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EXPRESSIONS AND CHARACTERIZATIONS FOR THE MOORE-PENROSE INVERSE OF OPERATORS AND MATRICES*

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Abstract. Under certain conditions, we prove that the Moore–Penrose inverse of a sum of operators is the sum of the Moore–Penrose inverses. From this, we derive expressions and characterizations for the Moore–Penrose inverse of an operator that are useful for its computation. We give formulations of them for finite matrices and study the Moore–Penrose inverse of circulant matrices and of distance matrices of certain graphs.

Key words. Hilbert space, Moore–Penrose inverse, Circulant matrix, Distance matrix, Weighted tree, Wheel graph.

AMS subject classifications. 47A05, 47B02, 15A09, 15B05, 05C50.

1. Introduction.

A generalization of the concept of inverse for matrices was first introduced by Moore [1, 2]. Then, this generalized inverse was independently reintroduced and studied by Bjerhammer [3] and Penrose [4]. The now commonly called *Moore–Penrose inverse* has been also defined and studied for operators in Hilbert spaces (see e.g., [5, 6] and the references therein). It has numerous applications to physics, statistics, optimization theory, solution of differential and integral equations, prediction theory, control system analysis, etc. For more details see, e.g., [7, 8, 9, 10].

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ denotes the set of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. The set of elements in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ with closed range will be denoted with $\mathcal{BC}(\mathcal{H}_1, \mathcal{H}_2)$. Given $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, we denote the adjoint, the null space, and the range of A by A^*, $\mathrm{N}(A)$ and $\mathrm{R}(A)$, respectively. If $A \in \mathcal{BC}(\mathcal{H}_1, \mathcal{H}_2)$, then $\mathcal{H}_1 = \mathrm{N}(A) \oplus \mathrm{R}(A^*)$ and $\mathcal{H}_2 = \mathrm{R}(A) \oplus \mathrm{N}(A^*)$, where $\oplus$ denotes orthogonal sum. If $\mathcal{W}$ is a closed subspace of a Hilbert space, then $P_{\mathcal{W}}$ denotes the orthogonal projection onto $\mathcal{W}$.

Definition 1.1. Let $A \in \mathcal{BC}(\mathcal{H}_1, \mathcal{H}_2)$. The unique solution $X \in \mathcal{BC}(\mathcal{H}_2, \mathcal{H}_1)$ of the system of operator equations

$$(1)\, AXA = A,\ (2)\, XAX = X,\ (3)\ (AX)^* = AX \text{ and } \ (4)\ (XA)^* = XA,$$

is called the *Moore–Penrose inverse* of A and is denoted by $A^\dagger$. Any $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ that satisfies (1), (3) and (4) is known as a $\{1, 3, 4\}$-*inverse* of A.

One of the most important properties used in the applications of the Moore–Penrose inverse is that the *minimal norm least squares problem*

$$\min \|x\| \text{ subject to } \|Ax - g\| = \min_{f \in \mathcal{H}_1} \|Af - g\|,$$

has the unique solution $x = A^\dagger g$.

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We note that if $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, then $A^\dagger$ exists if and only if $\mathrm{R}(A)$ is closed, and $\mathrm{R}(A)$ is closed if and only if $\mathrm{R}(A^*)$ is closed. Let $A \in \mathcal{BC}(\mathcal{H}_1, \mathcal{H}_2)$. Then $\left(A^\dagger\right)^\dagger = A$ and $(A^*)^\dagger = \left(A^\dagger\right)^*$. If A is invertible, then $A^\dagger = A^{-1}$. The Moore–Penrose inverse is characterized in the following theorem (see [5, Theorem] or [7, Theorem 9.3]):

Theorem 1.2. *If $A \in \mathcal{BC}(\mathcal{H}_1, \mathcal{H}_2)$, then $A^\dagger \in \mathcal{BC}(\mathcal{H}_2, \mathcal{H}_1)$ is the unique solution of anyone of the following equivalent systems:*

(i) $AX = P_{\mathrm{R}(A)}$, $\mathrm{N}(X^*) = \mathrm{N}(A)$.
(ii) $AX = P_{\mathrm{R}(A)}$, $XA = P_{\mathrm{R}(A^*)}$, $XAX = X$.
(iii) $XAA^* = A^*$, $XX^*A^* = X$.
(iv) $XAP_{\mathrm{R}(A^*)} = P_{\mathrm{R}(A^*)}$, $XP_{\mathrm{N}(A^*)} = 0$.
(v) $XA = P_{\mathrm{R}(A^*)}$, $\mathrm{N}(X) = \mathrm{N}(A^*)$.
(vi) $AX = P_{\mathrm{R}(A)}$, $XA = P_{\mathrm{R}(X)}$.

1.1. Our contributions and the organization of the paper. In Section 2, we prove that under certain conditions, the Moore–Penrose inverse of a sum of operators is the sum of the Moore–Penrose inverses (Theorem 2.1). From this, we give a result for $\{1, 3, 4\}$-inverses (Corollary 2.2) and derive expressions and characterizations for the Moore–Penrose inverse that are useful for its computation (Theorem 2.3, Corollary 2.4, Theorem 2.5 and Corollary 2.6). Part of them can be viewed as extensions of the following well-known result:

Proposition 1.3. *Let $A \in \mathcal{BC}(\mathcal{H}_1, \mathcal{H}_2)$. Then:*

(i) $A^\dagger = (A^*A)^\dagger A^* = A^* (AA^*)^\dagger$.
(ii) *If $\mathrm{N}(A) = \{0\}$, then A^*A is invertible and $A^\dagger = (A^*A)^{-1} A^*$.*
(iii) *If $\mathrm{R}(A) = \mathcal{H}_2$, then AA^* is invertible and $A^\dagger = A^* (AA^*)^{-1}$.*

In particular, we note that in Definition 1.1 and in Theorem 1.2, $A^\dagger$ is characterized as the solution of systems of equations, or of operators equations where the solution must satisfy some restrictions on its null space or on its range space. The importance of Proposition 1.3(ii)(iii) from the computational point of view is that if $A \in \mathcal{BC}(\mathcal{H}_1, \mathcal{H}_2)$ is injective (surjective), then $A^\dagger$ is the unique solution of the single equation $(A^*A)\, X = A^*$ (resp. $X\,(AA^*) = A^*$). Our results permit to address the computation of $A^\dagger$ in a similar manner as a solution of a single equation, in cases in which A is not necessarily injective or surjective.

Theorem 2.1(iv) extends [4, Lemma 1.7] about the Moore–Penrose inverse of sums of matrices to sums of operators with closed ranges. In Section 3, we consider other generalizations of [4, Lemma 1.7] given in [11, 12, 13] for sums of two matrices and in [14] for sums of two operators. We also mention other approaches to the study of the Moore–Penrose inverse of sums of matrices that appear in [15, 16, 17, 18, 19]. Then, we give formulations of results of Section 2 for finite matrices (Theorem 3.1, Corollary 3.2 and Theorem 3.4). These formulations provide methods for the obtention of the Moore–Penrose inverse in the finite-dimensional case. We show that results appeared in [7, 20, 21], which were proved with different approaches, are particular cases of Corollary 3.2(iii) (see Remark 3.3). We also show the relation of Theorem 2.1 with the singular value decomposition (Remarks 3.6 and 3.7).

In Sections 4 and 5, we use our previous results to determine closed-form expressions for the entries of the Moore–Penrose inverse of circulant matrices and of distance matrices of certain graphs.

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Circulant matrices arise in various areas of applied mathematics and science, such as statistics, physics, signal processing, and coding theory, among many others (see, e.g., [22, 23, 24]). Beside of its theoretical interest, having an explicit expression for the (Moore–Penrose) inverses of circulant matrices can reduce the computational cost in dealing with them in some applications. There are several papers that give expressions for the (Moore–Penrose) inverse of circulant matrices, see e.g. [25, 20, 26, 27, 28, 29, 30]. In Section 4, we consider this type of matrices finding the explicit expressions of the (Moore–Penrose) inverse of some families of complex circulant matrices (Lemma 4.2, Proposition 4.3, Example 4.5 and Proposition 4.7). Based on spectral properties of circulant matrices, Theorem 4.4 provides a way to obtain the (Moore–Penrose) inverse of circulant matrices in terms of the (Moore–Penrose) inverses of other circulant matrices. As a consequence, we get Proposition 4.6 that shows that we can easily have an explicit expression for the Moore–Penrose inverse of a circulant matrix for the case in which the generating vector has components with nonzero-sum from the case in which the generating vector has components with zero-sum, and vice versa. Hence, there is no need to separately consider the two cases. To our best knowledge, this fact was not noted so far in the literature. It will simplify the study and the application of the Moore–Penrose inverse of circulant matrices in future works.

Distance matrices of connected graphs have several interesting properties and have applications in, e.g., chemistry, biology, and data communication [31, 32]. In [33], Graham and Lovász express the inverse of the distance matrix of an unweighted tree in terms of the Laplacian matrix. Since then, there was interest in give expressions for the (Moore–Penrose) inverse of distance matrices of graphs using the Laplacian matrix of the graph or a generalization of it (see e.g. [34, 35, 36, 37]). Section 5 is devoted to find explicit expression of the Moore–Penrose inverse of distance matrices D of weighted trees and of wheel graphs with an odd number of vertices. Our expressions for $D^\dagger$ do not involve the Laplacian matrix or a generalization of it but an invertible matrix which is a $\{1,3,4\}$-inverse of D constructed directly from D and its null space (see (5.3), (5.4) and Theorem 5.6). These expressions give alternatives to the ones presented in [35, 36] for the computation of $D^\dagger$.

In Section 5.1, we consider distance matrices D of weighted trees with all the weights being nonzero and with sum equal to zero. We note that the expression of $D^\dagger$ given in [35, Theorem 11] can be viewed as an extension of [38, Theorems 3 and 4] for Euclidean distance matrices and the well-known formula due to Graham and Lovász [33]. In the expression for $D^\dagger$ provided in [35, Theorem 11] appears a vector that depends on $D^\dagger$. We show how our expressions (5.3) or (5.4) can be used to compute this vector without using $D^\dagger$. Moreover, for certain types of weighted trees, we give an explicit expression of this vector using only the Laplacian matrix and the degree vector of the tree (Proposition 5.1).

In Section 5.2, we deal with the Moore–Penrose inverse of distance matrices D of wheel graphs with an odd number of vertices. In [36], an explicit expression of $D^\dagger$ is given in terms of a generalized Laplacian matrix extending in this way the classical result of Graham and Lovász [33]. Here, we give a closed-form expression for each entry of the inverse of a $\{1,3,4\}$-inverse of D, and thus of $D^\dagger$. The principal result is Theorem 5.6. Lemmas 5.4 and 5.5, used to prove Theorem 5.6, are about these entries and describe properties of them. In Proposition 5.7, we give some properties of the inverse of the $\{1,3,4\}$-inverse of D which is used in our expression of $D^\dagger$ and of the generalized Laplacian matrix introduced in [36].

2. Moore–Penrose inverses of sums of operators. The following theorem gives conditions under which the Moore–Penrose inverse of a sum is the sum of the Moore–Penrose inverses.

THEOREM 2.1. *Let* $\{A_k\}_{k=1}^K \subset \mathcal{BC}(\mathcal{H}_1,\mathcal{H}_2)$. *Assume that* $\mathrm{R}(A_k) \subseteq \mathrm{N}(A_{k'}^*)$ *and* $\mathrm{R}(A_k^*) \subseteq \mathrm{N}(A_{k'})$ *for each* $k, k' = 1,\dots,K$, $k \neq k'$. *Then:*

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(i) $A_k A_{k'}^{\dagger} = 0$ *and* $A_k^{\dagger} A_{k'} = 0$ *for each* $k, k' = 1, \ldots, K$, $k \neq k'$.

(ii) $\mathrm{R}\Big(\sum_{k=1}^{K} A_k\Big) = \bigoplus_{k=1}^{K} \mathrm{R}(A_k)$ *and* $\mathrm{R}\Big(\sum_{k=1}^{K} A_k^*\Big) = \bigoplus_{k=1}^{K} \mathrm{R}(A_k^*)$.

(iii) $\mathrm{N}\Big(\sum_{k=1}^{K} A_k\Big) = \bigcap_{k=1}^{K} \mathrm{N}(A_k)$ *and* $\mathrm{N}\Big(\sum_{k=1}^{K} A_k^*\Big) = \bigcap_{k=1}^{K} \mathrm{N}(A_k^*)$.

(iv) $\Big(\sum_{k=1}^{K} A_k\Big)^{\dagger}$ *exists and* $\Big(\sum_{k=1}^{K} A_k\Big)^{\dagger} = \sum_{k=1}^{K} A_k^{\dagger}$.

Proof. Part (i) follows from the inclusions $\mathrm{R}\Big(A_{k'}^{\dagger}\Big) = \mathrm{R}(A_{k'}^*) \subseteq \mathrm{N}(A_k)$ and $\mathrm{R}(A_{k'}) \subseteq \mathrm{N}(A_k^*) = \mathrm{N}\Big(A_k^{\dagger}\Big)$ for each $k, k' = 1, \ldots, K$, $k \neq k'$.

Since $\mathrm{R}(A_k) \subseteq \mathrm{N}(A_{k'}^*)$ for each $k, k' = 1, \ldots, K$, $k \neq k'$, we get $\mathrm{R}(A_k) \perp \mathrm{R}(A_{k'})$ for each $k, k' = 1, \ldots, K$, $k \neq k'$. Then, we have the orthogonal sum $\bigoplus_{k=1}^{K} \mathrm{R}(A_k)$. Clearly, $\mathrm{R}\Big(\sum_{k=1}^{K} A_k\Big) \subseteq \bigoplus_{k=1}^{K} \mathrm{R}(A_k)$ and $\bigcap_{k=1}^{K} \mathrm{N}(A_k) \subseteq \mathrm{N}\Big(\sum_{k=1}^{K} A_k\Big)$.

