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FIELDS OF U-INVARIANTS OF MATRIX TUPLES *

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Abstract. The general linear group GL(n) acts on the direct sum of m copies of Mat(n) by the adjoint action. The action of GL(n) induces the action of the unitriangular subgroup U. We present the system of free generators of the field of U-invariants.

Key words. Theory of invariants, Adjoint representation, Matrix tuple, Unitriangular subgroup.

AMS subject classifications. 15A15, 15A72, 13A50.

1. Introduction. One of the classical problems of the theory of invariants is the problem of description of invariants of matrix tuples. Let K be an arbitrary field. Let $\mathrm{Mat}(n)$ stand for the linear space of $(n\times n)$-matrices with entries in the field K and $\mathrm{GL}(n)$ be the general linear group of order n. Let us consider the linear space

$$\mathcal{H} = \mathrm{Mat}(n)\oplus\ldots\oplus\mathrm{Mat}(n)$$

of m-tuples of $\mathrm{Mat}(n)$. The group $\mathrm{GL}(n)$ acts on $\mathcal{H}$ by the formula:

$$\mathrm{Ad}_g(X_1,\ldots,X_m) = (gX_1g^{-1},\ldots,gX_mg^{-1}),$$

where $g\in\mathrm{GL}(n)$ and $X_1,\ldots,X_m\in\mathrm{Mat}(n)$. The action of the group $\mathrm{GL}(n)$ on $\mathcal{H}$ defines the representation:

$$\rho(g)f(X_1,\ldots,X_m) = f(g^{-1}X_1g,\ldots,g^{-1}X_mg),$$

of the group $\mathrm{GL}(n)$ in the space of regular functions $K[\mathcal{H}]$. This representation is extended to the action of $\mathrm{GL}(n)$ on the field of rational functions $K(\mathcal{H})$. For a given subgroup $G\subseteq GL(n)$, the problem is to describe the algebra (respectively, the field) of invariants with respect to the action of G on $\mathcal{H}$.

In the case $G=\mathrm{GL}(n)$ (or $G=\mathrm{SL}(n)$), this problem is solved in the framework of the classical theory of invariants in tensors (see [1, 2, 3]). For K of zero characteristic, the algebra of $\mathrm{GL}(n)$-invariants is generated by the system of polynomials $\mathrm{Tr}(A_{i_1}\cdots A_{i_p})$, where $1\leqslant i_1,\ldots,i_p\leqslant m$.

The group $\mathrm{GL}(n)$ contains the subgroup of unitriangular matrices $U=\mathrm{UT}(n)$, which consists of the upper triangular matrices with ones on the diagonal. We consider the algebra of U-invariants $K[\mathcal{H}]^U$ and the field of U-invariants $K(\mathcal{H})^U$.

Little is known about the structure of the algebra of invariants $K[\mathcal{H}]^U$. It follows from [1, Theorem 3.13] that the algebra $K[\mathcal{H}]^U$ is finitely generated. For $K[\mathcal{H}]^U$, the problem of construction of system of generators with their relations is an unsolved problem even for $m=1$.

The structure of the field $K(\mathcal{H})^U$ is much simpler. The group U acts on $\mathcal{H}$ by unipotent transformations. It implies that

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(1) the field $K(\mathcal{H})^U$ is rational [4], that is, it is a pure transcendental extension over the main field K,
(2) the field $K(\mathcal{H})^U$ is a field of fractions of the algebra of invariants $K[\mathcal{H}]^U$ [1, Theorem 3.3].

The goal of this paper is to construct a system of free generators of the field of U-invariants $K(\mathcal{H})^U$. For the adjoint action of U on $\mathrm{Mat}(n)$ (the case $m = 1$), the system generators of the field of U-invariants is presented in Theorem 2.2. This system of generators is not unique. Another system of free generators of $K(\mathrm{Mat}(n))^U$ was constructed in [5, 6].

