Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 199-213, April 2023.

REFINEMENT OF VON NEUMANN-TYPE INEQUALITIES ON PRODUCT EATON TRIPLES*

MAREK NIEZGODA†

Abstract. In this paper, a von Neumann-type inequality is studied on an Eaton triple (V, G, D), where V is a real inner product space, G is a compact subgroup of the orthogonal group $O(V)$, and $D \subset V$ is a closed convex cone. By using an inner structure of an Eaton triple, a refinement of this inequality is shown. In the special case $G = O(V)$, a refinement of the Cauchy-Schwarz inequality is obtained.

Key words. Inner product space, Eaton triple, Normal map, Von Neumann-type inequality, Cauchy-Schwarz inequality.

AMS subject classifications. 15A30, 15A39, 15A18, 15B57.

1. Introduction. Throughout the paper, unless otherwise indicated, $(V, \langle\cdot,\cdot\rangle)$ is a finite-dimensional real inner product space, G is a compact subgroup of the orthogonal group $O(V)$ acting on V, and $D \subset V$ is a closed convex cone.

The group G induces the preorder $\prec_G$ on V, called *G-majorization*, as follows. For $x, y \in V$,

$$y \prec_G x \quad \text{iff} \quad y \in \operatorname{conv} Gx,$$

where $\operatorname{conv} Gx$ denotes the convex hull of the set $Gx = \{gx : g \in G\}$.

We say that the structure (V, G, D) is an *Eaton triple*, if

(A1) for each $x \in V$ there exist $g \in G$ and $y \in D$ such that $x = gy$,
(A2) $\langle x, gy\rangle \le \langle x, y\rangle$ for all $x, y \in D$ and $g \in G$.

The interested reader is referred to [3]–[10], [14]–[23], [25]–[29] for applications of Eaton triples. A generalization of the notion of Eaton triple is the concept of Fan-Theobald-von Neumann (FTvN) systems (see [10]).

For an Eaton triple (V, G, D) and for each $x \in V$, the set $D \cap Gx$ has the unique element denoted by $x_\downarrow$ (see [4, p. 15)], [17, p. 14]). This allows to define the *normal map* $(\cdot)_\downarrow$ of (V, G, D) by $V \ni x \mapsto x_\downarrow \in D$ [12].

It is readily seen that for each $x \in V$,

$$x = gx_\downarrow \quad \text{for some } g \in G. \tag{1.1}$$

Evidently, $0_\downarrow = 0$, because $0 \in D \cap (G\,0)$.

Also, the normal map $(\cdot)_\downarrow$ is G-invariant in the sense that

$$(gx)_\downarrow = x_\downarrow \quad \text{for any } x \in V \text{ and } g \in G. \tag{1.2}$$

*Received by the editors on September 4, 2022. Accepted for publication on April 10, 2023. Handling Editor: Tin-Yau Tam.
† Institute of Mathematics, Pedagogical University of Cracow, Podchorążych 2, 30-084 Kraków, Poland (bniezgoda@wp.pl).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 199-213, April 2023.

Because any two vectors $x, y \in V$ have the representations $x = g_1 x_\downarrow$ and $y = g_2 y_\downarrow$ for some $g_1, g_2 \in G \subset O(V)$, the inequality in condition **(A2)** can be restated equivalently as

$$\langle x, y\rangle \le \langle x_\downarrow, y_\downarrow\rangle \quad \text{for all } x, y \in V. \tag{1.3}$$

We shall call (1.3) *von Neumann-type inequality* for Eaton triple (V, G, D) with normal map $(\cdot)_\downarrow$.

In this paper, our goal is to refine inequality (1.3) and, in particular, to refine the standard Cauchy-Schwarz inequality in the spirit of results of Hovenier [11] and Abramovich, Mond and Pečarić [1].

We complete this section with some important examples of Eaton triples.

Example 1. Let $V = \mathbb{M}_n$, the space of all $n \times n$ complex matrices with the trace inner product: $\langle x, y\rangle = \operatorname{Re}\operatorname{tr} xy^*$ for $x, y \in \mathbb{M}_n$. Let G be the group of all linear operators of the form $x \to u_1 x u_2$ with $x \in \mathbb{M}_n$ and u_1, u_2 running over the unitary group $\mathbb{U}_n$. Put

$$D = \{\operatorname{diag}(d_1, \dots, d_n) : d_1 \ge \dots \ge d_n \ge 0\},$$

where for an n-tuple $d = (d_1, \dots, d_n) \in \mathbb{R}^n$, the symbol $\operatorname{diag} d$ denotes the diagonal matrix with the entries of d on the main diagonal.

Then, (V, G, D) is an Eaton triple, because condition **(A1)** is the Singular Value Decomposition Theorem for complex matrices and **(A2)** takes the form of the following *von Neumann trace inequality* (cf. [4, 6]):

$$\operatorname{Re}\operatorname{tr} x u_1 y u_2 \le \sum_{i=1}^{n} \sigma_i(x)\sigma_i(y) \quad \text{for } x, y \in \mathbb{M}_n \text{ and } u_1, u_2 \in \mathbb{U}_n,$$

where $\sigma(z) = (\sigma_1(z), \dots, \sigma_n(z))$ stands for the n-tuple of the singular values $\sigma_1(z) \ge \dots \ge \sigma_n(z) \ge 0$ of a complex matrix $z \in \mathbb{M}_n$ arranged in the descending order.

In this situation, the normal map is given by

$$x_\downarrow = \operatorname{diag}\sigma(x) \quad \text{for } x \in \mathbb{M}_n.$$

Example 2. Take V to be the space $\mathbb{H}_n$ of all $n \times n$ Hermitian matrices with the trace inner product: $\langle x, y\rangle = \operatorname{tr} xy$ for $x, y \in \mathbb{H}_n$. Let G be the group of all linear operators of the form $x \to uxu^*$ with $x \in \mathbb{H}_n$ and u running over the unitary group $\mathbb{U}_n$. Additionally, let

$$D = \{\operatorname{diag}(d_1, \dots, d_n) : d_1 \ge \dots \ge d_n\}.$$

Then, (V, G, D) is an Eaton triple. In fact, statement **(A1)** is the Spectral Theorem for Hermitian matrices, while **(A2)** reduces to Fan-Theobald's inequality (cf. [4, 6]):

$$\operatorname{tr} xuyu^* \le \sum_{i=1}^{n} \lambda_i(x)\lambda_i(y) \quad \text{for } x, y \in \mathbb{H}_n \text{ and } u \in \mathbb{U}_n,$$

where $\lambda(z) = (\lambda_1(z), \dots \lambda_n(z))$ stands for the n-tuple of the eigenvalues $\lambda_1(z) \ge \dots \ge \lambda_n(z)$ of a Hermitian matrix $z \in \mathbb{H}_n$ arranged in the descending order.

Here, the normal map is of the form

$$x_\downarrow = \operatorname{diag}\lambda(x) \quad \text{for } x \in \mathbb{H}_n.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 199-213, April 2023.

EXAMPLE 3. Let $(V, \langle\cdot,\cdot\rangle)$ be a real inner product space with norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$. Let $e \in V$ be a nonzero vector of norm one. We put $G = O(V)$, the full orthogonal group acting on V, and $D = \operatorname{cone} e$, the convex cone spanned by the vector e.

