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NULLITIES OF CYCLE-SPLICED BIPARTITE GRAPHS*

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Abstract. For a simple graph G, let $\eta(G)$ and $c(G)$ be the nullity and the cyclomatic number of G, respectively. A cycle-spliced bipartite graph is a connected graph in which every block is an even cycle. It was shown by Wong et al. (2022) that for every cycle-spliced bipartite graph G, $0 \leq \eta(G) \leq c(G)+1$. Moreover, the extremal graphs with $\eta(G) = c(G)+1$ and $\eta(G) = 0$, respectively, have been characterized. In this paper, we prove that there is no cycle-spliced bipartite graphs G of any order with nullity $\eta(G) = c(G)$. Furthermore, we also provide a structural characterization on cycle-spliced bipartite graphs G with nullity $\eta(G) = c(G) - 1$.

Key words. Cycle-spliced bipartite graph, Nullity, Cyclomatic number.

AMS subject classifications. 05C05; 05C50.

1. Introduction. All graphs considered in this paper are finite, undirected, and simple. Let G be a graph with $n(G)$ vertices and $e(G)$ edges. Let $\theta(G)$ be the number of connected components of G. The cyclomatic number of G is $c(G) = e(G) - n(G) + \theta(G)$. In particular, for a connected graph G, if $c(G) = 0$, $c(G) = 1$, or $c(G) = 2$, then G is a *tree*, a *unicyclic graph* or a *bicyclic graph*, respectively. A pendant vertex (i.e., a vertex of degree 1) is also called a leaf. A graph without leaves is said to be leaf-free. A *block* in a graph is a maximal connected subgraph with no cut vertex. A *block graph* is a graph in which all blocks are cliques (complete subgraphs). A tree can be viewed as a graph in which every block is K_2, thus it is a special block graph. We call a graph G to be a *cycle-spliced graph* if G is connected and every block in G is a cycle. A *cycle-spliced bipartite graph* is a cycle-spliced graph without odd cycle. An induced subgraph H of a graph G is called a *pendant subgraph* of G if H has at least two vertices and there is exactly one vertex in H, referred to as the root of H, that has at least one neighbor not in H. If, in addition, H is an induced cycle of G, then we refer to H as a *pendant cycle* of G. We call a pendant subgraph H of G a *maximal pendant subgraph* of G if there does not exist a pendant subgraph H' with $V(H') \supsetneqq V(H)$ (or, equivalently, $E(H') \supsetneqq E(H)$).

The adjacency matrix $A(G) = (a_{ij})$ of G with vertex set $V(G) = \{v_1, \ldots, v_n\}$ is defined to be an $n \times n$ symmetric matrix such that $a_{ij} = 1$ if vertices v_i and v_j are adjacent; and $a_{ij} = 0$, otherwise. The *rank* of a graph G, denoted by $r(G)$, is the rank of $A(G)$. The multiplicity of the eigenvalue zero of $A(G)$ is called the *nullity* of G and is denoted by $\eta(G)$. It is obvious that $\eta(G) = n(G) - r(G)$. The chemical importance of the nullity of graphs lies in the fact, that within the Hückel molecular orbital model, if $\eta(G) > 0$ for the molecular graph G, then the corresponding chemical compound is highly reactive and unstable, or

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nonexistent (see [1,6]). A graph is said to be singular (resp. nonsingular) if its adjacency matrix is singular (resp. nonsingular). In 1957, Collatz and Sinogowitz [3] first posed the problem of characterizing all singular graphs ($\eta(G) \neq 0$). Motivated by this problem, there have been lots of research work on bounding the nullities (or ranks) of graphs with given order in terms of various graph parameters. Ma *et al.* [9] proved that $\eta(G) \leq 2c(G) + p(G) - 1$ unless G is a cycle of order a multiple of 4, where $p(G)$ is the number of leaves in G. Chang *et al.* [4] characterized the leaf-free graphs with nullity $2c(G) - 1$. Wang [11] and Chang *et al.* [5] characterized all graphs with nullity $2c(G) + p(G) - 1$, respectively. Recently, Wong *et al.* [12] considered the singularity and the nullity of cycle-spliced bipartite graphs and proved that for every cycle-spliced bipartite graph G, $0 \leq \eta(G) \leq c(G) + 1$ and characterized all cycle-spliced bipartite graphs with $\eta(G) = c(G) + 1$ and $\eta(G) = 0$, respectively. Their result can be read as follows.

Theorem 1.1 ([12, Theorem 1.1]). *Let G be a cycle-spliced bipartite graph with $c(G)$ cycles. Then*

(i) $0 \leq \eta(G) \leq c(G) + 1$*;*
(ii) $\eta(G) = c(G) + 1$ *if and only if all cycles of G are with a multiple of 4;*
(iii) G is nonsingular (or $\eta(G) = 0$) if and only if G has a perfect matching, and G has a maximum matching M such that $M \bigcap E(C)$ is not a perfect matching of C for every 0-type cycle C in G, where 0-type cycle is a cycle whose order is equal to $0 (mod\ 4)$.

In this paper, we further consider the nullity of cycle-spliced bipartite graphs and prove that there is no cycle-spliced bipartite graphs G of any order with nullity $\eta(G) = c(G)$. Moreover, we also explore some structural characterization for cycle-spliced bipartite graphs G with nullity $\eta(G) = c(G) - 1$. Our main results can be read as follows, respectively.