For an arbitrary m, the system of free generators of $K(\mathcal{H})^U$ is presented in Theorem 3.1. We apply invariants $\{P_{ik}(X, Y)\}$ from (2.1). The similar invariants were earlier used in the papers [7, 8].

2. Field of U-invariants on $\mathrm{Mat}(n)$. For $m = 2$, we have $\mathcal{H}_2 = \mathrm{Mat}(n) \oplus \mathrm{Mat}(n)$. Let $\{x_{ij}\}_{i,j=1}^n$ and $\{y_{ij}\}_{i,j=1}^n$ be the systems of standard coordinate functions on the first and second components of $\mathcal{H}_2$. Consider two matrices:

$$X = (x_{ij})_{i,j=1}^n \qquad \text{and} \qquad Y = (y_{ij})_{i,j=1}^n.$$

For two positive integers a and b, we denote by $[a, b]$ the subset of integers $a \leqslant i \leqslant b$. For any integer $1 \leqslant i \leqslant n$, let i' be the symmetric number to i with respect to the center of the segment $[1, n]$. We have $i' = n - i + 1$.

For the pair $i' \leqslant j$ (i.e. (i, j) lies on or below the anti-diagonal), let $M_{ij}(X)$ be the minor of order i' of the matrix X with the system of rows $[i, n]$ and columns $[1, i' - 1] \sqcup \{j\}$.

For the pair $j \leqslant k$, let $N_{jk}(Y)$ be the minor of order k' of the matrix Y with the system of rows $\{j\} \sqcup [k + 1, n]$ and columns $[1, k']$.

Example. For $n = 5$, we have

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & x_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} \end{pmatrix}, \quad M_{34}(X) = \begin{vmatrix} x_{31} & x_{32} & x_{34} \\ x_{41} & x_{42} & x_{44} \\ x_{51} & x_{52} & x_{54} \end{vmatrix}, \quad M_{45}(X) = \begin{vmatrix} x_{41} & x_{45} \\ x_{51} & x_{55} \end{vmatrix}, \quad M_{53}(X) = x_{53},$$

$$Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{14} & y_{15} \\ y_{21} & y_{22} & y_{23} & y_{24} & y_{25} \\ y_{31} & y_{32} & y_{33} & y_{34} & y_{35} \\ y_{41} & y_{42} & y_{43} & y_{44} & y_{45} \\ y_{51} & y_{52} & y_{53} & y_{54} & y_{55} \end{pmatrix}, \quad N_{23}(Y) = \begin{vmatrix} y_{21} & y_{22} & y_{23} \\ y_{41} & y_{42} & y_{43} \\ y_{51} & y_{52} & y_{53} \end{vmatrix}, \quad N_{14}(Y) = \begin{vmatrix} y_{11} & y_{12} \\ y_{51} & y_{52} \end{vmatrix}, \quad N_{35}(Y) = y_{31}.$$

Let $i' < k$. This is equivalent to the pair (i, k) lies below the anti-diagonal. We define the polynomial:

$$P_{ik}(X, Y) = \sum_{i' \leqslant j \leqslant k} M_{ij}(X) N_{jk}(Y). \tag{2.1}$$

For each $1 \leqslant i \leqslant n$, let $D_k(X)$ stand for the lower left corner minor of order k' of the matrix X. Observe that

$$D_k(X) = M_{k,k'}(X) = N_{k,k}(X).$$

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Example. For $n = 3$, we have

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \qquad Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix},$$

$$D_3(X) = x_{31}, \quad D_2(X) = \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix}, \quad D_1(X) = \det(X),$$

$$D_3(Y) = y_{31}, \quad D_2(Y) = \begin{vmatrix} y_{21} & y_{22} \\ y_{31} & y_{32} \end{vmatrix}, \quad D_1(Y) = \det(Y),$$

$$P_{32}(X,Y) = \sum_{1\leqslant j\leqslant 2} M_{3j}(X)N_{j2}(Y) = M_{31}(X)N_{12}(Y) + M_{32}(X)N_{22}(Y) = x_{31}\begin{vmatrix} y_{11} & y_{12} \\ y_{31} & y_{32} \end{vmatrix} + x_{32}\begin{vmatrix} y_{21} & y_{22} \\ y_{31} & y_{32} \end{vmatrix},$$