Let $g \in \bigoplus_{k=1}^{K} \mathrm{R}(A_k)$. Then there exist $f_1, \ldots, f_K \in \mathcal{H}_1$ such that $g = \sum_{k=1}^{K} A_k f_k = \sum_{k=1}^{K} A_k P_{\mathrm{R}\left(A_k^*\right)} f_k$. Let $f = \sum_{k=1}^{K} P_{\mathrm{R}\left(A_k^*\right)} f_k$. Using that $\mathrm{R}(A_k^*) \subseteq \mathrm{N}(A_{k'})$ for each $k, k' = 1, \ldots, K$, $k \neq k'$, we get

$$\left(\sum_{k=1}^{K} A_k\right) f = \left(\sum_{k=1}^{K} A_k\right) \left(\sum_{k=1}^{K} P_{\mathrm{R}\left(A_k^*\right)} f_k\right) = \sum_{k=1}^{K} A_k P_{\mathrm{R}\left(A_k^*\right)} f_k = g.$$

Then $g \in \mathrm{R}\Big(\sum_{k=1}^{K} A_k\Big)$. This sows that $\bigoplus_{k=1}^{K} \mathrm{R}(A_k) \subseteq \mathrm{R}\Big(\sum_{k=1}^{K} A_k\Big)$. Hence, $\mathrm{R}\Big(\sum_{k=1}^{K} A_k\Big) = \bigoplus_{k=1}^{K} \mathrm{R}(A_k)$. The equality $\mathrm{R}\Big(\sum_{k=1}^{K} A_k^*\Big) = \bigoplus_{k=1}^{K} \mathrm{R}(A_k^*)$ can be proved similarly. Therefore, (ii) holds.

Let $k_0 \in \{1, \ldots, K\}$. We have $f \in \mathrm{N}\Big(\sum_{k=1}^{K} A_k\Big)$ if and only if

$$A_{k_0} f = -\left(\sum_{k=1, k\neq k_0}^{K} A_k\right) f \in \mathrm{R}(A_{k_0}) \cap \mathrm{R}\left(\sum_{k=1, k\neq k_0}^{K} A_k\right) \subseteq \mathrm{R}(A_{k_0}) \cap \mathrm{N}\big(A_{k_0}^*\big) = \{0\}.$$

Thus, $f \in \mathrm{N}(A_{k_0})$ for each $k_0 \in \{1, \ldots, K\}$. Therefore, $\bigcap_{k=1}^{K} \mathrm{N}(A_k) = \mathrm{N}\Big(\sum_{k=1}^{K} A_k\Big)$. In a similar way, we prove that $\mathrm{N}\Big(\sum_{k=1}^{K} A_k^*\Big) = \bigcap_{k=1}^{K} \mathrm{N}(A_k^*)$. This shows (iii).

By (ii), $\mathrm{R}\Big(\sum_{k=1}^{K} A_k\Big)$ is closed. Thus, $\Big(\sum_{k=1}^{K} A_k\Big)^{\dagger}$ exists. Using part now (i), the rest of (iv) follows from Definition 1.1. ∎

From the definition of $\{1, 3, 4\}$-inverse and Theorem 2.1(i)(iv), we obtain the following result:

Corollary 2.2. *Let* $\{A_k\}_{k=1}^{K} \subset \mathcal{BC}(\mathcal{H}_1, \mathcal{H}_2)$. *Assume that* $\mathrm{R}(A_k) \subseteq \mathrm{N}(A_{k'}^*)$ *and* $\mathrm{R}(A_k^*) \subseteq \mathrm{N}(A_{k'})$ *for each* $k, k' = 1, \ldots, K$, $k \neq k'$. *Then* $\Big(\sum_{k=1}^{K} A_k\Big)^{\dagger}$ *is a* $\{1, 3, 4\}$*-inverse of* A_{k_0} *for each* $k_0 = 1, \ldots, K$.

From Theorem 2.1, we now derive expressions and characterizations for the Moore–Penrose inverse of an operator. They are useful for its computation and part of them can be viewed as an extension of Proposition 1.3 to a sum of operators with closed range. Before we enunciate the results, we note that, under the hypothesis of Theorem 2.1, we always have $\mathrm{R}\big(A_{k_0}^*\big) \subseteq \mathrm{N}\Big(\sum_{k=1, k\neq k_0}^{K} A_k\Big)$ and $\mathrm{R}\Big(\sum_{k=1, k\neq k_0}^{K} A_k\Big) \subseteq \mathrm{N}\big(A_{k_0}^*\big)$ for each $k_0 \in \{1, \ldots, K\}$.

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THEOREM 2.3. *Let* $\{A_k\}_{k=1}^K \subset \mathcal{BC}(\mathcal{H}_1,\mathcal{H}_2)$. *Assume that* $\mathrm{R}(A_k) \subseteq \mathrm{N}(A_{k'}^*)$ *and* $\mathrm{R}(A_k^*) \subseteq \mathrm{N}(A_{k'})$ *for each* $k,k' = 1,2,\ldots,$ $k \neq k'$. *Let* $k_0 \in \{1,\ldots,K\}$. *The following assertions hold:*

(i) $A_{k_0}^\dagger = \left(\sum_{k=1}^K A_k^*A_k\right)^\dagger \left(\sum_{k=1}^K A_k^*\right) - \sum_{k=1,k\neq k_0}^K A_k^\dagger$.

(ii) $\left(\sum_{k=1}^K A_k^*A_k\right) A_{k_0}^\dagger = A_{k_0}^*$ *and* $A_{k_0}^\dagger = \left(\sum_{k=1}^K A_k^*A_k\right)^\dagger A_{k_0}^*$.

(iii) $\mathrm{N}\left(\sum_{k=1}^K A_k\right) = \{0\}$ *if and only if* $\mathrm{R}(A_{k_0}^*) = \mathrm{N}\left(\sum_{k=1,k\neq k_0}^K A_k\right)$.

(iv) *If* $\mathrm{N}\left(\sum_{k=1}^K A_k\right) = \{0\}$, *then* $\sum_{k=1}^K A_k^*A_k$ *is invertible.*

Proof. (i): From Theorem 2.1(iv) and Proposition 1.3(i), we obtain

$$\begin{aligned}
A_{k_0}^\dagger &= \left(\sum_{k=1}^K A_k\right)^\dagger - \left(\sum_{k=1,k\neq k_0}^K A_k\right)^\dagger \\
&= \left(\left(\sum_{k=1}^K A_k^*\right)\left(\sum_{k=1}^K A_k\right)\right)^\dagger \left(\sum_{k=1}^K A_k^*\right) - \sum_{k=1,k\neq k_0}^K A_k^\dagger \\
&= \left(\sum_{k=1}^K A_k^*A_k\right)^\dagger \left(\sum_{k=1}^K A_k^*\right) - \sum_{k=1,k\neq k_0}^K A_k^\dagger.
\end{aligned}$$

(ii): Using Theorem 2.1(i) we get,

$$\left(\sum_{k=1}^K A_k^*A_k\right) A_{k_0}^\dagger = A_{k_0}^* A_{k_0} A_{k_0}^\dagger = A_{k_0}^* P_{\mathrm{R}\left(A_{k_0}\right)} = A_{k_0}^*.$$

From this equality,

$$\left(\sum_{k=1}^K A_k^*A_k\right)^\dagger \left(\sum_{k=1}^K A_k^*A_k\right) A_{k_0}^\dagger = \left(\sum_{k=1}^K A_k^*A_k\right)^\dagger A_{k_0}^*. \tag{2.1}$$

Since

$$\left(\sum_{k=1}^K A_k^*A_k\right)^\dagger \left(\sum_{k=1}^K A_k^*A_k\right) = P_{\mathrm{R}\left(\sum_{k=1}^K A_k^*A_k\right)},$$

and, by Theorem 2.1(ii),

$$\mathrm{R}\left(\sum_{k=1}^K A_k^*A_k\right) = \mathrm{R}\left(\left(\sum_{k=1}^K A_k^*\right)\left(\sum_{k=1}^K A_k\right)\right) = \mathrm{R}\left(\sum_{k=1}^K A_k^*\right) = \bigoplus_{k=1}^K \mathrm{R}(A_k^*) \supseteq \mathrm{R}\left(A_{k_0}^*\right) = \mathrm{R}\left(A_{k_0}^\dagger\right),$$

equality (2.1) becomes

$$A_{k_0}^\dagger = \left(\sum_{k=1}^K A_k^*A_k\right)^\dagger A_{k_0}^*.$$

(iii): Assume that $\mathrm{N}\left(\sum_{k=1}^K A_k\right) = \{0\}$. Let $f \in \mathrm{N}\left(\sum_{k=1,k\neq k_0}^K A_k\right)$. Since $f = P_{\mathrm{R}\left(A_{k_0}^*\right)} f + P_{\mathrm{N}\left(A_{k_0}\right)} f$ and $\mathrm{R}\left(A_{k_0}^*\right) \subseteq \mathrm{N}(A_k)$ for $k \neq k_0$,

$$\sum_{k=1}^K A_k P_{\mathrm{N}\left(A_{k_0}\right)} f = \sum_{k=1,k\neq k_0}^K A_k P_{\mathrm{N}\left(A_{k_0}\right)} f = \sum_{k=1,k\neq k_0}^K A_k f = 0.$$

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Consequently, $P_{\mathrm{N}(A_{k_0})}f = 0$ and $f \in \mathrm{R}\big(A_{k_0}^*\big)$. This shows that $\mathrm{N}\Big(\sum_{k=1,k\neq k_0}^K A_k\Big) = \mathrm{R}\big(A_{k_0}^*\big)$.

Assume now that $\mathrm{R}\big(A_{k_0}^*\big) = \mathrm{N}\Big(\sum_{k=1,k\neq k_0}^K A_k\Big)$. By Theorem 2.1(iii),

$$\mathrm{N}\left(\sum_{k=1}^K A_k\right) = \bigcap_{k=1}^K \mathrm{N}(A_k) = \mathrm{N}(A_{k_0}) \cap \mathrm{N}\left(\sum_{k=1,k\neq k_0}^K A_k\right) = \mathrm{N}(A_{k_0}) \cap \mathrm{R}\big(A_{k_0}^*\big) = \{0\}\,.$$

(iv): It follows from Proposition 1.3(ii). ∎

For future references, we enunciate the following straightforward corollary.

Corollary 2.4. *Let $\{A_k\}_{k=1}^K \subset \mathcal{BC}(\mathcal{H}_1,\mathcal{H}_2)$. Assume that $\mathrm{R}(A_k) \subseteq \mathrm{N}(A_{k'}^*)$ and $\mathrm{R}(A_k^*) \subseteq \mathrm{N}(A_{k'})$ for each $k,k' = 1,2,\ldots,$ $k \neq k'$. Let $k_0 \in \{1,\ldots,K\}$. If $\sum_{k=1}^K A_k$ is injective, then $\sum_{k=1}^K A_k^* A_k$ is invertible and $A_{k_0}^\dagger$ is the unique solution of the operator equation $\Big(\sum_{k=1}^K A_k^* A_k\Big) X = A_{k_0}^*$.*

The following theorem can be proved in a similar manner as was proved Theorem 2.3 or can be proved applying Theorem 2.3 to $\{A_k^*\}_{k=1}^K$ and then taking adjoints.

Theorem 2.5. *Let $\{A_k\}_{k=1}^K \subset \mathcal{BC}(\mathcal{H}_1,\mathcal{H}_2)$. Assume that $\mathrm{R}(A_k) \subseteq \mathrm{N}(A_{k'}^*)$ and $\mathrm{R}(A_k^*) \subseteq \mathrm{N}(A_{k'})$ for each $k,k' = 1,2,\ldots,$ $k \neq k'$. Let $k_0 \in \{1,\ldots,K\}$. The following assertions hold:*

- (i) $A_{k_0}^\dagger = \Big(\sum_{k=1}^K A_k^*\Big)\Big(\sum_{k=1}^K A_k A_k^*\Big)^\dagger - \sum_{k=1,k\neq k_0}^K A_k^\dagger$.
- (ii) $A_{k_0}^\dagger \Big(\sum_{k=1}^K A_k A_k^*\Big) = A_{k_0}^*$ *and* $A_{k_0}^\dagger = A_{k_0}^* \Big(\sum_{k=1}^K A_k A_k^*\Big)^\dagger$.
- (iii) $\mathrm{R}\Big(\sum_{k=1}^K A_k\Big) = \mathcal{H}_2$ *if and only if* $\mathrm{R}\Big(\sum_{k=1,k\neq k_0}^K A_k\Big) = \mathrm{N}\big(A_{k_0}^*\big)$.
- (iv) *If* $\mathrm{R}\Big(\sum_{k=1}^K A_k\Big) = \mathcal{H}_2$, *then* $\sum_{k=1}^K A_k A_k^*$ *is invertible.*

We have the following immediate and useful consequence.

Corollary 2.6. *Let $\{A_k\}_{k=1}^K \subset \mathcal{BC}(\mathcal{H}_1,\mathcal{H}_2)$. Assume that $\mathrm{R}(A_k) \subseteq \mathrm{N}(A_{k'}^*)$ and $\mathrm{R}(A_k^*) \subseteq \mathrm{N}(A_{k'})$ for each $k,k' = 1,2,\ldots,$ $k \neq k'$. Let $k_0 \in \{1,\ldots,K\}$. If $\sum_{k=1}^K A_k$ is surjective, then $\sum_{k=1}^K A_k A_k^*$ is invertible and $A_{k_0}^\dagger$ is the unique solution of the operator equation $X\Big(\sum_{k=1}^K A_k A_k^*\Big) = A_{k_0}^*$.*

The next theorem address the case $\sum_{k=1}^K A_k$ invertible.