Theorem 1.2. *For any cycle-spliced bipartite graph G of order n with $c(G)$ cycles, $\eta(G) \neq c(G)$.*

Theorem 1.3. *Let G be a cycle-spliced bipartite graph with $c(G) \geq 2$ and all pendant cycles have length congruent to $2 (mod\ 4)$. Then $\eta(G) = c(G) - 1$ if and only if the distance between any two cut vertices of G is even.*

Theorem 1.4. *For any cycle-spliced bipartite graph G with $c(G)$ cycles, $\eta(G) = c(G) - 1$ if and only if G is a graph obtained from a cycle-spliced bipartite graph H with $\eta(H) = c(H) - 1$ in which every pendant cycle (if any) has length congruent to $2 (mod\ 4)$ by attaching $c(G) - c(H)$ cycles having length divisible by 4 on arbitrary vertex of H.*

The rest of this paper is organized as follows. In Section 2, we give some notations and preliminary lemmas which will be used in our proofs. In Section 3, the property on cycle-spliced bipartite graph with nullity $\eta(G) = c(G)+1$ is given and the proof of Theorem 1.2 is presented. In Section 4, a number of auxiliary results involving properties on cycle-spliced bipartite graph with nullity $\eta(G) = c(G) - 1$ are presented. In Section 5, we give the proofs for Theorems 1.3 and 1.4.

2. Preliminaries. For $v \in V(G)$, let $d_G(v)$ and $N_G(v)$ be the degree and the set of neighbors of v, respectively. Clearly, $d_G(v) = |N_G(v)|$. If H is an induced subgraph of G, we use $N_H(v)$ to denote the set of neighbors of v in H. If $S \subseteq V(G)$, we denote by $G - S$ the (induced) subgraph obtained from G by deleting vertices in S (together with the incident edges). If $S = \{u\}$ or $\{u, v\}$, then $G - S$ is abbreviated to $G - u$ or $G - u - v$. If H is an induced subgraph of G with $V(H) \bigcap S = \emptyset$, we use $H + S$ to denote the subgraph of G induced by the vertex set $V(H) \bigcup S$.

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The following is a frequently used result in this topic:

Lemma 2.1 ([8]). *For any vertex $v \in V(G)$, $\eta(G) - 1 \leq \eta(G - v) \leq \eta(G) + 1$.*

Lemma 2.2 ([6, 10]). *If v is a pendant vertex of a graph G and u is its unique neighbor, then $\eta(G) = \eta(G - u - v)$.*

Following [10], if v is a pendant vertex of G and u is its unique neighbor, we call the operation of obtaining $G - u - v$ from G a *pendant K_2 deletion*. Lemma 2.2 says that upon the application of a pendant K_2 deletion, the nullity of a graph is unchanged.

Lemma 2.3 ([2]). *Let P_n and C_n be the cycle and path of order n, respectively. Then*

$$\eta(P_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}, \quad \eta(C_n) = \begin{cases} 2 & \text{if } n \text{ is a multiple of } 4 \\ 0 & \text{otherwise} \end{cases}.$$

Recall that if H is a pendant subgraph of G with root u, then u is a cut vertex of G. Thus [7], Theorem 2.3 and Theorem 2.4] can be reformulated, respectively, as follows:

Lemma 2.4. *Let G_1 be a pendant subgraph of G with root u.*

(i) If $\eta(G_1 - u) = \eta(G_1) + 1$, then $\eta(G) = \eta(G_1) + \eta(G - G_1)$.
(ii) If $\eta(G_1 - u) = \eta(G_1) - 1$, then $\eta(G) = \eta(G_1) + \eta(G - G_1 + u) - 1$.

Corollary 2.5. *Let C be a pendant even cycle of G with root u.*

(i) If C has length divisible by 4, then $\eta(G) = \eta(G - C + u) + 1$.
(ii) If C has length congruent to $2(\text{mod } 4)$, then $\eta(G) = \eta(G - C)$.

3. A Proof of Theorem 1.2. For $u, v \in V(G)$, the *distance* between u and v, denoted by $d_G(u, v)$, is the shortest length of paths between u and v. The notation $d_G(v, S)$ to stand for the distance between a vertex $v \in V(G)$ and a subset $S \subseteq V(G)$, *i.e.*, the length of the shortest path from v to a vertex of S. In order to prove Theorem 1.2, the following property on a cycle-spliced bipartite graph with $\eta(G) = c(G) + 1$ is needed.

Lemma 3.1. *Let G be a cycle-spliced bipartite graph with $c(G)$ cycles. If $\eta(G) = c(G)+1$, then $\eta(G-x) = \eta(G) - 1$ for any $x \in V(G)$.*

Proof: We proceed by induction on $c(G)$ to prove $\eta(G - x) = \eta(G) - 1$ for any $x \in V(G)$. If $c(G) = 1$, since $\eta(G) = c(G) + 1 = 2$, then G is a cycle having length divisible by 4. It follows from Lemma 2.3 that $\eta(G - x) = \eta(G) - 1$ for any $x \in V(G)$, as required. Assume the assertion holds for cycle-spliced bipartite graphs with m cycles and let G have $m + 1$ cycles. Theorem 1.1(ii) implies that all cycles in G have length a multiple of 4. Let C be a pendant cycle of G with root u and $H = G - C + u$. Since H is a subgraph of G, H is a cycle-spliced bipartite graph with consists of cycles having length divisible by 4. By Theorem 1.1(ii), we have $\eta(H) = c(H)+1$. Note that $c(H) = m$. Then by the induction hypothesis, we have $\eta(H-v) = \eta(H)-1$ for any $v \in V(H)$. Let x be an arbitrary vertex in G. We now consider the following two cases according to the position of x in G.

Case 1. x does not lie on C.
In this case, C is also a pendant cycle of $G - x$ with root u. Let $H - x = (G - x) - C + u$. Then Corollary 2.5(i) implies that $\eta(G - x) = \eta(H - x) + 1$. Hence, $\eta(G - x) = (\eta(H) - 1) + 1 = \eta(G) - 1$, as desired.