$$P_{33}(X,Y) = \sum_{1\leqslant j\leqslant 3} M_{3j}(X)N_{j3}(Y) = x_{31}y_{11} + x_{32}y_{21} + x_{33}y_{31},$$

$$P_{23}(X,Y) = \sum_{2\leqslant j\leqslant 3} M_{2j}(X)N_{j3}(Y) = M_{22}(X)N_{23}(Y) + M_{23}(X)N_{33}(Y) = \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix} y_{21} + \begin{vmatrix} x_{21} & x_{23} \\ x_{31} & x_{33} \end{vmatrix} y_{31}.$$

Proposition 2.1. *The polynomials* $\{P_{ik}(X,Y) : \ i' < k\}$, *and* $D_k(X), \ D_k(Y)$, *where* $1 \leqslant k \leqslant n$, *are* U*-invariant.*

Proof. It is obvious that the corner minors $D_k(X), \ D_k(Y)$ are U-invariant.

For an arbitrary simple root $\alpha = (a, a+1)$, we consider the subgroup $s(t) = \exp(tE_\alpha)$, where $t \in K$ and $E_\alpha = E_{a,a+1}$ is the relative matrix unit. We denote by $\rho_\alpha(t)f(X)$ the action of $s_\alpha(t)$ on $f(X)$. We aim to show that $\rho_\alpha(t)P_{ik}(X,Y) = P_{ik}(X,Y)$, for each $i' < k$ and $\alpha = (a, a+1)$.

We obtain

$$\rho_\alpha(t)M_{ij}(X) = \begin{cases} M_{i,a+1}(X) + tM_{ia}(X), & \text{if } \ i' < j = a+1, \\ M_{ij}(X), & \text{in other cases,} \end{cases}$$

$$\rho_\alpha(t)N_{jk}(Y) = \begin{cases} N_{ak}(Y) - tN_{a+1,k}(Y), & \text{if } \ j = a < k, \\ N_{jk}(Y), & \text{in other cases.} \end{cases}$$

Suppose that $i' \leqslant a < a+1 \leqslant k$. Then,

$$\rho_\alpha(t)P_{ik}(X,Y) = M_{ia}(X)(N_{ak}(Y) - tN_{a+1,k}(Y)) + (M_{i,a+1}(X) + tM_{ia})(X)N_{a+1,k}(Y) + \sum_{\substack{i' \leqslant j \leqslant k, \\ j \neq a, a+1}} M_{ij}(X)N_{jk}(Y) =$$

$$M_{ia}(X)N_{ak}(Y) + M_{i,a+1}(X)N_{a+1,k}(Y) + \sum_{\substack{i' \leqslant j \leqslant k, \\ j \neq a, a+1}} M_{ij}(X)N_{jk}(Y) = P_{ik}(X,Y).$$

The polynomial $P_{ik}(X,Y)$, where $i' < k$, is U-invariant.

If the condition $i' \leqslant a < a+1 \leqslant k$ is not true, then all minors $M_{ij}(X), \ N_{jk}(Y)$, where $i' \leqslant j \leqslant k$, are U-invariant, and, therefore, $P_{ik}(X,Y), \ i' < k$, is U-invariant. ∎

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We denote

$$P_{ik}(X) = P_{ik}(X, X) = \sum_{i' \leqslant j \leqslant k} M_{ij}(X) N_{jk}(X). \tag{2.2}$$

COROLLARY 2.1. *The polynomials* $\{P_{ik}(X) : \ i' < k\}$ *are* U*-invariants.*

THEOREM 2.2. . *The field* $K(\mathrm{Mat}(n))^U$ *is freely generated over* K *by the system of polynomials:*

$$\{P_{ik}(X) : \ i' < k\} \sqcup \{D_k(X) : \ 1 \leqslant k \leqslant n\}.$$

Proof. Let B be the subgroup of upper triangular matrices and w_0 be the element of greatest length in the Weyl group.