Theorem 2.7. *Let $\{A_k\}_{k=1}^K \subset \mathcal{BC}(\mathcal{H}_1,\mathcal{H}_2)$. Assume that $\mathrm{R}(A_k) \subseteq \mathrm{N}(A_{k'}^*)$ and $\mathrm{R}(A_k^*) \subseteq \mathrm{N}(A_{k'})$ for each $k,k' = 1,2,\ldots,$ $k \neq k'$. Let $k_0 \in \{1,\ldots,K\}$. If $\sum_{k=1}^K A_k$ is invertible, then $A_{k_0}^\dagger$ is the solution of any of the equations $\Big(\sum_{k=1}^K A_k\Big) X = P_{\mathrm{N}\left(\sum_{k=1,k\neq k_0}^K A_k^*\right)}$, $X\Big(\sum_{k=1}^K A_k\Big) = P_{\mathrm{N}\left(\sum_{k=1,k\neq k_0}^K A_k\right)}$.*

Proof. Assume that $\sum_{k=1}^K A_k$ is invertible. From Theorem 2.1(iv),

$$A_{k_0}^\dagger = \left(\sum_{k=1}^K A_k\right)^\dagger - \sum_{k=1,k\neq k_0}^K A_k^\dagger = \left(\sum_{k=1}^K A_k\right)^{-1} - \sum_{k=1,k\neq k_0}^K A_k^\dagger.$$

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From here,

$$\begin{aligned}\left(\sum_{k=1}^{K} A_k\right) A_{k_0}^\dagger &= I_{\mathcal{H}_2} - \left(\sum_{k=1}^{K} A_k\right)\left(\sum_{k=1,k\neq k_0}^{K} A_k^\dagger\right)\\ &= I_{\mathcal{H}_2} - \sum_{k=1,k\neq k_0}^{K} A_k A_k^\dagger = I_{\mathcal{H}_2} - \sum_{k=1,k\neq k_0}^{K} P_{\mathrm{R}(A_k)}\\ &= I_{\mathcal{H}_2} - P_{\mathrm{R}\left(\sum_{k=1,k\neq k_0}^{K} A_k\right)} = P_{\mathrm{N}\left(\sum_{k=1,k\neq k_0}^{K} A_k^*\right)},\end{aligned}$$

and

$$\begin{aligned}A_{k_0}^\dagger \left(\sum_{k=1}^{K} A_k\right) &= I_{\mathcal{H}_1} - \left(\sum_{k=1,k\neq k_0}^{K} A_k^\dagger\right)\left(\sum_{k=1}^{K} A_k\right)\\ &= I_{\mathcal{H}_1} - \sum_{k=1,k\neq k_0}^{K} A_k^\dagger A_k = I_{\mathcal{H}_2} - \sum_{k=1,k\neq k_0}^{K} P_{\mathrm{R}\left(A_k^*\right)}\\ &= I_{\mathcal{H}_2} - P_{\mathrm{R}\left(\sum_{k=1,k\neq k_0}^{K} A_k^*\right)} = P_{\mathrm{N}\left(\sum_{k=1,k\neq k_0}^{K} A_k\right)}.\end{aligned}$$

□

It is important to note that in Definition 1.1 and in Theorem 1.2, $A^\dagger$ is characterized as the solution of systems of equations, or of operators equations where the solution must satisfy some restrictions on its null space or on its range space, whereas Corollary 2.4, Corollary 2.6 and Theorem 2.7 give characterizations of $A^\dagger$ as the solution of single equations.

The next result says that under hypotheses similar to the used for the previous characterizations, any series of operators, which converges in the operator norm, is a finite sum.

Proposition 2.8. *Let $\{A_k\}_{k=1}^\infty \subset \mathcal{BC}(\mathcal{H}_1,\mathcal{H}_2)$. Assume that $\mathrm{R}(A_k) \subseteq \mathrm{N}(A_{k'}^*)$ and $\mathrm{R}(A_k^*) \subseteq \mathrm{N}(A_{k'})$ for each $k,k' = 1,2,\ldots,$ $k \neq k'$. Assume that the series $\sum_{k=1}^\infty A_k$ converges in the operator norm. If $\mathrm{N}(\sum_{k=1}^\infty A_k) = \{0\}$ and $\mathrm{R}(\sum_{k=1}^\infty A_k)$ is closed, or $\mathrm{R}(\sum_{k=1}^\infty A_k) = \mathcal{H}_2$, then there exists $K_0 \in \mathbb{N}$ such that $A_k = 0$ for each $k > K_0$.*

Proof. Assume that $\mathrm{N}(\sum_{k=1}^\infty A_k) = \{0\}$ and $\mathrm{R}(\sum_{k=1}^\infty A_k)$ is closed. Then $(\sum_{k=1}^\infty A_k)^\dagger$ exists.

Let $K \in \mathbb{N}$. We have

$$\sum_{k=1}^{K} A_k = \sum_{k=1}^{\infty} A_k - \sum_{k=K+1}^{\infty} A_k. \tag{2.2}$$

Let $g \in \mathrm{R}\left(\sum_{k=K+1}^\infty A_k\right)$. There exists $f \in \mathcal{H}_1$ such that

$$g = \sum_{k=K+1}^{\infty} A_k f = \sum_{k=K+1}^{\infty} A_k P_{\overline{\mathrm{R}\left(\sum_{k=K+1}^\infty A_k^*\right)}} f.$$

Since $\overline{\mathrm{R}\left(\sum_{k=K+1}^\infty A_k^*\right)} \subseteq \mathrm{N}\left(\sum_{k=1}^K A_k\right)$,

$$g = \sum_{k=1}^{\infty} A_k P_{\overline{\mathrm{R}\left(\sum_{k=K+1}^\infty A_k^*\right)}} f.$$

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This shows that

$$\mathrm{R}\left(\sum_{k=K+1}^{\infty} A_k\right) \subseteq \mathrm{R}\left(\sum_{k=1}^{\infty} A_k\right), \tag{2.3}$$

for each $K \in \mathbb{N}$.

Let $K_0 \in \mathbb{N}$ be such that

$$\left\| \left(\sum_{k=1}^{\infty} A_k\right)^{\dagger} \sum_{k=K+1}^{\infty} A_k \right\| < 1,$$

for each $K \geq K_0$. By Theorem 2.1(ii), (2.2), (2.3) and [6, Lemma 3.3], if $K \geq K_0$, then

$$\bigoplus_{k=1}^{K} \mathrm{R}(A_k) = \mathrm{R}\left(\sum_{k=1}^{K} A_k\right) = \mathrm{R}\left(\sum_{k=1}^{\infty} A_k\right).$$

This implies that $\mathrm{R}(A_k) = \{0\}$ for each $k > K_0$.

Similarly, if $\mathrm{R}(\sum_{k=1}^{\infty} A_k) = \mathcal{H}_2$, we can apply the previous reasoning to $\{A_k^*\}_{k=1}^{\infty}$ to conclude that there exists $K_0 \in \mathbb{N}$ such that $A_k = 0$ for each $k > K_0$. ∎

3. Formulations for matrices. In this section, we specialize previous results for finite matrices and relate them with others in the literature. The elements of $\mathbb{F}^n$ will be consider as column vectors, and if $x \in \mathbb{F}^n$ then $x(i)$ denotes the ith component of x. The elements of the standard basis of $\mathbb{F}^n$ will be denoted by e_1, ..., e_n. We denote the vector with all its components equal to 1 with e. The set of $m \times n$ matrices over $\mathbb{F}$ is denoted by $\mathcal{M}_{m,n}$. If $m = n$ we write $\mathcal{M}_n$. If $A \in \mathcal{M}_{m,n}$, we denote the entry i, j, the ith row and the jth column of A with $A(i, j)$, $A(i, :)$ and $A(:, j)$, respectively. We note that if $x, y \in \mathbb{F}^n$, $x \neq 0$ and $y \neq 0$, then $(xy^*)^{\dagger} = \frac{1}{\|x\|^2\|y\|^2} yx^*$.

We begin noting that considering $\{A_k\}_{k=1}^{K} \subset \mathcal{M}_{m,n}$, Theorem 2.1(iv) is [4, Lemma 1.7]. For the case case $K = 2$, there are various papers that generalize [4, Lemma 1.7] considering weaker hypotheses. For example, if $A_1 A_2^* = 0$ and $C = \left(I - A_1 A_1^{\dagger}\right) A_2$, [11, Theorem 2] expresses $(A_1 + A_2)^{\dagger}$ in terms of A_1, A_2, A_1^*, A_2^*, C and their Moore–Penrose inverses. We can also mention [12, Theorem 1] which generalize [11, Theorem 2] and expresses $(A_1 + A_2)^{\dagger}$ in terms of A_1, A_2, A_1^*, A_2^*, other matrices and their Moore–Penrose inverses. In particular, by [13, Theorem 3], if $A_1, A_2 \in \mathcal{M}_n$ and $\operatorname{rank}(A_1 + A_2) = \operatorname{rank}(A_1) + \operatorname{rank}(A_2)$, then

$$(A_1 + A_2)^{\dagger} = \left(I - \left(P_{\mathrm{R}(A_2^*)} P_{\mathrm{N}(A_1)}\right)^{\dagger}\right) A_1^{\dagger} \left(I - \left(P_{\mathrm{N}(A_1^*)} P_{\mathrm{R}(A_2)}\right)^{\dagger}\right) + \left(P_{\mathrm{R}(A_2^*)} P_{\mathrm{N}(A_1)}\right)^{\dagger} A_2^{\dagger} \left(P_{\mathrm{N}(A_1^*)} P_{\mathrm{R}(A_2)}\right)^{\dagger}. \tag{3.1}$$

Equality (3.1) was proved for operators $A_1, A_2 \in \mathcal{BC}(\mathcal{H}_1, \mathcal{H}_2)$ such that $\mathrm{R}(A_1) \cap \mathrm{R}(A_2) = \mathrm{R}(A_1^*) \cap \mathrm{R}(A_2^*) = \{0\}$, $\mathrm{R}(A_1 + A_2) = \mathrm{R}(A_1) + \mathrm{R}(A_2)$ and $\mathrm{R}(A_1^* + A_2^*) = \mathrm{R}(A_1^*) + \mathrm{R}(A_2^*)$ (see [14, Theorem 5.2]). Note that if $A_1, A_2 \in \mathcal{BC}(\mathcal{H}_1, \mathcal{H}_2)$ satisfy the hypotheses of Theorem 2.1, they also satisfy the hypotheses of [14, Theorem 5.2] and from (3.1) we get $(A_1 + A_2)^{\dagger} = A_1^{\dagger} + A_2^{\dagger}$. In [15, 16] are considered sufficient conditions independent of the conditions of [4, Lemma 1.7] to have $(A_1 + A_2)^{\dagger} = A_1^{\dagger} + A_2^{\dagger}$ (see also Remark 3.6). For K arbitrary, we have [17, 18, 19] where $\left(\sum_{k=1}^{K} A_k\right)^{\dagger}$ is expressed in terms of the Moore–Penrose inverse of block circulant matrices.

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The following particular case of Theorem 2.1 permits us to compute $A^\dagger$ using the equality $A^\dagger = (A+B)^\dagger - B^\dagger$ having $B^\dagger$ an explicit expression.

THEOREM 3.1. *Let $A \in \mathcal{M}_{m,n}$, $r = \operatorname{rank}(A)$, $q = \min\{m,n\}$ and $r \le q' \le q$. Let $\{f_1, \ldots, f_{q'-r}\}$ be an orthonomal subset of $\mathrm{N}(A)$, $\{g_1, \ldots, g_{q'-r}\}$ be an orthonomal subset of $\mathrm{N}(A^*)$. Let $\{d_k\}_{k=1}^{q'-r} \subset \mathbb{F} \setminus \{0\}$. Then*

$$A^\dagger = \left(A + \sum_{k=1}^{q'-r} d_k g_k f_k^*\right)^\dagger - \sum_{k=1}^{q'-r} \frac{1}{d_k} f_k g_k^*. \tag{3.2}$$

As a consequence of Theorem 3.1, we get the following result which can be viewed as a particular case of Theorems 2.3(i) and 2.5(i):

COROLLARY 3.2. *Let $A \in \mathcal{M}_{m,n}$, $r = \operatorname{rank}(A)$ and $q = \min\{m,n\}$. Let $\{f_1, \ldots, f_{q-r}\}$ be an orthonomal subset of $\mathrm{N}(A)$ and $\{g_1, \ldots, g_{q-r}\}$ be an orthonomal subset of $\mathrm{N}(A^*)$. Let $\{d_k\}_{k=1}^{q-r} \subset \mathbb{F} \setminus \{0\}$. Then:*

(i) *If $m \ge n$, then $A^\dagger = (A^*A + \sum_{k=1}^{q-r} d_k^2 f_k f_k^*)^{-1}(A^* + \sum_{k=1}^{q-r} d_k f_k g_k^*) - \sum_{k=1}^{q-r} \frac{1}{d_k} f_k g_k^*$.*
(ii) *If $n \ge m$, then $A^\dagger = (A^* + \sum_{k=1}^{q-r} d_k f_k g_k^*)(AA^* + \sum_{k=1}^{q-r} d_k^2 g_k g_k^*)^{-1} - \sum_{k=1}^{q-r} \frac{1}{d_k} f_k g_k^*$.*
(iii) *If $m = n$, then $A^\dagger = (A + \sum_{k=1}^{q-r} d_k g_k f_k^*)^{-1} - \sum_{k=1}^{q-r} \frac{1}{d_k} f_k g_k^*$.*

Remark 3.3. From Corollary 3.2(iii), we obtain results that appear in the literature for the case $m = n$. First, we note that the result of Corollary 3.2(iii) appears in [7, Exercise 53]. The proof given there consists in a verification of the equality $\left(A + \sum_{k=1}^{q-r} d_k g_k f_k^*\right)\left(A^\dagger + \sum_{k=1}^{q-r} \frac{1}{d_k} f_k g_k^*\right) = I$.

If $A \in \mathcal{M}_n$ is a normal matrix and $d_k = 1$ for each $k = 1, \ldots, q-r$, from Corollary 3.2(iii) we obtain [20, Theorem 1] which is used to give expressions for the Moore–Penrose inverses of circulant matrices. The proof given in [20] is based on the spectral decomposition of A.

