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ON EIGENVALUES OF REAL SYMMETRIC INTERVAL MATRICES: SHARP BOUNDS AND DISJOINTNESS*

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Abstract. In this paper, the eigenvalue problem of real symmetric interval matrices is studied. First, in the case of 2×2 real symmetric interval matrices, all the four endpoints of the two eigenvalue intervals are determined. These are not necessarily eigenvalues of vertex matrices, but it is shown that such a real symmetric interval matrix can be constructed from the original one. Then, necessary and sufficient conditions are provided for the disjointness of eigenvalue intervals. In the general $n \times n$ case, due to Hertz, a set of special vertex matrices determines the maximal eigenvalue and a similar statement holds for the minimal one. In a special case, namely if the right endpoints of the off-diagonal intervals are not smaller than the absolute value of the left ones, he concluded the vertex matrix of the right endpoints provides the maximal eigenvalue. Generalizing it, it is shown that in the case of real symmetric interval matrices with special sign pattern, a single vertex matrix determines one of the extremal bounds.

Key words. Real symmetric interval matrix, Interval eigenvalue problem, Eigenvalue bounds, Vertex matrix of an interval matrix, Disjointness of eigenvalue intervals.

AMS subject classifications. 15A18, 65G99.

1. Introduction.

In many real-life problems, we encounter some kind of uncertainty which can be measurement error, manufacturing mistake or one originating from machine representation of numbers. Applying intervals can be one of the ways to handle these problems. Interval analysis ([9]) was proposed in the 1960s, and it has developed dynamically since then. One of its branches, the problem of computing the eigenvalue bounds of interval matrices, originating from the eigenvalue problem of perturbated matrices ([11]), has been studied since the 1990s. Deif ([1]) has first introduced the interval eigenvalue problem. Then, Rohn ([10]) gave bounds for the real and imaginary parts of complex eigenvalues. Hertz ([2, 3]) proposed a formula to determine the extremal eigenvalues of symmetric interval matrices, and Jian ([8]), in the tridiagonal case, also gave the other two exact bounds of the extremal eigenvalue intervals. Furthermore, Hladik et al. ([5, 6, 7]) and Su et al. ([12]) introduced several algorithms to approximate the eigenvalue set.

In this paper, we give the exact eigenvalue bounds of 2×2 real symmetric interval matrices and study the disjointness of eigenvalue intervals. Then in the general $n \times n$ case, we show that, concerning real symmetric interval matrices with special sign pattern, a single vertex matrix provides one of the extremal bounds.

2. Basic notations and lemmas.

First, some notations and lemmas are introduced concerning real symmetric interval matrices and basic linear algebra.

Let $A, B \in \mathbb{R}^{n \times m}$. Then, $A \leq B$ if $a_{ij} \leq b_{ij},\ 1 \leq i \leq n,\ 1 \leq j \leq m$. Define the real square interval matrix $\boldsymbol{A}$ and the real symmetric interval matrix $\boldsymbol{A}^S$ as

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$$\boldsymbol{A} := \left[\underline{A}, \overline{A}\right] = \left\{A \in \mathbb{R}^{n\times n} : \underline{A}, \overline{A} \in \mathbb{R}^{n\times n}, \underline{A} \le A \le \overline{A}\right\}$$

and

$$\boldsymbol{A}^S := \left\{A \in \boldsymbol{A} : A = A^T\right\}.$$

Let $\mathcal{S}\left[\underline{A}, \overline{A}\right]$ be an alternative notation for $\boldsymbol{A}^S$. By

$$A_c := \frac{1}{2}\left(\underline{A} + \overline{A}\right),$$

the midpoint matrix of $\boldsymbol{A}$ is denoted. Next, we recall some lemmas regarding basic linear algebra.

Lemma 2.1. *A symmetric matrix $A \in \mathbb{R}^{n\times n}$ has n real eigenvalues.*

The eigenvalues of a symmetric matrix $A \in \mathbb{R}^{n\times n}$ are ordered in a nonincreasing manner: $\lambda_1(A) \ge \lambda_2(A) \ge \cdots \ge \lambda_n(A)$. Henceforth, the first and the last of these, namely $\lambda_1(A)$ and $\lambda_n(A)$, are called extremal eigenvalues for which the following holds.

Lemma 2.2. $\max\limits_{\|x\|_2=1} x^T Ax = \lambda_1(A)$ *and* $\min\limits_{\|x\|_2=1} x^T Ax = \lambda_n(A)$ *where the matrix A is symmetric and the vectors, for which the extrema are attained, are the corresponding eigenvectors.*

Extending the notation of eigenvalues, the j-th eigenvalue set of the symmetric interval matrix $\boldsymbol{A}^S$ is defined as $\boldsymbol{\lambda}_j(\boldsymbol{A}^S) := \left\{\lambda_j(A) : A \in \boldsymbol{A}^S\right\}$. It is a compact interval which is stated in the following lemma.

Lemma 2.3. $\boldsymbol{\lambda}_j(\boldsymbol{A}^S) = [\underline{\boldsymbol{\lambda}_j}(\boldsymbol{A}^S), \overline{\boldsymbol{\lambda}_j}(\boldsymbol{A}^S)]$ $(j = 1, \ldots, n)$.

Proof. The roots of the characteristic polynomial vary continuously with their coefficients, and $\boldsymbol{A}^S$ is compact. Thus, the eigenvalue set is the union of n compact intervals. ∎

3. Hertz's theorem. In this section, Hertz's theorem, concerning the bounds $\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right)$ and $\underline{\boldsymbol{\lambda}_n}\left(\boldsymbol{A}^S\right)$, is introduced. It says the maximal and minimal eigenvalue of a real symmetric interval matrix coincide with the maximal and minimal eigenvalue of special sets of 2^{n-1} symmetric vertex matrices, respectively. To make this precise, the following notations are introduced.

Let $V[P, Q]$ be the set of symmetric vertex matrices determined by the symmetric matrices P and Q, with notation,

$$V[P, Q] := \{A \in \mathcal{S}[P, Q] : a_{kl} \in \{p_{kl}, q_{kl}\}\}.$$

Let $\{O_i\}_{i=1}^{2^{n-1}}$ denote the set of orthant pairs where orthants are paired with their opposites and are ordered in reverse binary order, that is, the first pair is $(+, +, \ldots, +) - (-, -, \ldots, -)$ and the last one is $(+, -, \ldots, -) - (-, +, \ldots, +)$. B^n stands for the n-dimensional unit sphere in Euclidean norm and let

$$B_i := B^n \cap O_i, \qquad 1 \le i \le 2^{n-1}.$$

Obviously,

$$B^n = \bigcup_{i=1}^{2^{n-1}} B_i.$$

The matrices $\overline{A}^i$ are defined as follows:

$$\overline{a}^i_{kl} := \begin{cases} q_{kk} & \text{if } l = k \\ q_{kl} & \text{if } x_k x_l \ge 0,\ l \ne k \text{ and } x \in B_i \\ p_{kl} & \text{if } x_k x_l < 0,\ l \ne k \text{ and } x \in B_i \end{cases} \quad 1 \le i \le 2^{n-1}, \tag{1}$$

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thus,

$$\overline{A}^i \in V[P,Q].$$

Since for a symmetric matrix A

$$x^T A x = \sum_{k=1}^{n} a_{kk} x_k^2 + 2 \sum_{1 \le k < l \le n} a_{kl} x_k x_l, \tag{2}$$

it can be noticed that

$$\overline{A}^i \in \operatorname*{argmax}_{A \in \boldsymbol{A}^S} x^T A x \quad \forall x \in B_i, \qquad 1 \le i \le 2^{n-1}. \tag{3}$$

Namely, $\overline{A}^i$ is a symmetric vertex matrix maximizing the quadratic form in (2) for each $x \in B_i$.