Let $\mathcal{S}$ stand for the subspace of matrices of the form:

$$S = \begin{pmatrix} 0 & 0 & \dots & s_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & s_{n-1,2} & \dots & s_{n-1,n} \\ s_{n1} & s_{n2} & \dots & s_{nn} \end{pmatrix}. \tag{2.3}$$

The subset $w_0 B$ is dense in $\mathcal{S}$. We denote by π the restriction map $K[\mathrm{Mat}(n)]^U \to K[\mathcal{S}]$. Since the Bruhat cell Bw_0B is dense in $\mathrm{GL}(n)$, the subset

$$\bigcup_{g \in U} g \mathcal{S} g^{-1}$$

is dense in $\mathrm{Mat}(n)$. It implies that π is an embedding $K[\mathrm{Mat}(n)]^U \hookrightarrow K[\mathcal{S}]$. The embedding π extends to the embedding of fields

$$K(\mathrm{Mat}(n))^U \hookrightarrow K(\mathcal{S}).$$

Let $\mathcal{F}_0$ be the subfield of $K(\mathrm{Mat}(n))^U$ generated by $\{D_k(X) : \ 1 \leqslant k \leqslant n\}$. Respectively, let $\mathcal{P}_0$ be the subfield of $K(\mathcal{S})$ generated by the elements $\{s_{k,k'} : 1 \leqslant k \leqslant n\}$ (i.e. the elements on the anti-diagonal). Since $\pi(D_k(X)) = \pm s_{n1} s_{n-1,2} \dots s_{k,k'}$ for each $1 \leqslant k \leqslant n$, the embedding π establishes the isomorphism $\mathcal{F}_0 \to \mathcal{P}_0$.

For each pair $j < k$, the minor $N_{jk}(S)$ of an arbitrary matrix S from $\mathcal{S}$ has the zero first row, and, therefore, it equals to zero. Then,

$$\pi(P_{ik}(X)) = \pi(M_{ik}(X) N_{kk}(X)) = M_{ik}(S) N_{kk}(S).$$

By definition $N_{kk}(S) = D_k(S)$. As $i' < k$,

$$M_{ik}(S) = \begin{vmatrix} 0 & \dots & 0 & s_{i,k} \\ 0 & \dots & s_{i+1,(i+1)'} & s_{i+1,k} \\ \vdots & \ddots & \vdots & \vdots \\ s_{n1} & \dots & s_{n,(i+1)'} & s_{n,k} \end{vmatrix} = \pm D_{i+1}(S) s_{ik}.$$

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We obtain

$$\pi(P_{ik}(X)) = \pm D_{i+1}(S)D_k(S)s_{ik}. \tag{2.4}$$

The elements $\{s_{ik} : \ i' < k\}$ freely generate the field $K(\mathcal{S})$ over the subfield $\mathcal{P}_0$. The formula (2.4) implies that π isomorphically maps $K(\mathrm{Mat}(n))^U$ onto $K(\mathcal{S})$, and the system of elements $\{P_{ik}(X) : \ i' < k\}$ freely generates the field $K(\mathrm{Mat}(n))^U$ over $\mathcal{F}_0$. It implies the claim of the theorem. ∎

We need the following definition.

Definition 2.3. *Let $\{\xi_\alpha : \alpha \in \mathfrak{A}\}$ and $\{\eta_\alpha : \alpha \in \mathfrak{A}\}$ be two finite systems of free generators of an extension F of the field K. Let $\prec$ be a linear order on $\mathfrak{A}$. We say that the second system of generators is obtained from the first one by a triangular transformation if each η_α can be presented in the form:*

$$\eta_\alpha = \phi_\alpha \xi_\alpha + \psi_\alpha, \tag{2.5}$$

where $\phi_\alpha \neq 0$ and $\phi_\alpha, \ \psi_\alpha$ belong to the subfield generated by $\{\xi_\beta : \ \beta \prec \alpha\}$.

