Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 1-16, January 2023.

MINIMAL RANK WEAK DRAZIN INVERSES: A CLASS OF OUTER INVERSES WITH PRESCRIBED RANGE*

CANG WU† AND JIANLONG CHEN†

Abstract. For any square matrix A, it is proved that minimal rank weak Drazin inverses (Campbell and Meyer, 1978) of A coincide with outer inverses of A with range $\mathcal{R}(A^k)$, where k is the index of A. It is shown that the minimal rank weak Drazin inverse behaves very much like the Drazin inverse, and many generalized inverses such as the core-EP inverse and the DMP inverse are its special cases.

Key words. Drazin inverse, Weak Drazin inverse, Outer inverse, Core-EP inverse, DMP inverse, Matrix decomposition.

AMS subject classifications. 15A09.

1. Introduction. Throughout this paper, the set of all $m \times n$ complex matrices is denoted by $\mathbb{C}^{m\times n}$, and the identity matrix of order n is denoted by I_n. For any $A \in \mathbb{C}^{m\times n}$, the symbols A^*, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $\mathrm{rk}(A)$ will represent the conjugate transpose, range, null space and rank of A, respectively. When A is square, the smallest nonnegative integer k satisfying $\mathrm{rk}(A^k) = \mathrm{rk}(A^{k+1})$ is called the index of A, and it is denoted by $\mathrm{Ind}(A)$.

In the study of generalized inverses, a common and practical method is to express generalized inverses of matrices as solutions to some matrix equations. For example, Penrose [17] showed that the *Moore-Penrose inverse* $A^\dagger$ for a given matrix A is the unique solution to the four equations:

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA.$$

Given any matrix A, first we recall that a solution to the equation $AXA = A$ is called an *inner inverse* (or a $\{1\}$-inverse) of A, and a solution to $XAX = X$ is called an *outer inverse* (or a $\{2\}$-inverse) of A. The *Drazin inverse* A^D is a special outer inverse of A which is defined (only if A is square) as the unique solution to the following equations [9]:

$$XAX = X, \quad XA = AX, \quad XA^{k+1} = A^k,$$

where k is the index of A. Also, it is well-known that A^D is the unique outer inverse of A with range $\mathcal{R}(A^k)$ and null space $\mathcal{N}(A^k)$.

Over the past few decades, Drazin inverses have been shown to be of great theoretical interest and have applications in many areas such as Differential equations, Statistics, Numerical analysis and Control theory (e.g., see [2, 3, 5, 7, 8, 13, 18] and references therein).

*Received by the editors on August 23, 2022. Accepted for publication on January 7, 2023. Handling Editor: Oskar Baksalary. Corresponding Author: Jianlong Chen.
†School of Mathematics, Southeast University, Nanjing 210096, China (cang_wu@hotmail.com, jlchen@seu.edu.cn).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 1-16, January 2023.

In [4], Campbell and Meyer proposed the notion of weak Drazin inverses, which are more easier to compute than Drazin inverses and can be used in place of Drazin inverses especially when studying systems of differential equations with singular coefficients or when studying Markov chains. Recall that a matrix X is called a *weak Drazin inverse* of A if it satisfies $XA^{k+1} = A^k$, where $k = \mathrm{Ind}(A)$. Moreover, three subclasses of weak Drazin inverses were introduced and investigated:

Definition 1.1. ([4, Definition 1]) *Let $A \in \mathbb{C}^{n\times n}$ have index k. Then, $X \in \mathbb{C}^{n\times n}$ is called*

- *a commuting weak Drazin inverse of A if $XA^{k+1} = A^k$ and $AX = XA$;*
- *a projective weak Drazin inverse of A if $XA^{k+1} = A^k$ and $\mathcal{R}(XA) = \mathcal{R}(A^DA)$;*
- *a minimal rank weak Drazin inverse of A if $XA^{k+1} = A^k$ and $\mathrm{rk}(X) = \mathrm{rk}(A^D)$.*

It was proved in [4] that every minimal rank weak Drazin inverse of A is a projective weak Drazin inverse and that the Drazin inverse A^D is the unique minimal rank commuting weak Drazin inverse of A.

This paper aims to investigate main properties and characterizations of the minimal rank weak Drazin inverse, as well as its relationships with some other generalized inverses that have been extensively studied in recent years.

In Section 2, for any $A \in \mathbb{C}^{n\times n}$ with index k, it is shown that X is a minimal rank weak Drazin inverse of A if and only if X is an outer inverse of A with range $\mathcal{R}(A^k)$ if and only if $XA^{k+1} = A^k$ and $AX^2 = X$. The construction of minimal rank weak Drazin inverses is given, and the set of all minimal rank weak Drazin inverses is described.

In Section 3, we give the formula to compute minimal rank weak Drazin inverses of triangular block matrices.

In Section 4, we show that many generalized inverses such as the core-EP inverse [1, 15], the DMP inverse [14], the weak group inverse [20] and the weak core inverse [10] are special cases of minimal rank weak Drazin inverses. Moreover, we consider decompositions of matrices based on minimal rank weak Drazin inverses.

2. Characterizations and properties of minimal rank weak Drazin inverses.

In this section, we develop some characterizations and properties of minimal rank weak Drazin inverses.

First, analogous to the concept of minimal rank weak Drazin inverses, one can also define matrices which satisfy the relations

$$A^{k+1}X = A^k \quad \text{and} \quad \mathrm{rk}(X) = \mathrm{rk}(A^D),$$

as "minimal rank right weak Drazin inverses of A" (considering that a right weak Drazin inverse is defined by the relation $A^{k+1}X = A^k$ [4, p. 169]). We shall mainly work with minimal rank weak Drazin inverses in what follows. It goes without saying that all results have analogues for minimal rank right weak Drazin inverses.

The following result reveals the relationships between minimal rank weak Drazin inverses and outer inverses with prescribed range and allows us to verify these generalized inverses by matrix equations. This result will be used freely in the sequel.

Theorem 2.1. *Let $A \in \mathbb{C}^{n\times n}$ have index k. Then, the following statements are equivalent for any $X \in \mathbb{C}^{n\times n}$:*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 1-16, January 2023.

(i) $XA^{k+1} = A^k$ *and* $\mathrm{rk}(X) = \mathrm{rk}(A^D)$*, that is, X is a minimal rank weak Drazin inverse of A.*
(ii) $XA^{k+1} = A^k$ *and* $\mathcal{R}(X) = \mathcal{R}(A^k)$.
(iii) $XAX = X$ *and* $\mathcal{R}(X) = \mathcal{R}(A^k)$.
(iv) $XA^{k+1} = A^k$ *and* $AX^2 = X$.

Proof. (i)⇒(ii). From $XA^{k+1} = A^k$, we can get $\mathcal{R}(A^k) \subseteq \mathcal{R}(X)$. Also, since A has index k, it follows that $\mathrm{rk}(A^D) = \mathrm{rk}(A^k)$, and thus, $\mathrm{rk}(X) = \mathrm{rk}(A^D) = \mathrm{rk}(A^k)$ by (i). So, $\mathcal{R}(X) = \mathcal{R}(A^k)$.