Similarly, let

$$\underline{a}^i_{kl} = \begin{cases} p_{kk} & \text{if } l = k \\ p_{kl} & \text{if } x_k x_l \ge 0,\ l \ne k \text{ and } x \in B_i \\ q_{kl} & \text{if } x_k x_l < 0,\ l \ne k \text{ and } x \in B_i \end{cases} \quad 1 \le i \le 2^{n-1}, \tag{4}$$

thus,

$$\underline{A}^i \in V[P,Q]$$

and

$$\underline{A}^i \in \operatorname*{argmin}_{A \in \boldsymbol{A}^S} x^T A x \quad \forall x \in B_i, \qquad 1 \le i \le 2^{n-1}. \tag{5}$$

To become familiar with these notations, see Examples 5.5, 5.6 and 5.7. Next, the theorem is announced.

Theorem 3.1 (Hertz's theorem [2, 4]).

$$\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right) = \max_{1 \le i \le 2^{n-1}} \lambda_1\left(\overline{A}^i\right) \tag{6}$$

$$\underline{\boldsymbol{\lambda}_n}\left(\boldsymbol{A}^S\right) = \min_{1 \le i \le 2^{n-1}} \lambda_n\left(\underline{A}^i\right) \tag{7}$$

4. Sharp bounds and disjointness of eigenvalue intervals of 2×2 real symmetric interval matrices. By Hertz's theorem, special sets of vertex matrices determine the sharp bounds $\underline{\boldsymbol{\lambda}_n}\left(\boldsymbol{A}^S\right)$ and $\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right)$.

In this section, regarding 2×2 real symmetric interval matrices, it is proposed that for each of the four endpoints of eigenvalue intervals, a single matrix can be chosen whose maximal or minimal eigenvalue coincides with that. In a certain case, these are vertex matrices. Otherwise, a real symmetric interval matrix can be constructed to do so.

In the remaining part of the section, the mutual position of eigenvalue intervals is studied. Depending on the relative positions of the diagonal intervals and the sign of the off-diagonal intervals, necessary and sufficient conditions are given for the disjointness of eigenvalue intervals.

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4.1. Middle bounds. In this subsection, results, regarding the endpoints $\overline{\lambda_2}\left(\boldsymbol{A}^S\right)$ and $\underline{\lambda_1}\left(\boldsymbol{A}^S\right)$, are proposed. Moreover, in the remaining part of the section, the following notation is used:

$$\boldsymbol{A}^S = \begin{pmatrix} [a_1, a_2] & [b_1, b_2] \\ [b_1, b_2] & [c_1, c_2] \end{pmatrix}.$$

THEOREM 4.1.

$$\overline{\boldsymbol{\lambda}_2}\left(\boldsymbol{A}^S\right) = \begin{cases} \min\{a_2, c_2\} & \textit{if } b_1 \le 0 \le b_2 \\ \lambda_2(A) & \textit{if } 0 < b_1 \le b_2 \textit{ where } A = \begin{pmatrix} a_2 & b_1 \\ b_1 & c_2 \end{pmatrix} \\ \lambda_2(A) & \textit{if } b_1 \le b_2 < 0 \textit{ where } A = \begin{pmatrix} a_2 & b_2 \\ b_2 & c_2 \end{pmatrix} \end{cases} \tag{8}$$

and

$$\underline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right) = \begin{cases} \max\{a_1, c_1\} & \textit{if } b_1 \le 0 \le b_2 \\ \lambda_1(A) & \textit{if } 0 < b_1 \le b_2 \textit{ where } A = \begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix} \\ \lambda_1(A) & \textit{if } b_1 \le b_2 < 0 \textit{ where } A = \begin{pmatrix} a_1 & b_2 \\ b_2 & c_1 \end{pmatrix} \end{cases}. \tag{9}$$

Proof.

(1) $b_1 \le 0 \le b_2$
By determining the roots of the characteristic polynomial, which are

$$\lambda_{1,2}(A) = \frac{a + c \pm \sqrt{4b^2 + (a-c)^2}}{2} \tag{10}$$

where $A = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \in \boldsymbol{A}^S$, the maximum of $\lambda_2(A)$ is attained at $b = 0$. Furthermore,

$$\begin{aligned} \overline{\boldsymbol{\lambda}_2}\left(\boldsymbol{A}^S\right) &= \max_{A \in \boldsymbol{A}^S} \lambda_2(A) = \max \frac{a + c - \sqrt{4b^2 + (a-c)^2}}{2} = \\ &= \max \frac{a + c - |a - c|}{2} = \begin{cases} a_2 & \text{if } a_2 \le c_2 \\ c_2 & \text{if } a_2 > c_2 \end{cases} = \min\{a_2, c_2\}. \end{aligned}$$

Similarly,

$$\underline{\lambda_1}\left(\boldsymbol{A}^S\right) = \max\{a_1, c_1\}.$$

(2) $0 < b_1 \le b_2$
Since $\frac{\partial \lambda_2}{\partial a} = \frac{1}{2}\left(1 - \frac{a-c}{\sqrt{4b^2 + (a-c)^2}}\right) > 0$ and $\frac{\partial \lambda_2}{\partial c} = \frac{1}{2}\left(1 + \frac{a-c}{\sqrt{4b^2 + (a-c)^2}}\right) > 0$ hold for the partial derivatives, the section functions of $\lambda_2(A)$, with respect to a and c, are monotone increasing. Therefore, the maximum is obtained at $a = a_2$ and $c = c_2$. Consequently, it is enough to optimize for b and b_1 maximizes. Hence,

$$\overline{\boldsymbol{\lambda}_2}\left(\boldsymbol{A}^S\right) = \lambda_2(A) \text{ where } A = \begin{pmatrix} a_2 & b_1 \\ b_1 & c_2 \end{pmatrix}.$$

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Analogically,

$$\underline{\lambda_1}\left(\boldsymbol{A}^S\right) = \lambda_1(A) \text{ where } A = \begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix}.$$

(3) $b_1 \le b_2 < 0$
It is similar to the previous case. □

Remark 4.2. If $0 \notin (b_1, b_2)$, then the middle bounds are attained by vertex matrices, given in Theorem 4.1. Moreover, based on (2) and (10), also the bounds in Hertz's theorem are determined by single well-chosen vertex matrices, one by one. These bounds are $\underline{\boldsymbol{\lambda}_2}\left(\boldsymbol{A}^S\right) = \lambda_2(A)$ and $\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right) = \lambda_1(B)$ where

$$A = \begin{cases} \begin{pmatrix} a_1 & b_2 \\ b_2 & c_1 \end{pmatrix} & \text{if } |b_1| \le |b_2| \\ \begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix} & \text{if } |b_1| > |b_2| \end{cases} \quad \text{and } B = \begin{cases} \begin{pmatrix} a_2 & b_2 \\ b_2 & c_2 \end{pmatrix} & \text{if } |b_1| \le |b_2| \\ \begin{pmatrix} a_2 & b_1 \\ b_1 & c_2 \end{pmatrix} & \text{if } |b_1| > |b_2| \end{cases}.$$

If $0 \in (b_1, b_2)$, then an interval matrix can be constructed whose vertex matrices determine its eigenvalue bounds which are the same as the original one's.