Then, (V, G, D) is an Eaton triple. In fact, condition **(A1)** holds, because each vector $x \in V$ has the representation $x = g_0(\|x\|e)$ for some orthogonal operator g_0 on V. Moreover, condition **(A2)** is fulfilled by the Cauchy-Schwarz inequality: for any $x, y \in V$ and $g \in G = O(V)$,

$$\langle x, gy\rangle \leq \|x\|\|gy\| = \|x\|\|y\| = \langle \|x\|e, \|y\|e\rangle = \langle x_\downarrow, y_\downarrow\rangle,$$

where $\langle e, e\rangle = \|e\|^2 = 1$, and $z_\downarrow = \|z\|e$ for $z \in V$ defines the normal map of (V, G, D).

EXAMPLE 4. Another important example of Eaton triple is related to the notion of (classical) *majorization* on $\mathbb{R}^n$ [13].

An n-tuple $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ is said to be *majorized* by an n-tuple $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ (which is written as $y \prec x$), if

$$\sum_{i=1}^{k} y_{[i]} \leq \sum_{i=1}^{k} x_{[i]} \quad \text{for all } k = 1, 2, \ldots, n, \quad \text{and} \quad \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i,$$

where $z_{[i]}$ stands for the ith largest entry of an n-tuple $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$ (see [13, p. 8], [2, p. 28])).

In this situation ,$(\mathbb{R}^n, \mathbb{P}_n, D)$ is an Eaton triple, where $\mathbb{P}_n$ is the group of all $n \times n$ permutation matrices and $D = \{(d_1, d_2, \ldots, d_n) \in \mathbb{R}^n : d_1 \geq d_2 \geq \ldots \geq d_n\}$. Here, the normal map is defined by $x = (x_1, x_2, \ldots, x_n) \mapsto x_\downarrow = (x_{[1]}, x_{[2]}, \ldots, x_{[n]})$ for $x \in \mathbb{R}^n$.

In the next section, for a given Eaton triple (V, G, D) we make use an inner structure of the linear space V in order to establish a refinement of the related inequality $\langle x, y\rangle \leq \langle x_\downarrow, y_\downarrow\rangle$ for $x, y \in V$. In doing so, we apply further Eaton triples defined on some subspaces of V and connected by some special relationships.

2. Refining von Neumann-type inequality. We begin this section with some properties of Eaton triples that will be engaged in refining von Neumann-type inequality $\langle x, y\rangle \leq \langle x_\downarrow, y_\downarrow\rangle$.

Firstly, it should be noticed that a product of Eaton triples can be made into another Eaton triple [24]. To see this, assume that $(V, \langle\cdot,\cdot\rangle)$ is a real (finite-dimensional) inner product space and V admits decomposition

$$V = V_1 + \cdots + V_m,$$

with mutually orthogonal nonzero subspaces V_i of V, $i = 1, \ldots, m$. Let (V_i, G_i, D_i) be an Eaton triple for some closed group $G_i \subset O(V_i)$ and closed convex cone $D_i \subset V_i$. We introduce

$$D := D_1 + \cdots + D_m,$$

and take

$$\widetilde{G} := G_1 \times \cdots \times G_m,$$

to be the group of all linear operators $(g_1, \ldots, g_m)$ with $g_i \in G_i$, $i = 1, \ldots, m$, acting on $V = V_1 + \cdots + V_m$ by

$$(g_1, \ldots, g_m)(x_1 + \cdots + x_m) := g_1x_1 + \cdots + g_mx_m \quad \text{for } x_i \in V_i.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 199-213, April 2023.

It is not hard to check that $\widetilde{G} \subset O(V)$ and

$$(V, \widetilde{G}, D) \quad \text{is an Eaton triple.} \tag{2.4}$$

We shall call $(V, \widetilde{G}, D)$ the *product* of Eaton triples (V_i, G_i, D_i), $i = 1, \ldots, m$ and write

$$(V, \widetilde{G}, D) = (V_1, G_1, D_1) \oplus \ldots \oplus (V_m, G_m, D_m).$$

Example 5. Referring to Example 2, we consider the following real linear spaces of $2n \times 2n$ matrices:

$$V_1 = \left\{ \begin{pmatrix} x_1 & 0 \\ 0 & 0 \end{pmatrix} : x_1 \in \mathbb{H}_n \right\} \quad \text{and} \quad V_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & x_2 \end{pmatrix} : x_2 \in \mathbb{H}_n \right\},$$

$$V = V_1 + V_2 = \left\{ \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{H}_n \right\}.$$

We equip V, V_1, and V_2 with the standard trace inner product of $\mathbb{H}_{2n}$.

We also introduce closed convex cones:

$$D_1 = \left\{ \begin{pmatrix} \operatorname{diag} d_1 & 0 \\ 0 & 0 \end{pmatrix} : d_1 \in \mathbb{R}^n_{\downarrow} \right\} \quad \text{and} \quad D_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & \operatorname{diag} d_2 \end{pmatrix} : d_2 \in \mathbb{R}^n_{\downarrow} \right\},$$

$$D = D_1 + D_2 = \left\{ \begin{pmatrix} \operatorname{diag} d_1 & 0 \\ 0 & \operatorname{diag} d_2 \end{pmatrix} : d_1, d_2 \in \mathbb{R}^n_{\downarrow} \right\},$$

where $\mathbb{R}^n_{\downarrow}$ denotes the set of all real n-tuples decreasingly ordered.

Additionally, we employ the following groups of unitary similarity operations:

$$G_1 = \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & I_n \end{pmatrix} (\cdot) \begin{pmatrix} u_1 & 0 \\ 0 & I_n \end{pmatrix}^* : u_1 \in \mathbb{U}_n \right\} \quad \text{and} \quad G_2 = \left\{ \begin{pmatrix} I_n & 0 \\ 0 & u_2 \end{pmatrix} (\cdot) \begin{pmatrix} I_n & 0 \\ 0 & u_2 \end{pmatrix}^* : u_2 \in \mathbb{U}_n \right\},$$

$$G = \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} (\cdot) \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}^* : u_1 \in \mathbb{U}_n \right\},$$

where I_n stands for the identity $n \times n$ matrix.

It easily seen that

$$\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}^* = \begin{pmatrix} u_1 x_1 u_1^* & 0 \\ 0 & u_2 x_2 u_2^* \end{pmatrix}$$

$$= \begin{pmatrix} u_1 & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} x_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ 0 & I_n \end{pmatrix}^* + \begin{pmatrix} I_n & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & u_2 \end{pmatrix}^*.$$

So, $G = G_1 \times G_2$ is the product group of G_1 and G_2. Therefore, (V, G, D) is the product of Eaton triples (V_1, G_1, D_1) and (V_2, G_2, D_2). This conclusion completes Example 5.

On the other hand, an Eaton triple (V, G, D) induces some further Eaton triples on its G-invariant subspaces, as shown in the below lemma.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 199-213, April 2023.

Remind that a subspace V_0 in V is said to be *G-invariant*, if $gx \in V_0$ for all $x \in V_0$ and $g \in G$.

In the sequel, the symbol $G_{|V_i}$ stands for the set $\{g_{|V_i} : g \in G\}$, where $g_{|V_i}$ denotes the restriction of a map g to a linear subspace V_i.

The following result is a slight extension of [24, Theorem 2.1].

Lemma 6 (Cf. [24, Theorem 2.1]). *Let (V, G, D) be an Eaton triple and*

$$V = V_1 + \ldots + V_m,$$

with G-invariant mutually orthogonal nonzero subspaces V_i, and $G_i = G_{|V_i}$ and $D_i = D \cap V_i$, $i = 1, \ldots, m$.