(ii)⇒(iii). Since $\mathcal{R}(X) = \mathcal{R}(A^k)$, there exists $W \in \mathbb{C}^{n\times n}$ such that $X = A^kW$. Thus, it follows from $XA^{k+1} = A^k$ that $XAX = XA(A^kW) = (XA^{k+1})W = A^kW = X$.

(iii)⇒(iv). Suppose that $XAX = X$ and $\mathcal{R}(X) = \mathcal{R}(A^k)$. From $\mathcal{R}(X) = \mathcal{R}(A^k)$, we can get $X = A^kW_1$ and $A^k = XW_2$ for some $W_1, W_2 \in \mathbb{C}^{n\times n}$. Since $XAX = X$, it follows that

$$XA^{k+1} = XA(XW_2) = (XAX)W_2 = XW_2 = A^k, \tag{2.1}$$

and therefore

$$\begin{aligned} AX^2 &= AX(A^kW_1) = AX(A^{k+1}A^D)W_1 \\ &= A(XA^{k+1})A^DW_1 \overset{(2.1)}{=} A^{k+1}A^DW_1 = A^kW_1 = X. \end{aligned} \tag{2.2}$$

(iv)⇒(i). It suffices to show $\mathrm{rk}(X) = \mathrm{rk}(A^D)$. Since $AX^2 = X$, we have

$$X = AX^2 = A(AX^2)X = A^2X^3 = \cdots = A^kX^{k+1}, \tag{2.3}$$

which implies that $\mathrm{rk}(X) \le \mathrm{rk}(A^k)$. Also, since $XA^{k+1} = A^k$, we can get $\mathrm{rk}(A^k) \le \mathrm{rk}(X)$. Thus, $\mathrm{rk}(X) = \mathrm{rk}(A^k)$. Finally, since A has index k, it follows that $\mathrm{rk}(X) = \mathrm{rk}(A^k) = \mathrm{rk}(A^D)$. □

According to [6, Theorem 2], if outer inverses of A with range $\mathcal{R}(U)$ exists, then X is such an outer inverse if and only if $X = U(AU)^-$ for some inner inverse $(AU)^-$ of AU. Now, applying Theorem 2.1 and [6, Theorem 2] to $U = A^k$, where $k = \mathrm{Ind}(A)$, we can get the following construction of minimal rank weak Drazin inverses.

Corollary 2.2. *Let $A \in \mathbb{C}^{n\times n}$ have index k. Then, X is a minimal rank weak Drazin inverse of A if and only if $X = A^k(A^{k+1})^-$ for some inner inverse $(A^{k+1})^-$ of A^{k+1}.*

Corollary 2.3. *Let $A \in \mathbb{C}^{n\times n}$ have index k. Then, the following statements are equivalent for any $X \in \mathbb{C}^{n\times n}$:*

(i) $X = A^D$.
(ii) *X is a minimal rank weak Drazin inverse of A and $XA = AX$.*
(iii) *X is a minimal rank weak Drazin inverse of A and $\mathcal{N}(X) = \mathcal{N}(A^k)$.*
(iv) *X is a minimal rank weak Drazin inverse of A and $\mathcal{N}(X) \subseteq \mathcal{N}(A^k)$.*
(v) *X is a minimal rank weak Drazin inverse of A and $\mathcal{N}(A^k) \subseteq \mathcal{N}(X)$.*
(vi) *X is a weak Drazin inverse of A and $\mathcal{N}(A^k) \subseteq \mathcal{N}(X)$.*

Proof. (i)⇔(ii)⇔(iii) follow from Theorem 2.1 directly.

(iii)⇒(iv) and (iii)⇒(v)⇒(vi) are clear.

(iv)⇒(i). Since X is a minimal rank weak Drazin inverse of A, we have $XA^{k+1} = A^k$ and $AX^2 = X = XAX$ by Theorem 2.1. Also, since $\mathcal{N}(X) \subseteq \mathcal{N}(A^k)$, there exists $W \in \mathbb{C}^{n\times n}$ such that $A^k = WX$, from which we can see

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 1-16, January 2023.

$$A^k = WX = W(XAX) = (WX)AX = A^{k+1}X.$$

Therefore,

$$\begin{aligned} X &\overset{(2.3)}{=} A^k X^{k+1} = \left[(A^D)^{k+1}A^{2k+1}\right]X^{k+1} \\ &= (A^D)^{k+1}(A^{2k+1}X^{k+1}) = (A^D)^{k+1}A^k = A^D. \end{aligned}$$

(vi)⇒(i). By (vi), we have $XA^{k+1} = A^k$ and $X = TA^k$ for some $T \in \mathbb{C}^{n\times n}$. Therefore,

$$\begin{aligned} X &= TA^k = T(A^{k+1}A^D) = (TA^k)AA^D = XAA^D \\ &= X\left[A^{k+1}(A^D)^{k+1}\right] = (XA^{k+1})(A^D)^{k+1} = A^k(A^D)^{k+1} = A^D. \end{aligned}$$ ◻

In view of Theorem 2.1, if X is a minimal rank weak Drazin inverse of A, then $AX^2 = X$, which means that A is a weak Drazin inverse of X. For the special case $\mathrm{Ind}(A) = 1$, we have the following.

Corollary 2.4. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = 1$. Then, X is a minimal rank weak Drazin inverse of A if and only if A is a minimal rank weak Drazin inverse of X.*

Proof. Since $\mathrm{Ind}(A) = 1$, it follows from Theorem 2.1 that X is a minimal rank weak Drazin inverse of A if and only if $XA^2 = A$ and $AX^2 = X$. Thus, the result follows by the symmetry between A and X. ◻

The next result shows how to get a minimal rank weak Drazin inverse from a given weak Drazin inverse.

Proposition 2.5. *Let $A \in \mathbb{C}^{n\times n}$ have index k. If $X_1, X_2, \cdots, X_{k+1}$ are weak Drazin inverses of A, then $A^k \prod_{i=1}^{k+1} X_i$ is a minimal rank weak Drazin inverse of A. In particular, A^kX^{k+1} is a minimal rank weak Drazin inverse of A for any weak Drazin inverse X of A.*

Proof. Since

$$\begin{aligned} A^{k+1}\left(\prod_{i=1}^{k+1} X_i\right)A^{k+1} &= A^{k+1}\left(\prod_{i=1}^{k+1} X_i\right)[A^{2k+1}(A^D)^k] \\ &= A^{k+1}\left[\left(\prod_{i=1}^{k+1} X_i\right)A^{2k+1}\right](A^D)^k \\ &= A^{k+1}A^k(A^D)^k = A^{k+1}, \end{aligned}$$

it follows that $\prod_{i=1}^{k+1} X_i$ is an inner inverse of A^{k+1}. Therefore, by Corollary 2.2, $A^k \prod_{i=1}^{k+1} X_i$ is a minimal rank weak Drazin inverse of A. ◻

Remark 2.6.