THEOREM 4.3. *Let* $0 \in (b_1, b_2)$ *and*

$$\tilde{\boldsymbol{A}}^S = \begin{pmatrix} [a_1, a_2] & [0, \max\{|b_1|, b_2\}] \\ [0, \max\{|b_1|, b_2\}] & [c_1, c_2] \end{pmatrix}.$$

Then, the eigenvalue bounds of $\boldsymbol{A}^S$ *and* $\tilde{\boldsymbol{A}}^S$ *are identical.*

Proof. Based on Remark 4.2, noticing that while alternating the sign of the off-diagonal elements of a 2×2 matrix, the eigenvalues remain the same; we get

$$\overline{\lambda_1}\left(\tilde{\boldsymbol{A}}^S\right) = \lambda_1(A_1) = \overline{\lambda_1}\left(\boldsymbol{A}^S\right), \quad \underline{\lambda_2}\left(\tilde{\boldsymbol{A}}^S\right) = \lambda_2(A_2) = \underline{\lambda_2}\left(\boldsymbol{A}^S\right)$$

where

$$A_1 = \begin{pmatrix} a_2 & \max\{|b_1|, b_2\} \\ \max\{|b_1|, b_2\} & c_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_1 & \max\{|b_1|, b_2\} \\ \max\{|b_1|, b_2\} & c_1 \end{pmatrix}.$$

By Theorem 4.1, the middle eigenvalue bounds of $\tilde{\boldsymbol{A}}^S$ are

$$\underline{\lambda_1}\left(\tilde{\boldsymbol{A}}^S\right) = \underline{\lambda_1}\left(\boldsymbol{A}^S\right) = \lambda_1(A_3), \quad \overline{\lambda_2}\left(\tilde{\boldsymbol{A}}^S\right) = \overline{\lambda_2}\left(\boldsymbol{A}^S\right) = \lambda_2(A_4)$$

where

$$A_3 = \begin{pmatrix} a_1 & 0 \\ 0 & c_1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} a_2 & 0 \\ 0 & c_2 \end{pmatrix}.$$

□

Overall, some special vertex matrices of $\tilde{\boldsymbol{A}}^S$ determine the eigenvalue bounds of $\boldsymbol{A}^S$.

4.2. Disjointness of the eigenvalue intervals. In the following part, the disjointness of eigenvalue intervals of 2×2 real symmetric interval matrices is studied. In this aspect, necessary and sufficient conditions are proposed.

THEOREM 4.4. *If* $0 \in [b_1, b_2]$, *then the disjointness of the intervals* $[a_1, a_2]$ *and* $[c_1, c_2]$ *is necessary and sufficient condition for the eigenvalue intervals not to overlap each other. If* $0 \notin [b_1, b_2]$, *then the previous condition is sufficient but not necessary.*

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Proof.

(1) $b_1 \leq 0 \leq b_2$
Necessity:

$$\min\{a_2, c_2\} = \overline{\boldsymbol{\lambda}_2}\left(\boldsymbol{A}^S\right) < \underline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right) = \max\{a_1, c_1\}$$

$$\Downarrow$$

$$a_2 < c_1 \text{ or } c_2 < a_1$$

$$\Downarrow$$

$$[a_1, a_2] \text{ and } [c_1, c_2] \text{ are disjoint.}$$

Sufficiency:
(I) $a_2 < c_1$

$$a_2 < c_1 \implies \overline{\boldsymbol{\lambda}_2}\left(\boldsymbol{A}^S\right) = a_2,\ \underline{\boldsymbol{\lambda}_1}(\boldsymbol{A}^S) = c_1$$

(II) $c_2 < a_1$

$$c_2 < a_1 \implies \overline{\boldsymbol{\lambda}_2}\left(\boldsymbol{A}^S\right) = c_2,\ \underline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right) = a_1$$

Thus, the eigenvalue intervals are disjoint.
(2) $0 < b_1 \leq b_2$
Sufficiency:
(I) $a_1 \leq a_2 < c_1 \leq c_2$

$$\begin{aligned}
c_2 - c_1 &< \sqrt{4b_1^2 + (a_2 - c_2)^2} \\
a_2 - a_1 &< \sqrt{4b_1^2 + (a_1 - c_1)^2} \\
c_2 - c_1 + a_2 - a_1 &< \sqrt{4b_1^2 + (a_2 - c_2)^2} + \sqrt{4b_1^2 + (a_1 - c_1)^2} \\
\frac{1}{2}\left(a_2 + c_2 - \sqrt{4b_1^2 + (a_2 - c_2)^2}\right) &< \frac{1}{2}\left(a_1 + c_1 + \sqrt{4b_1^2 + (a_1 - c_1)^2}\right) \\
\overline{\boldsymbol{\lambda}_2}\left(\boldsymbol{A}^S\right) &< \underline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right)
\end{aligned}$$

(II) $c_1 \leq c_2 < a_1 \leq a_2$
It is similar to the previous case.
(3) $b_1 \leq b_2 < 0$
It is similar to the previous case.

If $0 \notin [b_1, b_2]$, then the disjointness of the diagonal intervals is not a necessary condition for the disjointness of the eigenvalue intervals as the following example shows. Let

$$\boldsymbol{A}^S = \begin{pmatrix} [\frac{1}{2}, 2] & [2, 3] \\ [2, 3] & [1, 4] \end{pmatrix}.$$

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Then,

$$\underline{A}^1 = \begin{pmatrix} \frac{1}{2} & 2 \\ 2 & 1 \end{pmatrix} \qquad \lambda_2\left(\underline{A}^1\right) = \frac{1}{4}\left(3-\sqrt{65}\right)$$
$$\underline{A}^2 = \begin{pmatrix} \frac{1}{2} & 3 \\ 3 & 1 \end{pmatrix} \qquad \lambda_2\left(\underline{A}^2\right) = \frac{1}{4}\left(3-\sqrt{145}\right)$$
$$\overline{A}^1 = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \qquad \lambda_1\left(\overline{A}^1\right) = 3+\sqrt{10}$$
$$\overline{A}^2 = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix} \qquad \lambda_1\left(\overline{A}^2\right) = 3+\sqrt{5}.$$