Then,

$$(V_i, G_i, D_i) \text{ is an Eaton triple, } i = 1, \ldots, m, \tag{2.5}$$

$$D = D_1 + \ldots + D_m, \tag{2.6}$$

$$G_{|W} = (G_1 \times \ldots \times G_m)_{|W}, \quad \text{where } W = \operatorname{span} D, \tag{2.7}$$

$$\text{for each } x \in V,\ Gx = (G_1 \times \ldots \times G_m)x. \tag{2.8}$$

Proof. The assertions (2.5), (2.6), and (2.7) follow directly from [24, Theorem 2.1].

In order to prove (2.8), fix arbitrarily $x \in V$. Since (V, G, D) is an Eaton triple, we get $x = g_0 x_\downarrow$ for some $g_0 \in G$ and $x_\downarrow \in D$ by condition **(A1)**. Denote $D_0 = g_0 D$. Then, $x \in D_0$ and (V, G, D_0) is an Eaton triple, too. By applying [24, Theorem 2.1] for the Eaton triple (V, G, D_0) with $G_i = G_{|V_i}$ and $D_{i,0} = D_0 \cap V_i$, $i = 1, \ldots, m$, we obtain that

$$(V_i, G_i, D_{i,0}) \text{ is an Eaton triple, } i = 1, \ldots, m,$$

$$D_0 = D_{1,0} + \ldots + D_{m,0},$$

$$G_{|W_0} = (G_1 \times \ldots \times G_m)_{|W_0}, \quad \text{where } W_0 = \operatorname{span} D_0. \tag{2.9}$$

□

In particular, for $x \in D_0 \subset W_0$, we deduce from (2.9) that

$$Gx = (G_1 \times \ldots \times G_m)x,$$

which was to be proved.

In the context of Lemma 6, in general it holds that

$$G \subset \widetilde{G} = G_1 \times \ldots \times G_m. \tag{2.10}$$

Usually, the reverse inclusion does not hold; however, (2.7) is a partial converse of (2.10).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 199-213, April 2023.

Condition (2.7) guarantees that for $x_1 \in D_1, \dots , x_m \in D_m$

$$(G_1 \times \cdots \times G_m)(x_1 + \ldots + x_m) \subset G(x_1 + \ldots + x_m). \tag{2.11}$$

In the next lemma, we demonstrate a preliminary result about the expression $\langle x_\downarrow, y_\downarrow \rangle$. To do so, we use a collection of Eaton triples connected by condition (2.12) of type (2.11).

In what follows, $(\cdot)_{\downarrow i}$ stands for the normal map of an Eaton triple (V_i, G_i, D_i) for $i = 1, \dots, m$, while $(\cdot)_\downarrow$ is the normal map of an Eaton triple (V, G, D) (see Lemma 7). Additionally, $(\cdot)^{\downarrow(j)}$ denotes the normal map of yet another Eaton triple $(V^{(j)}, G^{(j)}, D^{(j)})$ for $j = 1, \dots, m$ (see Theorem 8).

LEMMA 7. *Let (V, G, D) be an Eaton triple with normal map $(\cdot)_\downarrow$. Suppose V has the orthogonal decomposition $V = V_1 + \cdots + V_m$, where V_i, $i = 1, \dots, m$, are nonzero subspaces of V.*

Let (V_i, G_i, D_i) be an Eaton triple with normal map $(\cdot)_{\downarrow i}$, where $G_i \subset O(V_i)$ is a closed group and $D_i \subset V_i$ is a closed convex cone, $i = 1, \dots, m$.

Assume that for any $z_1 \in V_1, \dots, z_m \in V_m$,

$$(G_1 \times \ldots \times G_m)\left(\sum_{i=1}^m z_i\right) \subset G\left(\sum_{i=1}^m z_i\right). \tag{2.12}$$

Then for each $x, y \in V$ with orthogonal decompositions

$$x = x_1 + \ldots + x_m \quad \text{and} \quad y = y_1 + \ldots + y_m \quad \text{with } x_i, y_i \in V_i,\ i = 1, \dots, m,$$

the following equality holds:

$$\left\langle \left(\sum_{i=1}^m (x_i)_{\downarrow i}\right)_\downarrow, \left(\sum_{i=1}^m (y_i)_{\downarrow i}\right)_\downarrow \right\rangle = \left\langle \left(\sum_{i=1}^m x_i\right)_\downarrow, \left(\sum_{i=1}^m y_i)\right)_\downarrow \right\rangle = \langle x_\downarrow, y_\downarrow \rangle. \tag{2.13}$$

Proof. It is enough to show that (2.12) implies $\left(\sum_{i=1}^m (x_i)_{\downarrow i}\right)_\downarrow = \left(\sum_{i=1}^m x_i\right)_\downarrow$ and $\left(\sum_{i=1}^m (y_i)_{\downarrow i}\right)_\downarrow = \left(\sum_{i=1}^m y_i)\right)_\downarrow$.

In fact, for $i = 1, \dots, m$, by condition **(A1)** for (V_i, G_i, D_i) there exist $g_i, h_i \in G_i$ such that

$$(x_i)_{\downarrow i} = g_i x_i \quad \text{and} \quad (y_i)_{\downarrow i} = h_i y_i. \tag{2.14}$$

By (2.12) there exist $\widetilde{g}, \widetilde{h} \in G$ such that

$$(g_1 \times \ldots \times g_m)\left(\sum_{i=1}^m x_i\right) = \widetilde{g}\left(\sum_{i=1}^m x_i\right), \tag{2.15}$$

$$(h_1 \times \ldots \times h_m)\left(\sum_{i=1}^m y_i\right) = \widetilde{h}\left(\sum_{i=1}^m y_i\right). \tag{2.16}$$

□

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 199-213, April 2023.

In light of (2.14), (2.15) and (2.16), we see that

$$\left\langle \left(\sum_{i=1}^{m}(x_i)_{\downarrow i}\right)_{\downarrow}, \left(\sum_{i=1}^{m}(y_i)_{\downarrow i}\right)_{\downarrow} \right\rangle = \left\langle \left(\sum_{i=1}^{m} g_i x_i\right)_{\downarrow}, \left(\sum_{i=1}^{m} h_i y_i\right)_{\downarrow} \right\rangle$$

$$= \left\langle \left((g_1 \times \ldots \times g_m)\left(\sum_{i=1}^{m} x_i\right)\right)_{\downarrow}, \left((h_1 \times \ldots \times h_m)\left(\sum_{i=1}^{m} y_i\right)\right)_{\downarrow} \right\rangle$$

$$= \left\langle \left(\widetilde{g}\sum_{i=1}^{m} x_i\right)_{\downarrow}, \left(\widetilde{h}\sum_{i=1}^{m} y_i\right)_{\downarrow} \right\rangle = \left\langle \left(\sum_{i=1}^{m} x_i\right)_{\downarrow}, \left(\sum_{i=1}^{m} y_i)\right)_{\downarrow} \right\rangle = \langle x_\downarrow, y_\downarrow\rangle.$$

This finishes the proof of (2.13).

In the next theorem, we refine the von Neumann-type inequality (1.3) for a given Eaton triple (V, G, D) by employing a collection of Eaton triples related to (V, G, D) and mutually connected by condition (2.17). This technique extends the method used by Hovenier [11] and Abramovich et al. [1] from the space $\mathbb{R}^n$ (equipped with the orthogonal group and its subgroups (see Theorem 14)) to an inner product space endowed with the structure of an Eaton triple and its subsystems.

