W_2^*D)$ . In this case, applying (1.6) to (2.3) gives

$$\begin{aligned} r(G_1) &= r \begin{bmatrix} W_2^*DW_2 & W_1^* \\ W_1 & 0 \end{bmatrix} - r(M) = r[W_2^*DW, W_1^*] + r(W_1) - r(M) \\ &= r[W_2^*D, W_1^*] + r(W_1) - r(M) \\ &= r \begin{bmatrix} A & BD \\ B^* & D^2 \end{bmatrix} + r(W_1) - r(M), \end{aligned}$$

as required for (2.33). Equation (2.34) can be shown similarly. Applying Lemma 1.4(a) to (2.6) gives

$$\begin{aligned} i_+(G_1) &= i_- \begin{bmatrix} W_2^*DW_2 & W_1^* \\ W_1 & 0 \end{bmatrix} - i_-(M) = r(W_1) - i_-(M) \\ &= r[A, B] - i_-(M), \\ i_-(G_1) &= i_+ \begin{bmatrix} W_2^*DW_2 & W_1^* \\ W_1 & 0 \end{bmatrix} - i_+(M) = r[W_2^*D, W_1^*] - i_+(M) \\ &= r \begin{bmatrix} A & BD \\ B^* & D^2 \end{bmatrix} - i_+(M), \end{aligned}$$

as required for (2.35). Equation (2.36) can be shown similarly.

If  $M \geq 0$ , then  $A \geq 0$ ,  $D \geq 0$ , and

$$\begin{aligned} r \begin{bmatrix} A & BD \\ B^* & D^2 \end{bmatrix} &= r \begin{bmatrix} A^2 & AB \\ B^* & D \end{bmatrix} = r(M) = i_+(M), \\ r[A, B] &= r(A), \quad r[B^*, D] = r(D). \end{aligned}$$

Hence (2.33)–(2.36) reduce to (2.37).  $\square$

We next obtain a group of inequalities for the rank and inertia of  $G_1$  in (1.2).

COROLLARY 2.6. *Let  $M$  and  $M^\dagger$  be given by (1.1) and (1.2). Then,*

$$r(G_1) \geq \max\{2r[A, B] - r(M), r(D) - 2r[B^*, D] + r(M)\}, \quad (2.38)$$

$$r(G_1) \leq r(D) + 2r[A, B] - r(M), \quad (2.39)$$

$$i_\pm(G_1) \geq \max\{r[A, B] - i_\mp(M), i_\mp(D) - r[B^*, D] + i_\pm(M)\}, \quad (2.40)$$

$$i_\pm(G_1) \leq i_\mp(D) + r[A, B] - i_\mp(M). \quad (2.41)$$

*Proof.* Applying (1.8) and (2.25) gives

$$2r(W_1) \leq r \begin{bmatrix} W_2 D W_2^* & W_1 \\ W_1^* & 0 \end{bmatrix} \leq r(W_2 D W_2^*) + 2r(W_1) = r(D) + 2r(W_1).$$

Substituting these two inequalities into (2.3), we obtain the first part of (2.38) and (2.39). Applying (1.9) to the first expression in (2.1), and simplifying by elementary block matrix operations, gives

$$\begin{aligned} r(G_1) &= r(P_1 M^\dagger P_1^*) \geq r \begin{bmatrix} M & P_1^* \\ P_1 & 0 \end{bmatrix} - 2r[M, P_1^*] + r(M) \\ &= r(D) - 2r[B^*, D] + r(M), \end{aligned}$$

establishing the second part of (2.38). Applying (1.18), (1.20) and (2.25) gives

$$\begin{aligned} r(W_1) &\leq i_\pm \begin{bmatrix} W_2 D W_2^* & W_1 \\ W_1^* & 0 \end{bmatrix} \leq i_\pm(W_2 D W_2^*) + r(W_1) = i_\pm(D) + r(W_1), \\ i_\pm(G_1) &= i_\pm(P_1 M^\dagger P_1^*) \geq i_\mp \begin{bmatrix} M & P_1^* \\ P_1 & 0 \end{bmatrix} - r[M, P_1^*] + i_\pm(M) \\ &= i_\mp(D) - r[B^*, D] + i_\pm(M). \end{aligned}$$

Substituting these inequalities into (2.6) leads to (2.40) and (2.41).  $\square$

Inequalities for the rank and inertia of the two submatrices  $G_2$  and  $G_3$  in (1.2) can be derived similarly. Setting  $D = 0$  in Theorem 2.1, and simplifying, yields the following result.