Therefore, due to Hertz's theorem,

$$\underline{\boldsymbol{\lambda}_2}\left(\boldsymbol{A}^S\right) = \min\left\{\frac{1}{4}\left(3-\sqrt{65}\right), \frac{1}{4}\left(3-\sqrt{145}\right)\right\} = \frac{1}{4}\left(3-\sqrt{145}\right)$$

and

$$\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right) = \max\left\{3+\sqrt{10}, 3+\sqrt{5}\right\} = 3+\sqrt{10}.$$

Furthermore, according to Theorem 4.1,

$$\overline{\boldsymbol{\lambda}_2}\left(\boldsymbol{A}^S\right) = \lambda_2\left(\begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}\right) = 3-\sqrt{5}$$

and

$$\underline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right) = \lambda_1\left(\begin{pmatrix} \frac{1}{2} & 2 \\ 2 & 1 \end{pmatrix}\right) = \frac{1}{4}\left(3+\sqrt{65}\right).$$

Thereby, the eigenvalue intervals are

$$\boldsymbol{\lambda}_2\left(\boldsymbol{A}^S\right) = \left[\frac{1}{4}\left(3-\sqrt{145}\right), 3-\sqrt{5}\right] \approx [-2.2604, 0.7639]$$

and

$$\boldsymbol{\lambda}_1\left(\boldsymbol{A}^S\right) = \left[\frac{1}{4}\left(3+\sqrt{65}\right), 3+\sqrt{10}\right] \approx [2.7656, 6.1623].$$

□

It has been shown before, in general, the disjointness of the diagonal intervals is not necessary. Next, necessary and sufficient condition, regarding the endpoints of the off-diagonal intervals, is provided for the disjointness of eigenvalue intervals, given the diagonal intervals $[a_1, a_2]$ and $[c_1, c_2]$ overlap each other.

Theorem 4.5. *If the nondegenerate intervals* $[a_1, a_2]$ *and* $[c_1, c_2]$ *overlap each other, then*

$$\overline{\boldsymbol{\lambda}_2}\left(\boldsymbol{A}^S\right) < \underline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right) \iff \max(b_1, -b_2) > \frac{\sqrt{(a_2-a_1)(a_2-c_1)(c_2-a_1)(c_2-c_1)}}{a_2-a_1+c_2-c_1}.$$

If $0 \notin [b_1, b_2]$ *and at least one of the diagonal intervals is degenerate, then the eigenvalue intervals are disjoint.*

Remark 4.6. The expression $\frac{\sqrt{(a_2-a_1)(a_2-c_1)(c_2-a_1)(c_2-c_1)}}{a_2-a_1+c_2-c_1}$ is well-defined because the product $(a_2-a_1)(a_2-c_1)(c_2-a_1)(c_2-c_1)$ is nonnegative, and the denominator is positive due to the overlapping and nondegenerate diagonal intervals.

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Proof.

(1) At least one of the diagonal intervals is degenerate.

(I) $0 < b_1 \leq b_2$

By Theorem 4.1,

$$\overline{\boldsymbol{\lambda}_2}\left(\boldsymbol{A}^S\right) = \lambda_2\left(\begin{pmatrix} a_2 & b_1 \\ b_1 & c_2 \end{pmatrix}\right), \text{ and } \underline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right) = \lambda_1\left(\begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix}\right).$$

(i) $a := a_1 = a_2$

$$\begin{aligned} c_2 - c_1 &\leq |a - c_2| + |a - c_1| = \sqrt{(a-c_2)^2} + \sqrt{(a-c_1)^2} \\ c_2 - c_1 &< \sqrt{4b_1^2 + (a-c_2)^2} + \sqrt{4b_1^2 + (a-c_1)^2} \\ \overline{\boldsymbol{\lambda}_2}\left(\boldsymbol{A}^S\right) = \frac{a + c_2 - \sqrt{4b_1^2 + (a-c_2)^2}}{2} &< \frac{a + c_1 + \sqrt{4b_1^2 + (a-c_1)^2}}{2} = \underline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right) \end{aligned}$$

(ii) $c := c_1 = c_2$

$$\begin{aligned} a_2 - a_1 &< \sqrt{4b_1^2 + (a_2-c)^2} + \sqrt{4b_1^2 + (a_1-c)^2} \\ \overline{\boldsymbol{\lambda}_2}\left(\boldsymbol{A}^S\right) = \frac{a_2 + c - \sqrt{4b_1^2 + (a_2-c)^2}}{2} &< \frac{a_1 + c + \sqrt{4b_1^2 + (a_1-c)^2}}{2} = \underline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right) \end{aligned}$$

(II) $b_1 \leq b_2 < 0$

It is similar to the previous case.

(2) The diagonal intervals are nondegenerate.

(I) $0 < b_1 \leq b_2$

(i) The diagonal intervals touch each other.

(a) $a_1 < a_2 = c_1 < c_2$

$$\begin{aligned} \overline{\boldsymbol{\lambda}_2}\left(\boldsymbol{A}^S\right) &< \underline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right) \\ \frac{a_2 + c_2 - \sqrt{4b_1^2 + (a_2-c_2)^2}}{2} &< \frac{a_1 + c_1 + \sqrt{4b_1^2 + (a_1-c_1)^2}}{2} \\ a_2 - a_1 + c_2 - c_1 &< \sqrt{4b_1^2 + (a_1-c_1)^2} + \sqrt{4b_1^2 + (a_2-c_2)^2} = \\ &= \sqrt{4b_1^2 + (a_1-a_2)^2} + \sqrt{4b_1^2 + (c_1-c_2)^2} \end{aligned}$$

which holds for every $b_1 > 0$.

(b) $c_1 < c_2 = a_1 < a_2$

It is similar to the previous case.

(ii) There are more than one point of intersection.

Starting from the disjointness of eigenvalue intervals, we would like to cancel the terms $(a_1 - c_1)^2$ and $(a_2 - c_2)^2$.

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$$
\begin{aligned}
\overline{\boldsymbol{\lambda}_2}\left(\boldsymbol{A}^S\right) &< \underline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right)\\
\frac{a_2+c_2-\sqrt{4b_1^2+(a_2-c_2)^2}}{2} &< \frac{a_1+c_1+\sqrt{4b_1^2+(a_1-c_1)^2}}{2}\\
a_2-a_1+c_2-c_1 &< \sqrt{4b_1^2+(a_1-c_1)^2}+\sqrt{4b_1^2+(a_2-c_2)^2} \quad /()^2\\
(a_2-a_1+c_2-c_1)^2 &< 8b_1^2+(a_1-c_1)^2+(a_2-c_2)^2+\\
&\quad +2\sqrt{(4b_1^2+(a_1-c_1)^2)\,(4b_1^2+(a_2-c_2)^2)}.
\end{aligned}
$$