Remark 2.4. *Using the induction method, it is easy to prove that if $\{\xi_\alpha : \alpha \in \mathfrak{A}\}$ is a system of free generators of a field F and the other system $\{\eta_\alpha : \alpha \in \mathfrak{A}\}$ is linked with the first one by formulas (2.5), then it also freely generates F.*

3. U-invariants of matrix tuples. As in the introduction $\mathcal{H} = \mathrm{Mat}(n) \oplus \ldots \oplus \mathrm{Mat}(n)$ is a sum of m copies of $\mathrm{Mat}(n)$. In this section, we aim to construct a system of free generators of $K(\mathcal{H})^U$. We consider the following systems of polynomials:

$$\mathbb{P}_{1,\ell} = \{P_{ik}(X_1, X_\ell) : \ \ 1 \leqslant i' < k \leqslant n\} \ \text{ for each } \ 2 \leqslant \ell \leqslant m,$$

$$\mathbb{P}_\ell = \{P_{ik}(X_\ell) : \ \ 1 \leqslant i' < k \leqslant n\} \ \text{ for each } \ 1 \leqslant \ell \leqslant m,$$

$$\mathbb{D}_\ell = \{D_k(X_\ell) : 1 \leqslant k \leqslant n\} \ \text{ for each } \ 1 \leqslant \ell \leqslant m.$$

Theorem 3.1. *The field $K(\mathcal{H})^U$ is freely generated over K by the system of polynomials:*

$$\mathbb{B} = \left(\bigcup_{\ell=2}^m \mathbb{P}_{1,\ell}\right) \cup \left(\bigcup_{\ell=1}^m \mathbb{P}_\ell\right) \cup \left(\bigcup_{\ell=1}^m \mathbb{D}_\ell\right).$$

Proof. The adjoint action $Ad_g, \ g \in U$, on $\mathcal{H}$ has the section $\mathcal{S}_\mathcal{H}$ that consists of matrix m-tuples of the form $(S, X_2, \ldots, X_m)$, where S is a matrix from (2.3) and $X_2, \ldots, X_m$ is an arbitrary $(n \times n)$-matrices. The restriction map $\pi : K[\mathcal{H}]^U \to K[\mathcal{S}_\mathcal{H}]$ is an embedding, and it extends to the embedding of fields:

$$\pi : K(\mathcal{H})^U \hookrightarrow K(\mathcal{S}_\mathcal{H}).$$

Let us show that the system of polynomials $\pi(\mathbb{B})$ in $\mathcal{S}_\mathcal{H}$ freely generates the field $K(\mathcal{S}_\mathcal{H})$. It follows from Theorem 2.2 that $\pi(\mathbb{P}_1 \cup \mathbb{D}_1)$ freely generates $K(\mathcal{S})$.

We denote $K_0 = K(\mathcal{S})$. It is sufficient to prove that for each $2 \leqslant \ell \leqslant m$ the system of polynomials

$$\pi(\mathbb{P}_{1,\ell}) \cup \mathbb{P}_\ell \cup \mathbb{D}_\ell$$

freely generates the field $K_0(X_\ell)$. We simplify notations $Y = X_\ell = (y_{ij})_{i,j=1}^n$. Respectively $K_0(X_\ell) = K_0(Y)$.

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We denote $P_{ik}(S,Y) = \pi(P_{ik}(X,Y))$, where $S = (s_{ij})$ is a matrix of the form (2.3) with elements $s_{ij} = \pi(x_{ij})$. The formula (2.1) takes the form:

$$P_{ik}(S,Y) = \sum_{i'\leqslant j\leqslant k} M_{ij}(S)N_{jk}(Y) = \tag{3.6}$$
$$M_{i,i'}(S)N_{i',k}(Y) + M_{i,i'+1}(S)N_{i'+1,k}(Y) + \ldots + M_{i,k}(S)N_{k,k}(Y).$$

Define the linear order on the set of pairs $\{(a,b) :\ 1 \leqslant a,b \leqslant m\}$ as follows:

$$(a,b) \prec (a_1,b_1), \quad \text{if} \quad b < b_1, \quad \text{or} \quad b = b_1 \quad \text{and} \quad a > a_1.$$

Consider the subfield $\mathcal{Q}_0$ in $K_0(Y)$ generated by $\mathbb{D}_\ell = \{D_k(Y) : 1 \leqslant k \leqslant n\}$. The field $\mathcal{Q}_0$ is freely generated by the system $\mathbb{D}_\ell$ over K_0.