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Case 2. x lies on C.
If $d_G(x,u)$ is even (possibly zero), applying pendant K_2 deletions on $G-x$, we have $\eta(G-x) = \eta(H-u)+1 = (\eta(H)-1)+1 = \eta(G)-1$; if $d_G(x,u)$ is odd, applying pendant K_2 deletions on $G-x$, we have $\eta(G-x) = \eta(H) = \eta(G)-1$, which completes the proof. □

Proof of Theorem 1.2: We proceed by induction on $c(G)$ to prove $\eta(G) \neq c(G)$. If $c(G)=1$, then G is a cycle. Clearly, G is an even cycle since G is bipartite. It follows from Lemma 2.3 that $\eta(G) \neq c(G)$, as required. Assume the assertion holds for cycle-spliced bipartite graphs with m cycles and let G have $m+1$ cycles. Let C be a pendant cycle of G with root u and $H = G-C+u$. Note that H is a cycle-spliced bipartite graph and $c(H)=m$. Then by the induction hypothesis, we have $\eta(H) \neq c(H)$. Thus Theorem 1.1(i) implies that $\eta(H) = c(H)+1$ or $\eta(H) \leq c(H)-1$. We now consider the following two cases.

Case 1. $\eta(H) = c(H)+1$.
Recall that C is an even cycle of G. If C has length a multiple of 4, by Corollary 2.5(i), then $\eta(G) = \eta(H)+1 = (c(H)+1)+1 = c(G)+1$. If C has length congruent to $2(mod\ 4)$, by Corollary 2.5(ii), then $\eta(G) = \eta(G-C) = \eta(H-u)$. Note that H is a cycle-spliced bipartite graphs with $\eta(H) = c(H)+1$, we know from Lemma 3.1 that $\eta(H-u) = \eta(H)-1$. Hence, $\eta(G) = \eta(H)-1 = (c(H)+1)-1 = c(G)-1$.

Case 2. $\eta(H) \leq c(H)-1$.
Let x be a vertex in C which adjacent with u. By Lemma 2.1, we have $\eta(G) \leq \eta(G-x)+1$. Applying pendant K_2 deletions on $G-x$, we have $\eta(G-x) = \eta(H)$. Hence, $\eta(G) \leq \eta(H)+1 \leq (c(H)-1)+1 = c(G)-1$.

By the above arguments, we see that $\eta(G) \neq c(G)$, which completes the proof of Theorem 1.2. □

4. Properties on cycle-spliced bipartite graphs G with $\eta(G) = c(G)-1$. In this section, we present some properties on cycle-spliced bipartite graphs G with $\eta(G) = c(G)-1$.

LEMMA 4.1. *Let G be a graph obtained from two cycle-spliced bipartite graphs G_1 and G_2 by identifying the unique common vertex u. If $\eta(G_1) \leq c(G_1)-k$ and $\eta(G_2) \leq c(G_2)-1$, then $\eta(G) \leq c(G)-k$.*

Proof: Firstly, Lemma 2.1 implies that $\eta(G_1)-1 \leq \eta(G_1-u) \leq \eta(G_1)+1$. We now consider the following three cases.

Case 1. $\eta(G_1-u) = \eta(G_1)+1$.
Clearly, G_1 is a pendant subgraph of G with root u. By Lemma 2.4(i), we have

$$\begin{aligned}\eta(G) &= \eta(G_1)+\eta(G-G_1)\\ &= \eta(G_1)+\eta(G_2-u)\\ &\leq \eta(G_1)+(\eta(G_2)+1)\\ &\leq (c(G_1)-k)+(c(G_2)-1)+1 = c(G)-k.\end{aligned}$$

Case 2. $\eta(G_1-u) = \eta(G_1)$.
Clearly, G_2 is also a pendant subgraph of G with root u. If $\eta(G_2-u) = \eta(G_2)+1$, then by Lemma 2.4(i), we have

$$\begin{aligned}\eta(G) &= \eta(G_2)+\eta(G-G_2)\\ &= \eta(G_2)+\eta(G_1-u)\\ &= \eta(G_2)+\eta(G_1)\\ &\leq (c(G_2)-1)+(c(G_1)-k) < c(G)-k;\end{aligned}$$

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if $\eta(G_2-u)\le\eta(G_2)$, then Lemma 2.1 implies that

$$\begin{aligned}\eta(G) &\le \eta(G-u)+1\\ &= \eta(G_1-u)+\eta(G_2-u)+1\\ &\le \eta(G_1)+\eta(G_2)+1\\ &\le (c(G_1)-k)+(c(G_2)-1)+1 = c(G)-k.\end{aligned}$$

Case 3. $\eta(G_1-u)=\eta(G_1)-1$.
Note that G_1 is a pendant subgraph of G with root u. Then Lemma 2.4(ii) implies that

$$\begin{aligned}\eta(G) &= \eta(G_1)+\eta(G_2)-1\\ &\le (c(G_1)-k)+(c(G_2)-1)-1 < c(G)-k,\end{aligned}$$

as desired. This completes the proof. □

In particular, for $k=2$ in Lemma 4.1, we immediately obtain the following corollary.

Corollary 4.2. *Let G be a graph obtained from two cycle-spliced bipartite graphs G_1 and G_2 by identifying the unique common vertex u. If $\eta(G_1)\le c(G_1)-2$ and $\eta(G_2)\le c(G_2)-1$, then $\eta(G)\le c(G)-2$.*

Lemma 4.3. *Let G be a graph obtained from two cycle-spliced bipartite graphs G_1 and G_2 by identifying the unique common vertex u. If $\eta(G)=c(G)-1$, then we have the following two items.*

(i) If $\eta(G_1)=c(G_1)+1$, then $\eta(G_2)=c(G_2)-1$;
(ii) if $\eta(G_i)\le c(G_i)-1$ for $i=1,2$, then $\eta(G_i)=c(G_i)-1$ for $i=1,2$.