If $A \in \mathcal{M}_n$ is symmetric, $r = n-1$, $d_1 = \alpha n$, $\alpha \ne 0$ and $f_1 = g_1 = \frac{1}{\sqrt{n}} e \in \mathrm{N}(A)$, from Corollary 3.2(iii) we obtain [21, Theorem 2.1]. In [21], the authors first prove that $A + \alpha e e^t$ is nonsingular by showing that all its eigenvalues are nonzero. Then, they prove that $X = (A + \alpha e e^t)^{-1} - \frac{1}{\alpha n^2} e e^t$ verifies the equations in Definition 1.1 and consequently $X = A^\dagger$.

In the finite dimensional case we can get a result similar to Theorem 2.3, Theorem 2.5 and Theorem 2.7 but with weaker assumptions.

THEOREM 3.4. *Let $A, B \in \mathcal{M}_{m,n}$. Then:*

(i) *If $\mathrm{R}(B^*) = \mathrm{N}(A)$, then $A^*A + B^*B$ is invertible and $A^\dagger$ is the unique solution of the equation $(A^*A + B^*B)X = A^*$.*
(ii) *If $\mathrm{R}(B) = \mathrm{N}(A^*)$, then $AA^* + BB^*$ is invertible and $A^\dagger$ is the unique solution of the equation $X(AA^* + BB^*) = A^*$.*
(iii) *If $m = n$, $\mathrm{R}(B^*) = \mathrm{N}(A)$ and $\mathrm{R}(B) \subseteq \mathrm{N}(A^*)$ (or, $\mathrm{R}(B^*) \subseteq \mathrm{N}(A)$ and $\mathrm{R}(B) = \mathrm{N}(A^*)$), then $A + B$ is invertible, $A^\dagger = (A+B)^{-1} - B^\dagger$ and $A^\dagger$ is the unique solution of any of the equations $(A+B)X = P_{\mathrm{N}(B^*)}$ and $X(A+B) = P_{\mathrm{N}(B)}$.*

Proof. Parts (i) and (ii) follow from [39, Equalities (8) and (9)].

Assume that $m = n$. If $\mathrm{R}(B^*) = \mathrm{N}(A)$ and $\mathrm{R}(B) \subseteq \mathrm{N}(A^*)$, by Theorem 2.3(iii), $A + B$ is injective and hence invertible. If $\mathrm{R}(B^*) \subseteq \mathrm{N}(A)$ and $\mathrm{R}(B) = \mathrm{N}(A^*)$, by Theorem 2.5(iii), $A + B$ is surjective and hence invertible. By Theorem 2.1(iv), $A^\dagger = (A+B)^{-1} - B^\dagger$. The rest of part (iii) follows from Theorem 2.7. ∎

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Remark 3.5. In [39] appears the following variant of parts (i) and (ii) of the previous theorem. Let $A \in \mathcal{M}_{m,n}$ and $r = \operatorname{rank}(A)$. Let $V \in M_{n-r,n}$, $\operatorname{rank}(V) = n - r$ and $\mathrm{R}(V^*) = \mathrm{N}(A)$. Let $W \in M_{m,m-r}$, $\operatorname{rank}(W) = m - r$ and $\mathrm{R}(W) = \mathrm{N}(A^*)$. By [39, Theorem 3], $A^\dagger = (A^*A + V^*V)^{-1} A^* = A^* (AA^* + WW^*)^{-1}$.

Theorem 3 in [39] is used to obtain condensed Cramer rules for the minimal-norm least-squares solution $x = A^\dagger b$ of linear equations $Ax = b$ and to give condensed determinantal expressions for $A^\dagger$, $AA^\dagger$ and $A^\dagger A$. This theorem in [39] is proved given before an explicit expression for the $\{2\}$-inverse of A with range T and null space S (see [39, Theorem 2]). Then the expressions for $A^\dagger$ are obtained considering $T = \mathrm{R}(A^*)$ and $S = \mathrm{N}(A^*)$.

We finish this section showing that the sufficient conditions of [15, Theorem 3.2] (see also [16, Proposition 2.3]) and of Theorem 2.1 are not necessary. This will also show the relation of Theorem 2.1 with the singular value decomposition. We recall first a property. Let $A = V\Sigma W^*$ where $V \in \mathcal{M}_m$ and $W \in \mathcal{M}_n$ are unitary matrices, and

$$\Sigma = \begin{pmatrix} \Sigma_r & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{pmatrix} \in \mathcal{M}_{m,n}. \tag{3.3}$$

with $O_{k,l}$ a zero matrix in $\mathcal{M}_{k,l}$ and Σ_r a diagonal matrix in $\mathcal{M}_r$ with nonzero diagonal elements. Let $\Sigma^\dagger$ obtained from Σ by first replacing each nonzero element with its inverse and then transposing. Then $A^\dagger = W\Sigma^\dagger V^*$.

Remark 3.6. Let $A, B \in \mathcal{M}_n$. By [15, Theorem 3.2], if $AB^* + BB^* = 0$ and $B^*A + B^*B = 0$ (see [16] for details about these conditions), then $(A + B)^\dagger = A^\dagger + B^\dagger$. Now, if $\mathrm{R}(B^*) \subseteq \mathrm{N}(A)$, $\mathrm{R}(A^*) \subseteq \mathrm{N}(B)$, $\mathrm{R}(B) \subseteq \mathrm{N}(A^*)$, $\mathrm{R}(A) \subseteq \mathrm{N}(B^*)$, $BB^* \neq 0$ and $B^*B \neq 0$, then $AB^* + BB^* \neq 0$ and $B^*A + B^*B \neq 0$ and, by Theorem 2.1, $(A + B)^\dagger = A^\dagger + B^\dagger$. We are going to see that $A, B \in \mathcal{M}_n$ with the previous properties exist.

Let $A \in \mathcal{M}_n$ be arbitrary, $r = \operatorname{rank}(A)$, $q = \min\{n\}$ and $r \leq q' \leq q$. Let $A = V\Sigma W^*$ be a singular value decomposition of A where Σ is as in (3.3) and the diagonal elements $\sigma_1, \ldots, \sigma_r$ of Σ_r are the *singular values* of A (see e.g. [40, 7.3.P7]). Let $B = VEW^*$ where

$$E = \begin{pmatrix} O_{r,r} & O_{r,q'-r} & O_{r,n-q'} \\ O_{q'-r,r} & E_{q'-r} & O_{q'-r,n-q'} \\ O_{n-q',r} & O_{n-q',q'-r} & O_{n-q',n-q'} \end{pmatrix},$$

and $E_{q'-r}$ is a diagonal matrix in $M_{q'-r}$ with positive diagonal elements e_1, ..., $e_{q'-r}$. Then $\mathrm{R}(B^*) \subseteq \mathrm{N}(A)$, $\mathrm{R}(A^*) \subseteq \mathrm{N}(B)$, $\mathrm{R}(B) \subseteq \mathrm{N}(A^*)$, $\mathrm{R}(A) \subseteq \mathrm{N}(B^*)$, $BB^* \neq 0$ and $B^*B \neq 0$ and

$$(A + B)^\dagger = W (\Sigma + E)^\dagger V^* = W \left(\Sigma^\dagger + E^\dagger\right) V^* = W\Sigma^\dagger V^* + WE^\dagger V^* = A^\dagger + B^\dagger.$$

This shows that the sufficient conditions of [15, Theorem 3.2] are not necessary.

Remark 3.7. Let $\alpha, \beta, \gamma \in \mathbb{C} \setminus \{0\}$ be such that $(\alpha + \beta + \gamma)^{-1} = \alpha^{-1} + \beta^{-1} + \gamma^{-1}$ (e.g., $\alpha = \beta = -\gamma$). Let $V \in \mathcal{M}_m$ and $W \in \mathcal{M}_n$ be unitary matrices. Let Σ_α, Σ_β and Σ_γ as in (3.3) with the r nonzero entries equal to α, β and γ, respectively. Thus, $(\Sigma_\alpha + \Sigma_\beta + \Sigma_\gamma)^\dagger = \Sigma_\alpha^\dagger + \Sigma_\beta^\dagger + \Sigma_\gamma^\dagger$ and if $A = V\Sigma_\alpha W^*$, $B = V\Sigma_\beta W^*$ and $C = V\Sigma_\gamma W^*$, then $(A + B + C)^\dagger = A^\dagger + B^\dagger + C^\dagger$. In this case, $\mathrm{R}(A) = \mathrm{R}(B) = \mathrm{R}(C)$, $\mathrm{N}(A) = \mathrm{N}(B) = \mathrm{N}(C)$, $\mathrm{R}(A^*) = \mathrm{R}(B^*) = \mathrm{R}(C^*)$ and $\mathrm{N}(A^*) = \mathrm{N}(B^*) = \mathrm{N}(C^*)$. This example shows that the conditions of Theorem 2.1 are not necessary.

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Note that $(\alpha+\beta)^{-1} = \alpha^{-1}+\beta^{-1}$ if and only if $\alpha^2+\alpha\beta+\beta^2 = 0$, or equivalently, $\alpha = \left(\frac{-1\pm i\sqrt{3}}{2}\right)\beta$. Hence, α and β cannot be both real numbers. See [41, 42] for interesting details about the equality $(\alpha+\beta)^{-1} = \alpha^{-1}+\beta^{-1}$.

4. Moore–Penrose inverse of circulant matrices. In this section, we consider circulant matrices of order $n \geq 2$, $C = \operatorname{circ}(c)$ where $c \in \mathbb{C}^n$ (see [22, 23, 24]). For example, if $n = 2, 3$ we have,

$$C = \begin{pmatrix} c(1) & c(2) \\ c(2) & c(1) \end{pmatrix}, \quad C = \begin{pmatrix} c(1) & c(2) & c(3) \\ c(3) & c(1) & c(2) \\ c(2) & c(3) & c(1) \end{pmatrix}.$$

If $\Pi = \operatorname{circ}(0,1,0,\dots,0)$ then $\Pi^{-1} = \Pi^{n-1} = \Pi^t = \operatorname{circ}(0,\dots,0,1)$ and $C = \operatorname{circ}(c) = \sum_{k=0}^{n-1} c(k+1)\Pi^k$. If $\rho : \mathbb{C}^n \to \mathbb{C}^n$ is given by $(\rho(c))(1) = c(1)$ and $(\rho(c))(k) = c(n-k+2)$ for $k = 2,\dots,n$, then $C^t = \operatorname{circ}(\rho(c))$.

If $\operatorname{circ}(c) = \operatorname{circ}(a)\operatorname{circ}(b)$, then $\operatorname{circ}(c) = \operatorname{circ}(b)\operatorname{circ}(a)$ and

$$c(l) = \sum_{k=1}^{l} a(k)b(l-k+1) + \sum_{k=l+1}^{n} a(k)b(n+l-k+1) \tag{4.1}$$

for $l = 1,\dots,n$, or equivalently, $c = \operatorname{circ}(\rho(a))\, b$.

Remark 4.1. If $0 \leq l \leq n-1$ and C is a circulant matrix, then $\mathrm{N}\left(\Pi^l C\right) = \mathrm{N}(C)$ and, by the reverse-order law for the Moore–Penrose inverse (see, e.g., [7, Chapter 4, Ex. 22]), $\left(\Pi^l C\right)^\dagger = C^\dagger \Pi^{n-l}$.

In [29, 30], the coefficients of the inverse and the group inverse of a circulant matrix depending on up to four complex parameters, i.e., $\operatorname{circ}(a,b,c,d,\dots,d)$, are expressed in terms of functions $k_j(a,b,c,d)$, $j = 1,\dots,n$. In particular, in the case of four parameters, these functions involves Chebyshev polynomials. The group inverse of a circulant matrix coincides with its Moore–Penrose inverse. The techniques used in these papers are related with the solution of boundary value problems associated to second-order linear difference equations. Here we use results of the previous sections to obtain properties and explicit expressions of the Moore–Penrose inverse of circulant matrices.

Consider circulant matrices of the form $C = \alpha\Pi^{k-1} + \beta\Pi^k$ where $\alpha,\beta \in \mathbb{C}\setminus\{0\}$ and $1 \leq k < n$. Then $C = \Pi^{k-1}(\alpha I + \beta\Pi)$. Similarly, $C = \alpha I + \beta\Pi^{n-1} = \Pi^{n-1}(\beta I + \alpha\Pi)$. Hence, by Remark 4.1, in both cases, in order to study $C^\dagger$ it is sufficient to consider circulant matrices of the form $C = \alpha I + \beta\Pi$. Part (ii) of the following lema can be obtained from [27, Theorem 2.3] with parameters $a = 2$, $b = 0$, and $c = 1$. We include here a direct brief proof.