Let $\mathcal{Q}_1$ be an extension of the field $\mathcal{Q}_0$ by the system of polynomials:

$$\mathbb{N} = \{N_{jk}(Y) :\ 1 \leqslant j < k \leqslant n\}.$$

The system of polynomials $\mathbb{N}$ is algebraically independent and freely generates the field $\mathcal{Q}_1$ over $\mathcal{Q}_0$.

The elements $\pi(\mathbb{P}_{1,\ell})$ belong to $\mathcal{Q}_1$, since all minors $N_{jk}(Y)$, $j \leqslant k$, belong to $\mathcal{Q}_1$, and $M_{ij}(S)$, $i' \leqslant j$, belong to K_0.
Item 1. Let us show that $\pi(\mathbb{P}_{1,\ell})$ freely generates the field $\mathcal{Q}_1$ over $\mathcal{Q}_0$.

Consider the linear order $\prec$ on $\mathbb{N}$. According to the formula (3.6), we get

$$P_{ik}(S,Y) = M_{i,i'}(S)N_{i',k}(Y) + \{\text{terms} \quad \text{of} \quad \text{lower} \quad \text{order }\}.$$

The greatest coefficient $M_{i,i'}(S)$ coincides with the corner minor $D_i(S) \neq 0$. The system of polynomials $\pi(\mathbb{P}_{1,\ell})$ is obtained from $\mathbb{N}$ by a triangular transformation. According to the remark at the end of previous section, the system of polynomials $\pi(\mathbb{P}_{1,\ell})$ freely generates the field $\mathcal{Q}_1$ over $\mathcal{Q}_0$.
Item 2. Let us show that $\pi(\mathbb{P}_\ell)$ freely generates $K_0(Y)$ over $\mathcal{Q}_1$.

Easy to see that the system of matrix entries $\mathbb{Y} = \{y_{i,k} : i' < k\}$ freely generates $K_0(Y)$ over $\mathcal{Q}_1$.

The formula (2.2) implies

$$P_{ik}(Y) = \sum_{i'\leqslant j\leqslant k} M_{ij}(Y)N_{jk}(Y) = \tag{3.7}$$
$$M_{i,i'}(Y)N_{i',k}(Y) + \ldots + M_{i,k-1}(Y)N_{k-1,k}(Y) + M_{ik}(Y)N_{k,k}(Y).$$

Consider the linear order $\prec$ on $\mathbb{Y}$. Expanding the minor $M_{ik}(Y)$ along its first row, we get

$$M_{ik}(Y) = \pm D_{i+1}(Y)y_{ik} + \{\text{expression} \ \ \text{over} \ \ K \ \ \text{in} \ \ y_{ab}, \ \ (a,b) \prec (i,k)\}.$$

Taking into account $N_{kk}(Y) = D_k(Y)$, we obtain

$$P_{ik}(Y) = \pm D_{i+1}(Y)D_k(Y)y_{ik} + \{\text{expression} \ \ \text{over} \ \ \mathcal{Q}_1 \ \ \text{in} \ \ y_{ab}, \ \ (a,b) \prec (i,k)\}.$$

It follows that the system of polynomials $\pi(\mathbb{P}_{1,\ell})$ is obtained from $\mathbb{Y}$ by a triangular transformation. Therefore, $\pi(\mathbb{P}_{1,\ell})$ freely generates the field $K_0(Y)$ over $\mathcal{Q}_1$. □

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