By

$$
\begin{aligned}
&(a_2-a_1+c_2-c_1)^2 = ((a_2-c_2)+c_2-a_1+c_2+(a_1-c_1)-a_1)^2 =\\
&=(((a_2-c_2)+(a_1-c_1))+2(c_2-a_1))^2 = (a_1-c_1)^2+(a_2-c_2)^2+\\
&+2(a_1-c_1)(a_2-c_2)+4(c_2-a_1)(a_1-c_1+a_2-c_2)+4(c_2-a_1)^2=\\
&=(a_1-c_1)^2+(a_2-c_2)^2+2(a_1-c_1)(a_2-c_2)+4(c_2-a_1)(a_2-c_1),
\end{aligned}
$$

we get

$$
\begin{aligned}
&(a_1-c_1)(a_2-c_2)+2(c_2-a_1)(a_2-c_1)-4b_1^2<\\
&\qquad < \sqrt{(4b_1^2+(a_1-c_1)^2)\,(4b_1^2+(a_2-c_2)^2)}.
\end{aligned}
\tag{11}
$$

If we know which side of (11) has greater absolute value, we can decide the sign of the difference of their squares. Next, it is proven that the absolute value of the right-hand side of (11) is greater than the left-hand side's. If the left-hand side is nonnegative, then it holds trivially. If it is negative, then, because $(c_2-a_1)(a_2-c_1)>0$, the inequality

$$
(a_1-c_1)(a_2-c_2)+2(c_2-a_1)(a_2-c_1)-4b_1^2 > (a_1-c_1)(a_2-c_2)-4b_1^2
$$

holds. By

$$
\begin{aligned}
&\left(4b_1^2+(a_1-c_1)^2\right)\left(4b_1^2+(a_2-c_2)^2\right)-\left((a_1-c_1)(a_2-c_2)-4b_1^2\right)^2=\\
&\quad=16b_1^4+(a_1-c_1)^2(a_2-c_2)^2+4b_1^2\left((a_1-c_1)^2+(a_2-c_2)^2\right)-\\
&\quad-\left(16b_1^4+(a_1-c_1)^2(a_2-c_2)^2-8b_1^2(a_1-c_1)(a_2-c_2)\right)=\\
&\quad=4b_1^2\left((a_1-c_1)^2+(a_2-c_2)^2+2(a_1-c_1)(a_2-c_2)\right)=\\
&\quad=4b_1^2\left((a_1-c_1)+(a_2-c_2)\right)^2\geq 0,
\end{aligned}
\tag{12}
$$

the right-hand side of (11) is greater than the absolute value of the left-hand side. Based on (11) and (12),

$$
\begin{aligned}
0&<\left(4b_1^2+(a_1-c_1)^2\right)\left(4b_1^2+(a_2-c_2)^2\right)-\\
&-\left((a_1-c_1)(a_2-c_2)+2(a_2-c_1)(c_2-a_1)-4b_1^2\right)^2=16b_1^4+\\
&+4b_1^2\left((a_1-c_1)^2+(a_2-c_2)^2\right)+(a_1-c_1)^2(a_2-c_2)^2-\\
&-\left((a_1-c_1)^2(a_2-c_2)^2+4(a_2-c_1)^2(c_2-a_1)^2+16b_1^4+\right.\\
&+4(a_1-c_1)(a_2-c_2)(a_2-c_1)(c_2-a_1)-\\
&\left.-8b_1^2(a_1-c_1)(a_2-c_2)-16b_1^2(a_2-c_1)(c_2-a_1)\right)\qquad /:4
\end{aligned}
$$

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$$
\begin{aligned}
0 <& b_1^2\left((a_1-c_1)^2+(a_2-c_2)^2\right)-(a_2-c_1)^2(c_2-a_1)^2-\\
&-(a_1-c_1)(a_2-c_2)(a_2-c_1)(c_2-a_1)+\\
&+2b_1^2(a_1-c_1)(a_2-c_2)+4b_1^2(a_2-c_1)(c_2-a_1)=\\
&=b_1^2\big((a_1-c_1)^2+(a_2-c_2)^2+\\
&+2(a_1-c_1)(a_2-c_2)+4(a_2-c_1)(c_2-a_1)\big)-\\
&-(a_2-c_1)^2(c_2-a_1)^2-(a_1-c_1)(a_2-c_2)(a_2-c_1)(c_2-a_1)=\\
&=b_1^2\big((a_1-c_1+a_2-c_2)^2+4(a_2-c_1)(c_2-a_1)\big)-\\
&-(a_2-c_1)(c_2-a_1)\big((a_2-c_1)(c_2-a_1)+(a_1-c_1)(a_2-c_2)\big)=\\
&=b_1^2\big((a_2-c_1)+(c_2-a_1)\big)^2-\\
&-(a_2-c_1)(c_2-a_1)(a_2c_2+a_1c_1-a_1c_2-a_2c_1)=\\
&=b_1^2\big((a_2-c_1)+(c_2-a_1)\big)^2-(a_2-c_1)(c_2-a_1)(a_2-a_1)(c_2-c_1).
\end{aligned} \tag{13}
$$

By rearranging (13) and taking square root,

$$b_1 > \frac{\sqrt{(a_2-a_1)(a_2-c_1)(c_2-a_1)(c_2-c_1)}}{a_2-a_1+c_2-c_1}.$$

The other direction of the equivalence can be proved using the previous steps.

(II) $b_1 \le b_2 < 0$
It is similar to the previous case. □

Example 4.7. Illustrating the results of Theorem 4.5, let

$$\boldsymbol{A}^S = \begin{pmatrix} [-1,1] & [b_1, \frac{8}{5}] \\ [b_1, \frac{8}{5}] & [0, 1+\sqrt{5}] \end{pmatrix}.$$

Then,

$$\frac{\sqrt{(a_2-a_1)(a_2-c_1)(c_2-a_1)(c_2-c_1)}}{a_2-a_1+c_2-c_1} = \frac{\sqrt{2\cdot 1\cdot(2+\sqrt{5})(1+\sqrt{5})}}{3+\sqrt{5}} = \frac{\sqrt{2(7+3\sqrt{5})}}{3+\sqrt{5}}$$
$$= \frac{\sqrt{14+6\sqrt{5}}}{3+\sqrt{5}} = 1,$$

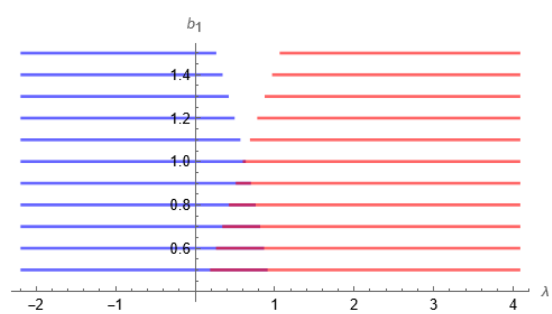

Fig. 1: Eigenvalue intervals of $\boldsymbol{A}^S$ for different values of b_1.

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thus, if $b_1 > 1$, then the eigenvalue intervals are disjoint, otherwise they overlap each other. It can be seen in Fig. 1.