We use the convention $\sum\limits_{i=1}^{0} a_i = 0$ and $\sum\limits_{i=m+1}^{m} b_i = 0$ for any scalars or vectors a_i and b_i.

Theorem 8. *Let (V, G, D) be an Eaton triple with normal map $(\cdot)_\downarrow$. Suppose V has the orthogonal decomposition $V = V_1 + \cdots + V_m$, where V_i, $i = 1, \ldots, m$, are nonzero subspaces of V.*

Let (V_i, G_i, D_i) be an Eaton triple with normal map $(\cdot)_{\downarrow i}$, where $G_i \subset O(V_i)$ is a closed group and $D_i \subset V_i$ is a closed convex cone, $i = 1, \ldots, m$.

For $j = 1, \ldots, m$, denote $V^{(j)} = V_j + \cdots + V_m$, and let $(V^{(j)}, G^{(j)}, D^{(j)})$ be an Eaton triple with normal map $(\cdot)^{\downarrow(j)}$ for some closed group $G^{(j)} \subset O(V^{(j)})$ and closed convex cone $D^{(j)} \subset V^{(j)}$ satisfying

$$G_{j-1}z_{j-1} + G^{(j)}(z_j + \ldots + z_m) \subset G^{(j-1)}(z_{j-1} + z_j + \ldots + z_m) \tag{2.17}$$

for any $z_{j-1} \in V_{j-1}$, $z_j \in V_j$, ... , $z_m \in V_m$, and $j = 2, \ldots, m$. Assume that $(V, G, D) = (V^{(1)}, G^{(1)}, D^{(1)})$ with $(\cdot)_\downarrow = (\cdot)^{\downarrow(1)}$, and $(V_m, G_m, D_m) = (V^{(m)}, G^{(m)}, D^{(m)})$ with $(\cdot)_{\downarrow m} = (\cdot)^{\downarrow(m)}$.

Then for each $x, y \in V$ with orthogonal decompositions

$$x = x_1 + \ldots + x_m \quad \text{and} \quad y = y_1 + \ldots + y_m \quad \text{with } x_i, y_i \in V_i,\ i = 1, \ldots, m,$$

and for any $j = 1, \ldots, m + 1$ the following inequalities hold:

$$\langle x, y\rangle \le \left\langle \sum_{i=1}^{j-1} x_i, \sum_{i=1}^{j-1} y_i \right\rangle + \left\langle \left(\sum_{i=j}^{m}(x_i)_{\downarrow i}\right)^{\downarrow(j)}, \left(\sum_{i=j}^{m}(y_i)_{\downarrow i}\right)^{\downarrow(j)} \right\rangle \le \langle x_\downarrow, y_\downarrow\rangle. \tag{2.18}$$

Proof. In the sequel, for $j = 1, \ldots, m + 1$, we use the notation

$$R_j = \left\langle \sum_{i=1}^{j-1} x_i, \sum_{i=1}^{j-1} y_i \right\rangle + \left\langle \left(\sum_{i=j}^{m}(x_i)_{\downarrow i}\right)^{\downarrow(j)}, \left(\sum_{i=j}^{m}(y_i)_{\downarrow i}\right)^{\downarrow(j)} \right\rangle. \tag{2.19}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 199-213, April 2023.

Therefore, we have $R_{m+1} = \langle x, y\rangle$. So, (2.18) is satisfied for $j = m+1$ by (1.3).

It is now sufficient to prove (2.18) in the case $1 \le j \le m$.

First, we shall show the inequality

$$\langle x, y\rangle \le R_j \quad \text{for } 1 \le j \le m. \tag{2.20}$$

Recall that $x_i, y_i \in V_i$ and $(x_i)_{\downarrow i}, (y_i)_{\downarrow i} \in D_i \subset V_i$ for $i = 1,\dots,m$. By (1.3) applied to the triple (V_i, G_i, D_i), we obtain

$$\langle x_i, y_i\rangle \le \langle (x_i)_{\downarrow i}, (y_i)_{\downarrow i}\rangle \quad \text{for } i = 1,\dots,m. \tag{2.21}$$

So, by the mutual orthogonality of $V_1,\dots,V_m$, and by (2.21), for $1 \le j \le m$ we derive

$$\langle x, y\rangle = \left\langle \sum_{i=1}^m x_i, \sum_{i=1}^m y_i \right\rangle = \sum_{i=1}^m \langle x_i, y_i\rangle = \sum_{i=1}^{j-1} \langle x_i, y_i\rangle + \sum_{i=j}^m \langle x_i, y_i\rangle$$

$$\le \sum_{i=1}^{j-1} \langle x_i, y_i\rangle + \sum_{i=j}^m \langle (x_i)_{\downarrow i}, (y_i)_{\downarrow i}\rangle = \left\langle \sum_{i=1}^{j-1} x_i, \sum_{i=1}^{j-1} y_i \right\rangle + \left\langle \sum_{i=j}^m (x_i)_{\downarrow i}, \sum_{i=j}^m (y_i)_{\downarrow i} \right\rangle$$

$$\le \left\langle \sum_{i=1}^{j-1} x_i, \sum_{i=1}^{j-1} y_i \right\rangle + \left\langle \left(\sum_{i=j}^m (x_i)_{\downarrow i}\right)^{\downarrow(j)}, \left(\sum_{i=j}^m (y_i)_{\downarrow i}\right)^{\downarrow(j)} \right\rangle = R_j,$$

as wanted. The last inequality follows from inequality (1.3) used for the Eaton triple $(V^{(j)}, G^{(j)}, D^{(j)})$ with normal map $(\cdot)^{\downarrow(j)}$.

Thus, we have proved inequality (2.20) for $1 \le j \le m$. Therefore, the left-hand side of the double inequality (2.18) holds for $1 \le j \le m$.

It remains to prove the right-hand side of (2.18) for $1 \le j \le m$. To this end, we shall show that $R_j \le R_1$ for $j = 2,\dots,m$, and $R_1 = \langle x_\downarrow, y_\downarrow\rangle$.

Assume that $2 \le j \le m$. By (2.19) and (1.3) applied to the Eaton triple $(V_{j-1}, G_{j-1}, D_{j-1})$, we get the following sequence of equalities/inequalities (for further explanations, see below)

$$R_j = \sum_{i=1}^{j-1} \langle x_i, y_i\rangle + \left\langle \left(\sum_{i=j}^m (x_i)_{\downarrow i}\right)^{\downarrow(j)}, \left(\sum_{i=j}^m (y_i)_{\downarrow i}\right)^{\downarrow(j)} \right\rangle$$

$$= \sum_{i=1}^{j-2} \langle x_i, y_i\rangle + \langle x_{j-1}, y_{j-1}\rangle + \left\langle \left(\sum_{i=j}^m (x_i)_{\downarrow i}\right)^{\downarrow(j)}, \left(\sum_{i=j}^m (y_i)_{\downarrow i}\right)^{\downarrow(j)} \right\rangle$$

$$\le \sum_{i=1}^{j-2} \langle x_i, y_i\rangle + \langle (x_{j-1})_{\downarrow j-1}, (y_{j-1})_{\downarrow j-1}\rangle + \left\langle \left(\sum_{i=j}^m (x_i)_{\downarrow i}\right)^{\downarrow(j)}, \left(\sum_{i=j}^m (y_i)_{\downarrow i}\right)^{\downarrow(j)} \right\rangle$$

$$= \sum_{i=1}^{j-2} \langle x_i, y_i\rangle + \left\langle (x_{j-1})_{\downarrow j-1} + \left(\sum_{i=j}^m (x_i)_{\downarrow i}\right)^{\downarrow(j)}, (y_{j-1})_{\downarrow j-1} + \left(\sum_{i=j}^m (y_i)_{\downarrow i}\right)^{\downarrow(j)} \right\rangle$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 199-213, April 2023.