(i) From (i)⇒(vi) in Corollary 2.3 (or from [4, Corollary 6]), it can be seen that the Drazin inverse A^D is exactly a weak Drazin inverse X of A which satisfies $X = X^{k+1}A^k$. Compared with this, we notice from Proposition 2.5 and equation (2.3) that a minimal rank weak Drazin inverse of A is a weak Drazin inverse X which satisfies $X = A^kX^{k+1}$.

(ii) For any weak Drazin inverse X_0 of A and every integer $1 \le i \le k$, let $X_i = AX_{i-1}^2$ be defined by induction on i, and let $Y_i = A^iX_0^{i+1}$. It is routine to check that X_i, Y_i are both weak Drazin inverses of A. Thus, $\mathrm{rk}(X_i) \ge \mathrm{rk}(A^k)$, $\mathrm{rk}(Y_i) \ge \mathrm{rk}(A^k)$. Moreover, from the construction of X_i, Y_i, one can see

$$\begin{aligned} &\mathrm{rk}(X_i) \le \mathrm{rk}(A^i) \quad \text{and} \quad \mathrm{rk}(X_k) \le \mathrm{rk}(X_{k-1}) \le \cdots \le \mathrm{rk}(X_1) \le \mathrm{rk}(X_0), \\ &\mathrm{rk}(Y_i) \le \mathrm{rk}(A^i) \quad \text{and} \quad \mathrm{rk}(Y_k) \le \mathrm{rk}(Y_{k-1}) \le \cdots \le \mathrm{rk}(Y_1) \le \mathrm{rk}(Y_0). \end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 1-16, January 2023.

Therefore, $\mathrm{rk}(X_k) = \mathrm{rk}(Y_k) = \mathrm{rk}(A^k) = \mathrm{rk}(A^D)$, which follows that X_k, Y_k are minimal rank weak Drazin inverses of A.

For a square matrix A, minimal rank weak Drazin inverses are in general not unique. The next result reveals the relationships between different minimal rank weak Drazin inverses.

PROPOSITION 2.7. *Let $A, X, Y \in \mathbb{C}^{n\times n}$. If X, Y are both minimal rank weak Drazin inverses of A, then $XY = Y^2$ and $XAY = AXY = Y$.*

Proof. We shall prove the more general fact: If X is a weak Drazin inverse of A and Y is a minimal rank weak Drazin inverse of A, then $XY = Y^2$ and $XAY = AXY = Y$. To this end, let $\mathrm{Ind}(A) = k$. Then from assumptions on X and Y, we can get

$$XA^{k+1} = A^k,\ YA^{k+1} = A^k \text{ and } AY^2 = Y.$$

Therefore,

$$\begin{aligned} XY &\overset{(2.3)}{=} X(A^{k+1}Y^{k+2}) = (XA^{k+1})Y^{k+2} = A^kY^{k+2} = Y^2,\\ XAY &\overset{(2.3)}{=} XA(A^kY^{k+1}) = (XA^{k+1})Y^{k+1} = A^kY^{k+1} = Y,\\ AXY &= A(XY) = AY^2 = Y. \end{aligned}$$ ◻

In [4, Theorem 2], it was proved that all minimal rank weak Drazin inverses of a matrix A can be given by the set $\{A^D + A^DAZ(I_n - A^DA) : Z \in \mathbb{C}^{n\times n}\}$. It is clear that

$$\{A^D + A^DAZ(I_n - A^DA) : Z \in \mathbb{C}^{n\times n}\} = \{A^D + A^DW(I_n - A^DA) : W \in \mathbb{C}^{n\times n}\}.$$

Now, given an arbitrary minimal rank weak Drazin inverse of A (not necessarily be the Drazin inverse A^D), we can also characterize the set of all minimal rank weak Drazin inverses of A.

THEOREM 2.8. *Let $A \in \mathbb{C}^{n\times n}$. If X is a minimal rank weak Drazin inverse of A, then the set of all minimal rank weak Drazin inverses of A is given by $\{X + XW(I_n - XA) : W \in \mathbb{C}^{n\times n}\}$ and also by $\{X + XW(I_n - AX) : W \in \mathbb{C}^{n\times n}\}$.*

Proof. Let $\mathrm{Ind}(A) = k$ and let X be a minimal rank weak Drazin inverse of A.

We first prove that $\{X + XW(I_n - XA) : W \in \mathbb{C}^{n\times n}\}$ is exactly the set of all minimal rank weak Drazin inverses of A. To this end, we see, for any $W \in \mathbb{C}^{n\times n}$,

$$\big[X + XW(I_n - XA)\big]A^{k+1} = XA^{k+1} + XW(A^{k+1} - XA^{k+2}) = A^k,$$

and

$$\begin{aligned} A\big[X + XW(I_n - XA)\big]^2 &= A\big[X + XW(I_n - XA)\big]X\big[I_n + W(I_n - XA)\big]\\ &= A\big[X^2 + XW(X - XAX)\big]\big[I_n + W(I_n - XA)\big]\\ &= AX^2\big[I_n + W(I_n - XA)\big] = X\big[I_n + W(I_n - XA)\big]\\ &= X + XW(I_n - XA). \end{aligned}$$

Thus, $X + XW(I_n - XA)$ is a minimal rank weak Drazin inverse of A. On the other hand, for any minimal rank weak Drazin inverse Y of A, by Proposition 2.7 we have $YX = X^2$ which implies that $(Y - X)X = 0$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 1-16, January 2023.

Thus,

$$Y - X = Y - X - (Y - X)XA = (Y - X)(I_n - XA).$$

So, $Y = X + (Y - X)(I_n - XA)$. Since we also have $XAY = Y$ by Proposition 2.7, it follows that

$$\begin{aligned} Y &= X + (XAY - X)(I_n - XA) \\ &= X + X(AY - I_n)(I_n - XA) \in \{X + XW(I_n - XA) : W \in \mathbb{C}^{n\times n}\}. \end{aligned}$$

Therefore, we complete the proof that $\{X + XW(I_n - XA) : W \in \mathbb{C}^{n\times n}\}$ is the set of all minimal rank weak Drazin inverses of A.

We next show that $\{X + XW(I_n - AX) : W \in \mathbb{C}^{n\times n}\}$ is also the set of all minimal rank weak Drazin inverses of A. Similar to the previous steps, it is straightforward to verify that $X + XW(I_n - AX)$ is a minimal rank weak Drazin inverse of A for any $W \in \mathbb{C}^{n\times n}$. On the other hand, for any minimal rank weak Drazin inverse Y of A, we have $XAY = Y$ and $YAX = X$ by Proposition 2.7. It follows that

$$Y = Y + (X - YAX) = X + Y(I_n - AX) = X + XAY(I_n - AX),$$

which implies that $Y \in \{X + XW(I_n - AX) : W \in \mathbb{C}^{n\times n}\}$, as needed. $\square$

We conclude this section with the next result which shows that minimal rank weak Drazin inverses behave very much like Drazin inverses. First recall that, when a matrix A has index 1, the Drazin inverse of A is also called the group inverse of A and denoted by $A^{\#}$.