Proof: (i) Firstly, Lemma 3.1 implies that $\eta(G_1-u)=\eta(G_1)-1$ since $\eta(G_1)=c(G_1)+1$. Moreover, by Lemma 2.4(ii), we have

$$\eta(G)=\eta(G_1)+\eta(G_2)-1 \text{ and } c(G)-1=c(G_1)+1+\eta(G_2)-1.$$

It follows that $\eta(G_2)=c(G_2)-1$ since $\eta(G)=c(G)-1$.

(ii) Since $\eta(G)=c(G)-1$ and $\eta(G_i)\le c(G_i)-1$ for $i=1,2$, by Corollary 4.2, we have $\eta(G_i)=c(G_i)-1$ for $i=1,2$. □

Lemma 4.4. *Let G be a cycle-spliced bipartite graph with $c(G)$ cycles. If $\eta(G)=c(G)-1$, then $\eta(G-x)\ne\eta(G)$ for any $x\in V(G)$.*

Proof: We proceed by induction on $c(G)$ to prove $\eta(G-x)\ne\eta(G)$ for any $x\in V(G)$. If $c(G)=1$, then G is a cycle of length congruent to $2(mod\ 4)$ since $\eta(G)=c(G)-1=0$. It follows from Lemma 2.3 that $\eta(G-x)=\eta(G)+1\ne\eta(G)$ for any $x\in V(G)$, as required. Assume the assertion holds for cycle-spliced bipartite graphs H with $\eta(H)=c(H)-1$ which have less cycles than G. Now we assume that $c(G)\ge 2$. Hence, G has at least one pendant cycle.

Case 1. There is a pendant cycle C of length a multiple of 4.
Let u be the root of pendant cycle C and $H=G-C+u$. Then Corollary 2.5(i) implies that $\eta(G)=\eta(H)+1$. Moreover, Lemma 4.3(i) implies that H is a cycle-spliced bipartite graph with $\eta(H)=c(H)-1$. As H has one cycle less than that of G, the induction hypothesis implies that $\eta(H-v)\ne\eta(H)$ for any $v\in V(H)$. Let x be an arbitrary vertex in G. We consider the following two subcases according to the position of x in G.

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Subcase 1. x does not lie on C.
Similar with the proof of Case 1 in Lemma 3.1, we have $\eta(G-x) = \eta(H-x)+1 \neq \eta(H)+1 = \eta(G)$.

Subcase 2. x lies on C.
If $d_G(x,u)$ is even (possibly zero), applying pendant K_2 deletions on $G-x$, we have $\eta(G-x) = \eta(H-u)+1 \neq \eta(H)+1 = \eta(G)$; if $d_G(x,u)$ is odd, applying pendant K_2 deletions on $G-x$, we have $\eta(G-x) = \eta(H) = \eta(G)-1 \neq \eta(G)$.

Case 2. All pendant cycles have length congruent to $2(mod\ 4)$.
Let C be a pendant cycle of G with root u and $H = G - C + u$. Note that C is a cycle of length congruent to $2(mod\ 4)$, then we have $\eta(C-u) = \eta(C)+1$. By Corollary 2.5(ii), we have $\eta(G) = \eta(H-u)$. On the other hand, since all pendant cycles of G have length congruent to $2(mod\ 4)$, H contains at least one cycle of length congruent to $2(mod\ 4)$. By Theorem 1.1(i), (ii) and Theorem 1.2, we have $\eta(H) \leq c(H)-1$. Moreover, by Lemma 4.3(ii), we have $\eta(H) = c(H)-1$ since $\eta(C) = c(C)-1$ and $\eta(G) = c(G)-1$. As H has one cycle less than that of G, the induction hypothesis implies that $\eta(H-v) \neq \eta(H)$ for any $v \in V(H)$. Let x be an arbitrary vertex in G. We consider the following two subcases according to the position of x in G.

Subcase 1. x lies on one pendant cycle, say C_i.
Let u_i be the root of pendant cycle C_i. If $d_G(x,u_i)$ is even (possibly zero), applying pendant K_2 deletions on $G-x$, we have $\eta(G-x) = \eta(H-u_i)+1 = \eta(G)+1 \neq \eta(G)$; if $d_G(x,u_i)$ is odd, applying pendant K_2 deletions on $G-x$, we have $\eta(G-x) = \eta(H) \neq \eta(H-u_i) = \eta(G)$.

Subcase 2. x does not lie on any pendant cycle.
Suppose x lies on cycle C'. We can assume that all cut vertices in C' are $u_1, \ldots, u_k$, where $k \geq 2$ since C' is not a pendant cycle. Let G_i be the maximal pendant subgraph of G with root u_i $(i = 1, \ldots, k)$. Then G can be seen as a graph obtained by a cycle C' with G_i attached at u_i $(i = 1, \ldots, k)$, respectively (see Fig. 1). Let $H_i = \eta(G - G_i + u_i)$.