$$\leq \sum_{i=1}^{j-2} \langle x_i, y_i \rangle$$

$$+ \left\langle \left((x_{j-1})_{\downarrow j-1} + \left(\sum_{i=j}^{m} (x_i)_{\downarrow i} \right)^{\downarrow (j)} \right)^{\downarrow (j-1)}, \left((y_{j-1})_{\downarrow j-1} + \left(\sum_{i=j}^{m} (y_i)_{\downarrow i} \right)^{\downarrow (j)} \right)^{\downarrow (j-1)} \right\rangle$$

$$(2.22) \qquad = \sum_{i=1}^{j-2} \langle x_i, y_i \rangle + \left\langle \left(\sum_{i=j-1}^{m} (x_i)_{\downarrow i} \right)^{\downarrow (j-1)}, \left(\sum_{i=j-1}^{m} (y_i)_{\downarrow i} \right)^{\downarrow (j-1)} \right\rangle = R_{j-1}.$$

The third equality is a consequence of the orthogonality of the subspaces V_{j-1} and $V^{(j)} = V_j + \ldots + V_m$. The last inequality is due to inequality (1.3) applied to the Eaton triple $(V^{(j-1)}, G^{(j-1)}, D^{(j-1)})$ with the normal map $(\cdot)^{\downarrow (j-1)}$, and to the fact that

$$(x_{j-1})_{\downarrow j-1} + \left(\sum_{i=j}^{m} (x_i)_{\downarrow i} \right)^{\downarrow (j)} \in V_{j-1} + V^{(j)} = V^{(j-1)},$$

and

$$(y_{j-1})_{\downarrow j-1} + \left(\sum_{i=j}^{m} (y_i)_{\downarrow i} \right)^{\downarrow (j)} \in V_{j-1} + V^{(j)} = V^{(j-1)}.$$

The last but one equality follows from (2.17). Indeed, (1.1) applied to $(V^{(j)}, G^{(j)}, D^{(j)})$ and (2.17) imply for $j = 2, \ldots, m$ and for any $z_{j-1} \in D_{j-1}$, $z_j \in D_j$, ... , $z_m \in D_m$ that

$$z_{j-1} + (z_j + \ldots + z_m)^{\downarrow (j)} = g^{(j-1)}(z_{j-1} + z_j + \ldots + z_m).$$

for some $g^{(j-1)} \in G^{(j-1)}$. Hence, we get

$$(2.23) \qquad \left(z_{j-1} + (z_j + \ldots + z_m)^{\downarrow (j)} \right)^{\downarrow (j-1)} = (z_{j-1} + z_j + \ldots + z_m)^{\downarrow (j-1)},$$

by the $G^{(j-1)}$-invariance of $(\cdot)_{\downarrow}^{j-1}$ (see (1.2)). So, the substitutions $z_k = (x_k)_{\downarrow k}$ and $z_k = (y_k)_{\downarrow k}$ for $k = j-1, j, \ldots, m$ into (2.23) ensure the validity of the last but one equality of (2.22).

In summary, we have proved that

$$R_j \leq R_{j-1} \quad \text{for } j = 2, \ldots, m,$$

(see (2.22)). Hence,

$$(2.24) \qquad R_m \leq R_{m-1} \leq \ldots \leq R_2 \leq R_1,$$

which gives $R_j \leq R_1$ for $j = 1, \ldots, m$, as wanted.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 199-213, April 2023.

In the sequel, we shall use Lemma 7. To do so, we must show that condition (2.12) is satisfied.

In fact, by using (2.17) successively for $j = 2, 3, \ldots, m$, we obtain for $z_1 \in V_1, \ldots, z_m \in V_m$,

$$\begin{aligned} G(z_1 + \ldots + z_m) = G^{(1)}(z_1 + \ldots + z_m) &\supset G_1 z_1 + G^{(2)}(z_2 + \ldots + z_m) \\ &\supset G_1 z_1 + G_2 z_2 + G^{(3)}(z_3 + \ldots + z_m) \\ \ldots &\supset G_1 z_1 + G_2 z_2 + G_3 z_3 + \ldots + G_{m-2} z_{m-2} + G^{(m-1)}(z_{m-1} + z_m) \\ &\supset G_1 z_1 + G_2 z_2 + G_3 z_3 + \ldots + G_{m-1} z_{m-1} + G^{(m)} z_m \\ &= G_1 z_1 + G_2 z_2 + \ldots + G_{m-1} z_{m-1} + G_m z_m = (G_1 \times \cdots \times G_m)(z_1 + \ldots + z_m). \end{aligned}$$

This proves (2.12).

Now, by employing Lemma 7, it follows that

$$R_1 = \langle x_\downarrow, y_\downarrow \rangle. \tag{2.25}$$ □

This completes the proof of Theorem 8.

Remark 9. Condition (2.17) in Theorem 8 says that

$$(G_{j-1} \times G^{(j)})(z_{j-1} + z_j + \ldots + z_m) \subset G^{j-1}(z_{j-1} + z_j + \ldots + z_m),$$

for $z_{j-1} + z_j + \ldots + z_m \in V^{(j-1)} = V_{j-1} + V^{(j)}$.

Corollary 10. *Under the assumptions of Theorem 8, it holds that*

$$\langle x, y \rangle = R_{m+1} \le R_m \le \ldots \le R_2 \le R_1 = \langle x_\downarrow, y_\downarrow \rangle,$$

where for $j = 1, \ldots, m, m+1$,

$$R_j = \sum_{i=1}^{j-1} \langle x_i, y_i \rangle + \left\langle \left(\sum_{i=j}^{m} (x_i)_{\downarrow i} \right)^{\downarrow(j)}, \left(\sum_{i=j}^{m} (y_i)_{\downarrow i} \right)^{\downarrow(j)} \right\rangle.$$

Proof. See the proof of Theorem 8. Clearly, $\langle x, y \rangle = R_{m+1}$. Next, apply (2.24) and (2.25).

Corollary 11. *Let (V, G, D) be an Eaton triple with normal map $(\cdot)_\downarrow$. Suppose V has the orthogonal decomposition $V = V_1 + V_2$, where V_i, $i = 1, 2$, are nonzero subspaces of V.*

Let (V_i, G_i, D_i) be an Eaton triple with normal map $(\cdot)_{\downarrow i}$, where $G_i \subset O(V_i)$ is a closed group and $D_i \subset V_i$ is a closed convex cone, $i = 1, 2$.

Assume

$$G_1 z_1 + G_2 z_2 \subset G(z_1 + z_2), \tag{2.26}$$

for any $z_1 \in V_1$, $z_2 \in V_2$.

Then for each $x, y \in V$ with orthogonal decompositions

$$x = x_1 + x_2 \quad \text{and} \quad y = y_1 + y_2 \quad \text{with } x_i, y_i \in V_i,\ i = 1, 2,$$

the following inequalities hold:

$$\langle x, y \rangle \le \langle x_1, y_1 \rangle + \langle (x_2)_{\downarrow 2}, (y_2)_{\downarrow 2} \rangle \le \langle x_\downarrow, y_\downarrow \rangle. \tag{2.27}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 199-213, April 2023.