Proposition 2.9. *Let $A \in \mathbb{C}^{n\times n}$ have index $k > 0$. If X is a minimal rank weak Drazin inverse of A, then*

(i) *AXA, A^2X, XA^2 and X are of index 1, and*
$$(AXA)^{\#} = X^2A, \quad (A^2X)^{\#} = X, \quad (XA^2)^{\#} = X^3A^2.$$
(ii) *$A - AXA$, $A - A^2X$ and $A - XA^2$ are nilpotent.*
(iii) *$A + I_n - AX$ and $A + I_n - XA$ are invertible, with*
$$(A + I_n - AX)^{-1} = X + (I_n - XA)\textstyle\sum_{i=0}^{k-1}(-A)^i,$$
$$(A + I_n - XA)^{-1} = X^2A + (I_n - X^2A^2)\textstyle\sum_{i=0}^{k-1}(-A)^i.$$

Proof. (i). Since

$$(AXA)(X^2A) = [AX(AX^2)]A = (AX^2)A = XA \text{ and}$$
$$(X^2A)(AXA) = (X^2A)[A(A^kX^{k+1})A] = (X^2A^{k+2})X^{k+1}A = A^kX^{k+1}A = XA,$$

it follows that

$$\begin{aligned} (AXA)(X^2A) &= (X^2A)(AXA), \\ (AXA)(X^2A)(AXA) &= (AXA)XA = A(XAX)A = AXA, \\ (X^2A)(AXA)(X^2A) &= XA(X^2A) = X(AX^2)A = X^2A. \end{aligned}$$

So, by definition, we have $(AXA)^{\#} = X^2A$ and $\mathrm{Ind}(AXA) = 1$. The rest part of (i) can be proved similarly.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 1-16, January 2023.

(ii). Since

$$\begin{aligned}(A-AXA)^2 &= (A-AXA)A-(A-AXA)(AXA)\\ &= (A-AXA)A-(A-AXA)(A^kX^kA) \qquad (\text{by } AX^2=X)\\ &= (A-AXA)A-A(A^k-XA^{k+1})(X^kA)\\ &= (A-AXA)A\end{aligned}$$

and similarly

$$(A-A^2X)^2=(A-A^2X)A, \quad (A-XA^2)^2=(A-XA^2)A,$$

it follows that

$$\begin{aligned}(A-AXA)^k &= (A-AXA)A^{k-1}\\ &= A^k-AXA^k = A^k-AXA^{k+1}A^D\\ &= A^k-A^{k+1}A^D=0,\end{aligned}$$

$$(A-A^2X)^{k+1}=(A-A^2X)A^k=A^{k+1}-A^2XA^k=0,$$

and

$$(A-XA^2)^k=(A-XA^2)A^{k-1}=A^k-XA^{k+1}=0.$$

Therefore, $A-AXA$, $A-A^2X$ and $A-XA^2$ are nilpotent.

(iii). First, a quick calculation gives

$$\begin{aligned}(A+I_n-AX)(X+I_n-AX)&=I_n+A-A^2X,\\ (A+I_n-XA)(X^2A+I_n-XA)&=I_n+A-AXA.\end{aligned}$$

Since $A-A^2X$ and $A-AXA$ are nilpotent by (ii), it follows that I_n+A-A^2X and $I_n+A-AXA$ are invertible, with

$$\begin{aligned}(I_n+A-A^2X)^{-1} &= I_n+\sum_{i=1}^{k}[-(A-A^2X)]^i\\ &= I_n-(A-A^2X)\sum_{i=0}^{k-1}(-A)^i,\\ (I_n+A-AXA)^{-1} &= I_n+\sum_{i=1}^{k}[-(A-AXA)]^i\\ &= I_n-(A-AXA)\sum_{i=0}^{k-1}(-A)^i.\end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 1-16, January 2023.

Consequently, $A+I_n-AX$ and $A+I_n-XA$ are invertible, with

$$\begin{aligned}
(A+I_n-AX)^{-1} &= (X+I_n-AX)(I_n+A-A^2X)^{-1}\\
&= (X+I_n-AX)\big[I_n-(A-A^2X)\sum_{i=0}^{k-1}(-A)^i\big]\\
&= (X+I_n-AX)-(XA-XA^2X+A-A^2X-AXA+AXA^2X)\sum_{i=0}^{k-1}(-A)^i\\
&= (X+I_n-AX)-(XA-AX+A-AXA)\sum_{i=0}^{k-1}(-A)^i\\
&\qquad \text{(since } XA^2X \overset{(2.3)}{=} XA^2(A^kX^{k+1})=(XA^{k+2})X^{k+1}=A^{k+1}X^{k+1}\overset{(2.3)}{=}AX)\\
&= (X+I_n-AX)-(XA+A)\sum_{i=0}^{k-1}(-A)^i+AX\sum_{i=0}^{k-1}(-A)^i-AX\sum_{i=0}^{k-1}(-A)^{i+1}\\
&= (X+I_n-AX)-XA\sum_{i=0}^{k-1}(-A)^i+\sum_{i=0}^{k-1}(-A)^{i+1}+AX-AX(-A)^k\\
&= X+I_n-XA\sum_{i=0}^{k-1}(-A)^i+\sum_{i=0}^{k-1}(-A)^{i+1}-(-A)^k\\
&= X-XA\sum_{i=0}^{k-1}(-A)^i+\sum_{i=0}^{k-1}(-A)^i\\
&= X+(I_n-XA)\sum_{i=0}^{k-1}(-A)^i,
\end{aligned}$$

and similarly

$$\begin{aligned}
(A+I_n-XA)^{-1} &= (X^2A+I_n-XA)(I_n+A-AXA)^{-1}\\
&= X^2A+(I_n-X^2A^2)\sum_{i=0}^{k-1}(-A)^i.
\end{aligned}$$

Thus, we complete the proof. ∎

3. Minimal rank weak Drazin inverses of block triangular matrices. In [4, Theorem 1], minimal rank weak Drazin inverses for the matrix $P\begin{bmatrix} A & 0\\ 0 & N\end{bmatrix}P^{-1}$ were given, where A is invertible and N is nilpotent. For a matrix $M\in\mathbb{C}^{n\times n}$ of the more general form

$$M=P\begin{bmatrix} A & B\\ 0 & C\end{bmatrix}P^{-1}, \tag{3.4}$$

where $A\in\mathbb{C}^{r\times r}$, $B\in\mathbb{C}^{r\times(n-r)}$ and $C\in\mathbb{C}^{(n-r)\times(n-r)}$ are arbitrary, it was proved in [16, Theorem 3.2] that

$$M^D=P\begin{bmatrix} A^D & S\\ 0 & C^D\end{bmatrix}P^{-1}, \tag{3.5}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 1-16, January 2023.

where

$$\begin{aligned} S = &- A^D BC^D + (I_r - AA^D) \left[\sum_{i=0}^{k-1} A^i B (C^D)^i \right] (C^D)^2 \\ &+ (A^D)^2 \left[\sum_{i=0}^{l-1} (A^D)^i B C^i \right] (I_{n-r} - CC^D), \qquad k = \mathrm{Ind}(A),\ l = \mathrm{Ind}(C). \end{aligned}$$

We now consider minimal rank weak Drazin inverses for such M.