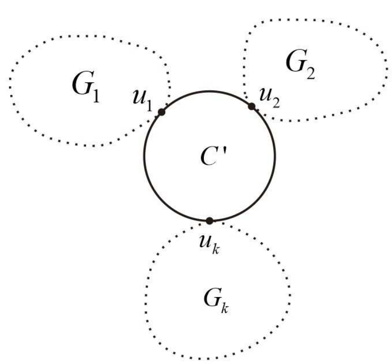

Fig. 1. *Graph G with a cycle C' which is not a pendant cycle.*

Since all pendant cycles of G have length congruent to $2(mod\ 4)$, G_i and H_i both contain at least one cycle of length congruent to $2(mod\ 4)$. By Theorem 1.1(i), (ii) and Theorem 1.2, we have $\eta(G_i) \leq c(G_i)-1$ and $\eta(H_i) \leq c(H_i)-1$. Moreover, by Lemma 4.3(ii), we have $\eta(G_i) = c(G_i)-1$ and $\eta(H_i) = c(H_i)-1$. As G_i has less cycles than that of G, the induction hypothesis implies that $\eta(G_i - u_i) \neq \eta(G_i)$. We contend

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that $\eta(G_i-u_i)=\eta(G_i)+1$. Otherwise, there is a graph G_i such that $\eta(G_i-u_i)=\eta(G_i)-1$. By Lemma 2.4(ii), we have $\eta(G)=\eta(G_i)+\eta(H_i)-1=(c(G_i)-1)+(c(H_i)-1)-1=c(G)-3$, which is a contradiction. Then Lemma 2.4(i) implies that

$$\begin{aligned}\eta(G)&=\eta(G_1)+\cdots+\eta(G_k)+\eta(G-G_1-\cdots-G_k)\\&=\sum_{i=1}^{k}\eta(G_i)+\eta(C'-u_1-\cdots-u_k)\\&=\sum_{i=1}^{k}(c(G_i)-1)+\eta(C'-u_1-\cdots-u_k)\\&=c(G)-1-k+\eta(C'-u_1-\cdots-u_k).\end{aligned}$$

It follows that $\eta(C'-u_1-\cdots-u_k)=k$ since $\eta(G)=c(G)-1$. It means that $d_G(u_i,u_j)$ is even for any $i,j=1,\ldots,k$ and $\eta(G)=\sum_{i=1}^{k}\eta(G_i)+k$.

If $x=u_i$, then we have $\eta(G-x)=\eta(G_i-u_i)+\eta(H_i-u_i)$. In this case, G_j $(j\neq i)$ is also a pendant subgraph of H_i-u_i with root u_j and $\eta(G_j-u_j)=\eta(G_j)+1$. Applying Lemma 2.4(i), we have

$$\begin{aligned}\eta(G-x)&=\eta(G_i-u_i)+\eta(H_i-u_i)\\&=(\eta(G_i)+1)+\sum_{j\neq i}\eta(G_j)+\eta(C'-u_1-\cdots-u_k)\\&=\sum_{i=1}^{k}\eta(G_i)+1+k=\eta(G)+1\neq\eta(G).\end{aligned}$$

Similarly, if $x\neq u_i$, then we have

$$\begin{aligned}\eta(G-x)&=\sum_{i=1}^{k}\eta(G_i)+\eta((C'-x)-u_1-\cdots-u_k)\\&\neq\sum_{i=1}^{k}\eta(G_i)+k=\eta(G).\end{aligned}$$

The inequality holds because $\eta(C'-x-u_1-\cdots-u_k)\neq k$ since $d_G(u_i,u_j)$ is even. $\square$

Lemma 4.5. *Let G be a graph obtained from two cycle-spliced bipartite graphs G_1 and G_2 by identifying the unique common vertex u. Then $\eta(G)=c(G)-1$ if and only if one of the following conditions is satisfied:*

(i) There is one of G_i, say G_1, such that $\eta(G_1)=c(G_1)+1$ and $\eta(G_2)=c(G_2)-1$;
(ii) $\eta(G_i)=c(G_i)-1$ and $\eta(G_i)=\eta(G_i-u)-1$ for $i=1,2$.

Proof: "Only if" part: If there is a graph, say G_1, such that $\eta(G_1)=c(G_1)+1$, then Lemma 4.3(i) implies that $\eta(G_2)=c(G_2)-1$, and (i) holds.

If $\eta(G_i)\neq c(G_i)+1$ for $i=1,2$, then Theorem 1.1(i) and Theorem 1.2 imply that $\eta(G_i)\leq c(G_i)-1$. Moreover, by Lemma 4.3(ii), we have $\eta(G_i)=c(G_i)-1$ for $i=1,2$. From Lemma 4.4, we have $\eta(G_i-u)\neq\eta(G_i)$ since $\eta(G_i)=c(G_i)-1$. We conclude that $\eta(G_i-u)=\eta(G_i)+1$ for $i=1,2$. Otherwise, there is a graph, say G_1, such that $\eta(G_1-u)=\eta(G_1)-1$. Then Lemma 2.4(ii) implies that $\eta(G)=\eta(G_1)+\eta(G_2)-1=(c(G_1)-1)+(c(G_2)-1)-1=c(G)-3$, which is a contradiction. Then (ii) holds.

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"If" part:(i) By Lemma 3.1, $\eta(G_1 - u) = \eta(G_1) - 1$ since $\eta(G_1) = c(G_1) + 1$. Therefore, by Lemma 2.4(ii), we have $\eta(G) = \eta(G_1) + \eta(G_2) - 1 = (c(G_1) + 1) + (c(G_2) - 1) = c(G) - 1$.

(ii) From Lemma 2.4(i), we have $\eta(G) = \eta(G_1) + \eta(G_2 - u) = \eta(G_1) + \eta(G_2) + 1 = (c(G_1) - 1) + (c(G_2) - 1) + 1 = c(G) - 1$. □

LEMMA 4.6. *Let G be a cycle-spliced bipartite graph with $c(G) \geq 2$ and all pendant cycles have length congruent to $2(mod\ 4)$. If $\eta(G) = c(G) - 1$, then*

(i) $\eta(G - u) = \eta(G) + 1$ for any cut vertex u of G;
(ii) $d_G(u, v)$ is even for any two cut vertices u and v in G;
(iii) $\eta(G - v) = \eta(G) + 1$ for $v \in V(G)$ such that the distance between v and any cut vertex is even;
(iv) $\eta(G - w) = \eta(G) - 1$ for $w \in V(G)$ such that the distance between w and any cut vertex is odd.