COROLLARY 2.7. *Let*

$$M_1 = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad (2.42)$$

where  $A \in \mathbb{C}_h^{m \times m}$  and  $B \in \mathbb{C}^{m \times n}$ , and denote its Moore–Penrose inverse by

$$M_1^\dagger = \begin{bmatrix} G_1 & G_2 \\ G_2^* & G_3 \end{bmatrix}, \quad (2.43)$$

where  $G_1 \in \mathbb{C}_h^{m \times m}$ ,  $G_2 \in \mathbb{C}^{m \times n}$  and  $G_3 \in \mathbb{C}_h^{n \times n}$ . Then,

$$r(G_1) = 2r[A, B] - r(M_1), \quad (2.44)$$

$$r(G_2) = r(B), \quad (2.45)$$

$$r(G_3) = r \begin{bmatrix} A^3 & A^2 B & B \\ B^* A^2 & B^* A B & 0 \\ B^* & 0 & 0 \end{bmatrix} - r(M_1), \quad (2.46)$$

$$i_\pm(G_1) = r[A, B] - i_\mp(M_1), \quad (2.47)$$

$$i_\pm(G_3) = i_\mp \begin{bmatrix} A^3 & A^2 B & B \\ B^* A^2 & B^* A B & 0 \\ B^* & 0 & 0 \end{bmatrix} - i_\mp(M_1). \quad (2.48)$$

Under the condition  $A \geq 0$ ,

$$r(G_1) = i_+(G_1) = r[A, B] - r(B), \quad (2.49)$$

$$r(G_3) = i_-(G_3) = r(A) + r(B) - r[A, B]. \quad (2.50)$$

Equalities for the rank and inertia of the submatrices in (2.42), (2.43) and their operations, such as,  $A - AG_1A$  and  $A + BG_3B^*$ , can also be derived. The following result was recently given in [18, Theorem 3.11].

**THEOREM 2.8.** *Let  $M_1$  and  $M_1^\dagger$  be given by (2.42) and (2.43). Then,*

$$i_\pm(M_1) = i_\pm(A) + r(B) - i_\pm(A - AG_1A), \quad (2.51)$$

$$i_\pm(M_1) = r(M) - i_\mp(A) - r(B) + i_\mp(A - AG_1A), \quad (2.52)$$

$$i_\pm(M_1) = r[A, B] - i_\mp(A + BG_3B^*), \quad (2.53)$$

$$i_\pm(M_1) = r(M) - r[A, B] + i_\pm(A + BG_3B^*), \quad (2.54)$$

$$r(M_1) = r(A) + 2r(B) - r(A - AG_1A), \quad (2.55)$$

$$r(M_1) = 2r[A, B] - r(A + BG_3B^*). \quad (2.56)$$

Hence,

$$(a) \ A \geq AG_1A \Leftrightarrow i_-(M_1) = i_-(A) + r(B).$$

$$(b) \ A \leq AG_1A \Leftrightarrow i_+(M_1) = i_+(A) + r(B).$$

$$(c) \ A = AG_1A \Leftrightarrow r(M_1) = r(A) + 2r(B).$$

- (d)  $A + BG_3B^* \geq 0 \Leftrightarrow i_+(M_1) = r[A, B]$ .  
 (e)  $A + BG_3B^* \leq 0 \Leftrightarrow i_-(M_1) = r[A, B]$ .

In addition to (1.2), other types of generalized inverses of  $M$  can also be written in partitioned forms. For example, we can partition the Hermitian  $g$ -inverse of  $M$  in (1.1) as

$$\begin{bmatrix} A & B \\ B^* & D \end{bmatrix}_h^- = \begin{bmatrix} G_1 & G_2 \\ G_2^* & G_3 \end{bmatrix},$$

where  $G_1 \in \mathbb{C}_h^{m \times m}$ ,  $G_2 \in \mathbb{C}^{m \times n}$  and  $G_3 \in \mathbb{C}_h^{n \times n}$ . Then, the rank and inertia of  $G_1$ ,  $G_2$  and  $G_3$  may vary with respect to the choice of  $M_h^-$ . In such a case, it would be of interest to consider the maximal and minimal possible ranks and inertias of  $G_1$ ,  $G_2$  and  $G_3$ .