THEOREM 3.1. *Let* $M = P \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} P^{-1}$ *be as in* (3.4). *If* X *and* Y *are minimal rank weak Drazin inverses of* A *and* C*, respectively, then* $Z = P \begin{bmatrix} X & S \\ 0 & Y \end{bmatrix} P^{-1}$ *is a minimal rank weak Drazin inverse of* M*, where*

$$S = -XBY + (I_r - XA) \left[\textstyle\sum_{i=0}^{k-1} A^i B Y^i \right] Y^2, \qquad k = \mathrm{Ind}(A).$$

Proof. Let C have index l and let $m > k + l$. A direct calculation shows

$$M^m = P \begin{bmatrix} A^m & \sum_{i=0}^{m-1} A^i B C^{m-1-i} \\ 0 & C^m \end{bmatrix} P^{-1},$$

and

$$M^{m+1} = P \begin{bmatrix} A^{m+1} & \sum_{i=0}^{m} A^i B C^{m-i} \\ 0 & C^{m+1} \end{bmatrix} P^{-1}.$$

Since X and Y are minimal rank weak Drazin inverses of A and C, respectively, it follows that

$$XA^{k+1} = A^k, \ \ AX^2 = X, \ \ YC^{l+1} = C^l \ \text{ and } \ CY^2 = Y.$$

Now, we have

$$\begin{aligned} ZM^{m+1} &= P \begin{bmatrix} X & S \\ 0 & Y \end{bmatrix} \begin{bmatrix} A^{m+1} & \sum_{i=0}^{m} A^i B C^{m-i} \\ 0 & C^{m+1} \end{bmatrix} P^{-1} \\ &= P \begin{bmatrix} A^m & X \sum_{i=0}^{m} A^i B C^{m-i} + SC^{m+1} \\ 0 & C^m \end{bmatrix} P^{-1}. \end{aligned}$$

Since

$$\begin{aligned} & X \sum_{i=0}^{m} A^i B C^{m-i} + SC^{m+1} \\ = {} & X \sum_{i=0}^{m} A^i B C^{m-i} - XBYC^{m+1} + (I_r - XA) \left[\sum_{i=0}^{k-1} A^i B Y^i \right] Y^2 C^{m+1} \\ = {} & X \sum_{i=0}^{m} A^i B C^{m-i} - XBC^m + (I_r - XA) \left[\sum_{i=0}^{k-1} A^i B Y^{i+2} C^{m+1} \right] \end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 1-16, January 2023.

$$
\begin{aligned}
&= X\sum_{i=1}^{m} A^iBC^{m-i} + (I_r - XA)\left[\sum_{i=0}^{k-1} A^iBC^{m-1-i}\right]\\
&= X\sum_{i=0}^{m-1} A^{i+1}BC^{m-(i+1)} + (I_r - XA)\left[\sum_{i=0}^{m-1} A^iBC^{m-1-i}\right] \quad (\text{by } (I_r - XA)A^i = 0 \text{ for all } i \geq k)\\
&= XA\sum_{i=0}^{m-1} A^iBC^{m-1-i} + (I_r - XA)\left[\sum_{i=0}^{m-1} A^iBC^{m-1-i}\right] = \sum_{i=0}^{m-1} A^iBC^{m-1-i},
\end{aligned}
$$

we obtain $ZM^{m+1} = M^m$, which implies that $ZM^{\mathrm{Ind(M)}+1} = M^{\mathrm{Ind(M)}}$.

Also, we have

$$
MZ^2 = P\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}\begin{bmatrix} X & S \\ 0 & Y \end{bmatrix}^2 P^{-1} = P\begin{bmatrix} X & AXS + ASY + BY^2 \\ 0 & Y \end{bmatrix} P^{-1}.
$$

Since

$$
\begin{aligned}
&\quad AXS + ASY + BY^2\\
&= \Big(-AX^2BY + (AX - AX^2A)\big[\sum_{i=0}^{k-1} A^iBY^i\big]Y^2\Big)\\
&\quad + \Big(-AXBY^2 + A(I_r - XA)\big[\sum_{i=0}^{k-1} A^iBY^i\big]Y^3\Big) + BY^2\\
&= -XBY + (AX - XA)\big[\sum_{i=0}^{k-1} A^iBY^i\big]Y^2 - AXBY^2 + (I_r - AX)A\big[\sum_{i=0}^{k-1} A^iBY^i\big]Y^3 + BY^2\\
&= -XBY + (AX - XA)\big[\sum_{i=0}^{k-1} A^iBY^i\big]Y^2 + (I_r - AX)\big[\sum_{i=0}^{k-1} A^{i+1}BY^{i+1}\big]Y^2 + (I_r - AX)BY^2\\
&= -XBY + (AX - XA)\big[\sum_{i=0}^{k-1} A^iBY^i\big]Y^2 + (I_r - AX)\big[\sum_{i=1}^{k} A^iBY^i\big]Y^2 + (I_r - AX)BY^2\\
&= -XBY + (AX - XA)\big[\sum_{i=0}^{k-1} A^iBY^i\big]Y^2 + (I_r - AX)\big[\sum_{i=0}^{k-1} A^iBY^i\big]Y^2 \quad (\text{by } (I_r - AX)A^k = 0)\\
&= -XBY + (I_r - XA)\big[\sum_{i=0}^{k-1} A^iBY^i\big]Y^2 = S,
\end{aligned}
$$

we obtain $MZ^2 = Z$. Therefore, Z is a minimal rank weak Drazin inverse of M. □

Recall from [12, Corollary 6] that, for any $M \in \mathbb{C}^{n\times n}$ with rank r, there exists a unitary matrix U such that

$$
M = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*, \tag{3.6}
$$

where $\Sigma = \mathrm{diag}(\sigma_1 I_{r_1}, \ldots, \sigma_t I_{r_t})$ is the diagonal matrix of singular values of M, $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + r_2 + \cdots + r_t = r$, and $K \in \mathbb{C}^{r\times r}, L \in \mathbb{C}^{r\times(n-r)}$ satisfy $KK^* + LL^* = I_r$. The equation (3.6) is called the Hartwig-Spindelböck decomposition of M.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 1-16, January 2023.

By [19], for any matrix $M \in \mathbb{C}^{n\times n}$, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$M = U \begin{bmatrix} T & B \\ 0 & N \end{bmatrix} U^*, \tag{3.7}$$

where T is an invertible upper triangular matrix whose diagonal entries are nonzero eigenvalues of M, and N is a nilpotent upper triangular matrix with $\mathrm{Ind}(N) = \mathrm{Ind}(M)$.