LEMMA 4.2. *The following assertions hold:*

(i) *If n even, $C_{1,1} = \operatorname{circ}(1,1,0,\dots,0)$, $v = (1,-1,1,-1,\dots,1,-1)$ and $w \in \mathbb{R}^n$ is given by*

$$w(k) = (-1)^{k+1}\left(n^2 - (2k-1)\,n + 2\right)$$

for $k = 1,\dots,n$, then $C_{1,1} + v * v^t = \operatorname{circ}(2,0,1,-1,\dots,1,-1)$ is invertible and

$$\left(C_{1,1} + v * v^t\right)^{-1} = \frac{1}{2n^2}\operatorname{circ}(w)\,.$$

(ii) *If $C_{1,-1} = \operatorname{circ}(1,-1,0,\dots,0)$, then $C_{1,-1} + ee^t = \operatorname{circ}(2,0,1,\dots,1)$ is invertible and*

$$\left(C_{1,-1} + ee^t\right)^{-1} = -\frac{1}{2n}\operatorname{circ}(1,3,5,\dots,2n-1) + \frac{n^2+2}{2n^2}ee^t.$$

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Proof. (i) Let $\operatorname{circ}(a) := (C_{1,1} + v * v^t) \operatorname{circ}(w)$. Using (4.1) we obtain,

$$\begin{aligned} a(1) &= 2\left(n^2 - n + 2\right) + \sum_{k=2}^{n-1} (-1)^{n-k+3} (-1)^{k+1}, \left(n^2 - (2k-1)\,n + 2\right) \\ &= 2\left(n^2 - n + 2\right) + (n-2)\left(n^2 + n + 2\right) - 2n\left(\frac{(n-1)n}{2} - 1\right) = 2n^2, \end{aligned}$$

and for $l = 2, \ldots, n$,

$$\begin{aligned} a(l) &= \sum_{k=1}^{l-2} (-1)^{l-k+2} (-1)^{k+1} \left(n^2 - (2k-1)\,n + 2\right) + 2\,(-1)^{l+1}\left(n^2 - (2l-1)\,n + 2\right) + \\ &\quad + \sum_{k=l+1}^{n} (-1)^{n+l-k+2} (-1)^{k+1} \left(n^2 - (2k-1)\,n + 2\right) \\ &= (-1)^{l+1}\left(\sum_{k=1}^{n}\left(n^2 - (2k-1)\,n + 2\right) + \left(n^2 - (2l-1)\,n + 2\right) - \left(n^2 - (2\,(l-1)-1)\,n + 2\right)\right) \\ &= (-1)^{l+1}\left(n\left(n^2 + n + 2\right) - 2n\frac{n(n+1)}{2} - 2n\right) = 0. \end{aligned}$$

This shows that $(C_{1,1} + v * v^t)^{-1} = \frac{1}{2n^2}\operatorname{circ}(w)$.

(ii) Let $B = -\frac{1}{2n}\operatorname{circ}(1, 3, 5, \ldots, 2n-1) + \frac{n^2+1}{2n^2} ee^t$. Using that $C_{1,-1}e = 0$ and (4.1), we obtain

$$\begin{aligned} \left(C_{1,-1} + ee^t\right) B &= -\frac{1}{2n} C_{1,-1}\operatorname{circ}(1, 3, 5, \ldots, 2n-1) - \frac{1}{2n}\sum_{k=1}^{n} (2k-1)\, ee^t + \frac{n^2+2}{2n} ee^t \\ &= -\frac{1}{2n}\operatorname{circ}(2-2n, 2, 2, \ldots, 2) + \frac{1}{n} ee^t = \operatorname{circ}(1, 0, \ldots, 0) = I. \end{aligned}$$

Hence $B = C^{-1}$. ∎

The following proposition can be derived from [29, Theorem 3.4.] considering parameters $a = \alpha$, $b = \beta$, and $c = 0$. Here, we present a short proof based on the previous lemma and Corollary 3.2(iii), giving the explicit expressions for the Moore–Penrose inverses.

Proposition 4.3. *Let $C = \alpha I + \beta \Pi$ with $\alpha, \beta \in \mathbb{C}\setminus\{0\}$. Then C is singular if and only if $\beta^n = (-1)^n \alpha^n$ and the following assertions hold:*

(i) *If n is even and $\alpha = \beta$, then $\mathrm{N}(C) = \operatorname{span}\{(1, -1, 1, -1, \ldots, 1, -1)\}$ and*

$$C^\dagger = \frac{1}{\alpha n^2}\left(\frac{1}{2}\operatorname{circ}(w) - \operatorname{circ}(1, -1, 1, -1, \ldots, 1, -1)\right),$$

where $w \in \mathbb{R}^n$ is given by $w(k) = (-1)^{k+1}\left(n^2 - (2k-1)\,n + 2\right)$ for $k = 1, \ldots, n$.

(ii) *If $\alpha = -\beta$, then $\mathrm{N}(C) = \operatorname{span}\{(1, 1, \ldots, 1)\}$ and*

$$\begin{aligned} C^\dagger &= \frac{1}{2\alpha}\left(-\frac{1}{n}\operatorname{circ}(1, 3, 5, \ldots, 2n-1) + ee^t\right) \\ &= \frac{1}{2nc_1}\operatorname{circ}(n-1, n-3, n-5, \ldots, 2-n, 1-n). \end{aligned}$$

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Proof. We have $Cx = 0$ if and only if $x(k+1) = (-1)^k \left(\frac{\alpha}{\beta}\right)^k x(1)$ for each $k = 1, \ldots, n-1$ and $x(n) = -\frac{\beta}{\alpha} x(1)$. Hence, C is singular if and only if $\beta^n = (-1)^n \alpha^n$.

(i) If n is even and $\alpha = \beta$, then $\mathrm{N}(C) = \operatorname{span}\{(1, -1, 1, -1, \ldots, 1, -1)\}$. Let $C_{1,1}$ and v as in Lemma 4.2. By Corollary 3.2(iii) and Lemma 4.2,

$$\begin{aligned} C^\dagger &= \frac{1}{\alpha} C_{1,1}^\dagger = \frac{1}{\alpha}\left(\left(C_{1,1} + vv^t\right)^{-1} - \frac{1}{\|v\|^2} vv^t\right) \\ &= \frac{1}{\alpha n^2}\left(\frac{1}{2}\operatorname{circ}(w) - \operatorname{circ}(1, -1, 1, -1, \ldots, 1, -1)\right). \end{aligned}$$

(ii) If $\alpha = -\beta$, then $\mathrm{N}(C) = \operatorname{span}\{(1, 1, \ldots, 1)\}$. Consider $C_{1,-1}$ as in Lemma 4.2. By Corollary 3.2(iii) and Lemma 4.2,

$$\begin{aligned} C^\dagger &= \frac{1}{\alpha} C_{1,-1}^\dagger = \frac{1}{\alpha}\left(\left(C_{1,-1} + ee^t\right)^{-1} - \frac{1}{n^2} ee^t\right) \\ &= \frac{1}{\alpha}\left(-\frac{1}{2n}\operatorname{circ}(1, 3, 5, \ldots, 2n-1) + \frac{n^2+2}{2n^2} ee^t - \frac{1}{n^2} ee^t\right) \\ &= \frac{1}{2\alpha}\left(-\frac{1}{n}\operatorname{circ}(1, 3, 5, \ldots, 2n-1) + ee^t\right). \end{aligned}$$

□

The Fourier matrix of order n denoted with F is given by $F(k,l) = \frac{1}{\sqrt{n}} e^{-\frac{2\pi i}{n}(k-1)(l-1)}$. If $C = \operatorname{circ}(c)$, then C is diagonalizable by F,

$$C = \overline{F} \Lambda F, \quad C^\dagger = \overline{F} \Lambda^\dagger F,$$

$$\lambda = \sqrt{n} F c \text{ and } c = \frac{1}{\sqrt{n}} F \lambda. \tag{4.2}$$

where λ is the diagonal of Λ. As a consequence of Theorem 2.1 and (4.2), we obtain the next theorem that provides a way to obtain the (Moore–Penrose) inverse of circulant matrices in terms of the (Moore–Penrose) inverses of other circulant matrices. The *support* of a vector c, denoted by $\operatorname{supp}(c)$, are the indices of the non-zero components of c.

THEOREM 4.4. *Let* $\{c_k\}_{k=1}^K \subseteq \mathbb{C}^n$, $\lambda_k = \sqrt{n}\overline{F} c_k$ *and* $\widetilde{\lambda_k} = \sqrt{n}\overline{F}\rho(c_k)$ *for each* $k = 1, \ldots, K$. *If* $\operatorname{supp}(\lambda_k) \cap \operatorname{supp}\left(\widetilde{\lambda_{k'}}\right) = \emptyset$ *for each* $k, k' = 1, \ldots, K$, $k \neq k'$, *then*

$$\left(\operatorname{circ}\left(\sum_{k=1}^K c_k\right)\right)^\dagger = \sum_{k=1}^K \left(\operatorname{circ}(c_k)\right)^\dagger.$$

Moreover, if $\bigcup_{k=1}^K \operatorname{supp}(\lambda_k) = \{1, \ldots, n\}$, *then* $\operatorname{circ}\left(\sum_{k=1}^K c_k\right)$ *is invertible.*

Example 4.5. Let n be even. Let $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Let $c_\beta = (0, \ldots, 0, \beta, \beta, 0, \ldots, 0)$ where β appears in the l-th position. Then

$$\lambda_\beta = \sqrt{n}\overline{F} c_\beta = \beta\left(e^{\frac{2\pi i}{n}(k-1)(l-1)} + e^{\frac{2\pi i}{n}(k-1)l}\right)_{k=1}^n = \beta\left(e^{\frac{2\pi i}{n}(k-1)l}\left(e^{-\frac{2\pi i}{n}(k-1)} + 1\right)\right)_{k=1}^n.$$

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Hence $\lambda_\beta(\frac{n}{2}+1)=0$ and $\lambda_\beta(k)\neq 0$ for $k\neq \frac{n}{2}+1$. If $\widetilde{\lambda_\alpha}=\alpha n e_{\frac{n}{2}+1}$, then

$$c_\alpha := \frac{1}{\sqrt{n}} F\widetilde{\lambda_\alpha} = \alpha\,(1,-1,\ldots,1,-1) = \rho(c_\alpha).$$

By Theorem 4.4,

$$(\operatorname{circ}(c_\alpha + c_\beta))^{-1} = (\operatorname{circ}(c_\alpha))^\dagger + (\operatorname{circ}(c_\beta))^\dagger .$$

If $c=(1,-1,\ldots,1,-1)$, then $(\operatorname{circ}(c_\alpha))^\dagger = \frac{1}{\alpha\|c\|^4} vv^t = \frac{1}{\alpha n^2}\operatorname{circ}(c)$. We also have

$$(\operatorname{circ}(c_\beta))^\dagger = \frac{1}{\beta}\,(\operatorname{circ}(1,1,0,\ldots,0))^\dagger\,\Pi^{n-l+1}$$

and the explicit expression of $(\operatorname{circ}(1,1,0,\ldots,0))^\dagger$ appears in Proposition 4.3(i).

We have $\operatorname{supp}\big(\sqrt{n}\overline{F}e\big)=\{1\}$, and if $a\in\mathbb{C}^n$ is such that $\sum_{j=1}^n a(j)=0$, then $1\notin\operatorname{supp}\big(\sqrt{n}\overline{F}a\big)$. Hence, by Theorem 4.4, we get the following proposition.

Proposition 4.6. *Let $c\in\mathbb{C}^n$. The following assertions hold:*

(i) *If $\sum_{j=1}^n c(j)\neq 0$, then*

$$\operatorname{circ}(c)^\dagger = \operatorname{circ}\left(c(1)-\frac{\sum_{j=1}^n c(j)}{n},\ldots,c(n)-\frac{\sum_{j=1}^n c(j)}{n}\right)^\dagger + \frac{1}{n\sum_{j=1}^n c(j)} ee^t.$$

(ii) *If $\sum_{j=1}^n c(j)=0$ and $\alpha\neq 0$, then*

$$\operatorname{circ}(c)^\dagger = \operatorname{circ}(c(1)+\alpha,\ldots,c(n)+\alpha)^\dagger - \frac{1}{n^2\alpha} ee^t.$$

By the previous proposition, it is very easy to obtain the Moore–Penrose inverse in the case $\sum_{j=1}^n c(j)\neq 0$ from the case $\sum_{j=1}^n c(j)=0$, and vice versa. As a consequence, in order to give explicit expressions, there is no need to separately consider each of the two cases. This fact will simplify future researchers in the subject.

Part (ii) of the following proposition can be used to compute explicitly and easily the Moore–Penrose inverse of

$$\operatorname{circ}\left(a,\underbrace{b,\ldots,b}_{k},a,\underbrace{b,\ldots,b}_{k},\ldots,a,\underbrace{b,\ldots,b}_{k}\right)$$

for each $a,b\in\mathbb{C}$ choosing $\alpha=\frac{a+kb}{k+1}$ and $\beta=\frac{a-b}{k+1}$.

Proposition 4.7. *Let k and q be positive integers and $n=q(k+1)$. The following assertions hold:*

(i) *If $C=\operatorname{circ}\left(k,\underbrace{-1,\ldots,-1}_{k},\ldots,k,\underbrace{-1,\ldots,-1}_{k}\right)$, then $C^2=nC$ and $C^\dagger=\frac{1}{n^2}C$.*

(ii) *If $\alpha,\beta\in\mathbb{C}\setminus\{0\}$, then*

$$\operatorname{circ}\left(\alpha+k\beta,\underbrace{\alpha-\beta,\ldots,\alpha-\beta}_{k},\ldots,\alpha+k\beta,\underbrace{\alpha-\beta,\ldots,\alpha-\beta}_{k}\right)^\dagger = \frac{1}{\alpha n^2}ee^t + \frac{1}{\beta n^2}C.$$

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Proof. For $j = 1, k+2, 2k+3, 3k+4, \ldots, n-k$,

$$C^2(1,j) = qk^2 + qk = qk(k+1) = nk = nC(1,j).$$

Otherwise,

$$C^2(1,j) = 2q(-k) + n - 2q = n - 2q(k+1) = n - 2n = -n = nC(1,j).$$

Hence, $C^2 = nC$. The rest of part (i) follows from this equality and Definition 1.1.

Taking into account that

$$\operatorname{circ}\left(\alpha + k\beta, \underbrace{\alpha-\beta, \ldots, \alpha-\beta}_{k}, \ldots, \alpha + k\beta, \underbrace{\alpha-\beta, \ldots, \alpha-\beta}_{k}\right) = \alpha e e^t + \beta C,$$

part (ii) follows from Proposition 4.6. □

5. Moore–Penrose inverse of distance matrices of certain graphs. In this section, we give applications to distance matrices of certain graphs. Specifically, we consider distance matrices of weighted trees and of wheel graphs with an odd number of vertices.