Proof: (i) Since $c(G) \geq 2$, G has at least one cut vertex. Let u be an arbitrary cut vertex of G. Then G can be seen as a graph obtained from two cycle-spliced bipartite graphs G_1 and G_2 by identifying the unique common vertex u. Since all pendant cycles of G have length congruent to $2(mod\ 4)$, G_i $(i = 1, 2)$ contains at least one cycle of length congruent to $2(mod\ 4)$. By Theorem 1.1(i), (ii) and Theorem 1.2, we have $\eta(G_i) \leq c(G_i) - 1$. Moreover, by Lemma 4.5(ii), we have $\eta(G_i) = c(G_i) - 1$ and $\eta(G_i - u) = \eta(G_i) + 1$ for $i = 1, 2$ since $\eta(G) = c(G) - 1$. In view of Lemma 2.4(i), we have $\eta(G) = \eta(G_1) + \eta(G - G_1) = \eta(G_1) + \eta(G_2 - u) = \eta(G_1) + \eta(G_2) + 1$. Thus, we have $\eta(G - u) = \eta(G_1 - u) + \eta(G_2 - u) = \eta(G_1) + 1 + \eta(G_2) + 1 = \eta(G) + 1$.

(ii) There is nothing to prove when G has only one cut vertex. Suppose G has at least two cut vertices. Let u and v be two arbitrary cut vertices of G. In order to prove that $d_G(u, v)$ is even, we only consider u and v lie on same cycle, say C'. It means that C' is not a pendant cycle. We can assume that all cut vertices in C' are $u_1, \ldots, u_k$ $(k \geq 2)$. Clearly, $u, v \in \{u_1, \ldots, u_k\}$. Let G_i be the maximal pendant subgraph of G with root u_i $(i = 1, \ldots, k)$. Then G can be seen as a graph obtained by a cycle C' with G_i attached at u_i $(i = 1, \ldots, k)$, respectively. By the argument given in the proof of Case 2 of Lemma 4.4, we have $d_G(u_i, u_j)$ $(i, j = 1, \ldots, k)$ is even. It means that $d_G(u, v)$ is even.

(iii) Let v be a vertex such that the distance between v and any cut vertex is even. Without loss of generality, we assume that v and a cut vertex u lie on same cycle C.

Case 1. C is a pendant cycle of G.
Clearly, u is the unique cut vertex in C. $\eta(C - u) = \eta(C) + 1$ since C has length congruent to $2(mod\ 4)$. It follows from Corollary 2.5(ii), $\eta(G) = \eta(C) + \eta(G - C) = \eta(G - C)$. Applying pendant K_2 deletions on $G - v$, we have $\eta(G - v) = 1 + \eta(G - C) = \eta(G) + 1$.

Case 2. C is not a pendant cycle of G.
In this case, there are at least two cut vertices in C. We can assume that all cut vertices in C are $u_1, \ldots, u_k$ $(k \geq 2)$. Let G_i be the maximal pendant subgraph of G with root u_i $(i = 1, \ldots, k)$. Then G can be seen as a graph obtained by a cycle C with G_i attached at u_i $(i = 1, \ldots, k)$, respectively. By the argument given in the proof of Case 2 of Lemma 4.4, we have $\eta(G) = \sum_{i=1}^{k} \eta(G_i) + k$. Since the distance between any two cut vertices is even, applying pendant K_2 deletions on $G - v$, we have $\eta(G - v) = 1 + \sum_{i=1}^{k} \eta(G_i) + k = \eta(G) + 1$.

(iv) Let w be a vertex such that the distance between w and any cut vertex is odd. Then Lemma 4.4 implies that $\eta(G - w) \neq \eta(G)$. To establish (iv), assume to the contrary that $\eta(G - w) = \eta(G) + 1$. Let G' be a graph obtained from G with a cycle C of length congruent to $2(mod\ 4)$ attached at w. By Lemma 4.5(ii), we have $\eta(G') = c(G') - 1$. Thus, G' is a cycle-spliced bipartite graph with $\eta(G') = c(G') - 1$ and

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all pendant cycles have length congruent to $2(mod\ 4)$. Note that w and u are cut vertices of G'. Then Lemma 4.6(ii) implies that $d_{G'}(w,u)$ is even. Also $d_G(w,u)$ is even, which contradicts the assumption. Then $\eta(G-w)=\eta(G)-1$, as desired. $\square$

REMARK 4.7. The condition that all pendant cycles of G have length congruent to $2(mod\ 4)$ in Lemma 4.6 is necessary. See Fig. 2, where the graph G is obtained by attaching C_4 and C_6 at vertices u_1 and u_2 of C_4, respectively. It is easy to see that $\eta(G)=c(G)-1$. But for u_1, we have $\eta(G-u_1)=\eta(G)-1$ and the distance between the two cut vertices (u_1 and u_2) is odd.