Notice that the Jordan normal form, the Hartwig-Spindelböck decomposition and the decomposition as in (3.7) of a matrix M are all special cases of (3.4) with C being nilpotent. For such cases, we have the following.

Proposition 3.2. *Let $M = P \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} P^{-1} \in \mathbb{C}^{n\times n}$, where $C \in \mathbb{C}^{(n-r)\times(n-r)}$ is nilpotent. Then, Z is a minimal rank weak Drazin inverse of M if and only if $Z = P \begin{bmatrix} X & W \\ 0 & 0 \end{bmatrix} P^{-1}$ for some minimal rank weak Drazin inverse X of A and some $W \in \mathbb{C}^{r\times(n-r)}$ satisfying $\mathcal{R}(W) \subseteq \mathcal{R}(X)$.*

Proof. Let $\mathrm{Ind}(A) = k$, $\mathrm{Ind}(C) = l$, and let $m > k + l$. Since $C^l = 0$, a direct calculation shows

$$M^m = P \begin{bmatrix} A^m & \sum_{i=m-l}^{m-1} A^i BC^{m-1-i} \\ 0 & 0 \end{bmatrix} P^{-1},$$

and

$$M^{m+1} = P \begin{bmatrix} A^{m+1} & \sum_{i=m-l+1}^{m} A^i BC^{m-i} \\ 0 & 0 \end{bmatrix} P^{-1} = P \begin{bmatrix} A^{m+1} & \sum_{i=m-l}^{m-1} A^{i+1} BC^{m-(i+1)} \\ 0 & 0 \end{bmatrix} P^{-1}.$$

We first prove the sufficiency. If X is a minimal rank weak Drazin inverse of A and $\mathcal{R}(W) \subseteq \mathcal{R}(X)$, then we have $XA^{k+1} = A^k$, $AX^2 = X$ and $W = XT$ for some $T \in \mathbb{C}^{n\times n}$. Thus, we obtain

$$\begin{aligned} ZM^{m+1} &= P \begin{bmatrix} XA^{m+1} & X\sum_{i=m-l}^{m-1} A^{i+1} BC^{m-(i+1)} \\ 0 & 0 \end{bmatrix} P^{-1} \\ &= P \begin{bmatrix} XA^{m+1} & \sum_{i=m-l}^{m-1} (XA^{i+1}) BC^{m-1-i} \\ 0 & 0 \end{bmatrix} P^{-1} \\ &= P \begin{bmatrix} A^m & \sum_{i=m-l}^{m-1} A^i BC^{m-1-i} \\ 0 & 0 \end{bmatrix} P^{-1} = M^m, \end{aligned}$$

which implies that $ZM^{\mathrm{Ind(M)}+1} = M^{\mathrm{Ind(M)}}$. Also, we have

$$\begin{aligned} MZ^2 &= P \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \begin{bmatrix} X & XT \\ 0 & 0 \end{bmatrix}^2 P^{-1} \\ &= P \begin{bmatrix} AX^2 & AX^2T \\ 0 & 0 \end{bmatrix} P^{-1} = P \begin{bmatrix} X & XT \\ 0 & 0 \end{bmatrix} P^{-1} = Z. \end{aligned}$$

Therefore, Z is a minimal rank weak Drazin inverse of M.

For the necessity, let X_0 be a minimal rank weak Drazin inverse of A. Since C is nilpotent, it follows that $0 \in \mathbb{C}^{(n-r)\times(n-r)}$ is a minimal rank weak Drazin inverse of C. Thus, by Theorem 3.1, $Z_0 = P \begin{bmatrix} X_0 & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 1-16, January 2023.

is a minimal rank weak Drazin inverse of M. Now, for any minimal rank weak Drazin inverse Z of M, we have $Z = Z_0 + Z_0G(I_n - MZ_0)$ for some $G \in \mathbb{C}^{n\times n}$ by Theorem 2.8. By writing $G = P\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}P^{-1}$, we can get

$$\begin{aligned} Z &= Z_0 + Z_0G(I_n - MZ_0) \\ &= P\Big(\begin{bmatrix} X_0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} X_0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}\begin{bmatrix} I_r - AX_0 & 0 \\ 0 & I_{n-r} \end{bmatrix}\Big)P^{-1} \\ &= P\begin{bmatrix} X_0 + X_0G_1(I_r - AX_0) & X_0G_2 \\ 0 & 0 \end{bmatrix}P^{-1}. \end{aligned}$$

Set $X = X_0 + X_0G_1(I_r - AX_0)$. Then, X is a minimal rank weak Drazin inverse of A by Theorem 2.8, and $\mathcal{R}(X_0G_2) \subseteq \mathcal{R}(X_0) = \mathcal{R}(A^D) = \mathcal{R}(X)$. So the necessity follows. $\square$

The next two results follow by Proposition 3.2 directly.

Corollary 3.3. *Let $M = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}U^*$ be as in (3.6). Then, Z is a minimal rank weak Drazin inverse of M if and only if $Z = U\begin{bmatrix} X & W \\ 0 & 0 \end{bmatrix}U^*$ for some minimal rank weak Drazin inverse X of ΣK and some W satisfying $\mathcal{R}(W) \subseteq \mathcal{R}(X)$.*

Corollary 3.4. *Let $M = U\begin{bmatrix} T & B \\ 0 & N \end{bmatrix}U^*$ be as in (3.7). Then, Z is a minimal rank weak Drazin inverse of M if and only if $Z = U\begin{bmatrix} T^{-1} & W \\ 0 & 0 \end{bmatrix}U^*$ for some $W \in \mathbb{C}^{r\times(n-r)}$.*

4. Relationships with some other generalized inverses and with matrix decompositions.

In this section, we first show the relationships between minimal rank weak Drazin inverses and some new types of generalized inverses that have been introduced in the recent decade. Then, we consider matrix decompositions based on minimal rank weak Drazin inverses.

For a square matrix A, recall that

- [15, 19] $A^{\textcircled{\dagger}} = A^k(A^{k+1})^\dagger$ is called the *core-EP inverse* of A, where k is the index of A;
- [14] $A^{D,\dagger} = A^DAA^\dagger$ is called the *DMP inverse* of A;
- [20] $(A^{\textcircled{\dagger}})^2A$ is called the weak group inverse of A;
- [10] $(A^{\textcircled{\dagger}})^2A^2A^\dagger$ is called the weak core inverse of A.

The next results show that all these four generalized inverses are special cases of minimal rank weak Drazin inverses.

Proposition 4.1. *Let $A \in \mathbb{C}^{n\times n}$ have index k. Then, the following statements are equivalent:*

(i) *X is the core-EP inverse of A.*
(ii) *X is a minimal rank weak Drazin inverse of A and $\mathcal{N}(X) = \mathcal{N}\big[(A^k)^*\big]$.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 1-16, January 2023.