5. Extremal eigenvalue bounds of $\boldsymbol{n \times n}$ real symmetric interval matrices. In this section, the extremal eigenvalue bounds are studied in the general $n \times n$ case. One of Hertz's works ([3]) motivated the achieved results. It says that in the case of a special sign pattern of an interval matrix, the vertex matrix, whose maximal eigenvalue coincides with $\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right)$, can be given. This can be generalized and extended to the bound $\underline{\boldsymbol{\lambda}_n}\left(\boldsymbol{A}^S\right)$, too.

Based on Hertz's theorem, by calculating the extremal eigenvalues of some vertex matrices, the sharp bounds $\underline{\boldsymbol{\lambda}_n}\left(\boldsymbol{A}^S\right)$ and $\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right)$ can be given. Motivated by Xin's works ([13]), in a special case, Hertz concluded that if the right endpoints of the off-diagonal intervals are not smaller than the absolute values of the left ones, then the bound $\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right)$ is the maximal eigenvalue of the matrix of the right endpoints, namely $\overline{A}^1$ ([3]). This result can be generalized by utilizing the sign pattern of the midpoint matrix. It is noticed that the orthant pairs O_i $(i = 1, \dots, 2^{n-1})$ reserve the sign of the outer product xx^T $(x \in B^n)$. That is, if we have two vectors from the same orthant pair, then the signs of their outer products with themselves are identical, while if they belong to different orthant pairs, then the corresponding signs differ. It is observed that if the sign of the midpoint matrix is similar to one of the sign patterns reserved by the orthant pairs, then the matrix (or possibly matrices) $\overline{A}^{j_0} \in \left\{\overline{A}^i\right\}_{i=1,\dots,2^{n-1}}$ can be given for which $\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right) = \lambda_1\left(\overline{A}^{j_0}\right)$ holds. ($\overline{A}^{j_0}$ maximizes in such orthant, with positive first coordinate, which has the same sign as the first row (or column) of that certain sign pattern.) As it is shown, a similar statement holds for the bound $\underline{\boldsymbol{\lambda}_n}\left(\boldsymbol{A}^S\right)$. Below, in Fig. 2, all the 3×3 special sign patterns are illustrated and ordered in accordance with the orthant pairs (1-black, -1-white).

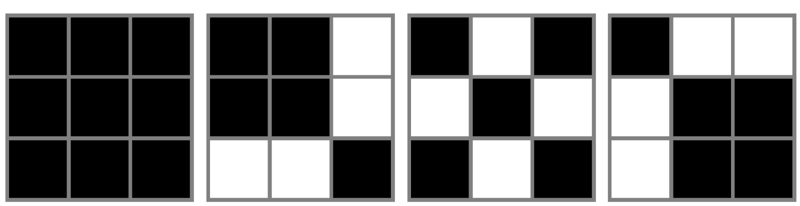

Fig. 2: All the special sign patterns reserved by the orthant pairs in $\mathbb{R}^{3\times 3}$.

THEOREM 5.1. *Let*

$$S^{\pm}_{kl} := \begin{cases} \operatorname{sign}((A_c)_{kl}) & \text{if } k \neq l \\ 1 & \text{if } k = l \end{cases}.$$

If the midpoint matrix A_c of the real symmetric interval matrix $\boldsymbol{A}^S = \mathcal{S}[P,Q]$ has a special sign pattern, except the diagonal intervals at most, that is, if $S^{\pm} \in \left\{xx^T : x \in H := \left\{\left(\{1\} \times \{\pm 1\}^{n-1}\right)^T\right\}\right\}$, then concerning the index $i \in \left\{1, 2, \dots, 2^{n-1}\right\}$ for which $S^{\pm}_{*1} \in H \cap O_i$, where $S^{\pm}_{*1}$ is the first column of $S^{\pm}$, it holds that

$$\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right) = \lambda_1\left(\overline{A}^i\right). \tag{14}$$

Remark 5.2. In Theorem 5.1, the main emphasis is on the signs of the off-diagonal elements of the midpoint matrix A_c which can be seen in the proof.

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Remark 5.3. Theorem 5.1 says that, in the case of special sign patterns, the single $\overline{A}^i$ matrix can be chosen which determines $\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right)$.

Proof. The proof will be carried out by contradiction. Since the quadratic form $x^T Ax$ is continuous on the compact set $\boldsymbol{A}^S \times B^n$; thus, it attains its maximum $\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right)$ for some $x^{(0)} \in B^n$ and $A^{(0)} \in \boldsymbol{A}^S$. Assume that (14) is incorrect, that is

$$\overline{x}^T \overline{A}^i \overline{x} < x^{(0)T} A^{(0)} x^{(0)},$$

where $S^{\pm}_{*1} \in H \cap O_i$ and $\overline{x}$ is the eigenvector corresponding to $\lambda_1\left(\overline{A}^i\right)$. Let $S^{\pm} = (s_{kl})$ and $y^{(0)} = \left(s_{11}\left|x_1^{(0)}\right|, s_{21}\left|x_2^{(0)}\right|, \ldots, s_{n1}\left|x_n^{(0)}\right|\right)^T$, then

$$y^{(0)T}\overline{A}^i y^{(0)} \leq \overline{x}^T \overline{A}^i \overline{x} < x^{(0)T} A^{(0)} x^{(0)} \leq y^{(0)T}\overline{A}^i y^{(0)}.$$

Only the last inequality has to be proved. It is carried out by overestimating each of the addends of the quadratic form

$$x^{(0)T} A^{(0)} x^{(0)} = \sum_{k=1}^{n} a_{kk}^{(0)} x_k^{(0)2} + 2\sum_{1\leq k<l\leq n} a_{kl}^{(0)} x_k^{(0)} x_l^{(0)}.$$

(1) $k = l$

$$a_{kk}^{(0)} x_k^{(0)2} \leq q_{kk} {y_k}^{(0)2} \tag{15}$$

(2) $k < l$

We utilize the fact that $S^{\pm}$ has a special sign pattern, that is, $s_{kl} = s_{k1}s_{l1}$.

(I) $s_{kl} = 1$

Then,

$$y_k{}^{(0)} y_l{}^{(0)} = s_{k1}\left|x_k^{(0)}\right| s_{l1}\left|x_l^{(0)}\right| = s_{kl}\left|x_k^{(0)}\right|\left|x_l^{(0)}\right| \geq 0$$

and

$$\left|a_{kl}^{(0)}\right| \in [p_{kl}, q_{kl}].$$

Thus,

$$a_{kl}^{(0)} x_k{}^{(0)} x_l{}^{(0)} \leq \left|a_{kl}^{(0)}\right| y_k^{(0)} y_l^{(0)} \leq q_{kl} y_k^{(0)} y_l^{(0)}. \tag{16}$$

(II) $s_{kl} = -1$

Then

$$y_k{}^{(0)} y_l{}^{(0)} = s_{k1}\left|x_k^{(0)}\right| s_{l1}\left|x_l^{(0)}\right| = s_{kl}\left|x_k^{(0)}\right|\left|x_l^{(0)}\right| \leq 0$$

and

$$-\left|a_{kl}^{(0)}\right| \in [p_{kl}, q_{kl}].$$

Thus,

$$a_{kl}^{(0)} x_k{}^{(0)} x_l{}^{(0)} \leq -\left|a_{kl}^{(0)}\right| y_k^{(0)} y_l^{(0)} \leq p_{kl} y_k^{(0)} y_l^{(0)}. \tag{17}$$

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Therefore, if $s_{kl} = 1$, then the right endpoint is chosen, otherwise, the left endpoint. Hence, we get the matrix $\overline{A}^i$. By (16) and (17), we get

$$x^{(0)T}A^{(0)}x^{(0)} \leq y^{(0)T}\overline{A}^i y^{(0)}$$

which leads to contradiction. □

Remark 5.4. If the centers of some intervals are zeros, but there is a sign pattern (or possibly more) which coincides with $S^{\pm}$ except the zeros, then, by choosing the endpoints corresponding to this sign pattern, we get the matrix $\overline{A}^i$ which maximizes since the inequalities (15), (16) and (17) still hold. (If there are more appropriate sign patterns, then all the corresponding matrices $\overline{A}^i$ maximize.)