Proof. Apply Theorem 8 for $m = 2$ and $j = 2$ with $(V^{(1)}, G^{(1)}, D^{(1)}) = (V, G, D)$ and $(V^{(2)}, G^{(2)}, D^{(2)}) = (V_2, G_2, D_2)$ with $(\cdot)^{\downarrow(2)} = (\cdot)_{\downarrow 2}$.

Example 12. (Cf. [24, Example 3.1]) Take $V = \mathbb{M}_n(\mathbb{R})$, the space of all $n \times n$ real matrices with the trace inner product: $\langle x, y\rangle = \operatorname{tr} xy^T$ for $x, y \in \mathbb{M}_n(\mathbb{R})$, and with even $n = 2k$.

We consider the orthogonal decomposition $V = V_1 + V_2$, where $V_1 = \mathbb{S}_n$ is the space of $n \times n$ real symmetric matrices and $V_2 = \mathbb{K}_n$ is the space of $n \times n$ real skew-symmetric matrices. Evidently,

$$x = \frac{x + x^T}{2} + \frac{x - x^T}{2} \quad \text{for } x \in \mathbb{M}_n(\mathbb{R}),$$

with $x_1 = \frac{x+x^T}{2} \in \mathbb{S}_n$ and $x_2 = \frac{x-x^T}{2} \in \mathbb{K}_n$.

Let G_1 be the group of all linear operators of the form $x \mapsto uxu^T$ with $x \in \mathbb{S}_n$ and $u \in \mathbb{O}_n$, where $\mathbb{O}_n$ stands for the group of $n \times n$ orthogonal matrices. Analogously, let G_2 be the group of all linear operators of the form $x \mapsto uxu^T$ with $x \in \mathbb{K}_n$ and $u \in \mathbb{O}_n$.

We set

$$D_1 = \{\operatorname{diag}(\lambda_1, \dots, \lambda_n) : \lambda_1 \geq \dots \geq \lambda_n\},$$

where for an n-tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, the symbol $\operatorname{diag} \lambda$ denotes the diagonal matrix with the entries of λ on the main diagonal.

Then, (V_1, G_1, D_1) is an Eaton triple with the normal map:

$$(x_1)_{\downarrow 1} = \operatorname{diag} \lambda(x_1) \quad \text{for } x_1 \in \mathbb{S}_n,$$

where $\lambda(x_1)$ is the vector of the eigenvalues of $x_1 \in \mathbb{S}_n$ (see [4, 6]).

On the other hand, we also set

$$D_2 = \{\operatorname{diag}(\sigma_1 e, \dots, \sigma_k e) : \sigma_1 \geq \dots \geq \sigma_k \geq 0\},$$

where $e = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$\operatorname{diag}(\sigma_1 e, \dots, \sigma_k e) = \operatorname{diag}\Big(\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}\Big),$$

and the symbol $\operatorname{diag}(e_1, \dots, e_k)$ stands for the block diagonal matrix with the 2×2 blocks $e_1, \dots, e_k$ on the main block diagonal.

Then, (V_2, G_2, D_2) is an Eaton triple with the normal map:

$$(x_2)_{\downarrow 2} = \operatorname{diag}\Big(\begin{pmatrix} 0 & \sigma_1(x_2) \\ -\sigma_1(x_2) & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \sigma_k(x_2) \\ -\sigma_k(x_2) & 0 \end{pmatrix}\Big),$$

for $x_2 \in \mathbb{K}_n$, where all the singular values of an $x_2 \in \mathbb{K}_n$ are

$$\sigma_1(x_2) = \sigma_1(x_2) \geq \sigma_2(x_2) = \sigma_2(x_2) \geq \dots \geq \sigma_k(x_2) = \sigma_k(x_2),$$

(see [4, 6]).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 199-213, April 2023.

Finally, we put

$$G = G_1 \times G_2 \quad \text{and} \quad D = D_1 + D_2.$$

Thus, the action of G on $V = \mathbb{M}_n(\mathbb{R})$ is

$$gx = u_1 \frac{x + x^T}{2} u_1^T + u_2 \frac{x - x^T}{2} u_2^T \quad \text{for } x \in \mathbb{M}_n(\mathbb{R}) \text{ and } u_1, u_2 \in \mathbb{O}_n,$$

where $g = g_1 \times g_2$ with $g_1 = u_1(\cdot)u_1^T$ and $g_2 = u_2(\cdot)u_2^T$. Clearly, condition (2.26) is fulfilled.

In consequence, (V, G, D) is a (product) Eaton triple with the normal map:

$$x_\downarrow = \operatorname{diag}\left(\begin{pmatrix} \lambda_1(x_1) & \sigma_1(x_2) \\ -\sigma_1(x_2) & \lambda_2(x_1) \end{pmatrix}, \dots, \begin{pmatrix} \lambda_{n-1}(x_1) & \sigma_k(x_2) \\ -\sigma_k(x_2) & \lambda_n(x_1) \end{pmatrix} \right) \quad \text{for } x \in \mathbb{M}_n(\mathbb{R}),$$

where $x = x_1 + x_2$ with $x_1 = \frac{x+x^T}{2}$, $x_2 = \frac{x-x^T}{2}$ and $n = 2k$.

So, by making use of inequality (2.27) in Corollary 11, we obtain the following upper estimation of $\operatorname{tr} xy^T$:

$$\operatorname{tr} xy^T \le \operatorname{tr} x_1 y_1 + \operatorname{tr} (x_2)_{\downarrow 2}((y_2)_{\downarrow 2})^T \le \sum_{i=1}^{n} \lambda_i(x_1)\lambda_i(y_1) + 2\sum_{i=1}^{k} \sigma_i(x_2)\sigma_i(y_2) \quad \text{for } x, y \in \mathbb{M}_n(\mathbb{R}),$$

where $x_1 = \frac{x+x^T}{2}$, $x_2 = \frac{x-x^T}{2}$, $y_1 = \frac{y+y^T}{2}$, $y_2 = \frac{y-y^T}{2}$.

We now show a construction of two needed sequences of Eaton triples (V_j, G_j, D_j) and $(V^{(j)}, G^{(j)}, D^{(j)})$ for $j = 1, 2, \dots, m$ satisfying the assumptions of Theorem 8. In doing so, we utilize the results of Lemma 6.

Let (V, G, D) be a fixed Eaton triple. We proceed by induction.

(i). We put $(V^{(1)}, G^{(1)}, D^{(1)}) := (V, G, D)$.

(ii). For given $2 \le j \le m$, if $(V^{(j-1)}, G^{(j-1)}, D^{(j-1)})$ is already defined, then we give further definitions, as follows (see Lemma 6).

Let V_{j-1} be a $G^{(j-1)}$-invariant subspace of $V^{(j-1)}$. Let $V^{(j)}$ denote the orthogonal complement of V_{j-1} to $V^{(j-1)}$, that is, $V^{(j-1)} = V_{j-1} \oplus V^{(j)}$. Then, $V^{(j)}$ is a $G^{(j-1)}$-invariant subspace of $V^{(j-1)}$, because $G^{(j-1)} \subset O(V^{(j-1)})$. We define

$$G_{j-1} := G^{(j-1)}|_{V_{j-1}} \quad \text{and} \quad D_{j-1} := D^{(j-1)} \cap V_{j-1}.$$

That is, G_{j-1} is defined to be the restriction of the group $G^{(j-1)}$ to V_{j-1}. Then, $(V_{j-1}, G_{j-1}, D_{j-1})$ is an Eaton triple (see Lemma 6).