In an earlier paper [10], Johnson and Lundquist defined the inertia of Hermitian operator in a Hilbert space, and gave some formulas for the inertias of Hermitian operators and their inverses. Under this general frame, it would be of interest to extend the results in this paper to inertias of Hermitian operators in a Hilbert space.

**Acknowledgments.** The author thanks the referee and Professor Moshe Goldberg for their helpful suggestions and comments.

#### REFERENCES

- [1] A. Ben-Israel and T.N.E. Greville. *Generalized Inverses: Theory and Applications*. Second ed., Springer, New York, 2003.
- [2] D.S. Bernstein. *Matrix Mathematics: Theory, Facts and Formulas*. Second ed., Princeton University Press, Princeton, 2009.
- [3] M. Fiedler and T.L. Markham. Completing a matrix when certain entries of its inverse are specified. *Linear Algebra Appl.*, 74:225–237, 1986.
- [4] W.H. Gustafson. A note on matrix inversion. *Linear Algebra Appl.*, 57:71–73, 1984.
- [5] E.V. Haynsworth. Determination of the inertia of a partitioned Hermitian matrix. *Linear Algebra Appl.*, 1:73–81, 1968.
- [6] E.V. Haynsworth and A.M. Ostrowski. On the inertia of some classes of partitioned matrices. *Linear Algebra Appl.*, 1:299–316, 1968.
- [7] L. Hogben. *Handbook of Linear Algebra*. Chapman & Hall/CRC, 2007.
- [8] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1985.
- [9] C.H. Hung and T.L. Markham. The Moore–Penrose inverse of a partitioned matrix  $\begin{bmatrix} A & C \\ B & D \end{bmatrix}$ . *Linear Algebra Appl.*, 11:73–86, 1975.
- [10] C.R. Johnson and M. Lundquist. Operator matrix with chordal inverse patterns. *Operator Theory Adv. Appl.*, 59:234–251, 1992.
- [11] C.R. Johnson and M. Lundquist. An inertia formula for Hermitian matrices with sparse inverses. *Linear Algebra Appl.*, 162-164:541–556, 1992.

- [12] G. Marsaglia and G.P.H. Styan. Equalities and inequalities for ranks of matrices. *Linear and Multilinear Algebra*, 2:269–292, 1974.
- [13] J. Miao. General expression for the Moore–Penrose inverse of a  $2 \times 2$  block matrix. *Linear Algebra Appl.*, 151:1–15, 1990.
- [14] D.W. Robinson. Nullities of submatrices of the Moore–Penrose inverse. *Linear Algebra Appl.*, 94:127–132, 1987.
- [15] G. Strang and T. Nguyen. The interplay of ranks of submatrices. *SIAM Review*, 46:637–646, 2004.
- [16] Y. Tian. The Moore–Penrose inverses of  $m \times n$  block matrices and their applications. *Linear Algebra Appl.*, 283:35–60, 1998.
- [17] Y. Tian. Eight expressions for generalized inverses of a bordered matrix. *Linear and Multilinear Algebra*, 58:203–220, 2010.
- [18] Y. Tian. Equalities and inequalities for inertias of Hermitian matrices with applications. *Linear Algebra Appl.*, 433:263–296, 2010.
- [19] Y. Tian and Y. Takane. The inverse of any two-by-two nonsingular partitioned matrix and three matrix inverse completion problems. *Comput. Math. Appl.*, 57:1294–1304, 2009.
- [20] R. Vandebril and M. Van Barel. A note on the nullity theorem. *J. Comput. Appl. Math.*, 189:179–190, 2006.
- [21] H.J. Woerdeman. A matrix and its inverse: revisiting minimal rank completions. *Operator Theory Adv. Appl.*, 179:329–338, 2007.