5.1. Moore–Penrose inverse of the distance matrix of a weighted tree. Let $T = (V, E)$ denotes a *weighted tree* with the set of *vertices* $V = \{1, \ldots, n\}$ and the set E of unordered pairs *edges* (i,j), $i \neq j$. To each $i, j \in V$ is assigned a *weight* $w_{ij} \neq 0$ if $i \neq j$ and (i,j) is an edge of T. If $i \neq j$ and (i,j) is not an edge of T then $w_{ij} = 0$. The *Laplacian matrix* L of T is the $n \times n$ positive definitive matrix given by $L(i,j) = -w_{ij}^{-1}$ if $i \neq j$ and (i,j) is an edge of T, $L(i,j) = 0$ if $i \neq j$ and (i,j) is not an edge of T, and $L(i,i) = -\sum_{j \neq i} L(i,j)$. The *distance matrix* D of T is the matrix with $D(i,j)$ equal to the *distance* between vertices i and j, defined to be the sum of the weights of the edges on the (unique) ij-path. We set $D(i,i) = 0, i = 1, \ldots, n$. We denote the degree of the vertex i by $\delta(i)$, $i = 1, \ldots, n$. Let δ be the vector with components $\delta(1), \ldots, \delta(n)$. We set $\tau = 2e - \delta$.

Assume that $\sum_{j=1}^{n-1} w_j = 0$ and that all the weights are nonzero. By [35, Theorem 11],

$$D^\dagger = -\frac{1}{2}L + u\tau^t + \tau u^t, \tag{5.1}$$

where

$$u = \frac{1}{2}\left(D^\dagger e + \frac{e^t D^\dagger e}{4}\tau\right). \tag{5.2}$$

From the proof of [35, Theorem 11], $\mathrm{N}(D) = \operatorname{span}\{\tau\}$. Thus, by Corollary 3.2(iii),

$$D^\dagger = \left(D + \alpha\tau\tau^t\right)^{-1} - \frac{1}{\alpha\left\|\tau\right\|^4}\tau\tau^t. \tag{5.3}$$

or equivalently, multiplying both sides from the left by $D + \alpha\tau\tau^t$, $D^\dagger$ is the unique solution of the equation

$$\left(D + \alpha\tau\tau^t\right)X = I - \frac{1}{\left\|\tau\right\|^2}\tau\tau^t. \tag{5.4}$$

By Corollary 2.2, the invertible matrix $D + \alpha\tau\tau^t$ is a $\{1,3,4\}$-inverse of D. Expressions (5.3) and (5.4) are alternatives for (5.1) to obtain $D^\dagger$. In order to compute u in (5.1), we note that from (5.3),

$$D^\dagger e = \left(D + \alpha\tau\tau^t\right)^{-1} e - \frac{2}{\alpha\left\|\tau\right\|^4}\tau, \tag{5.5}$$

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and from (5.4), $D^\dagger e$ is the unique solution of the equation
$$(D+\alpha\tau\tau^t)\,x = e - \tfrac{2}{\|\tau\|^2}\tau,$$
where $\alpha \neq 0$. If $\tau^t L\tau = \delta^t L\delta \neq 0$, choosing α appropriately, from (5.5) we get an expression of u in (5.2) that do not involve $D^\dagger$.

PROPOSITION 5.1. *Let T be a weighted tree with distance matrix D, Laplacian matrix L, degrees given by δ and $\tau = 2e-\delta$. Assume that $\sum_{j=1}^{n-1} w_j = 0$ and that all the weights are nonzero. If $\tau^t L\tau \neq 0$, then the vector u in (5.2) is given by*

$$u = \frac{1}{2}\left(\frac{1}{\|\tau\|^2}L\tau - \frac{3\tau^t L\tau}{2\,\|\tau\|^4}\tau\right). \tag{5.6}$$

Proof. By [35, Lemma 9], $DL = e\tau^t - 2I$. Hence,
$$\left(D+\frac{2}{\tau^t L\tau}\tau\tau^t\right)\left(\frac{1}{\|\tau\|^2}L\tau\right) = e.$$
From (5.5) with $\alpha = \frac{2}{\tau^t L\tau}$ and the previous equality we obtain,
$$D^\dagger e = \frac{1}{\|\tau\|^2}L\tau - \frac{\tau^t L\tau}{\|\tau\|^4}\tau.$$
Replacing this expression for $D^\dagger e$ in (5.2), we get (5.6). ∎

5.2. Moore–Penrose inverses of distance matrices of wheel graphs with an odd number of vertices. Let $n \geq 5$ be an odd integer. Let $W(n)$ be the wheel graph having n number of vertices, with the center labeled 1, the other vertices labeled 2, ..., n lie in a cycle of length $n-1$, and $(i,i+1)$ is an edge. Without loss of generality, we fix this labeling because any other labeling of $W(n)$ leads to a distance matrix which is a permutation similar to D. If $u = (0,1,2,\ldots,2,1) \in \mathbb{R}^{n-1}$, then the distance matrix D of the wheel graph is given by
$$D = \begin{pmatrix} 0 & e^t \\ e & \mathrm{circ}(u) \end{pmatrix}.$$
The following proposition is about $\mathrm{null}(D)$ and $D^\dagger$.

PROPOSITION 5.2. *Let $n \geq 5$ odd. Let D be the distance matrix of the wheel graph with n vertices and let $a = (0,-1,1,\ldots,-1,1) \in \mathbb{R}^n$. Then* $\mathrm{null}(D) = \mathrm{span}\{a\}$ *and*

$$D^\dagger = \left(D+aa^t\right)^{-1} - \frac{1}{(n-1)^2}aa^t. \tag{5.7}$$

Proof. Clearly, $\mathrm{span}\{a\} \subseteq \mathrm{null}(D)$. Let $x \in \mathrm{null}(D)$. Since $x^t D(1,:) = 0$, $\sum_{j=2}^n x(j) = 0$. We have $\sum_{i=1}^n D(i,:) = (n-1, 3+2(n-3),\ldots,3+2(n-3))$. Since $x^t D(i,:) = 0$ for $i = 2,\ldots,n$,

$$x(2) = -\frac{x(n)+x(3)}{2}, \tag{5.8}$$

$$x(j) = -\frac{x(j-1)+x(j+1)}{2},\ \text{for } j = 3,\ldots,n-1, \tag{5.9}$$

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$$x(n) = -\frac{x(n-1)+x(2)}{2}, \tag{5.10}$$

and

$$0 = x^t \left(\textstyle\sum_{i=1}^n D(i,:)\right) = (n-1)\,x(1) + (3+2(n-3)) \textstyle\sum_{j=2}^n x(j) = (n-1)\,x(1).$$

From the last equality, $x(1)=0$, and from (5.8)-(5.9), $x(j) = (-1)^j\,((j-1)\,x(2) + (j-2)\,x(n))$ for $j = 3,\ldots,n$. In particular, $x(n) = -\left((n-1)\,x(2) + (n-2)\,x(n)\right)$. Therefore, $x(n) = -x(2)$ and, more generally, $x(j) = (-1)^j\,x(2)$ for $j = 3,\ldots,n$. This shows that $\operatorname{null}(D) = \operatorname{span}\{a\}$. Now, from Corollary 3.2(iii), we obtain (5.7). ◻

In [36], the equality

$$D^{\dagger} = -\frac{1}{2}\widetilde{L} + \frac{4}{n-1}ww^t \tag{5.11}$$

where $\widetilde{L}$ is a semidefinite positive matrix, $\operatorname{rank}\left(\widetilde{L}\right) = n-2$, $\widetilde{L}e = 0$ and $w = \frac{1}{4}\,(5-n,1,\ldots,1)$, is obtained. The matrix $\widetilde{L}$ can be viewed as a special Laplacian matrix. Equating (5.7) and (5.11) we get

$$\left(D + aa^t\right)^{-1} = -\frac{1}{2}\widetilde{L} + \frac{4}{n-1}ww^t + \frac{1}{(n-1)^2}aa^t. \tag{5.12}$$

One expression of $(D+aa^t)^{-1}$ can be derived from (5.12) and the expression of $\widetilde{L}$ given in [36, Definition 1]. Next, we give a closed-form expression of each entry of $(D+aa^t)^{-1}$ and hence, of each entry of $D^{\dagger}$, based on numerical experiments. The principal result is Theorem 5.6. We establish before three auxiliary lemmas. The first of them is about sums. The other two lemmas are about the entries of $(D+aa^t)^{-1}$ and describe properties of them.

Lemma 5.3. *Let $n \geq 5$ odd. The following equalities hold:*

(i) *If $n = 5+4m$ for some $m \in \mathbb{N}$, $m \geq 0$, then*

$$\sum_{k=1,k\,\text{even}}^{(n-3)/2} k = \frac{(n-5)\,(n-1)}{16} = \frac{n^2-6n+5}{16},$$

$$\sum_{k=1,k\,\text{odd}}^{(n-3)/2} k = \frac{(n-1)^2}{16} = \frac{n^2-2n+1}{16},$$

$$\sum_{k=1,k\,\text{even}}^{(n-3)/2} k^2 = \frac{(n-5)\,(n-1)\,(n-3)}{48} = \frac{n^3-9n^2+23n-15}{48}$$

and

$$\sum_{k=1,k\,\text{odd}}^{(n-3)/2} k^2 = \frac{(n-1)\,(n-3)\,(n+1)}{48} = \frac{n^3-3n^2-n+3}{48}.$$

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(ii) *If $n = 7 + 4m$ for some $m \in \mathbb{N}$, $m \geq 0$, then*

$$\sum_{k=2, k \text{ even}}^{(n-3)/2} k = \frac{(n-3)(n+1)}{16} = \frac{n^2 - 2n - 3}{16},$$

$$\sum_{k=1, k \text{ odd}}^{(n-3)/2} k = \frac{(n-3)^2}{16} = \frac{n^2 - 6n + 9}{16},$$

$$\sum_{k=1, k \text{ even}}^{(n-3)/2} k^2 = \frac{(n-3)(n+1)(n-1)}{48} = \frac{n^3 - 3n^2 - n + 3}{48}$$

and

$$\sum_{k=1, k \text{ odd}}^{(n-3)/2} k^2 = \frac{(n-3)(n-1)(n-5)}{48} = \frac{n^3 - 9n^2 + 23n - 15}{48}.$$

Proof. (i): Assume that $n = 5 + 4m$ for some $m \in \mathbb{N}$, $m \geq 0$. We have

$$\begin{aligned}\sum_{k=2, k \text{ even}}^{(n-3)/2} k &= \sum_{k=1, k \text{ even}}^{2m+1} k = \sum_{k=1}^{m} 2k = 2\frac{m(m+1)}{2} \\ &= \frac{n-5}{4}\left(\frac{n-5}{4} + 1\right) = \frac{(n-5)(n-1)}{16} = \frac{n^2 - 6n + 5}{16},\end{aligned}$$

$$\begin{aligned}\sum_{k=1, k \text{ odd}}^{(n-3)/2} k &= \sum_{k=1, k \text{ odd}}^{2m+1} k = \sum_{k=1}^{m+1} (2k-1) = 2\frac{(m+1)(m+2)}{2} - (m+1) \\ &= \left(\frac{n-5}{4} + 1\right)\left(\frac{n-5}{4} + 2\right) - \left(\frac{n-5}{4} + 1\right) = \frac{(n-1)^2}{16} = \frac{n^2 - 2n + 1}{16},\end{aligned}$$

$$\begin{aligned}\sum_{k=2, k \text{ even}}^{(n-3)/2} k^2 &= \sum_{k=1, k \text{ even}}^{2m+1} k^2 = \sum_{k=1}^{m} (2k)^2 = 4\frac{m(m+1)(2m+1)}{6} \\ &= \frac{2}{3}\frac{n-5}{4}\left(\frac{n-5}{4} + 1\right)\left(\frac{n-5}{2} + 1\right) = \frac{(n-5)(n-1)(n-3)}{48} = \frac{n^3 - 9n^2 + 23n - 15}{48}\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1, k \text{ odd}}^{(n-3)/2} k^2 &= \sum_{k=1, k \text{ odd}}^{2m+1} k^2 = \sum_{k=1}^{m+1} (2k-1)^2 = 4\sum_{k=1}^{m+1} k^2 - 4\sum_{k=1}^{m+1} k + m + 1 \\ &= 4\frac{(m+1)(m+2)(2(m+1)+1)}{6} - 4\frac{(m+1)(m+2)}{2} + m + 1 \\ &= (m+1)\left((m+2)\frac{4}{3}m + 1\right) = \left(\frac{n-5}{4} + 1\right)\left(\left(\frac{n-5}{4} + 2\right)\frac{n-5}{3} + 1\right) \\ &= \frac{(n-1)(n-3)(n+1)}{48} = \frac{n^3 - 3n^2 - n + 3}{48}.\end{aligned}$$

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(ii): Assume now that $n = 7 + 4m$ for some $m \in \mathbb{N}$, $m \geq 0$. Then

$$\sum_{k=1, k\,\mathrm{even}}^{(n-3)/2} k = \sum_{k=1, k\,\mathrm{even}}^{2m+2} k = 2\sum_{k=1}^{m+1} k = (m+1)(m+2)$$
$$= \left(\frac{n-7}{4}+1\right)\left(\frac{n-7}{4}+2\right) = \frac{(n-3)(n+1)}{16} = \frac{n^2-2n-3}{16},$$

$$\sum_{k=1, k\,\mathrm{odd}}^{(n-3)/2} k = \sum_{k=1, k\,\mathrm{odd}}^{2m+2} k = \sum_{k=1}^{m+1}(2k-1)$$
$$= 2\frac{(m+1)(m+2)}{2} - (m+1) = (m+1)^2$$
$$= \left(\frac{n-7}{4}+1\right)^2 = \frac{(n-3)^2}{16} = \frac{n^2-6n+9}{16},$$

$$\sum_{k=1, k\,\mathrm{even}}^{(n-3)/2} k^2 = \sum_{k=1, k\,\mathrm{even}}^{2m+2} k^2 = \sum_{k=1}^{m+1}(2k)^2 = 4\sum_{k=1}^{m+1} k^2$$
$$= 4\frac{(m+1)(m+2)(2(m+1)+1)}{6} = \frac{2}{3}\left(\frac{n-7}{4}+1\right)\left(\frac{n-7}{4}+2\right)\left(\frac{n-7}{2}+3\right)$$
$$= \frac{(n-3)(n+1)(n-1)}{48} = \frac{n^3-3n^2-n+3}{48},$$

and

$$\sum_{k=1, k\,\mathrm{odd}}^{(n-3)/2} k^2 = \sum_{k=1, k\,\mathrm{odd}}^{2m+2} k^2 = \sum_{k=1}^{m+1}(2k-1)^2 = 4\sum_{k=1}^{m+1} k^2 - 4\sum_{k=1}^{m+1} k + m + 1$$
$$= 4\frac{(m+1)(m+2)(2(m+1)+1)}{6} - 4\frac{(m+1)(m+2)}{2} + m + 1$$
$$= \frac{n^3-9n^2+23n-15}{48} = \frac{(n-3)(n-1)(n-5)}{48}. \qquad \square$$

The vector z of the following two lemmas will appear in the expression of $D^\dagger$ given in Theorem 5.6 below.