(iii) *X is a minimal rank weak Drazin inverse of A and $\mathcal{N}(X) \subseteq \mathcal{N}\big[(A^k)^*\big]$.*
(iv) *X is a minimal rank weak Drazin inverse of A and $\mathcal{N}\big[(A^k)^*\big] \subseteq \mathcal{N}(X)$.*

Proof. Notice that the core-EP inverse $A^{\textcircled{\dagger}}$ of A was originally defined as the unique solution to the system: $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$ (see [15, Definition 3.1]).

(i)$\Leftrightarrow$(ii). Since $\mathcal{R}(X^*) = \mathcal{R}(A^k)$ if and only if $\mathcal{N}(X) = \mathcal{N}[(A^k)^*]$, the result follows directly from Theorem 2.1.

(ii)$\Rightarrow$(iii). It is clear.

(iii)$\Rightarrow$(iv). Since X is a minimal rank weak Drazin inverse of A, we have $XAX = X$ and $AX^2 = X$. Also, since $\mathcal{N}(X) \subseteq \mathcal{N}[(A^k)^*]$, we have $(A^k)^* = TX$ for some $T \in \mathbb{C}^{n\times n}$. Thus,

$$(A^k)^* = TX = T(XAX) = (TX)AX = (A^k)^*AX,$$

whence it follows $A^k = (AX)^*A^k$. Since $AX^2 = X$ implies $AX = A^kX^k$, we have

$$AX = A^kX^k = \big[(AX)^*A^k\big]X^k = (AX)^*AX,$$

from which it can be seen that $AX = A^kX^k$ is Hermitian. Therefore,

$$X = XAX = X(A^kX^k) = X(A^kX^k)^* = X(X^k)^*(A^k)^*,$$

which gives $\mathcal{N}\big[(A^k)^*\big] \subseteq \mathcal{N}(X)$.

(iv)$\Rightarrow$(ii). It suffices to prove $\mathcal{N}(X) \subseteq \mathcal{N}[(A^k)^*]$. Since X is a minimal rank weak Drazin inverse of A, we can get $AXA^k = A^k$. Also, since $\mathcal{N}[(A^k)^*] \subseteq \mathcal{N}(X)$, there is $W \in \mathbb{C}^{n\times n}$ such that $X = W(A^k)^*$. So we have

$$X = W(A^k)^* = W\big[(A^k)^*A^k(A^k)^\dagger\big] = [W(A^k)^*]A^k(A^k)^\dagger = XA^k(A^k)^\dagger, \qquad \square$$

and therefore, $AX = A[XA^k(A^k)^\dagger] = (AXA^k)(A^k)^\dagger = A^k(A^k)^\dagger$, which implies that AX is Hermitian. Now, by using $A^k = AXA^k$ and $(AX)^* = AX$, we can get $(A^k)^* = (AXA^k)^* = (A^k)^*(AX)^* = (A^k)^*AX$, which gives $\mathcal{N}(X) \subseteq \mathcal{N}[(A^k)^*]$.

For DMP inverses, weak group inverses and weak core inverses, we have the following corresponding results whose proofs are similar to that of Proposition 4.1 and are therefore omitted.

Proposition 4.2. *Let $A \in \mathbb{C}^{n\times n}$ have index k. Then, the following statements are equivalent:*

(i) *X is the DMP inverse of A.*
(ii) *X is a minimal rank weak Drazin inverse of A and $\mathcal{N}(X) = \mathcal{N}(A^kA^\dagger)$.*
(iii) *X is a minimal rank weak Drazin inverse of A and $\mathcal{N}(X) \subseteq \mathcal{N}(A^kA^\dagger)$.*
(iv) *X is a minimal rank weak Drazin inverse of A and $\mathcal{N}(A^kA^\dagger) \subseteq \mathcal{N}(X)$.*

Proposition 4.3. *Let $A \in \mathbb{C}^{n\times n}$ have index k. Then, the following statements are equivalent:*

(i) *X is the weak group inverse of A.*
(ii) *X is a minimal rank weak Drazin inverse of A and $\mathcal{N}(X) = \mathcal{N}\big[(A^k)^*A\big]$.*
(iii) *X is a minimal rank weak Drazin inverse of A and $\mathcal{N}(X) \subseteq \mathcal{N}\big[(A^k)^*A\big]$.*
(iv) *X is a minimal rank weak Drazin inverse of A and $\mathcal{N}\big[(A^k)^*A\big] \subseteq \mathcal{N}(X)$.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 1-16, January 2023.

Proposition 4.4. *Let $A \in \mathbb{C}^{n\times n}$ have index k. Then, the following statements are equivalent:*

(i) *X is the weak core inverse of A.*
(ii) *X is a minimal rank weak Drazin inverse of A and $\mathcal{N}(X) = \mathcal{N}\big[(A^k)^*A^2A^\dagger\big]$.*
(iii) *X is a minimal rank weak Drazin inverse of A and $\mathcal{N}(X) \subseteq \mathcal{N}\big[(A^k)^*A^2A^\dagger\big]$.*
(iv) *X is a minimal rank weak Drazin inverse of A and $\mathcal{N}\big[(A^k)^*A^2A^\dagger\big] \subseteq \mathcal{N}(X)$.*

In view of Corollary 3.4, if a matrix M is written in the form (3.7), then any minimal rank weak Drazin inverse Z of M is of the form $Z = U\begin{bmatrix} T^{-1} & W \\ 0 & 0 \end{bmatrix}U^*$ for some W. Moreover, by [19, 11, 20, 10],

- Z is the core-EP inverse of M when $W = 0$;
- Z is the DMP inverse of M when $W = T^{-(k+1)}\Big[\sum_{i=0}^{k-1} T^iBN^{k-1-i}\Big]NN^\dagger$, where $k = \mathrm{Ind}(M)$;
- Z is the weak group inverse of M when $W = T^{-2}B$;
- Z is the weak core inverse of M when $W = T^{-2}BNN^\dagger$.

Recall from [2, p. 169, Theorem 11] that any $A \in \mathbb{C}^{n\times n}$ with index $k > 0$ has a unique decomposition

$$A = C + N, \tag{4.8}$$

such that $\mathrm{Ind}(C) = 1$, N is nilpotent with index k, and $CN = NC = 0$. The equation (4.8) is called the core-nilpotent decomposition of A. Analogous to (4.8), two other decompositions were introduced in [19] and [21]: By [19, Theorem 2.1], A has a unique decomposition

$$A = A_1 + A_2, \tag{4.9}$$

such that $\mathrm{Ind}(A_1) = 1$, $A_2^k = 0$, and $A_1^*A_2 = A_2A_1 = 0$; by [21, Theorem 2.1], A has a unique decomposition

$$A = \tilde{A}_1 + \tilde{A}_2, \tag{4.10}$$

such that $\mathcal{R}(\tilde{A}_1) = \mathcal{R}(\tilde{A}_1^*)$ (hence $\mathrm{Ind}(\tilde{A}_1) = 1$), $\tilde{A}_2^{k+1} = 0$, and $\tilde{A}_2\tilde{A}_1 = 0$. The equations (4.9) and (4.10) are called the core-EP decomposition and the EP-nilpotent decomposition of A, respectively.