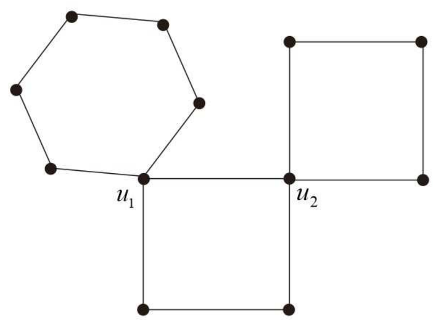

FIG. 2. *An example of a cycle-spliced bipartite graph with $\eta(G)=c(G)-1$.*

LEMMA 4.8. *Let G be a cycle-spliced bipartite graph in which every nonpendant cycle has exactly two cut vertices (see Fig. 3). If exactly one pendant cycle has length congruent to $2(mod\ 4)$, the other cycles all have length a multiple of 4 and the distance between any two cut vertices of G is even, then*

(i) $\eta(G)=c(G)-1$;
(ii) $\eta(G-u)=\eta(G)+1$ *for any cut vertex u of G;*
(iii) $\eta(G-v)=\eta(G)+1$ *for $v\in V(G)$ such that the distance between v and any cut vertex of G is even.*

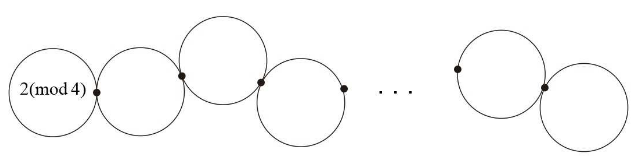

FIG. 3. *A cycle-spliced bipartite graph G in which every nonpendant cycle has exactly two cut vertices.*

Proof: (i) Let C be the pendant cycle of G with length congruent to 2(mod 4). Clearly, $\eta(C)=0=c(C)-1$. Let u be the root of C and $H=G-C+u$. Then G can be seen as a graph obtained from C and a cycle-spliced bipartite graph H by identifying an unique common vertex u. Theorem 1.1(ii) implies that $\eta(H)=c(H)+1$ since all cycles in H have length a multiple of 4. Then by Lemma 4.5(i), we immediately have $\eta(G)=c(G)-1$.

(ii) We proceed by induction on $c(G)$ to prove $\eta(G-u)=\eta(G)+1$ for any cut vertex u of G. If $c(G)=2$, then G is a graph obtained from two cycles C_1 and C_2 by identifying an unique common vertex u. Without loss of generality, we assume that C_1 has length congruent to $2(mod\ 4)$, C_2 has length a multiple

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of 4. Clearly, we have $\eta(G-u)=\eta(G)+1$. Assume the assertion holds for cycle-spliced bipartite graphs satisfied above assumptions with less cycles than G. Now we assume that $c(G)\geq 3$. Let u be an arbitrary cut vertex of G. Then G can be seen as a graph obtained from two cycle-spliced bipartite graphs G_1 and G_2 by identifying the unique common vertex u. Recall that G has exactly one pendant cycle with length congruent to $2(mod\ 4)$, the other cycles all have length a multiple of 4. Without loss of generality, we can assume that G_1 contains the pendant cycle of length congruent to $2(mod\ 4)$. Note that G_1 has less cycles than that of G and the distance between any two cut vertices of G_1 is even. Then by the induction hypothesis, we have $\eta(G_1-x)=\eta(G_1)+1$ for any cut vertex x in G_1. Let x' be the cut vertex of G_1 which lies on same cycle with u. Then $d_{G_1}(x',u)$ is even since x' and u are cut vertices of G. Applying pendant K_2 deletions on G_1-u, we have $\eta(G_1-u)=\eta(G_1-x')=\eta(G_1)+1$. Moreover, Theorem 1.1(ii) implies that $\eta(G_2)=c(G_2)+1$ since every cycle in G_2 has length a multiple of 4. By Lemma 3.1, we have $\eta(G_2-u)=\eta(G_2)-1$. Then Lemma 2.4(ii) implies that $\eta(G)=\eta(G_1)+\eta(G_2)-1$. Thus, we have $\eta(G-u)=\eta(G_1-u)+\eta(G_2-u)=\eta(G_1)+1+\eta(G_2)-1=\eta(G)+1$.

(iii) Let v be a vertex such that the distance between v and any cut vertex is even. Without loss of generality, we assume that v and the cut vertex, say u, lie on same cycle C.

Case 1. C is a pendant cycle of G.
Since C is even cycle, applying pendant K_2 deletions on $G-v$, we have $\eta(G-v)=\eta(G-u)=\eta(G)+1$.

Case 2. C is not a pendant cycle of G.
In this case, there is exactly two cut vertices in C, say u_1 and u_2. Without loss of generality, we assume that $u_1=u$. Let G_i be the maximal pendant subgraph of G with root u_i $(i=1,2)$. Then G can be seen as a graph obtained by a cycle C with G_i attached at u_i $(i=1,2)$, respectively. Recall that G has exactly one pendant cycle with length congruent to $2(mod\ 4)$, the other cycles all have length a multiple of 4. Without loss of generality, we can assume that G_1 contains the pendant cycle of length congruent to $2(mod\ 4)$. Note that G_1 is a cycle-spliced bipartite graph and the distance between any two cut vertices of G_1 is even. Then Lemma 4.8(ii) implies that $\eta(G_1-x)=\eta(G_1)+1$ for any cut vertex x in G_1. Let x' be the cut vertex of G_1 which lies on same cycle with u_1. Then $d_{G_1}(x',u_1)$ is even since x' and u_1 are cut vertices of G. Applying pendant K_2 deletions on G_1-u_1, we have $\eta(G_1-u_1)=\eta(G_1-x')=\eta(G_1)+1$. Moreover, Theorem 1.1(ii) implies that $\eta(G_2)=c(G_2)+1$ since all cycles in G_2 are length a multiple of 4. By Lemma 3.1, we have $\eta(G_2-u_2)=\eta(G_2)-1$. Then Lemma 2.4(ii) implies that $\eta(G)=\eta(G_1+C)+\eta(G_2)-1=\eta(G_1)+1+\eta(G_2)-1$. Since $d_G(v,u)$ and $d_G(u,u_2)$ are even, applying pendant K_2 deletions on $G-v$, we have $\eta(G-v)=1+\eta(G_1-u_1)+\eta(G_2-u_2)=1+(\eta(G_1)+1)+(\eta(G_2)-1)=\eta(G)+1$. □

5. Proofs of Theorems 1.3 and 1.4.

We now give the proofs of Theorems 1.3 and 1.4, respectively.