Likewise, we define

$$G^{(j)} := G^{(j-1)}|_{V^{(j)}} \quad \text{and} \quad D^{(j)} := D^{(j-1)} \cap V^{(j)}.$$

So, $G^{(j)}$ is defined to be the restriction of the group $G^{(j-1)}$ to the subspace $V^{(j)}$. Then, $(V^{(j)}, G^{(j)}, D^{(j)})$ is an Eaton triple (see Lemma 6).

It is easily seen that $V^{(j-1)} = V_{j-1} + V_j + \ldots + V_m$ for $2 \le j \le m + 1$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 199-213, April 2023.

By Lemma 6 applied for the Eaton triple $(V^{(j-1)}, G^{(j-1)}, D^{(j-1)})$, we have

$$G^{(j-1)}z^{(j-1)} = (G_{j-1} \times G^{(j)})z^{(j-1)},$$

for any $z^{(j-1)} \in V^{(j-1)}$.

Hence, for $z_{j-1} \in V_{j-1}$, $z^{(j)} \in V^{(j)}$ and $z^{(j-1)} = z_{j-1} + z^{(j)}$,

$$G_{j-1}z_{j-1} + G^{(j)}z^{(j)} = (G_{j-1} \times G^{(j)})(z_{j-1} + z^{(j)}) \subset G^{(j-1)}(z_{j-1} + z^{(j)}).$$

Because the Eaton triple $(V^{(j)}, G^{(j)}, D^{(j)})$ is already defined, we return to the item **(ii)** and continue the procedure.

The procedure ends at stage m whenever for the newly constructed subspace $V^{(m)}$ and group $G^{(m)}$, the next choice is $V_m := V^{(m)}$ and $V^{(m+1)} := \{0\}$. Then, $V^{(m)} = V_m \oplus V^{(m+1)}$. Also, we formally set $G^{(m+1)} := G^{(m)}|_{V^{(m+1)}} = \{\mathrm{id}|_{\{0\}}\}$ and $D^{(m+1)} := \{0\}$. Therefore, $(V_m, G_m, D_m) = (V^{(m)}, G^{(m)}, D^{(m)})$.

Corollary 13. *Let (V, G, D) be an Eaton triple with normal map $(\cdot)_\downarrow$. Assume that (V_i, G_i, D_i) and $(V^{(j)}, G^{(j)}, D^{(j)})$ are Eaton triples with normal maps $(\cdot)_{\downarrow i}$ and $(\cdot)^{\downarrow(j)}$, respectively, $j = 1, \ldots, m$, introduced by the above inductive procedure* **(i)**-**(ii)**.

Then for each $x, y \in V$ with orthogonal decompositions

$$x = x_1 + \ldots + x_m \quad \text{and} \quad y = y_1 + \ldots + y_m \quad \text{with } x_i, y_i \in V_i,\ i = 1, \ldots, m,$$

and for any $j = 1, \ldots, m+1$ the following inequalities hold:

$$\langle x, y\rangle \le \left\langle \sum_{i=1}^{j-1} x_i, \sum_{i=1}^{j-1} y_i \right\rangle + \left\langle \left(\sum_{i=j}^{m} (x_i)_{\downarrow i}\right)^{\downarrow(j)}, \left(\sum_{i=j}^{m} (y_i)_{\downarrow i}\right)^{\downarrow(j)} \right\rangle \le \langle x_\downarrow, y_\downarrow\rangle.$$

Proof. It is enough to use Theorem 8 for the Eaton triples defined by **(i)**–**(ii)**.

3. A refinement of Cauchy-Schwarz inequality. In the next theorem, we present a refinement of the classical Cauchy-Schwarz inequality $\langle x, y\rangle \le \|x\|\|y\|$ for $x, y \in V$.

As usual, $O(W)$ denotes the orthogonal group acting on an inner product space W.

Theorem 14. *Let $(V, \langle\cdot,\cdot\rangle)$ be a real inner product space with norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$, $\dim V < \infty$. Assume V has the orthogonal decomposition*

$$V = V_1 + \ldots + V_m,$$

where V_i, $i = 1, \ldots, m$, are nonzero subspaces of V.

Then for each $x, y \in V$ with orthogonal decompositions

$$x = x_1 + \ldots + x_m \quad \text{and} \quad y = y_1 + \ldots + y_m \quad \text{for } x_i, y_i \in V_i,\ i = 1, \ldots, m,$$

and for any $j = 1, \ldots, m, m+1$ the following inequalities holds:

$$\langle x, y\rangle \le \sum_{i=1}^{j-1} \langle x_i, y_i\rangle + \left(\sum_{i=j}^{m} \|x_i\|^2\right)^{1/2} \cdot \left(\sum_{i=j}^{m} \|y_i\|^2\right)^{1/2} \le \|x\|\|y\|. \tag{3.28}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 199-213, April 2023.

Proof. We employ Eaton triples of the type described in Example 3. Namely, for $i = 1, \ldots, m$, we set

$$G_i = O(V_i) \quad \text{and} \quad D_i = \operatorname{cone} e_i = \{te_i : 0 \le t \in \mathbb{R}\},$$

where $e_i \in V_i$ is a fixed vector of norm one. Then (V_i, G_i, D_i) is an Eaton triple with normal map $(\cdot)_{\downarrow i}$ given by $a_{\downarrow i} = \|a\| e_i$ for any $a \in V_i$ (see Example 3).

Likewise, for $j = 1, \ldots, m$ we set

$$V^{(j)} = V_j + \ldots + V_m, \quad G^{(j)} = O(V^{(j)}), \quad D^{(j)} = \operatorname{cone} e^{(j)} = \{te^{(j)} : 0 \le t \in \mathbb{R}\},$$

where $e^{(j)} \in V^{(j)}$ is a fixed vector of norm one. Then $(V^{(j)}, G^{(j)}, D^{(j)})$ is an Eaton triple with normal map $(\cdot)^{\downarrow(j)}$ given by $b^{\downarrow(j)} = \|b\| e^{(j)}$ for any $b \in V^{(j)}$ (see Example 3).

We also put $(V, G, D) = (V^{(1)}, G^{(1)}, D^{(1)})$ and $(\cdot)_{\downarrow} = (\cdot)^{\downarrow(1)}$. So, we have

$$\langle x_{\downarrow}, y_{\downarrow} \rangle = \langle x^{\downarrow(1)}, y^{\downarrow(1)} \rangle = \langle \|x\| e^{(1)}, \|y\| e^{(1)} \rangle = \|x\| \|y\|. \tag{3.29}$$

We assume that $e^{(m)} = e_m$. Hence, $D^{(m)} = D_m$ and $(V^{(m)}, G^{(m)}, D^{(m)}) = (V_m, G_m, D_m)$.

Also, condition (2.17) is met, because $O(V_{j-1}) \times O(V^{(j)}) \subset O(V^{(j-1)})$ on $V^{(j-1)} = V_{j-1} + V^{(j)}$.