Next, the results of Theorem 5.1 and Remark 5.4 are illustrated. For determining the sharp bounds $\overline{\boldsymbol{\lambda}}_1\left(\boldsymbol{A}^S\right)$, roots of third-order polynomials are calculated. Due to complicated expressions, these are given as Root$[p, k]$ objects, where p is a polynomial and k is the serial number of the root according to the increasing order, regarding real roots.

Example 5.5. Let

$$\boldsymbol{A}^S = \begin{pmatrix} [-3,-1] & [2,3] & [-2,1] \\ [2,3] & [2,4] & [-2,-1] \\ [-2,1] & [-2,-1] & [-3,4] \end{pmatrix}.$$

Then,

$$S^{\pm} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix},$$

which coincides with the second pattern of Fig. 2; thus based on Theorem 5.1, the largest eigenvalue of the matrix

$$\overline{A}^2 = \begin{pmatrix} -1 & 3 & -2 \\ 3 & 4 & -2 \\ -2 & -2 & 4 \end{pmatrix}$$

gives the sharp bound $\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right)$. It is reinforced by the fact that

$$\overline{A}^1 = \begin{pmatrix} -1 & 3 & 1 \\ 3 & 4 & -1 \\ 1 & -1 & 4 \end{pmatrix} \qquad \lambda_1\left(\overline{A}^1\right) = \text{Root}[x^3 - 7x^2 - 3x + 61, 3] \approx 5.58$$

$$\overline{A}^2 = \begin{pmatrix} -1 & 3 & -2 \\ 3 & 4 & -2 \\ -2 & -2 & 4 \end{pmatrix} \qquad \lambda_1\left(\overline{A}^2\right) = \text{Root}[x^3 - 7x^2 - 9x + 40, 3] \approx 7.49$$

$$\overline{A}^3 = \begin{pmatrix} -1 & 2 & 1 \\ 2 & 4 & -2 \\ 1 & -2 & 4 \end{pmatrix} \qquad \lambda_1\left(\overline{A}^3\right) = \text{Root}[x^3 - 7x^2 - x + 40, 3] \approx 6.08$$

$$\overline{A}^4 = \begin{pmatrix} -1 & 2 & -2 \\ 2 & 4 & -1 \\ -2 & -1 & 4 \end{pmatrix} \qquad \lambda_1\left(\overline{A}^4\right) = 2 + \sqrt{17} \approx 6.12,$$

thus according to Hertz's theorem,

$$\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right) = \max\left\{\lambda_1\left(\overline{A}^1\right), \lambda_1\left(\overline{A}^2\right), \lambda_1\left(\overline{A}^3\right), \lambda_1\left(\overline{A}^4\right)\right\} = \lambda_1\left(\overline{A}^2\right).$$

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Example 5.6. Let

$$\boldsymbol{A}^S = \begin{pmatrix} [-3,-1] & [-3,3] & [-2,1] \\ [-3,3] & [2,4] & [-5,5] \\ [-2,1] & [-5,5] & [-3,4] \end{pmatrix}.$$

Then,

$$S^{\pm} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

thus, there are two special sign patterns that are the same as $S^{\pm}$ except the zeros. These are

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

Based on Remark 5.4, the largest eigenvalues of the matrices

$$\overline{A}^2 = \begin{pmatrix} -1 & 3 & -2 \\ 3 & 4 & -5 \\ -2 & -5 & 4 \end{pmatrix} \text{ and } \overline{A}^4 = \begin{pmatrix} -1 & -3 & -2 \\ -3 & 4 & 5 \\ -2 & 5 & 4 \end{pmatrix}$$

coincide with the sharp bound $\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right)$. It holds since

$$\overline{A}^1 = \begin{pmatrix} -1 & 3 & 1 \\ 3 & 4 & 5 \\ 1 & 5 & 4 \end{pmatrix} \qquad \lambda_1\left(\overline{A}^1\right) = \mathrm{Root}[x^3 - 7x^2 - 27x + 1, 3] \approx 9.76$$

$$\overline{A}^2 = \begin{pmatrix} -1 & 3 & -2 \\ 3 & 4 & -5 \\ -2 & -5 & 4 \end{pmatrix} \qquad \lambda_1\left(\overline{A}^2\right) = \mathrm{Root}[x^3 - 7x^2 - 30x - 17, 3] \approx 10.13$$

$$\overline{A}^3 = \begin{pmatrix} -1 & -3 & 1 \\ -3 & 4 & -5 \\ 1 & -5 & 4 \end{pmatrix} \qquad \lambda_1\left(\overline{A}^3\right) = \mathrm{Root}[x^3 - 7x^2 - 27x + 1, 3] \approx 9.76$$

$$\overline{A}^4 = \begin{pmatrix} -1 & -3 & -2 \\ -3 & 4 & 5 \\ -2 & 5 & 4 \end{pmatrix} \qquad \lambda_1\left(\overline{A}^4\right) = \mathrm{Root}[x^3 - 7x^2 - 30x - 17, 3] \approx 10.13,$$

and by Hertz's theorem,

$$\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right) = \max\left\{\lambda_1\left(\overline{A}^1\right), \lambda_1\left(\overline{A}^2\right), \lambda_1\left(\overline{A}^3\right), \lambda_1\left(\overline{A}^4\right)\right\} = \lambda_1\left(\overline{A}^2\right) = \lambda_1\left(\overline{A}^4\right).$$

The following example shows that if the sign pattern of the interval matrix is general, then based merely on it, we cannot state stronger than Hertz's theorem.