Lemma 5.4. *Let* $z = (z(0), \ldots, z(n-2))$ *where*

$$z(0) = \frac{-n^3+3n^2+n+9}{12},$$

$$z(k) = z(n-1-k) = \begin{cases} \frac{-6(n-1)k^2+6(n-1)^2k-n^3+3n^2+n+9}{12}, & 1 \leq k \leq \frac{n-3}{2},\ k\ \textit{even}, \\ \frac{6(n-1)k^2-6(n-1)^2k+n^3-3n^2+5n-15}{12}, & 1 \leq k \leq \frac{n-3}{2},\ k\ \textit{odd}, \end{cases}$$

and

$$z((n-1)/2) = \begin{cases} \frac{n^3-3n^2+11n+15}{24}, & n = 5+4m,\ m = 0, 1, \ldots, \\ \frac{-n^3+3n^2+n-27}{24}, & n = 7+4m,\ m = 0, 1, \ldots. \end{cases}$$

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Then

$$\sum_{k=1,k\,\text{even}}^{(n-3)/2} z(k) = \begin{cases} \frac{n^3-3n^2-n-45}{48}, & n=5+4m,\ m=0,1,\ldots, \\ \frac{2n^3-6n^2+10n-30}{48}, & n=7+4m,\ m=0,1,\ldots, \end{cases} \tag{5.13}$$

$$\sum_{k=1,k\,\text{odd}}^{(n-3)/2} z(k) = \begin{cases} -\frac{n-1}{4}, & n=5+4m,\ m=0,1,\ldots, \\ \frac{n^3-3n^2-13n+39}{48}, & n=7+4m,\ m=0,1,\ldots, \end{cases} \tag{5.14}$$

$$\sum_{k=0,k\,\text{even}}^{n-2} z(k) = -\sum_{k=0,k\,\text{odd}}^{n-2} z(k) = \frac{n-1}{2}, \tag{5.15}$$

$$2\sum_{l=2,l\,\text{even}}^{n-3} z(l) - z(1) - z(n-2) = (n-1)(n-2), \tag{5.16}$$

$$2\sum_{l=1,l\,\text{odd}}^{n-4} z(l) - z(0) - z(n-3) = -(n-1), \tag{5.17}$$

$$2z(k) + z(k-1) + z(k+1) = 2(n-1) \ \textit{for } 2 \le k \le n-3 \textit{ and } k \textit{ even}, \tag{5.18}$$

$$2\sum_{l=0,l\,\text{even},l\neq k}^{n-2} z(l) - z(k-1) - z(k+1) = -(n-1) \ \textit{for } 2 \le k \le n-3 \textit{ and } k \textit{ even}, \tag{5.19}$$

$$2z(k) + z(k-1) + z(k+1) = 0 \ \textit{for } 1 \le k \le n-3 \textit{ and } k \textit{ odd}, \tag{5.20}$$

and

$$2\sum_{l=0,l\,\text{odd},l\neq k}^{n-2} z(l) - z(k-1) - z(k+1) = -(n-1) \ \textit{for } 1 \le k \le n-3 \textit{ and } k \textit{ odd}. \tag{5.21}$$

Proof. We first note that if $n = 5+4m$ for some $m \geq 0$, then $\frac{n-3}{2}$ is odd and $\frac{n-1}{2}$ is even, whereas if $n = 7+4m$ for some $m \geq 0$, then $\frac{n-3}{2}$ is even and $\frac{n-1}{2}$ is odd.

Assume that $n = 5+4m$ for some $m \geq 0$. By Lemma 5.3(i),

$$\begin{aligned} \sum_{k=1,k\,\text{even}}^{(n-3)/2} z(k) =& \frac{1}{12}\left(-6(n-1)\sum_{k=1,k\,\text{even}}^{(n-3)/2} k^2 + 6\left(n^2-2n+1\right)\sum_{k=1,k\,\text{even}}^{(n-3)/2} k + \left(-n^3+3n^2+n+9\right)\frac{n-5}{4}\right) \\ =& \frac{1}{12}\left(-6(n-1)\frac{n^3-9n^2+23n-15}{48} + 6\left(n^2-2n+1\right)\frac{n^2-6n+5}{16}\right) \\ &+ \frac{1}{12}\left(\left(-n^3+3n^2+n+9\right)\frac{n-5}{4}\right) \\ =& \frac{n^3-3n^2-n-45}{48} \end{aligned} \tag{5.22}$$

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and

$$\begin{aligned}\sum_{k=1,k\,\mathrm{odd}}^{(n-3)/2} z(k) =& \frac{-1}{12}\left(-6\left(n-1\right)\sum_{k=1,k\,\mathrm{odd}}^{(n-3)/2} k^2+6\left(n^2-2n+1\right)\sum_{k=1,k\,\mathrm{odd}}^{(n-3)/2} k+\left(-n^3+3n^2-5n+15\right)\frac{n-1}{4}\right)\\ =&\frac{-1}{12}\left(-6\left(n-1\right)\frac{n^3-3n^2-n+3}{48}+6\left(n^2-2n+1\right)\frac{n^2-2n+1}{16}\right)\\ &+\frac{-1}{12}\left(\left(-n^3+3n^2-5n+15\right)\frac{n-1}{4}\right)\\ =&-\frac{n-1}{4}. \end{aligned} \tag{5.23}$$

Using (5.22) and (5.23) we get

$$\begin{aligned}\sum_{k=0,k\,\mathrm{even}}^{n-2} z(k) &= z(0)+\sum_{k=1,k\,\mathrm{even}}^{(n-3)/2} z(k)+z((n-1)/2)+\sum_{k=(n+1)/2,k\,\mathrm{even}}^{n-2} z(k)\\ &=\frac{-n^3+3n^2+n+9}{12}+\frac{n^3-3n^2-n-45}{24}+\frac{n^3-3n^2+11n+15}{24}\\ &=\frac{n-1}{2},\end{aligned}$$

and

$$\sum_{k=0,k\,\mathrm{odd}}^{n-2} z(k)=\sum_{k=1,k\,\mathrm{odd}}^{(n-3)/2} z(k)+\sum_{k=(n+1)/2,k\,\mathrm{odd}}^{n-2} z(k)=-\frac{n-1}{2}.$$

Assume now that $n=7+4m$ for some $m\geq 0$. By Lemma 5.3(ii),

$$\begin{aligned}\sum_{k=1,k\,\mathrm{even}}^{(n-3)/2} z(k) =&\frac{1}{12}\left(-6\left(n-1\right)\sum_{k=1,k\,\mathrm{even}}^{(n-3)/2} k^2+6\left(n^2-2n+1\right)\sum_{k=1,k\,\mathrm{even}}^{(n-3)/2} k+\left(-n^3+3n^2+n+9\right)\frac{n-3}{4}\right)\\ =&\frac{1}{12}\left(-6\left(n-1\right)\frac{n^3-3n^2-n+3}{48}+6\left(n^2-2n+1\right)\frac{n^2-2n-3}{16}\right)\\ &+\frac{1}{12}\left(\left(-n^3+3n^2+n+9\right)\frac{n-3}{4}\right)\\ =&\frac{2n^3-6n^2+10n-30}{48}, \end{aligned} \tag{5.24}$$

and

$$\begin{aligned}\sum_{k=1,k\,\mathrm{odd}}^{(n-3)/2} z(k) =&\frac{-1}{12}\left(-6\left(n-1\right)\sum_{k=1,k\,\mathrm{odd}}^{(n-3)/2} k^2+6\left(n^2-2n+1\right)\sum_{k=1,k\,\mathrm{odd}}^{(n-3)/2} k+\left(-n^3+3n^2-5n+15\right)\frac{n-3}{4}\right)\\ =&\frac{1}{12}\left(-6\left(n-1\right)\frac{n^3-9n^2+23n-15}{48}+6\left(n^2-2n+1\right)\frac{n^2-6n+9}{16}\right)\\ &+\frac{1}{12}\left(\left(-n^3+3n^2-5n+15\right)\frac{n-3}{4}\right)\\ =&\frac{n^3-3n^2-13n+39}{48}. \end{aligned} \tag{5.25}$$

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From (5.24) and (5.25) we obtain,

$$\begin{aligned}\sum_{k=0,k\text{ even}}^{n-2} z(k) &= z(0) + \sum_{k=1,k\text{ even}}^{(n-3)/2} z(k) + \sum_{k=(n+1)/2,k\text{ even}}^{n-2} z(k)\\ &= \frac{-n^3+3n^2+n+9}{12} + \frac{2n^3-6n^2+10n-30}{24} = \frac{n-1}{2},\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0,k\text{ odd}}^{n-2} z(k) &= \sum_{k=1,k\text{ odd}}^{(n-3)/2} z(k) + z((n-1)/2) + \sum_{k=(n+1)/2,k\text{ odd}}^{n-2} z(k)\\ &= \frac{n^3-3n^2-13n+39}{24} + \frac{-n^3+3n^2+n-27}{24} = -\frac{n-1}{2}.\end{aligned}$$

Hence, we have proved (5.13), (5.14) and (5.15).

For each $n \geq 5$ odd, using (5.15) we obtain (5.16) and (5.17)

$$\begin{aligned}2\sum_{l=2,l\text{ even}}^{n-3} z(l) - z(1) - z(n-2) &= 2\sum_{l=0,l\text{ even}}^{n-3} z(l) - 2z(0) - z(1) - z(n-2)\\ &= 2\sum_{l=0,l\text{ even}}^{n-3} z(l) - 2z(0) - 2z(1)\\ &= n-1 - \frac{-n^3+3n^2+n+9}{6} - \frac{n^3-9n^2+23n-27}{6}\\ &= (n-1)(n-2),\end{aligned}$$

and

$$\begin{aligned}2\sum_{l=1,l\text{ odd}}^{n-4} z(l) - z(0) - z(n-3) =& 2\sum_{l=1,l\text{ odd}}^{n-2} z(l) - 2z(n-2) - z(0) - z(n-3)\\ &= 2\sum_{l=1,l\text{ odd}}^{n-2} z(l) - 2z(1) - z(0) - z(2)\\ =& 1-n-2\frac{6(n-1)-6(n-1)^2+n^3-3n^2+5n-15}{12}\\ &- \frac{-n^3+3n^2+n+9}{12} - \frac{-24(n-1)+12(n-1)^2-n^3+3n^2+n+9}{12}\\ &= -(n-1).\end{aligned}$$

Assume that $2 \leq k \leq n-3$ and k even. We note that $k-1$, $k+1$, $n-k-2$, and $n-k$ are odd and $n-k-1$ is even. If $k+1 < \frac{n-1}{2}$, then

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$$\begin{aligned}2z(k)+z(k-1)+z(k+1)=&2\frac{-6\left(n-1\right)k^2+6\left(n-1\right)^2k-n^3+3n^2+n+9}{12}\\&+\frac{6\left(n-1\right)\left(k-1\right)^2-6\left(n-1\right)^2\left(k-1\right)+n^3-3n^2+5n-15}{12}\\&+\frac{6\left(n-1\right)\left(k+1\right)^2-6\left(n-1\right)^2\left(k+1\right)+n^3-3n^2+5n-15}{12}\\&=2\left(n-1\right).\end{aligned}$$

If $k-1>\frac{n-1}{2}$, then

$$\begin{aligned}2z(k)+z(k-1)+z(k+1)=&2z(n-k-1)+z(n-k)+z(n-k-2)\\=&\frac{-12\left(n-1\right)\left(n-k-1\right)^2+12\left(n-1\right)^2\left(n-k-1\right)-2n^3+6n^2+2n+18}{12}\\&+\frac{6\left(n-1\right)\left(n-k\right)^2-6\left(n-1\right)^2\left(n-k\right)+n^3-3n^2+5n-15}{12}\\&+\frac{6\left(n-1\right)\left(n-k-2\right)^2-6\left(n-1\right)^2\left(n-k-2\right)+n^3-3n^2+5n-15}{12}\\=&2\left(n-1\right).\end{aligned}$$

If $n=5+4m$ for some $m\geq 0$ and $k=\frac{n-1}{2}$, then

$$\begin{aligned}2z(k)+z(k-1)+z(k+1)=&2z((n-1)/2)+z((n-3)/2)+z((n-3)/2+1)\\=&2\left(z((n-1)/2)+z((n-3)/2)\right)\\=&\frac{n^3-3n^2+11n+15}{12}\\&+\frac{3\left(n-1\right)\left(n-3\right)^2-6\left(n-1\right)^2\left(n-3\right)+2n^3-6n^2+10n-30}{12}\\=&2\left(n-1\right).\end{aligned}$$