So, in the above setting, we are allowed to utilize Theorem 8. By inequality (2.18), we obtain for any $j = 1, \ldots, m, m+1$,

$$\langle x, y \rangle \le \sum_{i=1}^{j-1} \langle x_i, y_i \rangle + \left\langle \left(\sum_{i=j}^{m} (x_i)_{\downarrow i} \right)^{\downarrow(j)}, \left(\sum_{i=j}^{m} (y_i)_{\downarrow i} \right)^{\downarrow(j)} \right\rangle \le \langle x_{\downarrow}, y_{\downarrow} \rangle. \tag{3.30}$$ ☐

Simultaneously, for any $z_i \in V_i$,

$$\left(\sum_{i=j}^{m} (z_i)_{\downarrow i} \right)^{\downarrow(j)} = \left(\sum_{i=j}^{m} \|z_i\| e_i \right)^{\downarrow(j)} = \left\| \sum_{i=j}^{m} \|z_i\| e_i \right\| e^{(j)},$$

with

$$\left\| \sum_{i=j}^{m} \|z_i\| e_i \right\| = \left\langle \sum_{i=j}^{m} \|z_i\| e_i, \sum_{i=j}^{m} \|z_i\| e_i \right\rangle^{1/2} = \left(\sum_{i=j}^{m} \|z_i\|^2 \right)^{1/2},$$

because $e_j \in V_i$, ... , $e_m \in V_m$, and the subspaces $V_j, \ldots, V_m$ are orthogonal, and $\langle e_i, e_i \rangle = \|e_i\|^2 = 1$.

In consequence, we have

$$\begin{aligned}\left\langle \left(\sum_{i=j}^{m} (x_i)_{\downarrow i} \right)^{\downarrow(j)}, \left(\sum_{i=j}^{m} (y_i)_{\downarrow i} \right)^{\downarrow(j)} \right\rangle &= \left\langle \left(\sum_{i=j}^{m} \|x_i\|^2 \right)^{1/2} e^{(j)}, \left(\sum_{i=j}^{m} \|y_i\|^2 \right)^{1/2} e^{(j)} \right\rangle \\ &= \left(\sum_{i=j}^{m} \|x_i\|^2 \right)^{1/2} \cdot \left(\sum_{i=j}^{m} \|y_i\|^2 \right)^{1/2},\end{aligned}$$

since $\langle e^{(j)}, e^{(j)} \rangle = \|e^{(j)}\|^2 = 1$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 199-213, April 2023.

This and statements (3.29) and (3.30) yield (3.28), completing the proof.

REMARK 15. In the special case when $V = \mathbb{R}^n$ with the standard inner product on $\mathbb{R}^n$, Theorem 14 leads to a result due to Hovenier [11] and to a variant of a result of Abramovich et al. [1] with $p = q = 2$.

Acknowledgements: The author would like to thank an anonymous referee for giving valuable comments that improved the previous version of the manuscript.

REFERENCES

[1] S. Abramovich, B. Mond, and J.E. Pečarić. Sharpening Hölder's inequality. *J. Math. Anal. Appl.*, 196:1131–1134, 1995.
[2] R. Bhatia. *Matrix Analysis*. Springer-Verlag, New York, 1997.
[3] C.L. Dolberry and T.Y. Tam. Ky Fan's dominance theorem and Eaton triple. *J. Convex Anal.*, 26:573–592, 2019.
[4] M.L. Eaton. On group induced orderings, monotone functions, and convolution theorems. In: Tong, Y.L. ed. *Inequalities in Statistics and Probability*. IMS Lectures Notes Monogr. Ser., vol. 5:13–25, 1984.
[5] M.L. Eaton. Lectures on topics in probability inequalities. *CWI Tract 35*. Centrum voor Wiskunde en Informatica, Amsterdam, 1987.
[6] M.L. Eaton. Group induced orderings with some applications in statistics. *CWI Newsletter*, 16:3–31, 1987.
[7] A.R. Francis and H.P. Wynn. Subgroup majorization. *Linear Algebra Appl.*, 444:53–66, 2014.
[8] M.S. Gowda. Commutation principles for optimization problems on spectral sets in Euclidean Jordan algebras. *Optim. Lett.*, 16:1119–1128, 2022.
[9] M.S. Gowda and J. Jeong. A pointwise weak-majorization inequality for linear maps over Euclidean Jordan algebras. *Linear and Multilinear Algebra*, 2021, published online, https://doi.org/10.1080/03081087.2020.1870096.
[10] M.S. Gowda and J. Jeong. Commutativity, majorization, and reduction in Fan-Theobald-von Neumann systems. *Results Math.*, 78 Paper No. 72, 42 p, 2023.
[11] J.W. Hovenier. Sharpening Cauchy's inequality. *J. Math. Anal. Appl.*, 186:156–160, 1994.
[12] A.S. Lewis. Convex analysis on Cartan subspaces. *Nonlinear Analysis*, 42:813–820, 2000.
[13] A.W. Marshall, I. Olkin, and B.C. Arnold. *Inequalities: Theory of Majorization and Its Applications*. 2nd Ed., Springer, New York, 2011.
[14] H.F. Miranda and R.C. Thompson. A trace inequality with a subtracted term. *Linear Algebra Appl.*, 185:165–172, 1993.
[15] H.F. Miranda and R.C. Thompson. Group majorization, the convex hull of sets of matrices, and the diagonal elements – singular values inequalities. *Linear Algebra Appl.*, 199:131–141, 1994.
[16] P. Niemiec and T.Y. Tam. A representation of G-invariant norms for Eaton triple. *J. Convex Analy.*, 18(2011):59–65, 2011.
[17] M. Niezgoda. Group majorization and Schur type inequalities. *Linear Algebra Appl.*, 268:9–30, 1998.
[18] M. Niezgoda. An analytical characterization of both effective and irreducible groups inducing cone orderings. *Linear Algebra Appl.*, 269:105–114, 1998.
[19] M. Niezgoda. On the structure of a class of Eaton triples. *Forum Math.*, 14:405–411, 2002.
[20] M. Niezgoda. Extended commutation principles for normal decomposition systems. *Linear Algebra Appl.*, 593C:251–273, 2018.
[21] M. Niezgoda. On convexity and ψ-uniform convexity of G-invariant functions on an Eaton triple. *J. Convex Anal.*, 26:1001–1019, 2019.
[22] M. Niezgoda. An extension of Schur-Ostrowski's condition, weak Eaton triples and generalized AI functions. *Linear Algebra Appl.*, 580:212–235, 2019.
[23] M. Niezgoda. Von Neumann-Davis type theorems for HLPK and Sherman functionals on Eaton triples. *Period. Math. Hung.*, 81:46–64, 2020.
[24] M. Niezgoda and T.-Y. Tam. On the norm property of $G(c)$-radii and Eaton triples. *Linear Algebra Appl.*, 336:119–130, 2001.
[25] T.Y. Tam. Group majorization, Eaton triples and numerical range. *Linear Multilinear Algebra*, 47:11-28, 2000.
[26] T.Y. Tam. An extension of a result of Lewis. *Electronic J. Linear Algebra*, 5:1–10, 1999.
[27] T.Y. Tam. Generalized Schur-concave functions and Eaton triples. *Linear Multilinear Algebra*, 50:113–120, 2002.
[28] T.Y. Tam and W.C. Hill. On G-invariant norms. *Linear Algebra Appl.*, 331:101–112, 2001.
[29] T.Y. Tam and W.C. Hill. Derivatives of orbital function and an extension of Berezin-Gel'fand's theorem. *Spec. Matrices*, 4:333–349, 2016.