Example 5.7. Let

$$\boldsymbol{A}^S = \begin{pmatrix} [-2,1] & [2,4] & [-2,1] \\ [2,4] & [-1,3] & [-2,5] \\ [-2,1] & [-2,5] & [2,4] \end{pmatrix} \text{ and } \boldsymbol{B}^S = \begin{pmatrix} [-2,1] & [2,4] & [-8,3] \\ [2,4] & [-1,3] & [-2,5] \\ [-8,3] & [-2,5] & [2,4] \end{pmatrix}.$$

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Then, in both cases

$$S^{\pm}=\begin{pmatrix}1&1&-1\\1&1&1\\-1&1&1\end{pmatrix}.$$

Since

$$\begin{aligned}
\overline{A}^1&=\begin{pmatrix}1&4&1\\4&3&5\\1&5&4\end{pmatrix} &\qquad \lambda_1\left(\overline{A}^1\right)&=\mathrm{Root}[x^3-8x^2-23x+40,3]\approx 9.91\\
\overline{A}^2&=\begin{pmatrix}1&4&-2\\4&3&-2\\-2&-2&4\end{pmatrix} &\qquad \lambda_1\left(\overline{A}^2\right)&=\mathrm{Root}[x^3-8x^2-5x+36,3]\approx 8.07\\
\overline{A}^3&=\begin{pmatrix}1&2&1\\2&3&-2\\1&-2&4\end{pmatrix} &\qquad \lambda_1\left(\overline{A}^3\right)&=\frac{1}{2}\left(9+\sqrt{5}\right)\approx 5.62\\
\overline{A}^4&=\begin{pmatrix}1&2&-2\\2&3&5\\-2&5&4\end{pmatrix} &\qquad \lambda_1\left(\overline{A}^4\right)&=\mathrm{Root}[x^3-8x^2-14x+81,3]\approx 8.53\\
\overline{B}^1&=\begin{pmatrix}1&4&3\\4&3&5\\3&5&4\end{pmatrix} &\qquad \lambda_1\left(\overline{B}^1\right)&=\mathrm{Root}[x^3-8x^2-31x-16,3]\approx 10.96\\
\overline{B}^2&=\begin{pmatrix}1&4&-8\\4&3&-2\\-8&-2&4\end{pmatrix} &\qquad \lambda_1\left(\overline{B}^2\right)&=\mathrm{Root}[x^3-8x^2-65x+120,3]\approx 12.45\\
\overline{B}^3&=\begin{pmatrix}1&2&3\\2&3&-2\\3&-2&4\end{pmatrix} &\qquad \lambda_1\left(\overline{B}^3\right)&=\mathrm{Root}[x^3-8x^2+2x+59,3]\approx 6.07\\
\overline{B}^4&=\begin{pmatrix}1&2&-8\\2&3&5\\-8&5&4\end{pmatrix} &\qquad \lambda_1\left(\overline{B}^4\right)&=\mathrm{Root}[x^3-8x^2-74x+381,3]\approx 11.55,
\end{aligned}$$

thus according to Hertz's theorem,

$$\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right)=\max\left\{\lambda_1\left(\overline{A}^1\right),\lambda_1\left(\overline{A}^2\right),\lambda_1\left(\overline{A}^3\right),\lambda_1\left(\overline{A}^4\right)\right\}=\lambda_1\left(\overline{A}^1\right)$$

and

$$\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{B}^S\right)=\max\left\{\lambda_1\left(\overline{B}^1\right),\lambda_1\left(\overline{B}^2\right),\lambda_1\left(\overline{B}^3\right),\lambda_1\left(\overline{B}^4\right)\right\}=\lambda_1\left(\overline{B}^2\right).$$

Consequently, despite the fact that the modified sign patterns of the interval matrices are the same, the maximizing matrices, at which the sharp bounds $\overline{\boldsymbol{\lambda}_1}$ are attained, belong to different orthant pairs.

Next, we provide a theorem for the sharp bound $\underline{\boldsymbol{\lambda}_n}\left(\boldsymbol{A}^S\right)$ similar to Theorem 5.1.

Theorem 5.8. *Let*

$$S^{\mp}_{kl}:=\begin{cases}\mathrm{sign}((A_c)_{kl}) & \textit{if } k\neq l\\ -1 & \textit{if } k=l\end{cases}.$$

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If the midpoint matrix A_c of the real symmetric interval matrix $\boldsymbol{A}^S = \mathcal{S}[P, Q]$ has a special sign pattern, except the diagonal intervals at most, that is, if $S^{\mp} \in \left\{-xx^T : x \in H := \left\{\left(\{-1\} \times \{\pm 1\}^{n-1}\right)^T\right\}\right\}$, then concerning the index $i \in \left\{1, 2, \ldots, 2^{n-1}\right\}$ for which $S^{\mp}_{*1} \in H \cap O_i$, it holds that

$$\underline{\boldsymbol{\lambda}_n}\left(\boldsymbol{A}^S\right) = \lambda_n\left(\underline{A}^i\right). \tag{18}$$

Proof. The proof is similar to the one after Theorem 5.1. ∎

Based on Theorems 5.1 and 5.8, the question arises as to what the proportion of the special sign patterns is. In the following remark we give the answer.

Remark 5.9. Assume that the midpoints of the diagonal intervals are positive, then there are 2^{n-1} special sign patterns, and in total, we have $2^{\frac{n(n-1)}{2}}$, concerning the bound $\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right)$. Thus, we get that the proportion of special sign patterns is

$$\frac{2^{n-1}}{2^{\frac{n(n-1)}{2}}} = \frac{1}{2^{\frac{(n-2)(n-1)}{2}}}.$$

Similarly, it is also the proportion of special sign patterns relating to the bound $\underline{\boldsymbol{\lambda}_n}\left(\boldsymbol{A}^S\right)$. Thus, in the case of 2×2 real symmetric interval matrices, all the patterns are special, consequently, for the sharp bounds $\underline{\boldsymbol{\lambda}_n}\left(\boldsymbol{A}^S\right)$ and $\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right)$ we can give the two vertex matrices whose corresponding extremal eigenvalues coincide with them. This agrees with the relevant part of Remark 4.2, and the example from the proof of Theorem 4.4 also indicates this. Furthermore, if $n = 3$, then half of the 8 sign patterns are special, regarding the bound $\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right)$, which is shown in Fig. 2. The same proportion holds concerning the bound $\underline{\boldsymbol{\lambda}_n}\left(\boldsymbol{A}^S\right)$. However, given a real symmetric interval matrix, both sign patterns $S^{\pm}$ and $S^{\mp}$ are special only in the 2×2 case. In the 3×3 case, only one of them, in the $n \times n$ case, $n > 3$, at most one of them is special.

6. Conclusion. First, in the case of 2×2 real symmetric interval matrices, we determined all the four endpoints of the eigenvalue intervals. Then, necessary and sufficient conditions were shown for the disjointness of the eigenvalue intervals. Finally, in the general $n \times n$ case, we presented that if the interval matrix follows a special sign pattern, then we can give the vertex matrix one of whose extremal eigenvalues provides the sharp bound $\underline{\boldsymbol{\lambda}_n}\left(\boldsymbol{A}^S\right)$ or $\overline{\boldsymbol{\lambda}_1}\left(\boldsymbol{A}^S\right)$.

However, the study of real symmetric interval matrices still has open problems. A similar construction to that in Theorem 4.3 and the further investigation of sign patterns and vertex matrices can lead to determine the other two endpoints of the extremal eigenvalue intervals. Moreover, it may help characterize the other eigenvalue intervals, and we can study their disjointness as well.

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