If $n=7+4m$ for some $m\geq 0$ and $k+1=\frac{n-1}{2}$, then

$$\begin{aligned}2z(k)+z(k-1)+z(k+1)=&2z((n-3)/2)+z((n-5)/2)+z((n-1)/2)\\=&2\frac{-6\left(n-1\right)\left(\frac{n-3}{2}\right)^2+6\left(n-1\right)^2\frac{n-3}{2}-n^3+3n^2+n+9}{12}\\&+\frac{6\left(n-1\right)\left(\frac{n-5}{2}\right)^2-6\left(n-1\right)^2\frac{n-5}{2}+n^3-3n^2+5n-15}{12}\\&+\frac{-n^3+3n^2+n-27}{24}\\=&2\left(n-1\right).\end{aligned}$$

If $n=7+4m$ for some $m\geq 0$ and $k-1=\frac{n-1}{2}$, using the above equality we get

$$\begin{aligned}2z(k)+z(k-1)+z(k+1)=&2z((n+1)/2)+z((n-1)/2)+z((n+3)/2)\\=&2z((n-3)/2)+z((n-1)/2)+z((n-5)/2)=2\left(n-1\right).\end{aligned}$$

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Thus, (5.18) is proved. Using (5.15) and (5.18) we obtain (5.19):

$$\begin{aligned}2\sum_{l=0,l\text{ even},l\neq k}^{n-2} z(l)-z(k-1)-z(k+1) &= 2\sum_{l=0,l\text{ even}}^{n-2} z(l)-2z(k)-z(k-1)-z(k+1)\\ &= n-1-2\left(n-1\right)=-\left(n-1\right).\end{aligned}$$

Assume that $1\leq k\leq n-3$, and k odd. We note that $k-1$, $k+1$, $n-k-2$, and $n-k$ are even, and $n-k-1$ is odd. If $k+1<\frac{n-1}{2}$, then

$$\begin{aligned}2z(k)+z(k-1)+z(k+1) =&2\frac{6\left(n-1\right)k^2-6\left(n-1\right)^2k+n^3-3n^2+5n-15}{12}\\ &+\frac{-6\left(n-1\right)\left(k-1\right)^2+6\left(n-1\right)^2\left(k-1\right)-n^3+3n^2+n+9}{12}\\ &+\frac{-6\left(n-1\right)\left(k+1\right)^2+6\left(n-1\right)^2\left(k+1\right)-n^3+3n^2+n+9}{12}\\ =&0.\end{aligned}$$

If $k-1>\frac{n-1}{2}$, then

$$\begin{aligned}2z(k)+z(k-1)+z(k+1) =&2z(n-k-1)+z(n-k)+z(n-k-2)\\ =&2\frac{6\left(n-1\right)\left(n-k-1\right)^2-6\left(n-1\right)^2\left(n-k-1\right)+n^3-3n^2+5n-15}{12}\\ &+\frac{-6\left(n-1\right)\left(n-k\right)^2+6\left(n-1\right)^2\left(n-k\right)-n^3+3n^2+n+9}{12}\\ &+\frac{-6\left(n-1\right)\left(n-k-2\right)^2+6\left(n-1\right)^2\left(n-k-2\right)-n^3+3n^2+n+9}{12}\\ =&0.\end{aligned}$$

If $n=5+4m$ for some $m\geq 0$ and $k+1=\frac{n-1}{2}$, then

$$\begin{aligned}2z(k)+z(k-1)+z(k+1) =&2z((n-3)/2)+z((n-5)/2)+z((n-1)/2)\\ =&2\frac{6\left(n-1\right)\left(\frac{n-3}{2}\right)^2-6\left(n-1\right)^2\left(\frac{n-3}{2}\right)+n^3-3n^2+5n-15}{12}\\ &+\frac{-6\left(n-1\right)\left(\frac{n-5}{2}\right)^2+6\left(n-1\right)^2\left(\frac{n-5}{2}\right)-n^3+3n^2+n+9}{12}\\ &+\frac{n^3-3n^2+11n+15}{24}\\ =&0.\end{aligned}$$

If $n=5+4m$ for some $m\geq 0$ and $k-1=\frac{n-1}{2}$, using the previous equality

$$\begin{aligned}2z(k)+z(k-1)+z(k+1) =&2z((n-2)/2+1)+z((n-1)/2)+z((n-1)/2+2)\\ =&2z(n-(n-1)/2-2)+z((n-1)/2)+z(n-(n-1)/2-3)\\ =&2z((n-3)/2)+z((n-1)/2)+z((n-5)/2)=0.\end{aligned}$$

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If $n = 7 + 4m$ for some $m \geq 0$ and $k = \frac{n-1}{2}$, then

$$\begin{aligned}2z(k) + z(k-1) + z(k+1) =& 2z((n-1)/2) + z((n-3)/2) + z((n-2)/2+1)\\ =& 2z((n-1)/2) + 2z((n-3)/2)\\ =& 2\frac{-n^3+3n^2+n-27}{24}\\ &+ 2\frac{-6(n-1)\left(\frac{n-3}{2}\right)^2 + 6(n-1)^2\left(\frac{n-3}{2}\right) - n^3 + 3n^2 + n + 9}{12}\\ =& 0.\end{aligned}$$

This proves (5.20). From (5.15) and (5.20) we get (5.21):

$$\begin{aligned}2\sum_{l=0, l\,\text{odd}, l\neq k}^{n-2} z(l) - z(k-1) - z(k+1) &= 2\sum_{l=0, l\,\text{odd}}^{n-2} z(l) - 2z(k) - z(k-1) - z(k+1)\\ &= 2\left(-\frac{n-1}{2}\right) - 0 = -(n-1).\end{aligned}$$ ∎

Lemma 5.5. *Let z as in Lemma 5.4. Then*

$$e^t \mathrm{circ}(z) = 0,$$

and

$$\mathrm{circ}(1,0,3,1,\ldots,3,1,3,0)\mathrm{circ}(z) = (n-1)^2 I - (n-1)ee^t.$$

Proof. Using Lemma 5.4,

$$e^t\mathrm{circ}(z) = \sum_{k=0}^{n-2} z(k) = \sum_{k=0, k\,\text{even}}^{n-2} z(k) + \sum_{k=0, k\,\text{odd}}^{n-2} z(k) = \frac{n-1}{2} - \frac{n-1}{2} = 0.$$

For the other equality we have,

$$\begin{aligned}&\mathrm{circ}(1,0,3,1,\ldots,3,1,3,0)\mathrm{circ}(z) =\\ &= \left(\Pi_{n-1}^0 + 3\left(\Pi_{n-1}^2 + \Pi_{n-1}^4 + \ldots + \Pi_{n-1}^{n-5} + \Pi_{n-1}^{n-3}\right) + \left(\Pi_{n-1}^3 + \Pi_{n-1}^5 + \ldots + \Pi_{n-1}^{n-6} + \Pi_{n-1}^{n-4}\right)\right)\\ &\quad\left(\sum_{k=0}^{n-2} z(k)\Pi_{n-1}^k\right)\\ &= \left(ee^t + 2\left(\Pi_{n-1}^2 + \Pi_{n-1}^4 + \ldots + \Pi_{n-1}^{n-5} + \Pi_{n-1}^{n-3}\right) - \Pi_{n-1}^1 - \Pi_{n-1}^{n-2}\right)\left(\sum_{k=0}^{n-2} z(k)\Pi_{n-1}^k\right)\\ &= 2\sum_{k=0}^{n-2} z(k) \sum_{k'=2, k',\text{even}}^{n-3} \Pi_{n-1}^{k+k'} - \sum_{k=0}^{n-2} z(k)\Pi_{n-1}^{k+1} - \sum_{k=0}^{n-2} z(k)\Pi_{n-1}^{n+k-2}\\ &= \left(2\sum_{l=2, l\,\text{even}}^{n-3} z(l)\right)\Pi_{n-1}^0 + \sum_{k=2, k\,\text{even}}^{n-3}\left(2\sum_{l=0, l\,\text{even}, l\neq k}^{n-2} z(l)\right)\Pi_{n-1}^k\\ &\quad + \sum_{k=1, k\,\text{odd}}^{n-3}\left(2\sum_{l=0, l\,\text{odd}, l\neq k}^{n-2} z(l)\right)\Pi_{n-1}^k + \left(2\sum_{l=1, l\,\text{odd}}^{n-4} z(l)\right)\Pi_{n-1}^{n-2}\end{aligned}$$

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$$
\begin{aligned}
&- \left(z(n-2)\Pi_{n-1}^{0} + \sum_{k=1}^{n-2} z(k-1)\Pi_{n-1}^{k} \right) - \left(\sum_{k=0}^{n-3} z(k+1)\Pi_{n-1}^{k} + z(0)\Pi_{n-1}^{n-2} \right) \\
= &\left(2 \sum_{l=2, l \text{ even}}^{n-3} z(l) - z(1) - z(n-2) \right) \Pi_{n-1}^{0} \\
&+ \sum_{k=2, k \text{ even}}^{n-3} \left(2 \sum_{l=0, l \text{ even}, l \neq k}^{n-2} z(l) - z(k-1) - z(k+1) \right) \Pi_{n-1}^{k} \\
&+ \sum_{k=1, k \text{ odd}}^{n-3} \left(2 \sum_{l=0, l \text{ odd}, l \neq k}^{n-2} z(l) - z(k-1) - z(k+1) \right) \Pi_{n-1}^{k} \\
&+ \left(2 \sum_{l=1, l \text{ odd}}^{n-4} z(l) - z(0) - z(n-3) \right) \Pi_{n-1}^{n-2}.
\end{aligned}
$$

Now, applying Lemma 5.4, we obtain

$$
\operatorname{circ}(1,0,3,1,\ldots,3,1,3,0)\operatorname{circ}(z) = (n-1)(n-2)I - (n-1)\sum_{k=1}^{n-2} \Pi_{n-1}^{k} = (n-1)^2 I - (n-1)ee^t.
$$

∎

Now we are in position to give our expression for $D^\dagger$ using the vector z of Lemma 5.4.

Theorem 5.6. *Let $n \geq 5$ odd. Let D be the distance matrix of the wheel graph with n vertices. Let $a = (0,-1,1,\ldots,-1,1) \in \mathbb{R}^n$ and z as in Lemma 5.4. Let $v = (1,-1,\ldots,1,-1) \in \mathbb{R}^{n-1}$. Then*

$$
\left(D + aa^t\right)^{-1} = \frac{1}{(n-1)^2} \begin{pmatrix} -2(n-1)(n-3) & (n-1)e^t \\ (n-1)e & \operatorname{circ}(z) \end{pmatrix}, \tag{5.26}
$$

and

$$
D^\dagger = \frac{1}{(n-1)^2} \begin{pmatrix} -2(n-1)(n-3) & (n-1)e^t \\ (n-1)e & \operatorname{circ}(z+v) \end{pmatrix}. \tag{5.27}
$$

Proof. We have

$$
aa^t = \begin{pmatrix} 0 & 0 \\ 0 & \operatorname{circ}(v) \end{pmatrix}, \tag{5.28}
$$

and

$$
D + aa^t = \begin{pmatrix} 0 & e^t \\ e & \operatorname{circ}(u+v) \end{pmatrix} = \begin{pmatrix} 0 & e^t \\ e & \operatorname{circ}(1,0,3,1,\ldots,3,1,3,0) \end{pmatrix}.
$$

Let

$$
X = \begin{pmatrix} -2(n-1)(n-3) & (n-1)e^t \\ (n-1)e & \operatorname{circ}(z) \end{pmatrix}.
$$

Since

$$
\left(D + aa^t\right) X = \begin{pmatrix} (n-1)^2 & e^t\operatorname{circ}(z) \\ 0 & (n-1)ee^t + \operatorname{circ}(1,0,3,1,\ldots,3,1,3,0)\operatorname{circ}(z) \end{pmatrix},
$$

applying Lemma 5.5, we get $\left(D + aa^t\right) X = (n-1)^2 I$. This shows (5.26). By Proposition 5.2, (5.26) and (5.28), we obtain (5.27). ∎

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We end this section with properties of $(D + aa^t)^{-1}$, $D^{\dagger}$, and the matrix $\widetilde{L}$ of (5.11).

PROPOSITION 5.7. *Let $n \geq 5$ odd. Let D be the distance matrix of the wheel graph with n vertices. Let $a = (0, -1, 1, \ldots, -1, 1) \in \mathbb{R}^n$. Let $w = \frac{1}{4}(5 - n, 1, \ldots, 1)$ and $\widetilde{L}$ be such that*

$$D^{\dagger} = -\frac{1}{2}\widetilde{L} + \frac{4}{n-1}ww^t.$$

Then

$$\left(D + aa^t\right)^{-1} a = \frac{1}{n-1}a, \tag{5.29}$$

$$D^{\dagger} = \left(D + aa^t\right)^{-1}\left(I - \frac{1}{n-1}aa^t\right), \tag{5.30}$$

$$\widetilde{L} = -2\left(\left(D + aa^t\right)^{-1} - \frac{4}{n-1}ww^t\right)\left(I - \frac{1}{n-1}aa^t\right), \tag{5.31}$$

and $(D + aa^t)^{-1} - \frac{4}{n-1}ww^t$ is negative semidefinite on $\mathrm{R}(D)$.

Proof. By the equality $(D + aa^t)\, a = n - 1a$, we get (5.29). Equality (5.30) is a consequence of (5.29) and (5.7). Using that $w^t a = 0$, from (5.11) and (5.30), (5.31) follows.

Since $\mathrm{N}(D) = \operatorname{span}\{a\}$, $P_{\mathrm{R}(D)}x = I - \frac{1}{n-1}aa^t$. By [36, Theorem 2], $\widetilde{L}$ is positive semidefinite. Hence, by (5.31) and noting that $\mathrm{R}\left(\widetilde{L}\right) \subseteq \mathrm{R}(D)$,

$$x^t\widetilde{L}x = -2\left(P_{\mathrm{R}(D)}x\right)^t\left(\left(D + aa^t\right)^{-1} - \frac{4}{n-1}ww^t\right)P_{\mathrm{R}(D)}x,$$

and we conclude that $(D + aa^t)^{-1} - \frac{4}{n-1}ww^t$ is negative semidefinite on $\mathrm{R}(D)$. ∎

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