For a concrete example, let $A = \begin{bmatrix} T & B \\ 0 & N \end{bmatrix}$, where T is invertible and N is nilpotent with index k. Then, (4.8), (4.9) and (4.10) become $A = \begin{bmatrix} T & \sum_{i=0}^{k} T^{-i}BN^i \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\sum_{i=1}^{k} T^{-i}BN^i \\ 0 & N \end{bmatrix}$, $A = \begin{bmatrix} T & B \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}$ and $A = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ 0 & N \end{bmatrix}$, respectively.

Now, for any minimal rank weak Drazin inverse X of A, consider the following decompositions of A with respect to X:

$$A = A_{\mathrm{x},1} + A_{\mathrm{x},2}, \tag{4.11}$$

where $A_{\mathrm{x},1} = AXA$ and $A_{\mathrm{x},2} = A - AXA$;

$$A = \tilde{A}_{\mathrm{x},1} + \tilde{A}_{\mathrm{x},2}, \tag{4.12}$$

where $\tilde{A}_{\mathrm{x},1} = A^2X$ and $\tilde{A}_{\mathrm{x},2} = A - A^2X$;

$$A = \hat{A}_{\mathrm{x},1} + \hat{A}_{\mathrm{x},2}, \tag{4.13}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 1-16, January 2023.

where $\hat{A}_{X,1} = XA^2$ and $\hat{A}_{X,2} = A - XA^2$. By Proposition 2.9, $A_{X,1}$, $\tilde{A}_{X,1}$, $\hat{A}_{X,1}$ are of index 1, and $A_{X,2}$, $\tilde{A}_{X,2}$, $\hat{A}_{X,2}$ are all nilpotent.

We close this section with the following simple observation which states that the core-nilpotent decomposition, the core-EP decomposition and the EP-nilpotent decomposition of A can be obtained from (4.11), (4.12) or (4.13) by choosing some specific minimal rank weak Drazin inverse X of A.

PROPOSITION 4.5. *Let $A, X \in \mathbb{C}^{n\times n}$ and X be a minimal rank weak Drazin inverse of A.*

(i) *If $X = A^D$, then (4.8), (4.11), (4.12) and (4.13) coincide.*
(ii) *If $X = A^{\textcircled{\dagger}}$, then (4.9) coincides with (4.11), and (4.10) coincides with (4.12).*

Proof. (i). By [2, p. 169, Theorem 11], the core-nilpotent decomposition (4.8) is given by $A = (AA^DA) + (A - AA^DA)$. Thus, if $X = A^D$, then (4.11), (4.12) and (4.13) coincide because $AX = XA$, and they become the core-nilpotent decomposition (4.8).

(ii). By [19, Theorem 3.4], the core-EP decomposition (4.9) is given by $A = (AA^{\textcircled{\dagger}}A) + (A - AA^{\textcircled{\dagger}}A)$; also, by [21, Theorem 2.6], the EP-nilpotent decomposition (4.10) is given by $A = (A^2A^{\textcircled{\dagger}}) + (A - A^2A^{\textcircled{\dagger}})$. Thus, if $X = A^{\textcircled{\dagger}}$, then (4.9) coincides with (4.11), and (4.10) coincides with (4.12). $\square$

Acknowledgments. We would like to thank the editor and reviewers for their valuable comments and suggestions which help us to improve the quality of the paper. The first author is supported by the Postgraduate Research and Practice Innovation Program of Jiangsu Province (No. KYCX21_0078). The second author is supported by the National Natural Science Foundation of China (Nos. 12171083, 11871145 and 12071070) and the Qing Lan Project of Jiangsu Province.

REFERENCES

[1] O.M. Baksalary and G. Trenkler. Core inverse of matrices. *Linear Multilinear Algebra*, 58(6):681–697, 2010.
[2] A. Ben-Israel and T.N.E. Greville. *Generalized Inverses: Theory and Applications*, 2nd edition. Springer, New York, 2003.
[3] S.L. Campbell. *Singular Systems of Differential Equations*. Pitman, London, 1980.
[4] S.L. Campbell and C.D. Meyer. Weak Drazin inverses. *Linear Algebra Appl.*, 20(2):167–178, 1978.
[5] S.L. Campbell and C.D. Meyer. *Generalized Inverses of Linear Transformations*. SIAM, Philadelphia, 2009.
[6] C.G. Cao and X. Zhang. The generalized inverse $A^{(2)}_{T,*}$ and its applications. *J. Appl. Math. Comput.*, 11(1–2):155–164, 2003.
[7] D.S. Cvetković-Ilić. A note on the representation for the Drazin inverse of 2×2 block matrices. *Linear Algebra Appl.*, 429:242–248, 2008.
[8] D.S. Cvetković-Ilić and Y.M. Wei. *Algebraic Properties of Generalized Inverses*. Springer, Singapore, 2017.
[9] M.P. Drazin. Pseudo-inverses in associative rings and semigroups. *Amer. Math. Monthly*, 65(7):506–514, 1958.
[10] D.E. Ferreyra, F.E. Levis, A.N. Priori, and N. Thome. The weak core inverse. *Aequationes Math.*, 95(2):351–373, 2021.
[11] D.E. Ferreyra, F.E. Levis, and N. Thome. Characterizations of k-commutative equalities for some outer generalized inverses. *Linear Multilinear Algebra*, 68(1):177–192, 2020.
[12] R.E. Hartwig and K. Spindelböck. Matrices for which A^* and $A^\dagger$ commute. *Linear Multilinear Algebra*, 14(3):241–256, 1983.
[13] R.E. Hartwig, G.R. Wang, and Y.M. Wei. Some additive results on Drazin inverse. *Linear Algebra Appl.*, 322:207–217, 2001.
[14] S.B. Malik and N. Thome. On a new generalized inverse for matrices of an arbitrary index. *Appl. Math. Comput.*, 226:575-580, 2014.
[15] K. Manjunatha Prasad and K.S. Mohana. Core-EP inverse. *Linear Multilinear Algebra*, 62(6):792–802, 2014.
[16] C.D. Meyer and N.J. Rose. The index and the Drazin inverse of block triangular matrices. *SIAM J. A. Math.*, 33(1):1–7, 1977.
[17] R. Penrose. A generalized inverse for matrices. *Proc. Cambridge Philos. Soc.*, 51:406–413, 1955.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 39, pp. 1-16, January 2023.

[18] G.R. Wang, Y.M. Wei, and S.Z. Qiao. *Generalized Inverses: Theory and Computations*, 2nd edition. Science Press, Beijing, 2018.

[19] H.X. Wang. Core-EP decomposition and its applications. *Linear Algebra Appl.*, 508:289–300, 2016.

[20] H.X. Wang and J.L. Chen. Weak group inverse. *Open Math.*, 16:1218–1232, 2018.

[21] H.X. Wang and X.J. Liu. EP-nilpotent decomposition and its applications. *Linear Multilinear Algebra*, 68(8):1682–1694, 2020.