Proof of Theorem 1.3: "If" part: We proceed by induction on $c(G)$ to prove $\eta(G)=c(G)-1$. If $c(G)=2$, then G is a graph obtained from two cycles C_1 and C_2 by identifying an unique common vertex u. And C_1 and C_2 both have length congruent to $2(mod\ 4)$ since they are pendant cycles. It is easy to calculate that $\eta(G)=c(G)-1$. Assume the assertion holds for cycle-spliced bipartite graphs satisfying above assumptions with less cycles than G. Now we assume that $c(G)\geq 3$. Let C be an arbitrary pendant cycle of G with root u. Then $G-C+u$ has at most one pendant cycle with length a multiple of 4. If all pendant cycles of $G-C+u$ have length congruent to $2(mod\ 4)$, then let $G_0=C$ and $H=G-G_0+u$. Otherwise, $G-C+u$ has exactly one pendant cycle with length a multiple of 4, say C_1 with root u_1. Let $G_1=C+C_1$, then $G-G_1+u_1$ has at most one pendant cycle with length a multiple of 4. If all pendant cycles of $G-G_1+u_1$ have length congruent to $2(mod\ 4)$ or $G-G_1+u_1$ is a cycle with length congruent

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to $2(mod\ 4)$, then let $H = G - G_1 + u$. With similar way, we must obtain $G_k = C + C_1 + \cdots + C_k$ and $H = G - G_k + u_k$. It is easy to see that G_k is a cycle-spliced bipartite graph in which each non-pendant cycle of G_k has exactly two cut vertices, only one pendant cycle of G_k, say C, has length congruent to $2(mod\ 4)$ and the other cycles of G_k all have length a multiple of 4. Moreover, the distance between any two cut vertices of G_k is even since G_k is induced subgraph of G. Then Lemma 4.8(i) implies that $\eta(G_k) = c(G_k) - 1$. Since u_k is a vertex such that the distance between u_k and the cut vertex u_{k-1} of G_k is even, by Lemma 4.8(iii), we have $\eta(G_k - u_k) = \eta(G_k) + 1$. If $c(H) \geq 2$, then $H = G - G_k + u_k$ is a cycle-spliced bipartite graph with $c(H) = c(G) - (k + 1) < c(G)$ and all pendant cycles of H have length congruent to $2(mod\ 4)$. The distance between any two cut vertices of H is even since H is induced subgraph of G. The induction hypothesis implies that $\eta(H) = c(H) - 1$. Since u_k is a vertex such that the distance between u_k and any cut vertex of H is even, by Lemma 4.6(iii), we have $\eta(H - u_k) = \eta(H) + 1$. Then Lemma 4.5(ii) implies that $\eta(G) = c(G) - 1$. If $c(H) = 1$, then $H = G - G_k + u_k$ is a cycle with length congruent to $2(mod\ 4)$. Clearly, $\eta(H) = c(H) - 1$ and $\eta(H - u_k) = \eta(H) + 1$. By Lemma 4.5(ii), we have $\eta(G) = c(G) - 1$.

"Only if" part: Follows from Lemma 4.6(ii). □

Proof of Theorem 1.4: "If" part: When $c(G) = 1$, $G = H$ is a cycle with length congruent to $2(mod\ 4)$ with attaching no cycles with length a multiple of 4 on arbitrary vertex. Clearly, $\eta(G) = c(G) - 1$. When $c(G) \geq 2$, if all pendant cycles of G have length congruent to $2(mod\ 4)$, then $G = H$ is a graph obtained from H with attaching no cycles with length a multiple of 4 on arbitrary vertex. Clearly, $\eta(G) = \eta(H) = c(H) - 1 = c(G) - 1$. Otherwise, G must contain a pendant cycle with length a multiple 4. By contracting all pendant cycles with length a multiple of 4 into a vertex, finally we have the graph H which is a cycle with length congruent to $2(mod\ 4)$ or a graph with all pendant cycles have length congruent to $2(mod\ 4)$. Then by Corollary 2.5(i), we have $\eta(G) = \eta(C) + (c(G) - 1) = c(G) - 1$ or $\eta(G) = \eta(H) + (c(G) - c(H)) = (c(H) - 1) + (c(G) - c(H)) = c(G) - 1$.

"Only if" part: When $c(G) = 1$, G is a cycle with length congruent to $2(mod\ 4)$ since $\eta(G) = c(G) - 1$. Then G is a cycle with length congruent to $2(mod\ 4)$ with attaching no cycles with length a multiple of 4 on arbitrary vertex. When $c(G) \geq 2$, if all pendant cycles of G have length congruent to $2(mod\ 4)$, then G is a graph obtained from $G = H$ with attaching no cycles with length a multiple of 4 on arbitrary vertex. Otherwise, G must contain a pendant cycle with length a multiple 4. By contracting all pendant cycles with length a multiple of 4 into a vertex, finally we have the graph H which is a cycle with length congruent to $2(mod\ 4)$ or a graph with all pendant cycles have length congruent to $2(mod\ 4)$. Then by Lemma 4.5(i), we have $\eta(H) = c(H) - 1$. Thus, G is a graph obtained from a cycle-spliced bipartite graph H with $\eta(H) = c(H) - 1$ and all pendant cycles have length congruent to $2(mod\ 4)$ or a cycle with length congruent to $2(mod\ 4)$ by attaching $c(G) - c(H)$ cycles with length a multiple of 4 on arbitrary vertex. □

Concluding remarks: In this paper, we only considered the nullity of a cycle-spliced graph whose cycles are all even. In [12], the case when their cycles are all odd was considered and the problem of identifying the nonsingular cycle-spliced graphs was only partly settled. The nullity of a cycle-spliced graph whose cycles are of any length demands further study.

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